MONODROMY AT INFINITY AND FOURIER TRANSFORM II

by

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Abstract. — For a regular twistor $\mathscr{D}$-module and for a given function f, we compare the nearby cycles at $f = \infty$ and the nearby or vanishing cycles at $\tau = 0$ for its partial Fourier-Laplace transform relative to the kernel $e^{-\tau f}$.

Contents

1. Introduction

The regular polarizable twistor $\mathscr{D}$-modules on a complex manifold form a category generalizing that of polarized Hodge $\mathscr{D}$-modules, introduced by M. Saito in [**6**]. This category, together with some of its properties, has been considered in [**3**]. A potential application is to produce a category playing the role, in complex algebraic geometry, of pure perverse ℓ-adic sheaves with wild ramification, that is, a category enabling meromorphic connections with *irregular* singularities together with a notion of weight, compatible with various functors as direct images by projective morphisms or nearby/vanishing cycles.

A way to obtain irregular singularity from a regular $\mathscr{D}$-module is to apply the functor that we call partial Laplace transform.

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In [**3**, Appendix], we have sketched some results concerning the behaviour of regular twistor $\mathscr{D}$-modules with respect to a partial Fourier-Laplace transform. We then have extensively used such results in [**2**] and [**5**]. In this article, we give details for the proof of the results which are not proved in [**3**, Appendix]. The proofs yet appeared in a preprint form in [**4**, Chap. 8]. As indicated in [**3**, Appendix], the goal is to analyze the behaviour of polarized regular twistor $\mathscr{D}$-modules under a partial (one-dimensional) Fourier-Laplace transform. We generalize to such objects the main result of [**1**], comparing, for a given function f, the nearby cycles at $f=\infty$ and the nearby or vanishing cycles for the partial Fourier-Laplace transform in the f-direction (Theorem 5.1).

A remark concerning the terminology. — We use the term (partial) *Laplace transform* when we consider the transform for $\mathscr{D}$-modules (or $\mathscr{R}$-modules). The effect of such a transform on a sesquilinear pairing is an ordinary Fourier transform. On a twistor object, consisting of a pair of $\mathscr{R}$-modules and a sesquilinear pairing between them with values in distributions, the corresponding transform is called *Fourier-Laplace.*

2. A quick review of polarizable twistor $\mathscr{D}$-modules

Let us quickly review some basic definitions and results concerning polarizable twistor $\mathscr{D}$-modules. We refer to [**3**] for details.

2.a. Some notation. — We denote by Ω_0 the complex line with coordinate z, and by $\mathbf{S}$ the unit circle $|z|=1$. In fact, one could also take for Ω_0 any open neighbourhood of the closed unit disc $\mathbf{D}=\{z\in\Omega_0 \mid |z|\leqslant 1\}$. For any $z_o\in\Omega_0$, we put

- $\zeta_o=\operatorname{Im} z_o$,
- $\ell_{z_o}:\mathbb{C}\to\mathbb{R}$ the function $(\alpha'+i\alpha'')\mapsto\alpha'-(\operatorname{Im} z_o)\alpha''$,
- $\alpha\star z_o=\alpha' z_o+i\alpha''(z_o^2+1)/2$.

(See [**3**, Chap. 0] for more notation and definitions.)

2.b. The category $\mathscr{R}$- $\mathrm{Triples}(X)$. — Given a n-dimensional complex manifold X, we denote by $\mathscr{X}$ the manifold $X\times\Omega_0$, by $\mathscr{O}_{\mathscr{X}}$ its structure sheaf, and by $\mathscr{R}_{\mathscr{X}}$ the sheaf of differential operators defined in local coordinates $x_1,\dots,x_n$ as $\mathscr{O}_{\mathscr{X}}\langle\eth_{x_1},\dots,\eth_{x_n}\rangle$, where we put $\eth_{x_i}=z\partial_{x_i}$.

A module over $\mathscr{O}_{\mathscr{X}}$ or $\mathscr{R}_{\mathscr{X}}$ is said to be *strict* if it has no $\mathscr{O}_{\Omega_0}$-torsion.

The objects of the category $\mathscr{R}$- $\mathrm{Triples}(X)$ are the triples $\mathscr{T}=(\mathscr{M}',\mathscr{M}'',C)$, where $\mathscr{M}',\mathscr{M}''$ are left $\mathscr{R}_{\mathscr{X}}$-modules and $C:\mathscr{M}'_{|\mathbf{S}}\otimes_{\mathscr{O}_{\mathbf{S}}}\overline{\mathscr{M}''_{|\mathbf{S}}}\to\mathfrak{Db}_{X\times\mathbf{S}/\mathbf{S}}$ is a sesquilinear pairing. Here, $\mathscr{O}_{\mathbf{S}}$ means $\mathscr{O}_{\Omega_0|\mathbf{S}}$, $\mathfrak{Db}_{X\times\mathbf{S}/\mathbf{S}}$ is the sheaf of distributions on $X\times\mathbf{S}$ which are continuous with respect to $z\in\mathbf{S}$, and the conjugation is taken in the twistor sense (*cf.* [**3**, § 1.5.a]): it is the usual conjugation functor in the X direction, and is the involution $z\mapsto -z^{-1}$ in the z-direction.

The morphisms are pairs (φ', φ'') of morphisms, contravariant on the "prime" side, and covariant on the "double-prime" side, which satisfy the compatibility relation $C_1(\varphi' m_2', \overline{m_1''}) = C_2(m_2', \overline{\varphi'' m_1''})$.

For any $k \in \frac{1}{2}\mathbb{Z}$, the Tate twist (k) is defined by $\mathscr{T}(k) = (\mathscr{M}', \mathscr{M}'', (iz)^{-2k}C)$, and the adjoint $\mathscr{T}^*$ of $\mathscr{T}$ is $(\mathscr{M}'', \mathscr{M}', C^*)$, with $C^*(m'', \overline{m'}) = \overline{C(m', \overline{m''})}$

A sesquilinear duality $\mathscr{S}$ of weight $w \in \mathbb{Z}$ on $\mathscr{T}$ is a morphism $\mathscr{S} : \mathscr{T} \to \mathscr{T}^*(-w)$.

There is a natural notion of direct image by a morphism f between smooth complex manifolds, which is denoted by $f_\dagger$.

2.c. Specialization along a smooth hypersurface. — We consider the following situation: the manifold X is an open set in the product $\mathbb{C} \times X'$ of the complex line with some complex manifold X', we regard the coordinate t on $\mathbb{C}$ as a function on X, and we put $X_0 = t^{-1}(0)$. There is a corresponding derivation $\eth_t$, and $\mathscr{R}_{\mathscr{X}}$ is equipped with an increasing filtration $V_\bullet\mathscr{R}_{\mathscr{X}}$, for which $\eth_t^k$ has degree k, t^k has degree $-k$ (for any $k \in \mathbb{N}$), and any local section of $\mathscr{R}_{\mathscr{X}_0}$ has degree 0.

A coherent left $\mathscr{R}_{\mathscr{X}}$-module $\mathscr{M}$ is said to be strictly specializable along $\mathscr{X}_0$ if there exist, near any $(x_o, z_o) \in \mathscr{X}$, a finite set $A \subset \mathbb{C}$ and a good V-filtration indexed by $\ell_{z_o}(A + \mathbb{Z}) \subset \mathbb{R}$, denoted by $V_\bullet^{(z_o)}\mathscr{M}$, such that, for any $a \in \ell_{z_o}(A + \mathbb{Z})$,

– each graded piece $\mathrm{gr}_a^{V^{(z_o)}}\mathscr{M}$ is a strict $\mathscr{R}_{\mathscr{X}_0}$-module;

– on each $\mathrm{gr}_a^{V^{(z_o)}}\mathscr{M}$, the operator $\eth_t t$ has a minimal polynomial which takes the form

$$\prod_{\substack{\alpha \in A+\mathbb{Z} \\ \ell_{z_o}(\alpha) = a}} [-(s + \alpha \star z)]^{\nu_\alpha},$$

where the integers ν_α only depend on $\alpha \bmod \mathbb{Z}$;

– if we denote by $\psi_{t,\alpha}\mathscr{M}$ the kernel of a sufficiently large power of $\eth_t t + \alpha \star z$ acting on $\mathrm{gr}_a^{V^{(z_o)}}\mathscr{M}$, with $a = \ell_{z_o}(\alpha)$, then

- $t : \psi_{t,\alpha}\mathscr{M} \to \psi_{t,\alpha-1}\mathscr{M}$ is onto for $\ell_{z_o}(\alpha) \leqslant 0$,
- $\eth_t : \psi_{t,\alpha}\mathscr{M} \to \psi_{t,\alpha+1}\mathscr{M}$ is onto for $\ell_{z_o}(\alpha) \geqslant -1$, but $\alpha \neq -1$.

We say that the strictly specializable module $\mathscr{M}$ is regular along $\mathscr{X}_0$ if each $V_a^{(z_o)}\mathscr{M}$ is $\mathscr{R}_{\mathscr{X}/\mathbb{C}}$-coherent (*cf.* [**3**, § 3.1.d]).

Given an object $\mathscr{T}$ of $\mathscr{R}$- Triples(X) for which $\mathscr{M}'$ and $\mathscr{M}''$ are strictly specializable along $\mathscr{X}_0$, and any $\alpha \in \mathbb{C}$, the specialization $\psi_{t,\alpha}C$ is defined by

$$\begin{aligned} \psi_{t,\alpha}\mathscr{M}'_{|\mathbf{S}} \underset{\mathscr{O}_{\mathbf{S}}}{\otimes} \overline{\psi_{t,\alpha}\mathscr{M}''_{|\mathbf{S}}} &\xrightarrow{\ \psi_{t,\alpha}C\ } \mathfrak{Db}_{X_0\times\mathbf{S}/\mathbf{S}} \\ ([m'], \overline{[m'']}) &\longmapsto \operatorname{Res}_{s=\alpha\star z/z} \big\langle |t|^{2s} C(m', \overline{m''}), \bullet \wedge \chi(t)\, \tfrac{i}{2\pi} dt \wedge d\bar{t} \big\rangle, \end{aligned} \tag{2.1}$$

where m', m'' are local liftings of $[m'], [m'']$. In such a way, we get an object $\psi_{t,\alpha}\mathscr{T}$ of $\mathscr{R}$- Triples(X_0).

We also define the objects $\Psi_{t,\alpha}\mathscr{T}$ by starting from the *localization* of $\mathscr{T}$ along $\mathscr{X}_0$ (*cf.* [**3**, § 3.4]).

2.d. Polarizable twistor $\mathscr{D}$-modules. — Let w be an integer. The category $\mathrm{MT}^{(\mathrm{r})}(X,w)$ of regular twistor $\mathscr{D}$-modules is defined in [**3**, Def. 4.1.2]. It is a full subcategory of $\mathscr{R}$-Triples(X). Each object of $\mathrm{MT}^{(\mathrm{r})}(X,w)$ is, in particular, strictly specializable along any local analytic hypersurface, as well as all its successive specializations.

The Tate twist by $(-w/2)$ is an equivalence between $\mathrm{MT}^{(\mathrm{r})}(X,w)$ and $\mathrm{MT}^{(\mathrm{r})}(X,0)$. If X is reduced to a point, the category $\mathrm{MT}^{(\mathrm{r})}(\mathrm{pt},0)$ (the regularity condition is now empty) was defined by C. Simpson in [**7**] as the category of twistor structures, which is equivalent to the category of trivializable vector bundles on $\mathbb{P}^1$, or the category of $\mathbb{C}$-vector spaces.

A polarization of an object of $\mathrm{MT}^{(\mathrm{r})}(X,w)$ is a sesquilinear duality $\mathscr{S}$ of weight w which induces, by any successive specializations ending to a point, and gradation by the successive monodromy filtrations, a polarization of the punctual twistor structures (*cf.* [**3**, § 4.2]). The subcategory $\mathrm{MT}^{(\mathrm{r})}(X,w)^{(\mathrm{p})}$ consisting of polarizable regular twistor $\mathscr{D}$-modules is semisimple (*cf.* Prop. 4.2.5 in *loc. cit.*).

3. Partial Laplace transform of $\mathscr{R}_{\mathscr{X}}$-modules

3.a. The setting. — We consider the product $\mathbb{A}^1 \times \widehat{\mathbb{A}}^1$ of two affine lines with coordinates (t,τ), and the partial compactification $\mathbb{P}^1 \times \widehat{\mathbb{A}}^1$, covered by two affine charts, with respective coordinates (t,τ) and (t',τ), where we put $t' = 1/t$. We denote by ∞ the divisor $\{t=\infty\}$ in $\mathbb{P}^1$, defined by the equation $t'=0$, as well as its inverse image in $\mathbb{P}^1 \times \widehat{\mathbb{A}}^1$.

Let Y be a complex manifold. We put $X = Y\times\mathbb{P}^1$, $\widehat{X} = Y\times\widehat{\mathbb{A}}^1$ and $Z = Y\times\mathbb{P}^1\times\widehat{\mathbb{A}}^1$. The manifolds X and Z are equipped with a divisor (still denoted by) ∞. We have projections

$$\begin{array}{ccccc} & & Z & & \\ & \overset{p}{\swarrow} & & \overset{\widehat{p}}{\searrow} & \\ X & & & & \widehat{X} \\ & \underset{q}{\searrow} & & \underset{\widehat{q}}{\swarrow} & \\ & & Y & & \end{array} \tag{3.1}$$

Let $\mathscr{M}$ be a left $\mathscr{R}_{\mathscr{X}}$-module. We denote by $\widetilde{\mathscr{M}}$ the localized module $\mathscr{R}_{\mathscr{X}}[*\infty]\otimes_{\mathscr{R}_{\mathscr{X}}}\mathscr{M}$. Then $p^+\widetilde{\mathscr{M}}$ is a left $\mathscr{R}_{\mathscr{Z}}[*\infty]$-module. We denote by $p^+\widetilde{\mathscr{M}}\otimes\mathcal{E}^{-t\tau/z}$ or, for short, by ${}^{\mathscr{F}}\!\mathscr{M}$, the $\mathscr{O}_{\mathscr{Z}}[*\infty]$-module $p^+\widetilde{\mathscr{M}}$ equipped with the twisted action of $\mathscr{R}_{\mathscr{Z}}$ described by the exponential factor: the $\mathscr{R}_{\mathscr{Y}}$-action is unchanged, and, for any local section m of $\mathscr{M}$,

– in the chart (t, τ),

$$\begin{aligned}\eth_t(m \otimes \mathcal{E}^{-t\tau/z}) &= [(\eth_t - \tau)m] \otimes \mathcal{E}^{-t\tau/z},\\ \eth_\tau(m \otimes \mathcal{E}^{-t\tau/z}) &= -tm \otimes \mathcal{E}^{-t\tau/z},\end{aligned} \tag{3.2}$$

– in the chart (t', τ),

$$\begin{aligned}\eth_{t'}(m \otimes \mathcal{E}^{-t\tau/z}) &= [(\eth_{t'} + \tau/t'^2)m] \otimes \mathcal{E}^{-t\tau/z},\\ \eth_\tau(m \otimes \mathcal{E}^{-t\tau/z}) &= -m/t' \otimes \mathcal{E}^{-t\tau/z},\end{aligned} \tag{3.3}$$

Definition 3.4. — The partial Laplace transform $\widehat{\mathscr{M}}$ of $\mathscr{M}$ is the complex of $\mathscr{R}_{\widehat{\mathscr{X}}}$-modules

$$\widehat{p}_+ {}^{\mathscr{F}}\!\mathscr{M} = \widehat{p}_+(p^+ \widetilde{\mathscr{M}} \otimes \mathcal{E}^{-t\tau/z}).$$

Recall (*cf.* [**3**, Prop. A.2.7]) that we have:

Proposition 3.5. — *Let $\mathscr{M}$ be a coherent $\mathscr{R}_{\mathscr{X}}$-module. Then ${}^{\mathscr{F}}\!\mathscr{M}$ is $\mathscr{R}_{\mathscr{Z}}$-coherent. If moreover $\mathscr{M}$ is good, then so is ${}^{\mathscr{F}}\!\mathscr{M}$, and therefore $\widehat{\mathscr{M}} = \widehat{p}_+ {}^{\mathscr{F}}\!\mathscr{M}$ is $\mathscr{R}_{\widehat{\mathscr{X}}}$-coherent.* □

Let us also recall the definition of the Fourier transform of a sesquilinear pairing. Assume that $\mathscr{M}', \mathscr{M}''$ are good $\mathscr{R}_{\mathscr{X}}$-modules. Let $C : \mathscr{M}'_{|\mathbf{S}} \otimes_{\mathscr{O}_\mathbf{S}} \overline{\mathscr{M}''_{|\mathbf{S}}} \to \mathfrak{Db}_{X\times\mathbf{S}/\mathbf{S}}$ be a sesquilinear pairing. We will define a sesquilinear pairing between the corresponding Laplace transforms:

$$\widehat{C} : \widehat{\mathscr{M}'_{|\mathbf{S}}} \otimes_{\mathscr{O}_\mathbf{S}} \overline{\widehat{\mathscr{M}''_{|\mathbf{S}}}} \longrightarrow \mathfrak{Db}_{\widehat{X}\times\mathbf{S}/\mathbf{S}}\,.$$

Given local sections m', m'' of $p^+\mathscr{M}'_{|\mathbf{S}}, p^+\mathscr{M}''_{|\mathbf{S}}$, which can be written as $m' = \sum_i \phi_i \otimes m'_i$, $m'' = \sum_j \psi_j \otimes m''_j$ with ϕ_i, ψ_j holomorphic functions on $\mathscr{Z}$ and m'_i, m''_j local sections of $\mathscr{M}'_{|\mathbf{S}}, \mathscr{M}''_{|\mathbf{S}}$, let φ be a C^∞ relative form of maximal degree on $Z \times \mathbf{S}$ with compact support. We define the sesquilinear pairing ${}^{\mathscr{F}}C : {}^{\mathscr{F}}\!\mathscr{M}'_{|\mathbf{S}} \otimes_{\mathscr{O}_\mathbf{S}} \overline{{}^{\mathscr{F}}\!\mathscr{M}''_{|\mathbf{S}}} \to \mathfrak{Db}_{Z\times\mathbf{S}/\mathbf{S}}$ by the following formula:

$$\langle {}^{\mathscr{F}}C(m', \overline{m''}), \varphi\rangle := \sum_{i,j} \Big\langle \widetilde{C}(m'_i, \overline{m''_j}), \int_p e^{z\overline{t\tau} - t\tau/z} \phi_i \overline{\psi_j} \varphi \Big\rangle.$$

This is meaningful, as, for any $z \in \mathbf{S}$, the expression $z\overline{t\tau} - t\tau/z$ is purely imaginary, so the integral is a (partial) Fourier transform of a function having compact support with respect to τ, hence defines a function having rapid decay as well as all its derivatives along $t = \infty$; we can apply to it $\widetilde{C}(m'_i, \overline{m''_j})$, which is *a priori* a distribution on $Y \times \mathbb{A}^1 \times \mathbf{S}$, tempered in the t-direction and continuous with respect to z.

We can now define, using the direct image defined in [**3**, § 1.6.d],

$$\widehat{C} = \widehat{p}^{\,0}_{\dagger} {}^{\mathscr{F}}C.$$

4. Partial Laplace transform and specialization

Denote by i_∞ the inclusion $Y\times\{\infty\}\hookrightarrow X$. We will consider the functors $\psi_{\tau,\alpha}$ and $\psi_{t',\alpha}$, as well as the functors $\Psi_{\tau,\alpha}$ and $\Psi_{t',\alpha}$ of Definition 3.4.3 in [**3**] We denote by $\mathrm{N}_\tau,\mathrm{N}_{t'}$ the natural nilpotent endomorphisms on the corresponding nearby cycles modules. We denote by $\mathrm{M}_\bullet(\mathrm{N})$ the monodromy filtration of the nilpotent endomorphism N and by $\mathrm{gr}\,\mathrm{N}:\mathrm{gr}^{\mathrm{M}}_\bullet\to\mathrm{gr}^{\mathrm{M}}_{\bullet-2}$ the morphism induced by N. For $\ell\geqslant 0$, $P\,\mathrm{gr}^{\mathrm{M}}_\ell$ denotes the primitive part $\ker(\mathrm{gr}\,\mathrm{N})^{\ell+1}_{|\mathrm{gr}^{\mathrm{M}}_\ell}$ of $\mathrm{gr}^{\mathrm{M}}_\ell$ and $P\mathrm{M}_\ell$ the inverse image of $P\,\mathrm{gr}^{\mathrm{M}}_\ell$ by the natural projection $\mathrm{M}_\ell\to\mathrm{gr}^{\mathrm{M}}_\ell$. Recall that, in an abelian category, the primitive part $P\,\mathrm{gr}^{\mathrm{M}}_0$ is equal to $\ker\mathrm{N}/(\ker\mathrm{N}\cap\mathrm{Im}\,\mathrm{N})$. We will also denote by $\widetilde{\mathscr{M}}_{\min}$ the minimal extension of $\widetilde{\mathscr{M}}$ (*cf.* § 3.4.b in *loc. cit.*).

Given a finite set of points with multiplicities in Ω_0, we will consider the corresponding divisor D and the corresponding sheaf $\mathscr{O}_{\Omega_0}(-D)$. Given a $\mathscr{R}$-module $\mathscr{N}$, we will put as usual $\mathscr{N}(-D)=\mathscr{O}_{\Omega_0}(-D)\otimes_{\mathscr{O}_{\Omega_0}}\mathscr{N}$.

Proposition 4.1 **(*cf.*** [**3**, Prop. A.3.1]**). —** *Assume that $\mathscr{M}$ is strictly specializable and regular along $t'=0$. Then,*

(i) *for any $\tau_o\neq 0$, the $\mathscr{R}_{\mathscr{X}}$-module $\widetilde{\mathscr{M}}\otimes\mathcal{E}^{-t\tau_o/z}$ is $\mathscr{R}_{\mathscr{X}}$-coherent; it is also strictly specializable (but not regular in general) along $t'=0$, with a constant V-filtration, so that all $\psi_{t',\alpha}(\widetilde{\mathscr{M}}\otimes\mathcal{E}^{-t\tau_o/z})$ are identically 0.*

Assume moreover that $\mathscr{M}$ is strict. Then,

(ii) *the $\mathscr{R}_{\mathscr{X}}$-module ${}^{\mathscr{F}}\!\mathscr{M}:=p^+\widetilde{\mathscr{M}}\otimes\mathcal{E}^{-t\tau/z}$ is strictly specializable and regular along $\tau=\tau_o$ for any $\tau_o\in\widehat{\mathbb{A}}^1$; it is equal to the* minimal extension *of its localization along $\tau=0$;*

(iii) *if $\tau_o\neq 0$, the V-filtration of ${}^{\mathscr{F}}\!\mathscr{M}$ along $\tau-\tau_o=0$ is given by*

$$V_k{}^{\mathscr{F}}\!\mathscr{M}=\begin{cases}{}^{\mathscr{F}}\!\mathscr{M} & \text{if } k\geqslant -1,\\ (\tau-\tau_o)^{-k+1}\,{}^{\mathscr{F}}\!\mathscr{M} & \text{if } k\leqslant -1;\end{cases}$$

we have

$$\psi_{\tau-\tau_o,\alpha}{}^{\mathscr{F}}\!\mathscr{M}=\begin{cases}0 & \text{if } \alpha\notin -\mathbb{N}-1,\\ \widetilde{\mathscr{M}}\otimes\mathcal{E}^{-t\tau_o/z} & \text{if } \alpha\in -\mathbb{N}-1.\end{cases}$$

(iv) *If $\tau_o=0$, we have:*

(a) *for any $\alpha\neq -1$ with $\mathrm{Re}\,\alpha\in[-1,0[$, a functorial isomorphism on some neighbourhood of $\mathbf{D}:=\{|z|\leqslant 1\}$,*

$$\big(\Psi_{\tau,\alpha}{}^{\mathscr{F}}\!\mathscr{M}_{|\mathbf{D}},\mathrm{N}_\tau\big)\xrightarrow{\sim} i_{\infty,+}\big(\psi_{t',\alpha}\widetilde{\mathscr{M}}(-D_\alpha)_{|\mathbf{D}},\mathrm{N}_{t'}\big),$$

where D_α is the divisor $1\cdot i$ if $\alpha'=-1$ and $\alpha''>0$, the divisor $1\cdot(-i)$ if $\alpha'=-1$ and $\alpha''<0$, and the empty divisor otherwise;

(b) *for $\alpha=0$, a functorial isomorphism*

$$\big(\psi_{\tau,0}{}^{\mathscr{F}}\!\mathscr{M},\mathrm{N}_\tau\big)\xrightarrow{\sim} i_{\infty,+}\big(\psi_{t',-1}\widetilde{\mathscr{M}},\mathrm{N}_{t'}\big),$$

(c) *for* $\alpha = -1$*, two functorial exact sequences*

$$0 \longrightarrow i_{\infty,+} \ker \mathrm{N}_{t'} \longrightarrow \ker \mathrm{N}_\tau \longrightarrow \widetilde{\mathscr{M}}_{\min} \longrightarrow 0$$
$$0 \longrightarrow \widetilde{\mathscr{M}}_{\min} \longrightarrow \operatorname{coker} \mathrm{N}_\tau \longrightarrow i_{\infty,+} \operatorname{coker} \mathrm{N}_{t'} \longrightarrow 0,$$

inducing isomorphisms

$$i_{\infty,+} \ker \mathrm{N}_{t'} \xrightarrow{\sim} \ker \mathrm{N}_\tau \cap \operatorname{Im} \mathrm{N}_\tau \subset \ker \mathrm{N}_\tau$$
$$\widetilde{\mathscr{M}}_{\min} \xrightarrow{\sim} \ker \mathrm{N}_\tau / (\ker \mathrm{N}_\tau \cap \operatorname{Im} \mathrm{N}_\tau) \subset \operatorname{coker} \mathrm{N}_\tau,$$

such that the natural morphism $\ker \mathrm{N}_\tau \to \operatorname{coker} \mathrm{N}_\tau$ *induces the identity on* $\widetilde{\mathscr{M}}_{\min}$.

Proof of 4.1(i). — Let us first prove the $\mathscr{R}_{\mathscr{X}}$-coherence of $\widetilde{\mathscr{M}} \otimes \mathcal{E}^{-t\tau_o/z}$ when $\tau_o \neq 0$. As this $\mathscr{R}_{\mathscr{X}}$-module is $\mathscr{R}_{\mathscr{X}}[*\infty]$-coherent by construction, it is enough to prove that it is locally finitely generated over $\mathscr{R}_{\mathscr{X}}$, and the only problem is at $t' = 0$. We also work locally near $z_o \in \Omega_0$ and forget the exponent (z_o) in the V-filtration along $t' = 0$. Then, $\widetilde{\mathscr{M}} = \mathscr{O}_{\mathscr{X}}[1/t'] \otimes_{\mathscr{O}_{\mathscr{X}}} V_{<0}\mathscr{M}$, equipped with its natural $\mathscr{R}_{\mathscr{X}}$-structure. By the regularity assumption, $V_{<0}\mathscr{M}$ is $\mathscr{R}_{\mathscr{X}/\mathbb{A}^1}$-coherent, so we can choose finitely many $\mathscr{R}_{\mathscr{X}/\mathbb{A}^1}$-generators m_i of $V_{<0}\mathscr{M}$.

The regularity assumption implies that, for any i,

$$t'\eth_{t'} m_i \in \sum_j \mathscr{R}_{\mathscr{X}/\mathbb{A}^1} \cdot m_j.$$

In $\widetilde{\mathscr{M}} \otimes \mathcal{E}^{-t\tau_o/z}$, using (3.3), this is written as

$$(t'\eth_{t'} - \tau_0/t')(m_i \otimes \mathcal{E}^{-t\tau_o/z}) \in \sum_j \mathscr{R}_{\mathscr{X}/\mathbb{A}^1} \cdot (m_j \otimes \mathcal{E}^{-t\tau_o/z}), \tag{4.2}$$

and therefore

$$(\tau_o/t')(m_i \otimes \mathcal{E}^{-t\tau_o/z}) \in \sum_j V_0\mathscr{R}_{\mathscr{X}} \cdot (m_j \otimes \mathcal{E}^{-t\tau_o/z}).$$

It follows that $\widetilde{\mathscr{M}} \otimes \mathcal{E}^{-t\tau_o/z}$ is $V_0\mathscr{R}_{\mathscr{X}}$-coherent, generated by the $m_i \otimes \mathcal{E}^{-t\tau_o/z}$. It is then obviously $\mathscr{R}_{\mathscr{X}}$-coherent. The previous relation also implies that $\tau_o(m_i \otimes \mathcal{E}^{-t\tau_o/z}) \in t'\widetilde{\mathscr{M}} \otimes \mathcal{E}^{-t\tau_o/z}$. Therefore, the constant V-filtration, defined by $V_a(\widetilde{\mathscr{M}} \otimes \mathcal{E}^{-t\tau_o/z}) = \widetilde{\mathscr{M}} \otimes \mathcal{E}^{-t\tau_o/z}$ for any a, is good and has a Bernstein polynomial equal to 1.

Proof of 4.1(ii) *for* $\tau_o \neq 0$ *and* 4.1(iii). — The analogue of Formula (4.2) now reads

$$(t'\eth_{t'} + \tau\eth_\tau)(m_i \otimes \mathcal{E}^{-t\tau/z}) \in \sum_j \mathscr{R}_{\mathscr{X}/\mathbb{A}^1} \cdot (m_j \otimes \mathcal{E}^{-t\tau/z}).$$

Therefore, the $\mathscr{R}_{\mathscr{Z}/\widehat{\mathbb{A}}^1}$-module generated by the $m_j \otimes \mathcal{E}^{-t\tau/z}$ is $V_0\mathscr{R}_{\mathscr{Z}}$-coherent, where V denotes the filtration relative to $\tau - \tau_o$. It is even $\mathscr{R}_{\mathscr{Z}}$-coherent if $\tau_o \neq 0$, as τ is a unit near τ_o, and this easily gives 4.1(iii), therefore also 4.1(ii) when $\tau_o \neq 0$.

Proof of 4.1(ii) *for* $\tau_o = 0$. — Let us now consider the case where $\tau_o = 0$. Then the previous argument gives the regularity of ${}^{\mathcal{F}}\!\mathscr{M}$ along $\tau = 0$. We will now show the strict specializability along $\tau = 0$. We will work near $z_o \in \Omega_0$ and forget the exponent (z_o) in the V-filtrations relative to $\tau = 0$ and to $t' = 0$.

Away from $t' = 0$ the result is easy: near $t = t_o$, Formula (3.2), together with the strictness of $\mathscr{M}$, implies that ${}^{\mathcal{F}}\!\mathscr{M}$ is strictly noncharacteristic along $\tau = 0$, hence ${}^{\mathcal{F}}\!\mathscr{M} = V_{-1}{}^{\mathcal{F}}\!\mathscr{M}$ and $\psi_{\tau,-1}{}^{\mathcal{F}}\!\mathscr{M} = \mathscr{M}$ (*cf.* [**3**, § 3.7]).

We will now focus on $t' = 0$. Denote by $V_\bullet\mathscr{M}$ the V-filtration of $\mathscr{M}$ relative to t' and put, for any $a \in [-1, 0[$,

$$V_{a+k}\widetilde{\mathscr{M}} = t'^{-k}V_a\mathscr{M} \ (= V_{a+k}\mathscr{M} \text{ if } k \leqslant 0).$$

Each $V_a\widetilde{\mathscr{M}}$ is a $V_0\mathscr{R}_{\mathscr{X}}$-coherent module and, by regularity, is also $\mathscr{R}_{\mathscr{X}/\mathbb{A}^1}$-coherent. We will now construct the V-filtration of ${}^{\mathcal{F}}\!\mathscr{M}$ along $\tau = 0$. For any $a \in \mathbb{R}$, put

$$U_a{}^{\mathcal{F}}\!\mathscr{M} = \sum_{p\geqslant 0} \eth_{t'}^p\big[(p^*V_a\widetilde{\mathscr{M}}) \otimes \mathcal{E}^{-t\tau/z}\big],$$

i.e., U_a is the $\mathscr{R}_{\mathscr{X}/\widehat{\mathbb{A}}^1}$-module generated by $(p^*V_a\widetilde{\mathscr{M}}) \otimes \mathcal{E}^{-t\tau/z}$ in ${}^{\mathcal{F}}\!\mathscr{M}$. Notice that, when we restrict to $t' \neq 0$, we have for any $a \in \mathbb{R}$,

$$U_{a|t'\neq 0} = {}^{\mathcal{F}}\!\mathscr{M}_{|t'\neq 0}.$$

(ii)(1) Clearly, $U_\bullet$ is an increasing filtration of ${}^{\mathcal{F}}\!\mathscr{M}$ and each U_a is $\mathscr{R}_{\mathscr{X}/\widehat{\mathbb{A}}^1}$-coherent for every $a \in \mathbb{R}$.

(ii)(2) U_a is stable by $\tau\eth_\tau$: indeed, for any local section m of $V_a\widetilde{\mathscr{M}}$, we have by (3.3):

$$\begin{aligned}(\tau\eth_\tau)\eth_{t'}^p(m \otimes \mathcal{E}^{-t\tau/z}) &= \eth_{t'}^p(\tau\eth_\tau)(m \otimes \mathcal{E}^{-t\tau/z})\\ &= \eth_{t'}^p\big[t'\eth_{t'}(m \otimes \mathcal{E}^{-t\tau/z}) - (t'\eth_{t'}m) \otimes \mathcal{E}^{-t\tau/z}\big]\\ &= \eth_{t'}^{p+1}(t'm \otimes \mathcal{E}^{-t\tau/z}) - \eth_{t'}^p\big[(\eth_{t'}t'm) \otimes \mathcal{E}^{-t\tau/z}\big].\end{aligned}$$

The first term in the RHS is in U_{a-1} and the second one is in U_a, as $V_a\widetilde{\mathscr{M}}$ is stable by $\eth_{t'}t'$.

(ii)(3) For any $a \in \mathbb{R}$, we have $U_{a+1} = U_a + \eth_\tau U_a$: indeed, for m as above, we have

$$\eth_\tau \cdot \eth_{t'}^p(m \otimes \mathcal{E}^{-t\tau/z}) = -\eth_{t'}^p\Big(\frac{1}{t'}m \otimes \mathcal{E}^{-t\tau/z}\Big) \in U_{a+1},$$

hence $\eth_\tau U_a \subset U_{a+1}$; applying this equality the in the other way gives the desired equality. This also shows that $\eth_\tau : \mathrm{gr}_a^U{}^{\mathcal{F}}\!\mathscr{M} \to \mathrm{gr}_{a+1}^U{}^{\mathcal{F}}\!\mathscr{M}$ is an isomorphism for any $a \in \mathbb{R}$.

(ii)(4) For any $a \in \mathbb{R}$, we have $\tau U_a \subset U_{a-1}$: indeed, one has, for m as above

$$\begin{aligned}\tau(m \otimes \mathcal{E}^{-t\tau/z}) &= t'^2\eth_{t'}(m \otimes \mathcal{E}^{-t\tau/z}) - (t'^2\eth_{t'}m) \otimes \mathcal{E}^{-t\tau/z}\\ &= \eth_{t'}(t'^2m \otimes \mathcal{E}^{-t\tau/z}) - (\eth_{t'}t'^2m) \otimes \mathcal{E}^{-t\tau/z};\end{aligned}$$

the first term of the RHS clearly belongs to U_{a-2} and the second one to U_{a-1}.

(ii)(5) Denote by $b_a(s)$ the minimal polynomial of $-\eth_{t'}t'$ on $\mathrm{gr}_a^V\widetilde{\mathscr{M}}$. Then, for m as above, we have

$$-(\eth_{t'}t' + \tau\eth_\tau)(m \otimes \mathcal{E}^{-t\tau/z}) = -(\eth_{t'}t'm) \otimes \mathcal{E}^{-t\tau/z}$$

after (3.3). Therefore, we have $b_a(-[\eth_{t'}t' + \tau\eth_\tau])(m \otimes \mathcal{E}^{-t\tau/z}) \in U_{<a}$. Using that $\eth_{t'}t'(m \otimes \mathcal{E}^{-t\tau/z}) = \eth_{t'}(t'm \otimes \mathcal{E}^{-t\tau/z}) \in U_{a-1}$ by definition, we deduce that $b_a(-\tau\eth_\tau)(m \otimes \mathcal{E}^{-t\tau/z}) \in U_{<a}$. Therefore, $b_a(-\tau\eth_\tau)U_a \subset U_{<a}$.

(ii)(6) We will now identify $U_a/U_{<a}$ with $\mathrm{gr}_a^V\widetilde{\mathscr{M}}[\eta] := \mathbb{C}[\eta] \otimes_\mathbb{C} \mathrm{gr}_a^V\widetilde{\mathscr{M}}$, where η is a new variable. Notice first that both objects are supported on $\{t' = 0\}$. Consider the map

$$\begin{aligned} V_a\widetilde{\mathscr{M}}[\eta] &\longrightarrow U_a \\ \sum_p m_p\eta^p &\longmapsto \sum_p \eth_{t'}^p(m_p \otimes \mathcal{E}^{-t\tau/z}). \end{aligned}$$

Its composition with the natural projection $U_a \to U_a/U_{<a}$ induces a surjective mapping $\mathrm{gr}_a^V\widetilde{\mathscr{M}}[\eta] \to U_a/U_{<a}$. In order to show that it is injective, it is enough to show that, if $\sum_p \eth_{t'}^p(m_p \otimes \mathcal{E}^{-t\tau/z})$ belongs to $U_{<a}$, then each m_p belongs to $V_{<a}\widetilde{\mathscr{M}}$. For that purpose, it is enough to work with an algebraic version of U_a, where "p^*" means "$\otimes_\mathbb{C}\mathbb{C}[\tau]$". Notice that, if a local section $\sum_{\ell=0}^r \tau^\ell(n_\ell \otimes \mathcal{E}^{-t\tau/z})$ of $\widetilde{\mathscr{M}}[\tau] \otimes \mathcal{E}^{-t\tau/z}$ belongs to U_a, then the leading coefficient n_r is a local section of $V_{a+2r}\widetilde{\mathscr{M}}$ (by using that $\eth_{t'}(n \otimes \mathcal{E}^{-t\tau/z}) = (\eth_{t'}n) \otimes \mathcal{E}^{-t\tau/z} - \tau((n/t'^2) \otimes \mathcal{E}^{-t\tau/z})$). Remark then that, using (3.3), $\sum_{p=0}^q \eth_{t'}^p(m_p \otimes \mathcal{E}^{-t\tau/z})$ is a polynomial of degree q in τ with leading coefficient $\pm(\tau^q/t'^{2q})(m_q \otimes \mathcal{E}^{-t\tau/z})$. If the sum belongs to $U_{<a}$, this implies that $m_q/t'^{2q} \in V_{<a+2q}\widetilde{\mathscr{M}}$, *i.e.*, $m_q \in V_{<a}\widetilde{\mathscr{M}}$. Therefore, by induction on q, all coefficients m_p are local sections of $V_{<a}\widetilde{\mathscr{M}}$, as was to be shown.

Let us describe the $\mathscr{R}_\mathscr{X}[\tau\eth_\tau]$-module structure on $\mathrm{gr}_a^V\widetilde{\mathscr{M}}[\eta]$ coming from the identification with $U_a/U_{<a}$. First, the $\mathscr{R}_\mathscr{Y}$-module structure is the natural one on $\mathrm{gr}_a^V\widetilde{\mathscr{M}}$, naturally extended to $\mathrm{gr}_a^V\widetilde{\mathscr{M}}[\eta]$. Then one checks that

$$\eth_{t'}\sum_p m_p\eta^p = \eta\sum_p m_p\eta^p, \quad t'\sum_p m_p\eta^p = -\eth_\eta\sum_p m_p\eta^p, \tag{4.3}$$

$$\tau\eth_\tau\sum_p m_p\eta^p = \sum_p(\eth_t't')(m_p)\eta^p. \tag{4.4}$$

If we denote by i_∞ the inclusion $Y \times \{\infty\} \hookrightarrow X$, the $\mathscr{R}_\mathscr{X}$-module $\mathrm{gr}_a^V\widetilde{\mathscr{M}}[\eta]$ that these formulas define is nothing but $i_{\infty,+}\mathrm{gr}_a^V\widetilde{\mathscr{M}}$, so we have obtained an isomorphism of $\mathscr{R}_\mathscr{X}$-modules:

$$(i_{\infty,+}\mathrm{gr}_a^V\widetilde{\mathscr{M}}, \eth_{t'}t') \xrightarrow{\sim} (\mathrm{gr}_a^U{}^\mathscr{F}\!\mathscr{M}, \tau\eth_\tau) \tag{4.5}$$

(ii)(7) Consider the filtration $V_\bullet{}^{\mathscr F}\!\mathscr M$ defined for $a \in [-1, 0[$ and $k \in \mathbb Z$ by

$$V_{a+k}{}^{\mathscr F}\!\mathscr M = \begin{cases} U_{a+1+k} & \text{if } k \geqslant 0, \\ \tau^{-k} U_{a+1} & \text{if } k \leqslant 0. \end{cases}$$

This is a V-filtration relative to τ on ${}^{\mathscr F}\!\mathscr M$, by (ii)(1), (ii)(2), (ii)(3) and (ii)(4). It is good, by the equality in (ii)(3) and because $\tau V_a{}^{\mathscr F}\!\mathscr M = V_{a-1}{}^{\mathscr F}\!\mathscr M$ for $a < 0$ by definition. Notice that, for $a > -1$, we have $\mathrm{gr}^V_a{}^{\mathscr F}\!\mathscr M = \mathrm{gr}^U_{a+1}{}^{\mathscr F}\!\mathscr M$.

For $a > -1$, we can use (4.5) to get a minimal polynomial of the right form for $-\eth_\tau \tau$ acting on $\mathrm{gr}^V_a{}^{\mathscr F}\!\mathscr M$ (here is the need for a shift by 1 between U and V), and strictness follows from (4.5) and the strictness of $\mathrm{gr}^V_a \widetilde{\mathscr M}$, which is by assumption.

It therefore remains to analyze $\mathrm{gr}^V_a{}^{\mathscr F}\!\mathscr M$ for $a \leqslant -1$.

(ii)(8) We will analyze $\mathrm{gr}^V_{-1}{}^{\mathscr F}\!\mathscr M = U_0/\tau U_{<1}$ through the following two diagrams of exact sequences, where the non labelled maps are the natural ones:

$$\tag{4.6} \begin{array}{ccccccccc} & & 0 & & & & & & \\ & & \downarrow & & & & & & \\ & & (U_{<0} \cap \tau{}^{\mathscr F}\!\mathscr M)/\tau U_{<1} & & & & & & \\ & & \downarrow & & & & & & \\ 0 & \longrightarrow & U_{<0}/\tau U_{<1} & \longrightarrow & U_0/\tau U_{<1} & \longrightarrow & U_0/U_{<0} & \longrightarrow & 0 \\ & & \downarrow & & & & & & \\ & & U_{<0}/(U_{<0} \cap \tau{}^{\mathscr F}\!\mathscr M) & & & & & & \\ & & \downarrow & & & & & & \\ & & 0 & & & & & & \end{array}$$

and

$$\tag{4.7} \begin{array}{ccccccccc} & & & & & & 0 & & \\ & & & & & & \downarrow & & \\ & & & & & & U_{<0}/(U_{<0} \cap \tau U_1) & & \\ & & & & & & \downarrow & & \\ 0 & \longrightarrow & U_0/U_{<0} & \xrightarrow{\tau\eth_\tau} & U_0/\tau U_{<1} & \longrightarrow & U_0/\tau U_1 & \longrightarrow & 0 \\ & & & & & & \downarrow & & \\ & & & & & & U_0/(\tau U_1 + U_{<0}) & & \\ & & & & & & \downarrow & & \\ & & & & & & 0 & & \end{array}$$

Notice that, in (4.7), $\tau\eth_\tau$ is injective because it is the composition

$$\tag{4.8} U_0/U_{<0} \xrightarrow{\eth_\tau} U_1/U_{<1} \xrightarrow{\tau} U_0/\tau U_{<1},$$

$\eth_\tau$ is an isomorphism (*cf.* (ii)(3)) and τ is injective, as it acts injectively on ${}^{\mathscr F}\!\mathscr M$. Recall that $(\mathrm{gr}^U_0{}^{\mathscr F}\!\mathscr M, \tau\eth_\tau)$ is identified, by (ii)(6), with $i_{\infty,+}(\mathrm{gr}^V_0 \widetilde{\mathscr M}, \eth_{t'} t')$. Notice also that

$\tau\eth_\tau$ vanishes on $U_{<0}/\tau U_{<1}$ (resp. on $U_0/\tau U_1$), as $\eth_\tau U_{<0} \subset U_{<1}$ (resp. $\eth_\tau U_0 \subset U_1$). It remains therefore to prove the strictness of $U_{<0}/\tau U_{<1}$ to get the desired properties for $\mathrm{gr}^V_{-1}\,{}^{\mathcal{F}}\!\mathscr{M}$. We denote by $\mathrm{N}_{t'}$ the action of $-t'\eth_{t'}$ on $\mathrm{gr}^V_{-1}\widetilde{\mathscr{M}}$ (by strictness, $\ker \mathrm{N}_{t'}$ is equal to the kernel of $-t'\eth_{t'}$ acting on $\psi_{t',-1}\widetilde{\mathscr{M}} \subset \mathrm{gr}^V_{-1}\widetilde{\mathscr{M}}$). The strictness of $\mathrm{gr}^V_{-1}\,{}^{\mathcal{F}}\!\mathscr{M}$ follows then from the strictness of $i_{\infty,+}\psi_{t',-1}\widetilde{\mathscr{M}}$, that of $\widetilde{\mathscr{M}}_{\min}$ (defined in [**3**, Def. 3.4.7]) and the first two lines of the lemma below, applied to the diagram (4.6).

Lemma 4.9. — *We have functorial isomorphisms of $\mathscr{R}_{\mathscr{X}}$-modules:*

$$\begin{aligned} U_{<0}/(U_{<0}\cap\tau U_1) = U_{<0}/(U_{<0}\cap\tau\,{}^{\mathcal{F}}\!\mathscr{M}) &\xrightarrow{\ \sim\ } \widetilde{\mathscr{M}}_{\min}\\ i_{\infty,+}\ker \mathrm{N}_{t'} &\xrightarrow{\ \sim\ } (U_{<0}\cap\tau\,{}^{\mathcal{F}}\!\mathscr{M})/\tau U_{<1}\\ i_{\infty,+}\operatorname{coker} \mathrm{N}_{t'} &\xrightarrow{\ \sim\ } U_0/(\tau U_1+U_{<0}).\end{aligned}$$

Proof. — For $m_0,\dots,m_p\in\widetilde{\mathscr{M}}$, we can write

$$\begin{aligned} (4.10)\quad & m_0\otimes\mathcal{E}^{-t\tau/z}+\eth_{t'}(m_1\otimes\mathcal{E}^{-t\tau/z})+\cdots+\eth_{t'}^p(m_p\otimes\mathcal{E}^{-t\tau/z})\\ &= n_0\otimes\mathcal{E}^{-t\tau/z}-\tau\Big[(n_1/t'^2)\otimes\mathcal{E}^{-t\tau/z}+\cdots+\eth_{t'}^{p-1}\big((n_p/t'^2)\otimes\mathcal{E}^{-t\tau/z}\big)\Big]\end{aligned}$$

with

$$(4.11)\qquad \begin{aligned} n_p &= m_p & m_p &= n_p\\ n_{p-1} &= m_{p-1}+\eth_{t'}m_p & m_{p-1} &= n_{p-1}-\eth_{t'}n_p\\ &\ \vdots & &\ \vdots\\ n_1 &= m_1+\eth_{t'}m_2+\cdots+\eth_{t'}^{p-1}m_p & m_1 &= n_1-\eth_{t'}n_2\\ n_0 &= m_0+\eth_{t'}m_1+\cdots+\eth_{t'}^{p}m_p & m_0 &= n_0-\eth_{t'}n_1\end{aligned}$$

Sending an element to its constant term in its τ expansion gives an injective morphism $U_{<0}/(U_{<0}\cap\tau\,{}^{\mathcal{F}}\!\mathscr{M})\to\widetilde{\mathscr{M}}$. Formulas (4.10) and (4.11) show that the image of this morphism is the $\mathscr{R}_{\mathscr{X}}$-submodule of $\widetilde{\mathscr{M}}$ generated by $V_{<0}\widetilde{\mathscr{M}}$: this is by definition the minimal extension of $\widetilde{\mathscr{M}}$ across $t'=0$.

Let us show that

$$(4.12)\qquad U_{<0}\cap\tau U_1 = U_{<0}\cap\tau\,{}^{\mathcal{F}}\!\mathscr{M}.$$

Consider a local section of $U_{<0}\cap\tau\,{}^{\mathcal{F}}\!\mathscr{M}$, written as in (4.10); it satisfies thus $m_0,\dots,m_p\in V_{<0}\widetilde{\mathscr{M}}$ and $n_0=0$; then $\eth_{t'}n_1=-m_0\in V_{<0}\widetilde{\mathscr{M}}$. This implies that n_1 is a local section of $V_{-1}\widetilde{\mathscr{M}}$: indeed, the condition on n_1 is equivalent to $t'\eth_{t'}n_1\in V_{<-1}\widetilde{\mathscr{M}}$; use then that, by strictness of $\mathrm{gr}^V_a\widetilde{\mathscr{M}}$, $t'\eth_{t'}$ acts injectively on $\mathrm{gr}^V_a\widetilde{\mathscr{M}}$ if $a\neq-1$. Therefore, $(n_1/t'^2)\otimes\mathcal{E}^{-t\tau/z}\in U_1$. We can now assume that $n_1=0$ and thus $\eth_{t'}n_2\in V_{<0}\widetilde{\mathscr{M}}$... hence (4.12), and the first line of the lemma. Notice moreover that the class of each n_j in $\mathrm{gr}^V_{-1}\widetilde{\mathscr{M}}$ is in $\ker \mathrm{N}_{t'}$.

Let η be a new variable. We define a morphism

$$\ker \mathrm{N}_{t'}[\eta] \longrightarrow U_{<0}/\tau U_{<1}$$

by the rule

$$\sum_{j\geqslant 1}[n_j]\eta^{j-1} \longmapsto -\tau\Big[(n_1/t'^2)\otimes \mathcal{E}^{-t\tau/z} + \cdots + \eth_{t'}^{p-1}\big((n_p/t'^2)\otimes \mathcal{E}^{-t\tau/z}\big)\Big], \tag{4.13}$$

by taking some lifting n_j of each $[n_j]\in \ker \mathrm{N}_{t'} \subset \mathrm{gr}_{-1}^V \widetilde{\mathscr{M}}$ in $V_{-1}\widetilde{\mathscr{M}}$.

– This morphism is well defined: using (4.10), write

$$-\tau\eth_{t'}^{j-1}\big((n_j/t'^2)\otimes \mathcal{E}^{-t\tau/z}\big) = \eth_{t'}^{j}\big(n_j\otimes \mathcal{E}^{-t\tau/z}\big) - \eth_{t'}^{j-1}\big((\eth_{t'}n_j)\otimes \mathcal{E}^{-t\tau/z}\big);$$

that $[n_j]$ belongs to $\ker \mathrm{N}_{t'}$ is equivalent to $t'\eth_{t'}n_j \in V_{<-1}\widetilde{\mathscr{M}}$; therefore, both n_j and $\eth_{t'}n_j$ belong to $V_{<0}\widetilde{\mathscr{M}}$; moreover, if $n_j\in V_{<-1}\widetilde{\mathscr{M}}$, so that $n_j/t'^2 \in V_{<1}\widetilde{\mathscr{M}}$, the image is in $\tau U_{<1}$.

– This morphism is injective: as we have seen in (ii)(6), the term between brackets in (4.13) belongs to $U_{<1}$ if and only if each n_j/t'^2 belongs to $V_{<1}\widetilde{\mathscr{M}}$, *i.e.*, each n_j is in $V_{<-1}\widetilde{\mathscr{M}}$.

– The image of this morphism is equal to $(U_{<0}\cap \tau{}^{\mathscr{F}}\!\mathscr{M})/\tau U_{<1}$: this was shown in the proof of (4.12).

As in (ii)(6), we can identify $\ker \mathrm{N}_{t'}[\eta]$ with $i_{\infty,+}\ker \mathrm{N}_{t'}$ and the morphism is seen to be $\mathscr{R}_{\mathscr{X}}$-linear.

Let us now consider the third line of the lemma. We identify $U_0/(\tau U_1 + U_{<0})$ with the cokernel of $\tau : \mathrm{gr}_1^U \to \mathrm{gr}_0^U$ or, equivalently, to that of $\tau\eth_\tau : \mathrm{gr}_0^U \to \mathrm{gr}_0^U$. By (ii)(6), it is identified with $i_{\infty,+}\operatorname{coker} \eth_{t'}t'$ acting on $i_{\infty,+}\mathrm{gr}_0^V\widetilde{\mathscr{M}}$. Use now the isomorphism $t' : \mathrm{gr}_0^V\widetilde{\mathscr{M}} \to \mathrm{gr}_{-1}^V\widetilde{\mathscr{M}}$ to conclude. □

(ii)(9) We will now prove that all the $\mathrm{gr}_a^V{}^{\mathscr{F}}\!\mathscr{M}$ for $a\leqslant -1$ are strict and have a Bernstein polynomial. In (ii)(8) we have proved this for $a=-1$.

Choose $a<-1$. It follows from the definition of $V_\bullet{}^{\mathscr{F}}\!\mathscr{M}$ that

$$\tau : \mathrm{gr}_{a+1}^V{}^{\mathscr{F}}\!\mathscr{M} \longrightarrow \mathrm{gr}_a^V{}^{\mathscr{F}}\!\mathscr{M} \tag{4.14}$$

is onto. Therefore, by decreasing induction on a and using (ii)(7), we have a Bernstein relation on each $\mathrm{gr}_a^V{}^{\mathscr{F}}\!\mathscr{M}$. It remains to prove the strictness of such a module. This is also done by decreasing induction on a, as it is now known to be true for any $a\in[-1,0[$. It is enough to show that (4.14) is also injective for any $a<-1$, and it is also enough to show that

$$\eth_\tau\tau : \mathrm{gr}_{a+1}^V{}^{\mathscr{F}}\!\mathscr{M} \longrightarrow \mathrm{gr}_{a+1}^V{}^{\mathscr{F}}\!\mathscr{M}.$$

is injective. If a section m satisfies $\eth_\tau\tau m = 0$ then, according to the Bernstein relation that we previously proved, it also satisfies $\prod(\alpha\star z)^{\nu_\alpha}m=0$, where the product is taken on a set of $\alpha\in\mathbb{C}$ with $\ell_{z_o}(\alpha)=a+1<0$ and $\nu_\alpha\in\mathbb{N}$. Such a set does not contain 0 and the function $z \mapsto \prod(\alpha \star z)^{\nu_\alpha}$ is not identically 0. By induction, $\mathrm{gr}^V_{a+1} {}^{\mathscr{F}}\!\mathscr{M}$ is strict. Therefore, $m = 0$, hence the injectivity.

(ii)(10) By construction, the filtration $V_\bullet {}^{\mathscr{F}}\!\mathscr{M}$ satisfies moreover that

- $\tau : \mathrm{gr}^V_a {}^{\mathscr{F}}\!\mathscr{M} \to \mathrm{gr}^V_{a-1} {}^{\mathscr{F}}\!\mathscr{M}$ is onto for any $a < 0$,
- $\eth_\tau : \mathrm{gr}^V_a {}^{\mathscr{F}}\!\mathscr{M} \to \mathrm{gr}^V_{a+1} {}^{\mathscr{F}}\!\mathscr{M}$ is onto for any $a \geqslant -1$.

This implies that all the conditions for strict specializability (*cf.* [**3**, Def. 3.3.8]) are satisfied, and that moreover the morphism can_τ introduced in [**3**, Rem. 3.3.6(6)] is *onto*. Notice also that the morphism var_τ is injective: indeed, this means that $\tau : \mathrm{gr}^V_0 {}^{\mathscr{F}}\!\mathscr{M} \to \mathrm{gr}^V_{-1} {}^{\mathscr{F}}\!\mathscr{M}$ is injective, or equivalently that $\tau : U_1/U_{<1} \to U_0/\tau U_{<1}$ is injective, which has been seen after (4.8).

In other words, we have shown that ${}^{\mathscr{F}}\!\mathscr{M}$ is strictly specializable along $\tau = 0$ and that it is equal to the *minimal extension* of its localization along $\tau = 0$, as defined in [**3**, § 3.4.b].

Proof of 4.1(iv). — Now that ${}^{\mathscr{F}}\!\mathscr{M}$ is known to be strictly specializable along $\tau = 0$, the $\mathscr{R}_{\mathscr{X}}$-modules $\psi_{\tau,\alpha} {}^{\mathscr{F}}\!\mathscr{M}$ (*cf.* Lemma 3.3.4 in *loc. cit.*) are defined. We can compare them with $i_{\infty,+}\psi_{t',\alpha}\widetilde{\mathscr{M}}$.

(iv)(1) For any $z_o \in \Omega_0$, we have a natural morphism, defined locally near z_o (putting $a = \ell_{z_o}(\alpha)$)

$$\psi_{\tau,\alpha} {}^{\mathscr{F}}\!\mathscr{M} \hookrightarrow \mathrm{gr}^V_a {}^{\mathscr{F}}\!\mathscr{M} \longrightarrow \mathrm{gr}^U_{a+1} {}^{\mathscr{F}}\!\mathscr{M} \xrightarrow{\sim} i_{\infty,+} \mathrm{gr}^V_{a+1} \widetilde{\mathscr{M}} \xrightarrow[\sim]{i_{\infty,+}t'} i_{\infty,+} \mathrm{gr}^V_a \widetilde{\mathscr{M}}, \tag{4.15}$$

which takes values in $i_{\infty,+}\psi_{t',\alpha}\widetilde{\mathscr{M}}$. One verifies that the various morphisms glue together in a well defined morphism $\psi_{\tau,\alpha} {}^{\mathscr{F}}\!\mathscr{M} \to i_{\infty,+}\psi_{t',\alpha}\widetilde{\mathscr{M}}$.

Lemma 4.16. — *Near any $z_o \in \mathbf{D}$, the natural morphism $\psi_{\tau,\alpha} {}^{\mathscr{F}}\!\mathscr{M} \to \mathrm{gr}^U_{a+1} {}^{\mathscr{F}}\!\mathscr{M}$ ($a = \ell_{z_o}(\alpha)$) is injective for any $\alpha \in \mathbb{C} \smallsetminus (-\mathbb{N}^*)$ and, if $a \geqslant -1$, $\psi_{\tau,\alpha} {}^{\mathscr{F}}\!\mathscr{M} \to i_{\infty,+}\psi_{t',\alpha}\widetilde{\mathscr{M}}$ is an isomorphism near z_o.*

Proof. — If $a > -1$, this has been proved in (4.5). Assume that $a = -1$ (and $\alpha \notin -\mathbb{N}^*$). If we decompose the horizontal sequence (4.6) with respect to the eigenvalues of $-\tau\eth_\tau$, we get that, for any $\alpha \neq -1$ with $\ell_{z_o}(\alpha) = -1$, the natural morphism

$$\psi_{\tau,\alpha} {}^{\mathscr{F}}\!\mathscr{M} \longrightarrow U_0/U_{<0}$$

is an isomorphism onto $(U_0/U_{<0})_{\alpha+1}$ and, according to (4.5), we have an isomorphism

$$\psi_{\tau,\alpha} {}^{\mathscr{F}}\!\mathscr{M} \xrightarrow{\sim} i_{\infty,+}\psi_{t',\alpha+1}\widetilde{\mathscr{M}} \xrightarrow[\sim]{i_{\infty,+}t'} i_{\infty,+}\psi_{t',\alpha}\widetilde{\mathscr{M}}.$$

Assume now that $a < -1$. Let $k \geqslant 0$ be such that $b = a + k \in [-1, 0[$. We prove the result by induction on k, knowing that it is true for $k = 0$. By induction, we have a commutative diagram

$$\begin{array}{ccc} \psi_{\tau,\alpha+1}\mathscr{F}\!\mathscr{M} & \hookrightarrow & \mathrm{gr}^U_{\alpha+2}\,\mathscr{F}\!\mathscr{M} \\ {\scriptstyle\tau}\big\downarrow\wr & & \wr\big\uparrow{\scriptstyle\eth_\tau} \\ \psi_{\tau,\alpha}\mathscr{F}\!\mathscr{M} & \longrightarrow & \mathrm{gr}^U_{\alpha+1}\,\mathscr{F}\!\mathscr{M} \end{array}$$

showing that the lower horizontal arrow is injective if and only if $\eth_\tau\tau$ is injective on $\psi_{\tau,\alpha+1}\mathscr{F}\!\mathscr{M}$, which follows from strictness if $(\alpha+1)\star z\not\equiv 0$, that is, if $\alpha\neq -1$. □

(iv)(2) Proof of 4.1(ivb). When $\alpha=0$, the proof follows from Lemma 4.16.

(iv)(3) Assume now that $\alpha\neq -1$ satisfies $\operatorname{Re}\alpha\in[-1,0[$. We wish to show that (4.15) induces an isomorphism

$$\psi_{\tau,\alpha}\mathscr{F}\!\mathscr{M}_{|\mathbf{D}}\xrightarrow{\ \sim\ } i_{\infty,+}\psi_{t',\alpha}\widetilde{\mathscr{M}}(-D_\alpha)_{|\mathbf{D}}. \tag{4.17}$$

This is a local question with respect to $z\in\mathbf{D}$.

Clearly, the image of $\psi_{\tau,\alpha}\mathscr{F}\!\mathscr{M}\to\mathrm{gr}^U_{\alpha+1}\,\mathscr{F}\!\mathscr{M}$ is contained in $\ker[(\eth_\tau\tau+\alpha\star z)^N:\mathrm{gr}^U_{\alpha+1}\,\mathscr{F}\!\mathscr{M}\to\mathrm{gr}^U_{\alpha+1}\,\mathscr{F}\!\mathscr{M}]$, for $N\gg 0$ and is equal to this submodule if $a\geqslant -1$.

If $a<-1$ and if $k\geqslant 1$ is such that $a+k\in[-1,0[$, the image is identified with

$$\operatorname{Im}(\tau^k\eth_\tau^k):\ker(\eth_\tau\tau+\alpha\star z)^N\longrightarrow\ker(\eth_\tau\tau+\alpha\star z)^N,$$

and it is identified with the image of the multiplication by $\prod_{j=1}^k(\alpha+j)\star z$ on this module. For $j=1,\dots,k$, the number $\beta=\alpha+j$ satisfies $\operatorname{Re}\beta\geqslant 0$, $\beta\neq 0$ and $\ell_{z_o}(\beta)<0$. Then $\beta\star z=0$ has a solution z in $\mathbf{D}$ iff $\operatorname{Re}\beta=0$, and this solution is $z=\pm i$. This occurs iff $\operatorname{Re}\alpha=-1$ and $j=1$. In conclusion, the image of $\psi_{\tau,\alpha}\mathscr{F}\!\mathscr{M}_{|\mathbf{D}}$ in $i_{\infty,+}\psi_{\tau,\alpha}\widetilde{\mathscr{M}_{|\mathbf{D}}}$, is equal to the image of the multiplication by $(\alpha+1)\star z$ on $i_{\infty,+}\psi_{\tau,\alpha}\widetilde{\mathscr{M}_{|\mathbf{D}}}$. As we assume that $\ell_{z_o}(\alpha)<-1$, the divisor of $z\mapsto(\alpha+1)\star z$ coincides, near z_o, with the divisor D_α, hence (4.17).

(iv)(4) We now show that there is no difference between $\psi_{\tau,\alpha}\mathscr{F}\!\mathscr{M}$ and $\Psi_{\tau,\alpha}\mathscr{F}\!\mathscr{M}$ on some neighbourhood of $\mathbf{D}$.

Lemma 4.18. — *Assume that $\alpha\neq -1$ and $\alpha':=\operatorname{Re}\alpha\in[-1,0[$. Then the natural inclusion $\psi_{\tau,\alpha}\mathscr{F}\!\mathscr{M}_{\mathbf{D}}\hookrightarrow\Psi_{\tau,\alpha}\mathscr{F}\!\mathscr{M}_{\mathbf{D}}$ is an isomorphism.*

Note that the existence of an inclusion is proved in [**3**, Lemma 3.4.2(1)].

Proof. — The question is local near points $z\in\mathbf{D}$ such that $\ell_z(\alpha)\geqslant 0$, otherwise the result follows from Lemma 3.4.1 in *loc. cit.* Fix z_o such that $\ell_{z_o}(\alpha)\geqslant 0$ and let $k\geqslant 1$ be such that $\ell_{z_o}(\alpha-k)\in[-1,0[$. We have a commutative diagram

$$\begin{array}{ccc} \psi_{\tau,\alpha}\mathscr{F}\!\mathscr{M} & \hookrightarrow & \Psi_{\tau,\alpha}\mathscr{F}\!\mathscr{M} \\ {\scriptstyle\tau^k}\big\downarrow & & \wr\big\downarrow{\scriptstyle\tau^k} \\ \psi_{\tau,\alpha-k}\mathscr{F}\!\mathscr{M} & \xrightarrow{\ \sim\ } & \Psi_{\tau,\alpha-k}\mathscr{F}\!\mathscr{M} \end{array}$$

and, as $a := \ell_{z_o}(\alpha)$ and $a-k$ are $\geqslant -1$ and $\alpha \neq -1$, $\psi_{\tau,\alpha}{}^{\mathscr{F}}\!\mathscr{M}$ (resp. $\psi_{\tau,\alpha-k}{}^{\mathscr{F}}\!\mathscr{M}$) is contained in $\mathrm{gr}^U_{a+1}{}^{\mathscr{F}}\!\mathscr{M}$ (resp. in $\mathrm{gr}^U_{a+1-k}{}^{\mathscr{F}}\!\mathscr{M}$), using the local filtration U near z_o. It follows (*cf.* (ii)(3)) that $\eth_\tau^k : \psi_{\tau,\alpha-k}{}^{\mathscr{F}}\!\mathscr{M} \to \psi_{\tau,\alpha}{}^{\mathscr{F}}\!\mathscr{M}$ is an isomorphism. Therefore, the image of $\psi_{\tau,\alpha}{}^{\mathscr{F}}\!\mathscr{M}$ in $\Psi_{\tau,\alpha}{}^{\mathscr{F}}\!\mathscr{M}$ is identified with the image of $\eth_\tau^k\tau^k$ acting on $\Psi_{\tau,\alpha}{}^{\mathscr{F}}\!\mathscr{M}$. Using the nilpotent endomorphism $\mathrm{N}_\tau = -(\eth_\tau\tau + \alpha\star z)$, we write $\eth_\tau^k\tau^k$ as $(-1)^k(\mathrm{N}_\tau + \alpha\star z)\cdots(\mathrm{N}_\tau + (\alpha-k+1)\star z)$. The proof of the lemma will be complete if we show that none of the $(\alpha-j)\star z_o$ $(j=0,\dots,k-1)$ vanishes (assuming that $z_o \in \mathbf{D}$).

Notice that $\beta := \alpha - j$ satisfies $\beta' < 0$ and $\beta' - \zeta_o\beta'' \geqslant 0$. Assume that $\beta\star z_o = 0$. By the previous conditions, we must have $\beta'' \neq 0$ and $z_o \neq 0$, and the only possibility for z_o is then $z_o = i\zeta_o$ and $\zeta_o = \frac{\beta' - \sqrt{\beta'^2+\beta''^2}}{\beta''}$. Now, the condition $\beta' < 0$ implies $|\zeta_o| > 1$, so $z_o \notin \mathbf{D}$. □

(iv)(5) Proof of 4.1(iva). It follows from (4.17) and Lemma 4.18 that we have a functorial isomorphism

$$\Psi_{\tau,\alpha}{}^{\mathscr{F}}\!\mathscr{M}_{|\mathbf{D}} \longrightarrow i_{\infty,+}\psi_{\tau,\alpha}\widetilde{\mathscr{M}}(-D_\alpha)_{|\mathbf{D}} \tag{4.19}$$

when $\alpha \neq -1$ satisfies $\operatorname{Re}\alpha \in [-1,0[$. This ends the proof of 4.1(iv) when $\alpha \neq -1$.

(iv)(6) Proof of 4.1(ivc). Let us now consider the case when $\alpha = -1$. The two exact sequences that we consider are the vertical exact sequences in (4.6) and (4.7), according to Lemma 4.9.

For the second assertion, notice first that, as the image of $\operatorname{Im}\mathrm{N}_\tau \cap \ker \mathrm{N}_\tau$ in $\widetilde{\mathscr{M}}_{\min}$ is supported on $\{t'=0\}$, it is zero by the definition of the minimal extension, hence we have an inclusion $\operatorname{Im}\mathrm{N}_\tau \cap \ker\mathrm{N}_\tau \subset i_{\infty,+}\ker\mathrm{N}_{t'}$. To prove $i_{\infty,+}\ker\mathrm{N}_{t'} \subset \operatorname{Im}\mathrm{N}_\tau$, remark that the image of (4.13) is in $\tau(U_1/U_{<1})$, hence in $\tau\psi_{\tau,0}{}^{\mathscr{F}}\!\mathscr{M}$, that is, in $\operatorname{Im}\mathrm{var}_\tau$, hence in $\operatorname{Im}\mathrm{N}_\tau$.

The last assertion is nothing but the identification $U_{<0} \cap \tau^{\mathscr{F}}\!\mathscr{M} = U_{<0}\cap\tau U_1$ of Lemma 4.9. □

5. Partial Fourier-Laplace transform of regular twistor $\mathscr{D}$-modules

The main result of this article is (*cf.* [**3**, Th. A.4.1]):

Theorem 5.1. — *Let* $(\mathscr{T},\mathscr{S}) = (\mathscr{M}',\mathscr{M}'',C,\mathscr{S})$ *be an object of* $\mathrm{MT}^{(\mathrm{r})}(X,w)^{(\mathrm{p})}$. *Then, along* $\tau=0$, $\widehat{\mathscr{M}'}$ *and* $\widehat{\mathscr{M}''}$ *are strictly specializable, regular and S-decomposable. Moreover,* $\Psi_{\tau,\alpha}(\widehat{\mathscr{T}},\widehat{\mathscr{S}})$, *with* $\operatorname{Re}\alpha\in[-1,0[$, *and* $\phi_{\tau,0}(\widehat{\mathscr{T}},\widehat{\mathscr{S}})$ *induce, by grading with respect to the monodromy filtration* $\mathrm{M}_\bullet(\mathrm{N}_\tau)$, *an object of* $\mathrm{MLT}^{(\mathrm{r})}(\widehat{X},w;-1)^{(\mathrm{p})}$.

Note that the definition of S-decomposability is given in [**3**, Def. 3.5.1], and that of the category $\mathrm{MLT}^{(\mathrm{r})}$ in § 4.1.f of *loc. cit.* In particular, all conditions of Definition 4.1.2 in *loc. cit.* are satisfied along the hypersurface $\tau = 0$.

This theorem is a generalization of [**1**, Th. 5.3], without the $\mathbb{Q}$-structure however. In fact, we give a precise comparison with nearby cycles of $(\mathscr{T}, \mathscr{S})$ at $t = \infty$ as in [**1**, Th. 4.3].

In order to prove Theorem 5.1, we need to extend the results of Proposition 4.1 to objects with sesquilinear pairings.

5.a. "Positive" functions of z. — Recall that we denote by $\mathbf{D}$ the disc $|z| \leqslant 1$ and by $\mathbf{S}$ its boundary. Let $\lambda(z)$ be a meromorphic function defined in some neighbourhood of $\mathbf{S}$. If the neighbourhood is sufficiently small, it has zeros and poles at most on $\mathbf{S}$. We say that λ is "real" if it satisfies $\overline{\lambda} = \lambda$, where $\overline{\lambda}(z)$ is defined as $c(\lambda(-1/c(z)))$ and c is the usual complex conjugation. For instance, if $\alpha \in \mathbb{C}$, the function $z \mapsto \alpha \star z/z$ is "real". If $\lambda(z)$ is "real" and if ψ is a meromorphic function on $\mathbb{C}$ which is real (in the usual sense, *i.e.*, $\psi c = c\psi$), then $\psi \circ \lambda$ is "real". In particular, for any $\alpha \in \mathbb{C}^*$, the function $z \mapsto \Gamma(\alpha \star z/z)$ is "real".

Lemma 5.2. — *Let $\lambda(z)$ be a "real" invertible holomorphic function in some neighbourhood of $\mathbf{S}$. Then there exists an invertible holomorphic function $\mu(z)$ in some neighbourhood of $\mathbf{D}$ such that $\lambda = \pm\mu\overline{\mu}$ in some neighbourhood of $\mathbf{S}$. Moreover, such a function μ is unique up to multiplication by a complex number having modulus equal to* 1.

Definition 5.3. — Let λ be as in the lemma. We say that λ is "positive" if $\lambda = \mu\overline{\mu}$, with μ invertible on $\mathbf{D}$, and "negative" if $\lambda = -\mu\overline{\mu}$.

***Remark 5.4* (Positive or negative "real" meromorphic functions)**

Assume that λ is a nonzero "real" *meromorphic* function in some neighbourhood of $\mathbf{S}$. Then λ can be written as $\prod_i [(z - z_i)\overline{(z - z_i)}]^{m_i} \cdot h$ with $z_i \in \mathbf{S}$, h holomorphic invertible near $\mathbf{S}$ and $\overline{h} = h$: indeed, one shows that, if $z_o \in \mathbf{S}$, then $\overline{z - z_o} = (z + z_o) \cdot (-1/z_o z)$; therefore, if $z_o \in \mathbf{S}$ is a pole or a zero of λ with order $m_o \in \mathbb{Z}$, then $-z_o$ has the same order, hence the product decomposition of λ.

It follows from Lemma 5.2 that $\lambda = \pm g\overline{g}$, with $g = \mu \prod_i (z - z_i)^m_i$, $z_i \in \mathbf{S}$, $m_i \in \mathbb{Z}$ and μ holomorphic invertible on $\mathbf{D}$. This decomposition is not unique, as one may change some z_i with $-z_i$. The sign is also non uniquely determined, as we have, for any $z_o \in \mathbf{S}$,

$$-1 = \Big(\frac{z - z_o}{z + z_o}\Big) \cdot \overline{\Big(\frac{z - z_o}{z + z_o}\Big)}.$$

Nevertheless, the decomposition and the sign are uniquely defined (up to a multiplicative constant) if we fix a choice of a "square root" of the divisor of λ so that no two points in the support of this divisor are opposed, and if we impose that the divisor of g is contained in this "square root". The sign does not depend on the choice of such a "square root". We say that λ is "positive" if the sign is $+$, and "negative" if the sign is $-$.

Proof of Lemma 5.2. — One can write $\lambda = \nu \cdot \overline{\mu}$ with μ holomorphic invertible near $\mathbf{D}$ and ν meromorphic in some neighbourhood of $\mathbf{D}$ and having poles or zeros at 0 at most. The function $c(z) = \nu/\mu = \overline{\nu}/\overline{\mu}$ defines a meromorphic function on $\mathbb{P}^1$ with divisor supported by $\{0, \infty\}$. Thus, $c(z) = c \cdot z^k$ with $c \in \mathbb{C}$ and $k \in \mathbb{Z}$, so $\lambda = cz^k\mu\overline{\mu}$. Moreover, the equality $\overline{\lambda} = \lambda$ implies that $c \in \mathbb{R}$ and $k = 0$. Changing notation for μ gives $\lambda = \pm\mu\overline{\mu}$, with μ invertible on $\mathbf{D}$.

For uniqueness, assume that $\mu\overline{\mu} = \pm 1$ with μ holomorphic invertible in some neighbourhood of $\mathbf{D}$. Then $\pm 1/\overline{\mu}$ is also holomorphic in some neighbourhood of $|z| \geqslant 1$, so μ extends as a holomorphic function on $\mathbb{P}^1$ and thus is constant. This implies that $\mu\overline{\mu} = 1$. □

Lemma 5.5. — *Let $\alpha \in \mathbb{C}$ be such that $\operatorname{Re}\alpha \in [0, 1[$ and $\alpha \neq 0$. Then the meromorphic function*

$$\lambda(z) = \frac{\Gamma(\alpha \star z/z)}{\Gamma(1 - \alpha \star z/z)}$$

is "real" and "positive" (it is holomorphic invertible near $\mathbf{S}$ if $\operatorname{Re}\alpha \neq 0$).

Proof. — That this function is "real" has yet been remarked. The only possible pole/zero of λ on $\mathbf{S}$ is $\pm i$, which occurs if there exists $k \in \mathbb{Z}$ such that $\operatorname{Re}\alpha + k = 0$. It is a simple pole (resp. a simple zero) if $k \geqslant 0$ (resp. $k \leqslant -1$). As we assume $\operatorname{Re}\alpha \in [0, 1[$, the only possibility is when $\operatorname{Re}\alpha = 0$, with $k = 0$ (hence a pole).

Write $\lambda(z)$ as $\Gamma(\alpha \star z/z)^2 \cdot (1/\pi) \sin \pi(\alpha \star z/z)$. It is then equivalent to showing that $(1/\pi) \sin \pi(\alpha \star z/z)$ is "positive" for α as above.

Write $\alpha = \alpha' + i\alpha''$. The result is clear if $\alpha'' = 0$, as we then have $\alpha \star z/z = \alpha' \in]0, 1[$. We thus assume now that $\alpha'' \neq 0$.

For any $\beta \in \mathbb{C}$ with $\beta'' \neq 0$, we put $b = \dfrac{\beta' + \sqrt{\beta'^2 + \beta''^2}}{\beta''}$ and we can write

$$\frac{\beta \star z}{z} = \frac{\beta'' b}{2}\Big(1 + \frac{iz}{b}\Big)\overline{\Big(1 + \frac{iz}{b}\Big)}.$$

If α is as above, we have $n - \alpha', n + \alpha' > 0$ for any $n \geqslant 1$ and we put for $n \geqslant 0$

$$b_n = -\frac{n - \alpha' + \sqrt{(n - \alpha')^2 + \alpha''^2}}{\alpha''}, \quad c_n = \frac{n + \alpha' + \sqrt{(n + \alpha')^2 + \alpha''^2}}{\alpha''}.$$

For $n \geqslant 1$, we have $|b_n|, |c_n| > 1$ and

$$\frac{(n - \alpha) \star z}{z} = \frac{n - \alpha' + \sqrt{(n - \alpha')^2 + \alpha''^2}}{2}\Big(1 + \frac{iz}{b_n}\Big)\overline{\Big(1 + \frac{iz}{b_n}\Big)},$$

$$\frac{(n + \alpha) \star z}{z} = \frac{n + \alpha' + \sqrt{(n + \alpha')^2 + \alpha''^2}}{2}\Big(1 + \frac{iz}{c_n}\Big)\overline{\Big(1 + \frac{iz}{c_n}\Big)}.$$

The number

$$c(\alpha) = \prod_{n \geqslant 1} \frac{(n - \alpha' + \sqrt{(n - \alpha')^2 + \alpha''^2})(n + \alpha' + \sqrt{(n + \alpha')^2 + \alpha''^2})}{4n^2}$$

is (finite and) positive. On the other hand, as $\dfrac{1}{b_n}+\dfrac{1}{c_n}=-\dfrac{\alpha'\alpha''}{n^2}+O(1/n^3)$, the infinite product

$$\prod_{n\geqslant 1}\Bigl(1+\frac{iz}{b_n}\Bigr)\Bigl(1+\frac{iz}{c_n}\Bigr)$$

defines an invertible holomorphic function in some neighbourhood of $\mathbf{D}$. Put

$$g(z)=\Bigl(\frac{c(\alpha)(\alpha'+\sqrt{\alpha'^2+\alpha''^2})}{2}\Bigr)^{1/2}\cdot\Bigl(1+\frac{iz}{c_0}\Bigr)\prod_{n\geqslant 1}\Bigl(1+\frac{iz}{b_n}\Bigr)\Bigl(1+\frac{iz}{c_n}\Bigr).$$

Then we have $(1/\pi)\sin\pi(\alpha\star z/z)=g(z)\overline{g}(z)$. □

If μ is a meromorphic function on some neighbourhood of $\mathbf{D}$, we denote by D_μ its divisor on $\mathbf{D}$. If $\mathscr{M}$ is a $\mathscr{R}_{\mathscr{X}}$-module, we put $\mathscr{M}(D_\mu)=\mathscr{O}_{\mathscr{X}}(D_\mu)\otimes_{\mathscr{O}_{\mathscr{X}}}\mathscr{M}$ with its natural $\mathscr{R}_{\mathscr{X}}$-structure.

Lemma 5.6. — *Let $(\mathscr{T},\mathscr{S})=(\mathscr{M}',\mathscr{M}'',C,\mathscr{S})$ be an object of $\mathrm{MT}(X,w)^{(\mathrm{p})}$. Then, for each μ as above, $(\mathscr{M}'(D_\mu),\mathscr{M}''(D_\mu),\mu\overline{\mu}C,\mathscr{S})$ is an object of $\mathrm{MT}(X,w)^{(\mathrm{p})}$ isomorphic to $(\mathscr{T},\mathscr{S})$.*

Remark 5.7. — We only assume here that $\mathscr{M}',\mathscr{M}''$ are defined in some neighbourhood of $\mathbf{D}$, and not necessarily on Ω_0. This does not change the category $\mathrm{MT}(X,w)^{(\mathrm{p})}$.

Proof. — The isomorphism is given by $\cdot\mu:\mathscr{M}'(D_\mu)\to\mathscr{M}'$ and $\cdot(1/\mu):\mathscr{M}''\to\mathscr{M}''(D_\mu)$. □

5.b. Exponential twist and specialization of a sesquilinear pairing

We now come back to our original situation of § 3.a. Let $\mathscr{T}=(\mathscr{M}',\mathscr{M}'',C)$ be an object of $\mathscr{R}$- Triples(X). We have defined the object ${}^{\mathscr{F}}\mathscr{T}=({}^{\mathscr{F}}\mathscr{M}',{}^{\mathscr{F}}\mathscr{M}'',{}^{\mathscr{F}}C)$ of $\mathscr{R}$- Triples(Z). If we assume that $\mathscr{M}',\mathscr{M}''$ are strict and strictly specializable along $t'=0$, then ${}^{\mathscr{F}}\mathscr{M}',{}^{\mathscr{F}}\mathscr{M}''$ are strictly specializable along $\tau=0$. Then, for $\operatorname{Re}\alpha\in[-1,0[$, $\Psi_{\tau,\alpha}{}^{\mathscr{F}}\mathscr{T}$ is defined as in [**3**, § 3.6]. Recall (*cf.* (3.6.2) in *loc. cit.*) that we denote by $\mathscr{N}_\tau:\Psi_{\tau,\alpha}{}^{\mathscr{F}}\mathscr{T}\to\Psi_{\tau,\alpha}{}^{\mathscr{F}}\mathscr{T}(-1)$ the morphism $(-i\mathrm{N}_\tau,i\mathrm{N}_\tau)$. If $\alpha=-1$ (more generally if α is real) we have $\Psi_{\tau,\alpha}{}^{\mathscr{F}}\mathscr{T}=\psi_{\tau,\alpha}{}^{\mathscr{F}}\mathscr{T}$. We also consider, as in § 3.6.b of *loc. cit.*, the vanishing cycle object $\phi_{\tau,0}{}^{\mathscr{F}}\mathscr{T}$.

The purpose of this section is to extend Proposition 4.1(iv) to objects of $\mathscr{R}$- Triples. It will be convenient to assume, in the following, that $\mathscr{M}'=\mathscr{M}'_{\min}$ and $\mathscr{M}''=\mathscr{M}''_{\min}$; with such an assumption, we will not have to define a sesquilinear pairing on the minimal extensions used in Proposition 4.1(iv), as we can use the given C.

Proposition 5.8 **(*cf.* [3, Prop. A.4.2]).** — *For $\mathscr{T}$ as above, we have isomorphisms in $\mathscr{R}$- Triples(X):*

$$\begin{aligned}\bigl(\Psi_{\tau,\alpha}{}^{\mathscr{F}}\mathscr{T},\mathscr{N}_\tau\bigr)&\xrightarrow{\ \sim\ }i_{\infty,+}\bigl(\Psi_{t',\alpha}\mathscr{T},\mathscr{N}_{t'}\bigr),\quad\forall\alpha\neq-1\ \textit{with}\ \operatorname{Re}\alpha\in[-1,0[,\\ \bigl(\phi_{\tau,0}{}^{\mathscr{F}}\mathscr{T},\mathscr{N}_\tau\bigr)&\xrightarrow{\ \sim\ }i_{\infty,+}\bigl(\psi_{t',-1}\mathscr{T},\mathscr{N}_{t'}\bigr),\end{aligned}$$

and an exact sequence

$$0 \longrightarrow i_{\infty,+} \ker \mathscr{N}_{t'} \longrightarrow \ker \mathscr{N}_\tau \longrightarrow \mathscr{T} \longrightarrow 0$$

inducing an isomorphism $P\,\mathrm{gr}_0^{\mathrm{M}}\,\psi_{\tau,-1}{}^{\mathscr{F}}\mathscr{T} \xrightarrow{\sim} \mathscr{T}$.

***Corollary 5.9 (cf.** [**3**, Cor. A.4.3]).* — *Assume that* $\mathscr{T}$ *is an object of* $\mathrm{MT}^{(\mathrm{r})}(X,w)$ *(*resp. $(\mathscr{T},\mathscr{S})$ *is an object of* $\mathrm{MT}^{(\mathrm{r})}(X,w)^{(\mathrm{p})}$*). Then, for any* $\alpha \in \mathbb{C}$ *with* $\mathrm{Re}\,\alpha \in [-1,0[$*,* $(\Psi_{\tau,\alpha}{}^{\mathscr{F}}\mathscr{T},\mathscr{N}_\tau)$ *induces by gradation an object of* $\mathrm{MLT}^{(\mathrm{r})}(X,w;-1)$ *(*resp. *an object of* $\mathrm{MLT}^{(\mathrm{r})}(X,w;-1)^{(\mathrm{p})}$*).*

Proof of Corollary 5.9. — Suppose that Proposition 5.8 is proved. Assume first that $\mathscr{T}$ is an object of $\mathrm{MT}^{(\mathrm{r})}(X,w)$. Then, by definition, $i_{\infty,+}\big(\mathrm{gr}_\bullet^{\mathrm{M}}\,\Psi_{t',\alpha}\mathscr{T},\mathrm{gr}_{-2}^{\mathrm{M}}\,\mathscr{N}_{t'}\big)$ is an object of $\mathrm{MLT}^{(\mathrm{r})}(X,w;-1)$ for any α with $\mathrm{Re}\,\alpha \in [-1,0[$; therefore, so is $\big(\mathrm{gr}_\bullet^{\mathrm{M}}\,\Psi_{\tau,\alpha}{}^{\mathscr{F}}\mathscr{T},\mathrm{gr}_{-2}^{\mathrm{M}}\,\mathscr{N}_\tau\big)$ for any such $\alpha \neq -1$. When $\alpha = -1$, as ${}^{\mathscr{F}}\mathscr{M}', {}^{\mathscr{F}}\mathscr{M}''$ are equal to their minimal extension along $\tau = 0$ (*cf.* Proposition 4.1) the morphism

$$\mathscr{C}an : \big(\psi_{\tau,-1}{}^{\mathscr{F}}\mathscr{T},\mathrm{M}_\bullet(\mathscr{N}_\tau)\big) \longrightarrow \big(\phi_{\tau,0}{}^{\mathscr{F}}\mathscr{T}(-1/2),\mathrm{M}_{\bullet-1}(\mathscr{N}_\tau)\big),$$

(*cf.* § 3.6.b in *loc. cit.*) is onto. It is strictly compatible with the monodromy filtrations (*cf.* [**6**, Lemme 5.1.12]), and induces an isomorphism $P\,\mathrm{gr}_\ell^{\mathrm{M}}\,\psi_{\tau,-1}{}^{\mathscr{F}}\mathscr{T} \xrightarrow{\sim} P\,\mathrm{gr}_{\ell-1}^{\mathrm{M}}\,\phi_{\tau,0}{}^{\mathscr{F}}\mathscr{T}(-1/2)$ for any $\ell \geqslant 1$, hence an isomorphism

$$P\,\mathrm{gr}_\ell^{\mathrm{M}}\,\psi_{\tau,-1}{}^{\mathscr{F}}\mathscr{T} \xrightarrow{\sim} i_{\infty,+}P\,\mathrm{gr}_{\ell-1}^{\mathrm{M}}\,\psi_{t',-1}\mathscr{T}(-1/2).$$

By assumption on $\mathscr{T}$, the right-hand term is an object of $\mathrm{MT}^{(\mathrm{r})}(X,w+\ell)$, hence so is the left-hand term. Moreover, $P\,\mathrm{gr}_0^{\mathrm{M}}\,\psi_{\tau,-1}{}^{\mathscr{F}}\mathscr{T} \simeq \mathscr{T}$ is in $\mathrm{MT}^{(\mathrm{r})}(X,w)$. This gives the claim when $\alpha = -1$.

In the polarized case, we can reduce to the case where $w = 0$, $\mathscr{M}' = \mathscr{M}''$, $\mathscr{S} = (\mathrm{Id},\mathrm{Id})$ and $C^* = C$. Then these properties are satisfied by the objects above, and the polarizability on the τ-side follows from the polarizability on the t'-side. □

The proof of the proposition will involve the computation of a Mellin transform with kernel given by a function $I_{\widehat{\chi}}(t,s,z)$. We first analyze this Mellin transform.

The function $I_{\widehat{\chi}}(t,s,z)$. — Let $\widehat{\chi} \in C_c^\infty(\widehat{\mathbb{A}}^1,\mathbb{R})$ be such that $\widehat{\chi}(\tau) \equiv 1$ near $\tau = 0$. For any $z \in \mathbf{S}$, $t \in \mathbb{A}^1$ and $s \in \mathbb{C}$ such that $\mathrm{Re}(s+1) > 0$, put

$$I_{\widehat{\chi}}(t,s,z) = \int_{\widehat{\mathbb{A}}^1} e^{z\overline{t\tau}-t\tau/z}\,|\tau|^{2s}\,\widehat{\chi}(\tau)\,\tfrac{i}{2\pi}\,d\tau \wedge d\overline{\tau}. \tag{5.10}$$

We also write $I_{\widehat{\chi}}(t',s,z)$ when working in the coordinate t' on $\mathbb{P}^1$. We will use the following coarse properties (they are similar to the properties described for the function $\widehat{I}_\chi$ of § 3.6.b of *loc. cit.*).

(i) Denote by $I_{\widehat{\chi},k,\ell}(t,s,z)$ $(k,\ell \in \mathbb{Z})$ the function obtained by integrating $|\tau|^{2s}\,\tau^k\overline{\tau}^\ell$. Then, for any $s \in \mathbb{C}$ with $\mathrm{Re}(s+1+(k+\ell)/2) > 0$ and any $z \in \mathbf{S}$, the function $(t,s,z) \to I_{\widehat{\chi},k,\ell}(t,s,z)$ is C^∞, depends holomorphically on s, and satisfies $\lim_{t\to\infty} I_{\widehat{\chi},k,\ell}(t,s,z) = 0$ locally uniformly with respect to s,z.

(ii) We have

$$\begin{aligned} tI_{\widehat{\mathcal{X}},k,\ell} &= z(s+k)I_{\widehat{\mathcal{X}},k-1,\ell} + zI_{\partial\widehat{\mathcal{X}}/\partial\tau,k,\ell} & \eth_t I_{\widehat{\mathcal{X}},k,\ell} &= -I_{\widehat{\mathcal{X}},k+1,\ell}\\ \overline{t}I_{\widehat{\mathcal{X}},k,\ell} &= \overline{z}(s+\ell)I_{\widehat{\mathcal{X}},k,\ell-1} + \overline{z}I_{\partial\widehat{\mathcal{X}}/\partial\overline{\tau},k,\ell} & \overline{\eth}_t I_{\widehat{\mathcal{X}},k,\ell} &= -I_{\widehat{\mathcal{X}},k,\ell+1},\end{aligned}$$

where the equalities hold on the common domain of definition (with respect to s) of the functions involved. Notice that the functions $I_{\partial\widehat{\mathcal{X}}/\partial\tau,k,\ell}$ and $I_{\partial\widehat{\mathcal{X}}/\partial\overline{\tau},k,\ell}$ are C^∞ on $\mathbb{P}^1\times\mathbb{C}\times\mathbf{S}$, depend holomorphically on s, and are infinitely flat at $t=\infty$.

It follows that, for $\mathrm{Re}(s+1)>0$, we have

$$\begin{aligned} t\eth_t I_{\widehat{\mathcal{X}}} &= -z(s+1)I_{\widehat{\mathcal{X}}} + zI_{\partial\widehat{\mathcal{X}}/\partial\tau,1,0},\\ \overline{t\eth_t} I_{\widehat{\mathcal{X}}} &= -\overline{z}(s+1)I_{\widehat{\mathcal{X}}} + \overline{z}I_{\partial\widehat{\mathcal{X}}/\partial\overline{\tau},0,1}.\end{aligned}\tag{5.11}$$

(iii) Moreover, for any $p\geqslant 0$, any $s\in\mathbb{C}$ with $\mathrm{Re}(s+1+(k+\ell)/2)>p$ and any $z\in\mathbf{S}$, all derivatives up to order p of $I_{\widehat{\mathcal{X}},k,\ell}(t',s,z)$ with respect to t' tend to 0 when $t'\to 0$, locally uniformly with respect to s,z; therefore, $I_{\widehat{\mathcal{X}},k,\ell}(t,s,z)$ extends as a function of class C^p on $\mathbb{P}^1\times\{\mathrm{Re}(s+1+(k+\ell)/2)>p\}\times\mathbf{S}$, holomorphic with respect to s.

Mellin transform with kernel $I_{\widehat{\mathcal{X}}}(t,s,z)$. — We will work near $z_o\in\mathbf{S}$. For any local sections μ',μ'' of $\mathscr{M}',\mathscr{M}''$ and any C^∞ relative form φ of maximal degree on $X\times\mathbf{S}$ with compact support contained in the open set where μ',μ'' are defined, the function

$$(s,z)\longmapsto\big\langle C(\mu',\overline{\mu''}),\varphi I_{\widehat{\mathcal{X}}}(t,s,z)\big\rangle$$

is holomorphic with respect to s for $\mathrm{Re}\,s\gg 0$ (according to (i)), continuous with respect to z. One shows as in Lemma 3.6.6 of *loc. cit.*, using (iii), that it extends as a meromorphic function on the whole complex plane, with poles on sets $s=\alpha\star z/z$.

This result can easily be extended to local sections μ',μ'' of $\widetilde{\mathscr{M}'},\widetilde{\mathscr{M}''}$: indeed, this has to be verified only near $t=\infty$; there exists $p\geqslant 0$ such that, in the neighbourhood of the support of φ, $t'^p\mu',t'^p\mu''$ are local sections of $\mathscr{M}',\mathscr{M}''$; apply then the previous argument to the kernel $|t|^{2p}I_{\widehat{\mathcal{X}}}(t,s,z)$. In the following, we will write $\big\langle C(\mu',\overline{\mu''}),\varphi I_{\widehat{\mathcal{X}}}(t,s,z)\big\rangle$ instead of $\big\langle C(t'^p\mu',t'^p\overline{\mu''}),\varphi\,|t|^{2p}\,I_{\widehat{\mathcal{X}}}(t,s,z)\big\rangle$ near $t=\infty$.

Lemma 5.12. — *Assume that φ is compactly supported on $(X\smallsetminus\infty)\times\mathbf{S}$. Then, for μ',μ'' as above, we have*

$$\mathrm{Res}_{s=-1}\big\langle C(\mu',\overline{\mu''}),\varphi I_{\widehat{\mathcal{X}}}(t,s,z)\big\rangle=\big\langle C(\mu',\overline{\mu''}),\varphi\big\rangle.$$

Proof. — The function $(s+1)I_{\widehat{\mathcal{X}}}(t,s,z)$ can be extended to the domain $\mathrm{Re}(s+1)>-1/2$ as C^∞ function of (t,s,z), holomorphic with respect to s: use (ii) with $k=1$, $\ell=0$ to write $(s+1)I_{\widehat{\mathcal{X}}}(t,s,z)=(t/z)I_{\widehat{\mathcal{X}},1,0}-I_{\partial\widehat{\mathcal{X}}/\partial\tau,1,0}$. It is then enough to show that this C^∞ function, when restricted to $s=-1$, is identically equal to 1. It amounts to proving that, for any t,z, $\lim_{\substack{s\to-1\\ \mathrm{Re}\,s>-1}}\big[(s+1)I_{\widehat{\mathcal{X}}}(t,s,z)\big]=1$. For $\mathrm{Re}\,s>-1$ we have

$I_{\widehat{\chi}}(t,s,z)=J(t,s,z)+H_{\widehat{\chi}}(t,s,z)$, with

$$J(t,s,z)=\int_{|\tau|\leqslant 1} e^{-2i\operatorname{Im} t\tau/z}\,|\tau|^{2s}\,\tfrac{i}{2\pi}\,d\tau\wedge d\overline{\tau},$$

and $H_{\widehat{\chi}}$ extends as a C^∞ function on $\mathbb{A}^1\times\mathbb{C}\times\mathbf{S}$, holomorphic with respect to s. It is therefore enough to work with $J(t,s,z)$ instead of $I_{\widehat{\chi}}$. We now have

$$\begin{aligned}J(t,s,z)&=|t|^{-2(s+1)}\int_{|u|<|t|} e^{-2i\operatorname{Im} u}\,|u|^{2s}\,\tfrac{i}{2\pi}\,du\wedge d\overline{u}\\&=\frac{1}{\pi}\,|t|^{-2(s+1)}\int_0^{2\pi}\!\!\int_0^{|t|} e^{-2i\rho\sin\theta}\rho^{2s+1}\,d\rho d\theta.\end{aligned}$$

Now, integrating by part, we get

$$\int_0^{|t|} e^{-2i\rho\sin\theta}\rho^{2s+1}\,d\rho=\frac{|t|^{2s+2}\,e^{-2i|t|\sin\theta}}{2s+2}+\frac{2i\sin\theta}{2s+2}\int_0^{|t|} e^{-2i\rho\sin\theta}\rho^{2s+2}\,d\rho,$$

and the second integral is holomorphic near $s=-1$. Therefore,

$$(s+1)J(t,s)=\frac{|t|^{-2(s+1)}}{2\pi}\int_0^{2\pi}\Big[|t|^{2s+2}\,e^{-2i|t|\sin\theta}+2i\sin\theta\int_0^{|t|} e^{-2i\rho\sin\theta}\rho^{2s+2}\,d\rho\Big]d\theta.$$

Taking $s\to -1$ gives

$$\lim_{\substack{s\to -1\\ \operatorname{Re}s>-1}}\,[(s+1)J(t,s)]=\frac{1}{2\pi}\int_0^{2\pi}\Big[e^{-2i|t|\sin\theta}+2i\sin\theta\int_0^{|t|} e^{-2i\rho\sin\theta}\,d\rho\Big]d\theta.$$

Now,

$$2i\sin\theta\int_0^{|t|} e^{-2i\rho\sin\theta}\,d\rho=-\int_0^{|t|}\frac{d}{d\rho}\big(e^{-2i\rho\sin\theta}\big)d\rho=1-e^{-2i|t|\sin\theta},$$

hence $\lim_{\substack{s\to -1\\ \operatorname{Re}s>-1}}[(s+1)J(t,s)]=1$. □

Remark 5.13. — To simplify notation, we now put

$$J_{\widehat{\chi}}(t,s,z)=\frac{1}{\Gamma(s+1)}\,I_{\widehat{\chi}}(t,s,z).$$

Using (ii) as in the previous lemma, one obtains that there exists a C^∞ function on $\mathbb{A}^1\times\mathbb{C}\times\mathbf{S}$, holomorphic with respect to s, which coincides with $J_{\widehat{\chi}}$ when $\operatorname{Re}(s+1)>0$. This implies that, when the support of φ does not contain ∞, the meromorphic function $s\mapsto\big\langle C(\mu',\overline{\mu''}),\varphi J_{\widehat{\chi}}(t,s,z)\big\rangle$ is entire.

We now work near ∞ with the coordinate t'. Assume that μ' is a local section of $V^{(z_o)}_{a_1+1}\widetilde{\mathscr{M}'}$ and that μ'' is a local section of $V^{(-z_o)}_{a_2+1}\widetilde{\mathscr{M}''}$. Assume moreover that the class of μ' in $\operatorname{gr}^{V^{(z_o)}}_{a_1+1}\widetilde{\mathscr{M}'}$ is in $\psi_{t',\alpha_1+1}\widetilde{\mathscr{M}'}$, and that the class of μ'' in $\operatorname{gr}^{V^{(-z_o)}}_{a_2+1}\widetilde{\mathscr{M}''}$ is in $\psi_{t',\alpha_2+1}\widetilde{\mathscr{M}''}$. Then one proves as in Lemma 3.6.6 of *loc. cit.* that $\big\langle C(\mu',\overline{\mu''}),\varphi J_{\widehat{\chi}}(t',s,z)\big\rangle$ has poles on sets $s=\gamma\star z/z$ with γ such that $2\operatorname{Re}\gamma<a_1+a_2$ or $\gamma=\alpha_1=\alpha_2$.

Let us then consider the case where $\alpha_1=\alpha_2:=\alpha$. Then, if ψ has compact support and vanishes along $t'=0$, the previous result shows that $\langle C(\mu',\overline{\mu''}),\psi J_{\widehat{\chi}}(t',s,z)\rangle$ has no pole along $s=\alpha\star z/z$. It follows that $\operatorname{Res}_{s=\alpha\star z/z}\langle C(\mu',\overline{\mu''}),\varphi J_{\widehat{\chi}}(t',s,z)\rangle$ only depends on the restriction of φ to $t'=0$; in other words, it is the direct image of a distribution on $t'=0$ by the inclusion i_∞. We will identify this distribution with $\psi_{t',\alpha+1}C$. We will put

$$i_\infty^*\varphi=\frac{\varphi_{|\infty}}{\frac{i}{2\pi}dt'\wedge d\overline{t'}}.$$

Lemma 5.14. — *For any $\alpha\in\mathbb{C}$ with $\operatorname{Re}\alpha\notin\mathbb{N}$, and μ',μ'' lifting local sections $[\mu'],[\mu'']$ of $\psi_{t',\alpha+1}\widetilde{\mathscr{M}}'$, $\psi_{t',\alpha+1}\widetilde{\mathscr{M}}''$, we have, when the support of φ is contained in the open set where μ',μ'' are defined,*

$$\operatorname{Res}_{s=\alpha\star z/z}\langle C(\mu',\overline{\mu''}),\varphi J_{\widehat{\chi}}(t',s,z)\rangle=\frac{1}{\Gamma(-\alpha\star z/z)}\langle\psi_{t',\alpha+1}C([\mu'],\overline{[\mu'']}),i_\infty^*\varphi\rangle.$$

Proof. — Let $\chi(t')$ be a C^∞ function which has compact support and is $\equiv 1$ near $t'=0$. As $\varphi-i_\infty^*\varphi\wedge\chi(t')\frac{i}{2\pi}dt'\wedge d\overline{t'}$ vanishes along $t'=0$, the left-hand term in the lemma is equal to

$$\operatorname{Res}_{s=\alpha\star z/z}\langle C(\mu',\overline{\mu''}),J_{\widehat{\chi}}(t',s,z)i_\infty^*\varphi\wedge\chi(t')\tfrac{i}{2\pi}dt'\wedge d\overline{t'}\rangle. \tag{5.15}$$

On the other hand, as $\operatorname{Re}\alpha\notin\mathbb{N}$, we have $\alpha\star z/z\notin\mathbb{N}$ for any $z\in\mathbf{S}$, and the function $1/\Gamma(-s)$ does not vanish when $s=\alpha\star z/z$ for any such α and z. Therefore, by definition of $\psi_{t',\alpha+1}C$, the right-hand term is equal to

$$\operatorname{Res}_{s=\alpha\star z/z}\frac{1}{\Gamma(-s)}\langle C(\mu',\overline{\mu''}),|t'|^{2(s+1)}\,i_\infty^*\varphi\wedge\chi(t')\tfrac{i}{2\pi}dt'\wedge d\overline{t'}\rangle. \tag{5.16}$$

Put $\widetilde{J}_{\widehat{\chi}}(t,s,z)=|t|^{2(s+1)}J_{\widehat{\chi}}(t,s,z)$. Then, by (5.11) expressed in the coordinate t', we have

$$t'\frac{\partial\widetilde{J}_{\widehat{\chi}}}{\partial t'}=-\widetilde{J}_{\partial\widehat{\chi}/\partial\tau,1,0},\quad \overline{t'}\frac{\partial\widetilde{J}_{\widehat{\chi}}}{\partial\overline{t'}}=-\widetilde{J}_{\partial\widehat{\chi}/\partial\overline{\tau},0,1},$$

and both functions $\widetilde{J}_{\partial\widehat{\chi}/\partial\tau,1,0}$ and $\widetilde{J}_{\partial\widehat{\chi}/\partial\overline{\tau},0,1}$ extend as C^∞ functions, infinitely flat at $t'=0$ and holomorphic with respect to $s\in\mathbb{C}$. Put

$$\widetilde{K}_{\widehat{\chi}}(t',s,z)=-\int_0^1\big[\widetilde{J}_{\partial\widehat{\chi}/\partial\tau,1,0}(\lambda t',s,z)+\widetilde{J}_{\partial\widehat{\chi}/\partial\overline{\tau},0,1}(\lambda t',s,z)\big]d\lambda.$$

Then $\widetilde{K}_{\widehat{\chi}}$ is of the same kind. Notice now that, for any $s\in\mathbb{C}$ with $\operatorname{Re}(s+1)\in\,]0,1/4[$ and any $z\in\mathbf{S}$, we have

$$\lim_{t\to\infty}(|t|^{2(s+1)}\,I_{\widehat{\chi}}(t,s,z))=\frac{\Gamma(s+1)}{\Gamma(-s)}. \tag{5.17}$$

[Let us sketch the proof of this statement. We assume for instance that $\widehat{\chi}\equiv 1$ when $|\tau|\leqslant 1$. We can replace $I_{\widehat{\chi}}(t,s,z)$ with

$$\int_{|\tau|\leqslant 1}e^{z\overline{t\tau}-t\tau/z}\,|\tau|^{2s}\,\tfrac{i}{2\pi}\,d\tau\wedge d\overline{\tau}$$

without changing the limit, and we are reduced to computing

$$\frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty} e^{-2i\rho\sin\theta}\rho^{2s+1}d\rho\,d\theta.$$

Using the Bessel function $J_0(r)=\frac{1}{2\pi}\int_0^{2\pi}e^{-ir\sin\theta}d\theta$, this integral is written as

$$2\int_0^{\infty}\rho^{2s+1}J_0(2\rho)d\rho=\frac{1}{2^{2s+1}}\int_0^{\infty}r^{2s+1}J_0(r)dr,$$

and it is known (*cf.* [**8**, § 13.24, p. 391]) that, on the strip $\operatorname{Re}(s+1)\in\,]0,1/4[$, the latter integral is equal to $2^{2s+1}\Gamma(s+1)/\Gamma(-s)$.]

On this strip, we can therefore write $\widetilde{J}_{\widehat{\chi}}(t',s,z)=\widetilde{K}_{\widehat{\chi}}(t',s,z)+1/\Gamma(-s)$, by Taylor's formula. For fixed $t'\neq 0$ and $z\in\mathbf{S}$, both functions are holomorphic for $\operatorname{Re}(s+1)>0$, hence they coincide when $\operatorname{Re}(s+1)>0$ and we thus have on this domain

$$J_{\widehat{\chi}}(t',s,z)=\frac{|t'|^{2(s+1)}}{\Gamma(-s)}+K_{\widehat{\chi}}(t',s,z).$$

By the properties of $K_{\widehat{\chi}}$, this implies that the function

$$s\longmapsto\big\langle C(\mu',\overline{\mu''}),K_{\widehat{\chi}}(t',s,z)i_\infty^*\varphi\wedge\chi(t')\tfrac{i}{2\pi}dt'\wedge d\overline{t'}\big\rangle$$

is entire for any $z\in\mathbf{S}$. Hence, there exists an entire function of s such that the difference of the meromorphic functions considered in (5.15) and (5.16), when restricted to the half-plane $\operatorname{Re}(s+1)>p$ (with p large enough so that they are holomorphic on the half-plane), coincides with this entire function. This difference is therefore identically equal to this entire function of s, and (5.15) and (5.16) coincide. This proves the lemma. □

Proof of Proposition 5.8. — We will work near $z_o\in\mathbf{S}$. By definition (*cf.* § 2.c), given any local sections $[m'],[m'']$ of $\psi_{\tau,\alpha}{}^{\mathscr{F}}\!\mathscr{M}',\psi_{\tau,\alpha}{}^{\mathscr{F}}\!\mathscr{M}''$ and local liftings m',m'' in $V_{a'}{}^{\mathscr{F}}\!\mathscr{M}',V_{a''}{}^{\mathscr{F}}\!\mathscr{M}''$ with $a'=\ell_{z_o}(\alpha)$ and $a''=\ell_{-z_o}(\alpha)$, we have, for any C^∞ relative form φ of maximal degree on $X\times\mathbf{S}$,

$$\big\langle\psi_{\tau,\alpha}{}^{\mathscr{F}}C([m'],\overline{[m'']}),\varphi\big\rangle=\operatorname{Res}_{s=\alpha\star z/z}\big\langle{}^{\mathscr{F}}C(m',\overline{m''}),\varphi\,|\tau|^{2s}\,\widehat{\chi}(\tau)\tfrac{i}{2\pi}\,d\tau\wedge d\overline{\tau}\big\rangle,\tag{5.18}$$

where $\widehat{\chi}\equiv 1$ near $\tau=0$. In particular, for sections m',m'' of the form $\mu'\otimes\mathcal{E}^{-t\tau/z}$, $\mu''\otimes\mathcal{E}^{-t\tau/z}$ with μ',μ'' local sections of $\widetilde{\mathscr{M}}$, the definition of ${}^{\mathscr{F}}C$ implies that the right-hand term above can be written as

$$\operatorname{Res}_{s=\alpha\star z/z}\big\langle C(\mu',\overline{\mu''}),\varphi I_{\widehat{\chi}}(t,s,z)\big\rangle.\tag{5.19}$$

[Here, we mean that both functions

$$\big\langle C(\mu',\overline{\mu''}),\varphi I_{\widehat{\chi}}(t,s,z)\big\rangle\quad\text{and}\quad\big\langle{}^{\mathscr{F}}C(\mu'\otimes\mathcal{E}^{-t\tau/z},\overline{\mu''\otimes\mathcal{E}^{-t\tau/z}}),\varphi\,|\tau|^{2s}\,\widehat{\chi}(\tau)\tfrac{i}{2\pi}\,d\tau\wedge d\overline{\tau}\big\rangle,$$

a priori defined for $\operatorname{Re}s\gg 0$, are extended as meromorphic functions of s on the whole complex plane.] Moreover, by $\mathscr{R}_{\mathscr{X}}$-linearity, it is enough to prove Proposition 5.8 for such sections.

Proof of Proposition 5.8 *away from* ∞. — This is the easy part of the proof. We only have to consider $\alpha=-1$ and, for φ compactly supported on $(X\smallsetminus\infty)\times\mathbf{S}$, we are reduced to proving that

$$\operatorname{Res}_{s=-1}\big\langle C(\mu',\overline{\mu''}),\varphi I_{\widehat\chi}(t,s,z)\big\rangle=\big\langle C(\mu',\overline{\mu''}),\varphi\big\rangle,$$

for local sections μ',μ'' of $\widetilde{\mathscr{M}'},\widetilde{\mathscr{M}''}$. This is Lemma 5.12 □

Proof of Proposition 5.8 *near* ∞ *for* $\alpha\neq-1,0$. — The question is local on $\mathbf{D}$. We can compute (5.18) by using liftings of m',m'' in $\operatorname{gr}^U_{a'+1}{}^{\mathscr{F}}\!\mathscr{M}',\operatorname{gr}^U_{a''+1}{}^{\mathscr{F}}\!\mathscr{M}''$, according to (4.15). By $\mathscr{R}$-linearity, we only consider sections $m'=t'^{-1}\mu'\otimes\mathcal{E}^{-t\tau/z}$, $m''=t'^{-1}\mu''\otimes\mathcal{E}^{-t\tau/z}$, where μ' is a local section of $V_{a'}\widetilde{\mathscr{M}'}$ and μ'' of $V_{a''}\widetilde{\mathscr{M}''}$. According to (5.19), we have

$$\big\langle\psi_{\tau,\alpha}{}^{\mathscr{F}}C([m'],\overline{[m'']}),\varphi\big\rangle=\operatorname{Res}_{s=\alpha\star z/z}\big\langle C(t'^{-1}\mu',\overline{t'^{-1}\mu''}),\varphi I_{\widehat\chi}(t,s,z)\big\rangle,$$

and, from Lemma 5.14, this is

$$\begin{aligned}\frac{\Gamma(1+\alpha\star z/z)}{\Gamma(-\alpha\star z/z)}&\big\langle\psi_{t',\alpha+1}C([t'^{-1}\mu'],\overline{[t'^{-1}\mu'']}),i_\infty^*\varphi\big\rangle\\&=\frac{\Gamma(1+\alpha\star z/z)}{\Gamma(-\alpha\star z/z)}\big\langle\psi_{t',\alpha}C([\mu'],\overline{[\mu'']}),i_\infty^*\varphi\big\rangle.\end{aligned}$$

By Lemma 5.5 and its proof, we have $\Gamma(1+\alpha\star z/z)/\Gamma(-\alpha\star z/z)=\mu\overline{\mu}$, with $D_\mu=-D_\alpha$ (recall that D_α was defined in Proposition 4.1(iva)), as we assume $\operatorname{Re}\alpha\in[-1,0[$. We then apply Lemma 5.6. □

Proof of Proposition 5.8 *near* ∞ *for* $\alpha=0$. — By the same reduction as above, we consider local sections m'_0,m''_0 of $V_0{}^{\mathscr{F}}\!\mathscr{M}',V_0{}^{\mathscr{F}}\!\mathscr{M}''$ of the form $m'_0=\mu'_1\otimes\mathcal{E}^{-t\tau/z}$, $m''_0=\mu''_1\otimes\mathcal{E}^{-t\tau/z}$, where μ'_1,μ''_1 are local sections of $V_1\widetilde{\mathscr{M}'},V_1\widetilde{\mathscr{M}''}$. We notice[1] that $\eth_\tau(-t'm''_0)=m''_0$ by (3.3) and, using [**3**, (3.6.23)] with $m''_{-1}=-t'm''_0$ (and replacing there t with τ), we get

$$\begin{aligned}\big\langle\phi_{\tau,0}{}^{\mathscr{F}}C([m'_0],\overline{[m''_0]}),\varphi\big\rangle&=\big\langle\phi_{\tau,0}{}^{\mathscr{F}}C([m'_0],\overline{[\eth_\tau m''_{-1}]}),\varphi\big\rangle\\&=-z^{-1}\big\langle\psi_{\tau,-1}{}^{\mathscr{F}}C([\tau m'_0],\overline{[m''_{-1}]}),\varphi\big\rangle\\&=z^{-1}\operatorname{Res}_{s=-1}\big\langle{}^{\mathscr{F}}C(m'_0,\overline{t'm''_0}),\varphi\tau\,|\tau|^{2s}\,\widehat\chi(\tau)\tfrac{i}{2\pi}d\tau\wedge d\overline{\tau}\big\rangle\\&=z^{-1}\operatorname{Res}_{s=-1}\big\langle C(\mu'_1,\overline{\mu''_1}),\varphi\overline{t'}I_{\widehat\chi,1,0}\big\rangle,\end{aligned}$$

1. I thank the referee for correcting a previous wrong proof and pointing out that, in the formula of [**3**, Lemma 3.6.33] which was previously used here, the term $|t|^{2s}$ has to be replaced with $|t|^{2s}-s$, making the right-hand term in this formula independent of χ.

by definition of ${}^{\mathscr{F}}C$. Now, by (ii) after (5.10), we have $z^{-1}\overline{t'}I_{\widehat{\mathcal{X}},1,0} = (s+1)|t'|^2 I_{\widehat{\mathcal{X}}} + |t'|^2 I_{\partial\widehat{\mathcal{X}}/\partial\tau,1,0}$, and the second term will not contribute to the residue, so

$$\begin{aligned}\big\langle \phi_{\tau,0}{}^{\mathscr{F}}C([m_0'],\overline{[m_0'']}),\varphi\big\rangle &= \operatorname{Res}_{s=-1}\big\langle C(\mu_1',\overline{\mu_1''}),\varphi(s+1)|t'|^2 I_{\widehat{\mathcal{X}}}\big\rangle\\ &= \operatorname{Res}_{s=-1}\big\langle C(t'\mu_1',\overline{t'\mu_1''}),\varphi J_{\widehat{\mathcal{X}}}\big\rangle \quad \text{by (5.13)}\\ &= \big\langle \psi_{t',0}C(t'\mu_1',\overline{t'\mu_1''}),i_\infty^*\varphi\big\rangle \quad \text{by Lemma 5.14 with } \alpha=-1\\ &= \psi_{t',-1}C(\mu_{-1}',\overline{\mu_{-1}''}),i_\infty^*\varphi\big\rangle,\end{aligned}$$

if we put $\mu_{-1} = t'^2\mu_1$. □

Proof of Proposition 5.8 *near* ∞ *for* $\alpha=-1$. — Let us first explain how $\psi_{\tau,-1}{}^{\mathscr{F}}C$ is defined and how it induces a sesquilinear pairing on $P\operatorname{gr}_0^{\mathrm{M}}\psi_{\tau,-1}{}^{\mathscr{F}}\mathscr{M}', P\operatorname{gr}_0^{\mathrm{M}}\psi_{\tau,-1}{}^{\mathscr{F}}\mathscr{M}''$.

In order to compute $\psi_{\tau,-1}{}^{\mathscr{F}}C$, we lift local sections $[m'],[m'']$ of $\psi_{\tau,-1}{}^{\mathscr{F}}\mathscr{M}',\psi_{\tau,-1}{}^{\mathscr{F}}\mathscr{M}''$ in $U_0{}^{\mathscr{F}}\mathscr{M}',U_0{}^{\mathscr{F}}\mathscr{M}''$ and compute (5.18) for $\alpha=-1$. We know, by [**3**, Lemma 3.6.6], that this is well defined.

To compute the induced form on $P\operatorname{gr}_0^{\mathrm{M}}$, we use (4.6) and (4.7) and, arguing as above, we have to consider sections m',m'' of $U_{<0}{}^{\mathscr{F}}\mathscr{M}',U_{<0}{}^{\mathscr{F}}\mathscr{M}''$. We are then reduced to proving that, for local sections μ',μ'' of $V_{<0}\widetilde{\mathscr{M}'},V_{<0}\widetilde{\mathscr{M}''}$, we have

$$\operatorname{Res}_{s=-1}\frac{\Gamma(s+1)}{\Gamma(-s)}\big\langle C(\mu',\overline{\mu''}),|t'|^{2(s+1)}\varphi\big\rangle = \big\langle C(\mu',\overline{\mu''}),\varphi\big\rangle.$$

By [**3**, Lemma 3.6.6], the meromorphic function $s\mapsto\big\langle C(\mu',\overline{\mu''}),|t'|^{2(s+1)}\varphi\big\rangle$ has poles along sets $s+1=\gamma\star z/z$ with $\operatorname{Re}\gamma<0$. For such a γ and for $z\in\mathbf{S}$, we cannot have $\gamma\star z/z=0$. Therefore, $s\mapsto\big\langle C(\mu',\overline{\mu''}),|t'|^{2(s+1)}\varphi\big\rangle$ is holomorphic near $s=-1$ and its value at $s=-1$ is $\big\langle C(\mu',\overline{\mu''}),\varphi\big\rangle$. The assertion follows. □

5.c. Proof of Theorem 5.1. — We first reduce to weight 0, and assume that $w=0$. It is then possible to assume that $(\mathscr{T},\mathscr{S})=(\mathscr{M},\mathscr{M},C,\mathrm{Id})$. We may also assume that $\mathscr{M}$ has strict support. Then, in particular, we have $\mathscr{M}=\widetilde{\mathscr{M}}_{\min}$, as defined above.

According to Corollary 5.9 (and to Proposition 5.8 for $\phi_{\tau,0}$), we can apply the arguments given in [**3**, § 6.3] to the direct image by q. □

Notice that we also get:

Corollary 5.20. — *Let* $(\mathscr{T},\mathscr{S})=(\mathscr{M}',\mathscr{M}'',C,\mathscr{S})$ *be an object of* $\mathrm{MT}^{(\mathrm{r})}(X,w)^{(\mathrm{p})}$. *Then, we have isomorphisms in* $\mathscr{R}$-$\mathrm{Triples}(X)$*:*

$$\begin{aligned}&\big(\Psi_{\tau,\alpha}\widehat{\mathscr{T}},\mathscr{N}_\tau\big)\xrightarrow{\sim}\big(\Psi_{t',\alpha}\mathscr{T},\mathscr{N}_{t'}\big),\quad \forall\alpha\neq-1 \textit{ with } \operatorname{Re}\alpha\in[-1,0[,\\ &\big(\phi_{\tau,0}\widehat{\mathscr{T}},\mathscr{N}_\tau\big)\xrightarrow{\sim}\big(\psi_{t',-1}\mathscr{T},\mathscr{N}_{t'}\big).\end{aligned}$$

□

5.d. A complement in dimension one. — Let first us indicate some shortcut to obtain the S-decomposability of $\widehat{\mathscr{M}}$ when Y is reduced to a point, so that $X = \mathbb{P}^1$. First, without any assumption on Y, we have exact sequences, according to Proposition 4.1,

$$
\begin{gathered}
0 \longrightarrow \ker \mathrm{N}_\tau \longrightarrow \psi_{\tau,-1}{}^{\mathscr{F}}\!\mathscr{M} \xrightarrow{\ \mathrm{can}_\tau\ } i_{\infty,+}\psi_{t',-1}\mathscr{M} \longrightarrow 0,\\
0 \longrightarrow i_{\infty,+}\psi_{t',-1}\mathscr{M} \xrightarrow{\ \mathrm{var}_\tau\ } \psi_{\tau,-1}{}^{\mathscr{F}}\!\mathscr{M} \longrightarrow \operatorname{coker} \mathrm{N}_\tau \longrightarrow 0,
\end{gathered}
\tag{5.21}
$$

and

$$
\begin{gathered}
0 \longrightarrow i_{\infty,+} \ker \mathrm{N}_{t'} \longrightarrow \ker \mathrm{N}_\tau \longrightarrow \mathscr{M} \longrightarrow 0\\
0 \longrightarrow \mathscr{M} \longrightarrow \operatorname{coker} \mathrm{N}_\tau \longrightarrow i_{\infty,+} \operatorname{coker} \mathrm{N}_{t'} \longrightarrow 0.
\end{gathered}
\tag{5.22}
$$

It follows that $\mathscr{H}^1 q_+ \ker \mathrm{can}_\tau = \mathscr{H}^1 q_+\mathscr{M}$ and $\mathscr{H}^{-1} q_+ \operatorname{coker} \mathrm{var}_\tau = \mathscr{H}^{-1} q_+\mathscr{M}$. By the first part of the proof, we then have exact sequences

$$
\begin{gathered}
\psi_{\tau,-1}\widehat{\mathscr{M}} \xrightarrow{\ \mathrm{can}_\tau\ } \psi_{\tau,0}\widehat{\mathscr{M}} = \psi_{t',-1}\mathscr{M} \longrightarrow \mathscr{H}^1 q_+\mathscr{M} \longrightarrow 0\\
0 \longrightarrow \mathscr{H}^{-1} q_+\mathscr{M} \longrightarrow \psi_{t',-1}\mathscr{M} = \psi_{\tau,0}\widehat{\mathscr{M}} \xrightarrow{\ \mathrm{var}_\tau\ } \psi_{\tau,-1}\widehat{\mathscr{M}}.
\end{gathered}
$$

Therefore, if $q_+\mathscr{M}$ has cohomology in degree 0 only, $\widehat{\mathscr{M}}$ is a minimal extension along $\tau = 0$. Such a situation occurs if Y is reduced to a point, so that $X = \mathbb{P}^1$: indeed, as $(\mathscr{T}, \mathscr{S})$ is an object of $\mathrm{MT}^{(\mathrm{r})}(\mathbb{P}^1, 0)^{(\mathrm{p})}$, we can assume that $\mathscr{T}$ is simple (*cf.* [**3**, Prop. 4.2.5]); denote by M the restriction of $\mathscr{M}$ to $z = 1$, *i.e.*, $M = \mathscr{M}/(z-1)\mathscr{M}$; by Theorem 5.0.1 of *loc. cit.*, M is an irreducible regular holonomic $\mathscr{D}_{\mathbb{P}^1}$-module;

- if M is not isomorphic to $\mathscr{O}_{\mathbb{P}^1}$, then q_+M has cohomology in degree 0 only [use duality to reduce to the vanishing of $\mathscr{H}^{-1}q_+M$, which is nothing but the space of global sections of the local system attached to M away from its singular points]; by Theorem 6.1.1 of *loc. cit.*, each cohomology $\mathscr{H}^j q_+\mathscr{M}$ is strict and its fibre at $z = 1$ is $\mathscr{H}^j q_+M$; therefore, $\mathscr{H}^j q_+\mathscr{M} = 0$ if $j \neq 0$;
- otherwise, M it isomorphic to $\mathscr{O}_{\mathbb{P}^1}$ with its usual $\mathscr{D}_{\mathbb{P}^1}$ structure, and $\widehat{\mathscr{M}}$ is $\mathscr{O}_{\mathscr{P}^1}$ (where $\mathscr{P}^1$ denotes $\mathbb{P}^1 \times \Omega_0$, *cf.* § 2.b), so $\widehat{\mathscr{M}}$ is supported on $\tau = 0$ and $\psi_{\tau,-1}\widehat{\mathscr{M}} = 0$;

in conclusion, the S-decomposability of $\widehat{\mathscr{M}}$ along $\tau = 0$ is true in both cases.

Corollary 5.20 does not give information on $\psi_{\tau,-1}\mathscr{T}$. We will derive it now in dimension one.

Proposition 5.23. — *Let* $(\mathscr{T}, \mathscr{S}) = (\mathscr{M}', \mathscr{M}'', C, \mathscr{S})$ *be an object of* $\mathrm{MT}^{(\mathrm{r})}(\mathbb{P}^1, w)^{(\mathrm{p})}$. *Assume that* $\mathscr{T}$ *is simple and not isomorphic to* $(\mathscr{O}_{\mathbb{P}^1}, \mathscr{O}_{\mathbb{P}^1}, C, \mathrm{Id})(-w/2)$. *Then, if* $q : \mathbb{P}^1 \to \mathrm{pt}$ *denotes the constant map, the complex* $q_+\mathscr{T}$ *has cohomology in degree* 0 *only and we have natural isomorphisms*

$$\begin{aligned} &\mathrm{gr}^{\mathrm{M}}_{\ell}\,\psi_{\tau,-1}(\widehat{\mathscr{T}},\widehat{\mathscr{S}}) \xrightarrow[\sim]{\mathrm{can}_\tau} \mathrm{gr}^{\mathrm{M}}_{\ell-1}\,\phi_{\tau,0}(\widehat{\mathscr{T}},\widehat{\mathscr{S}})(-1/2) \qquad \textit{for all } \ell \geqslant 1,\\ &\mathrm{gr}^{\mathrm{M}}_{\ell}\,\psi_{\tau,-1}(\widehat{\mathscr{T}},\widehat{\mathscr{S}}) \xleftarrow[\sim]{\mathrm{var}_\tau} \mathrm{gr}^{\mathrm{M}}_{\ell+1}\,\phi_{\tau,0}(\widehat{\mathscr{T}},\widehat{\mathscr{S}})(1/2) \qquad \textit{for all } \ell \leqslant -1,\\ &P\,\mathrm{gr}^{\mathrm{M}}_{0}\,\psi_{\tau,-1}(\widehat{\mathscr{T}},\widehat{\mathscr{S}}) \xrightarrow{\sim} \mathscr{H}^0 q_+(\mathscr{T},\mathscr{S}). \end{aligned}$$

(The gluing C for the trivial twistor $(\mathscr{O}_{\mathbb{P}^1},\mathscr{O}_{\mathbb{P}^1},C,\mathrm{Id})$ is given by $f\otimes\overline{g}\mapsto f\overline{g}$.)

Proof. — We first reduce to weight 0 and take $\mathscr{T}=(\mathscr{M},\mathscr{M},C)$ with $\mathscr{S}=(\mathrm{Id},\mathrm{Id})$. We *a priori* know by [**3**] that the morphisms can_τ and var_τ in the proposition are morphisms in $\mathrm{MT}^{(\mathrm{r})}(\mathbb{P}^1,w)^{(\mathrm{p})}$, so we only need to show the isomorphism at the level of $\mathscr{M}$. Notice that, by Proposition 4.1(iv), the exact sequences (5.22) induce isomorphisms

$$P\,\mathrm{gr}^{\mathrm{M}}_{0}\,\psi_{\tau,-1}{}^{\mathscr{F}}\!\mathscr{M} \xrightarrow{\sim} \mathscr{M} \quad\text{and}\quad \mathscr{M} \xrightarrow{\sim} P\,\mathrm{gr}^{\mathrm{M}}_{0}\,\psi_{\tau,-1}{}^{\mathscr{F}}\!\mathscr{M}. \tag{5.24}$$

The first point ($\mathscr{H}^i q_+\mathscr{T}=0$ for $i\neq 0$) is shown in the preliminary remark above under the assumption on $\mathscr{T}$ made in the proposition. Notice also that we have shown, as a consequence, that $\mathscr{H}^i q_+\ker\mathrm{N}_\tau$ and $\mathscr{H}^i q_+\operatorname{coker}\mathrm{N}_\tau$ also vanish for $i\neq 0$. With the exact sequences (5.21), this implies that

$$\mathscr{H}^0 q_+\ker\mathrm{N}_\tau=\ker\widehat{\mathrm{N}}_\tau \quad\text{and}\quad \mathscr{H}^0 q_+\operatorname{coker}\mathrm{N}_\tau=\operatorname{coker}\widehat{\mathrm{N}}_\tau, \tag{5.25}$$

where $\widehat{\mathrm{N}}_\tau$ denotes (here, in order to avoid confusion) the nilpotent endomorphism on $\mathscr{H}^0 q_+\psi_{\tau,-1}{}^{\mathscr{F}}\!\mathscr{M}=\psi_{\tau,-1}\widehat{\mathscr{M}}$. We then have exact sequences

$$\begin{gathered} 0\longrightarrow\ker\widehat{\mathrm{N}}_\tau\longrightarrow\psi_{\tau,-1}\widehat{\mathscr{M}}\xrightarrow{\mathrm{can}_\tau}\psi_{\tau,0}\widehat{\mathscr{M}}\longrightarrow 0,\\ 0\longrightarrow\psi_{\tau,0}\widehat{\mathscr{M}}\xrightarrow{\mathrm{var}_\tau}\psi_{\tau,-1}\widehat{\mathscr{M}}\longrightarrow\operatorname{coker}\widehat{\mathrm{N}}_\tau\longrightarrow 0. \end{gathered}$$

As can_τ and var_τ are strictly compatible with the monodromy filtration after a shift by 1 (*cf.* [**6**, Lemme 5.1.12]), and as $\ker\widehat{\mathrm{N}}_\tau$ is contained in $\mathrm{M}_0\psi_{\tau,-1}\widehat{\mathscr{M}}$, we get the first isomorphism for $\ell\geqslant 1$. Similarly, use that $\mathrm{M}_{-1}\psi_{\tau,-1}\widehat{\mathscr{M}}$ is contained in $\operatorname{Im}\widehat{\mathrm{N}}_\tau=\operatorname{Im}\mathrm{var}_\tau$ to get the second isomorphism for $\ell\leqslant -1$.

To get the third isomorphism, we only have to show that $\mathscr{H}^0 q_+$ commutes with taking $P\,\mathrm{gr}^{\mathrm{M}}_0$ because of (5.24). We deduce first from the previous results that we also have $\mathscr{H}^i q_+\operatorname{Im}\mathrm{N}_\tau=0$ for $i\neq 0$ and $\mathscr{H}^0 q_+\operatorname{Im}\mathrm{N}_\tau=\operatorname{Im}\widehat{\mathrm{N}}_\tau$. Then, the injective morphism

$$0\longrightarrow\operatorname{Im}\mathrm{N}_\tau\longrightarrow\operatorname{Im}\mathrm{N}_\tau+\ker\mathrm{N}_\tau$$

remains injective after applying $\mathscr{H}^0 q_+$ and, as the $\mathscr{H}^i q_+$ vanish for $i\neq 0$, we conclude that the cokernel satisfies

$$\mathscr{H}^i q_+P\,\mathrm{gr}^{\mathrm{M}}_0\,\psi_{\tau,-1}{}^{\mathscr{F}}\!\mathscr{M}=0 \text{ for } i\neq 0 \text{ and } \mathscr{H}^0 q_+P\,\mathrm{gr}^{\mathrm{M}}_0\,\psi_{\tau,-1}{}^{\mathscr{F}}\!\mathscr{M}=P\,\mathrm{gr}^{\mathrm{M}}_0\,\psi_{\tau,-1}\widehat{\mathscr{M}}. \qquad\square$$

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