HODGE THEORY OF KLOOSTERMAN CONNECTIONS

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Abstract. We construct motives over the rational numbers associated with symmetric power moments of Kloosterman sums, and prove that their L-functions extend meromorphically to the complex plane and satisfy a functional equation conjectured by Broadhurst and Roberts. Although the motives in question turn out to be "classical", the strategy consists in first realizing them as exponential motives and computing their Hodge numbers by means of the irregular Hodge filtration. We show that all Hodge numbers are either zero or one, which implies potential automorphy thanks to recent results of Patrikis and Taylor.

Contents

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1. Introduction

This paper is devoted to the study of a family of global L-functions built up by assembling symmetric power moments of Kloosterman sums over finite fields. We prove that they arise from potentially automorphic motives over the rational numbers, hence admit a meromorphic extension to the complex plane that satisfies the expected functional equation. The exact shape of the latter was conjectured by Broadhurst and Roberts.

1.1. L-functions of symmetric power moments of Kloosterman sums.

Let p be a prime number, $\mathbf{F}_p$ the finite field with p elements, and $\overline{\mathbf{F}}_p$ an algebraic closure of $\mathbf{F}_p$. If q is a power of p, we denote by $\mathbf{F}_q$ the subfield of $\overline{\mathbf{F}}_p$ with q elements and by $\mathrm{tr}_{\mathbf{F}_q/\mathbf{F}_p}\colon \mathbf{F}_q \to \mathbf{F}_p$ its trace map. Let $\psi\colon \mathbf{F}_p \to \mathbf{C}^\times$ be a non-trivial additive character. For each $a \in \mathbf{F}_q^\times$, the *Kloosterman sum* is the real number

$$\mathrm{Kl}_2(a;q) = \sum_{x\in\mathbf{F}_q^\times} \psi(\mathrm{tr}_{\mathbf{F}_q/\mathbf{F}_p}(x + a/x)). \tag{1.1}$$

As an application of the Riemann hypothesis for curves over finite fields [54], Weil proved that there exist algebraic integers α_a, β_a of absolute value $\sqrt{q}$ such that $\mathrm{Kl}_2(a;q) = -(\alpha_a + \beta_a)$ and $\alpha_a\beta_a = q$. For each integer $k \geqslant 1$, we define k-th symmetric powers of Kloosterman sums

$$\mathrm{Kl}_2^{\mathrm{Sym}^k}(a;q) = \sum_{i=0}^{k} \alpha_a^i \beta_a^{k-i}$$

and, summing over all a, we form the *moments*

$$m_2^k(q) = \sum_{a\in\mathbf{F}_q^\times} \mathrm{Kl}_2^{\mathrm{Sym}^k}(a;q).$$

Contrary to $\mathrm{Kl}_2(a;q)$ and its symmetric powers, the moments are rational integers that do not depend on the choice of the additive character. We pack them into the generating series

$$Z_k(p;T) = \exp\Bigl(\sum_{n=1}^{\infty} m_2^k(p^n)\frac{T^n}{n}\Bigr),$$

which in fact turns out to be a polynomial with integer coefficients. The first few cases are easy to compute: both $Z_1(p;T)$ and $Z_2(p;T)$ are equal to $1-T$, and one has

$$Z_3(p;T) = (1-T)(1-\left(\tfrac{3}{p}\right)p^2T), \qquad Z_4(p;T) = \begin{cases} 1-T & \text{if } p=2, \\ (1-T)(1-p^2T) & \text{if } p>2, \end{cases}$$

where in the first formula $(3/p)$ stands for the Legendre symbol. From this one may already infer that $Z_k(p;T)$ is always divisible by $1-T$. Other so-called "trivial factors" appear when k is a multiple of 4 or when k is even and p is small compared with k, see Section 5.1.1 *infra*. Better behaved than $Z_k(p;T)$ is the polynomial $M_k(p;T)$ obtained by removing these trivial factors, since all its roots then have the same absolute value $p^{-(k+1)/2}$.

We shall now build a global L-function over $\mathbf{Q}$ with the above polynomials as local Euler factors. We first discuss the case of odd symmetric powers, say of the form $k=2m+1$. Let S be the set of odd prime numbers smaller than or equal to k. For all $p\notin S$, define the local factor at p as $L_k(p;s) = M_k(p;p^{-s})^{-1}$ and consider the Euler product

$$L_k(s) = \prod_{p\notin S} L_k(p;s),$$

which by the previous remark about the roots of $M_k(p;T)$ converges absolutely on the half plane $\mathrm{Re}(s) > (k+3)/2$. This function is expected to have meromorphic continuation to the entire complex plane and satisfy a functional equation relating its values at s and $k+2-s$. As usual, the functional equation has a neat formulation only after completing the L-function by adding local factors at $p\in S$, as we shall do in (5.11) *infra*, and gamma factors at infinity. We set $\mathfrak{N}_k = 1_{\mathrm{s}}3_{\mathrm{s}}5_{\mathrm{s}}\cdots k_{\mathrm{s}}$, where n_{s} denotes the square-free part of an integer n (i.e., the product of all prime numbers with odd p-ordinal $\mathrm{ord}_p(n)$), and we consider the complete L-function

$$\Lambda_k(s) = \Bigl(\frac{\mathfrak{N}_k}{\pi^m}\Bigr)^{s/2}\prod_{j=1}^{m}\Gamma\Bigl(\frac{s-j}{2}\Bigr)\prod_{p \text{ prime}} L_k(p;s).$$

Theorem 1.2. *Assume $k=2m+1$ is an odd integer. The function $L_k(s)$ admits a meromorphic continuation to the complex plane and satisfies the functional equation*

$$\Lambda_k(s) = \Lambda_k(k+2-s).$$

A similar result holds for even symmetric powers, except that we were unable to make the local invariants explicit at $p=2$. To formulate the statement, we write either $k=2m+4$ or $k=2m+2$ with m an even integer, and we define S as the set of all prime numbers smaller than or equal to $k/2$. The local factors at odd primes in S are described in (5.19) *infra*. Besides, we put $\mathfrak{N}'_k = 2_{\mathrm{u}}4_{\mathrm{u}}6_{\mathrm{u}}\cdots k_{\mathrm{u}}$, where n_{u} denotes the odd part of the radical of an integer n (i.e., the product of all odd primes dividing n). We then complete the L-function outside the prime 2 as follows:

$$\Lambda'_k(s) = \Big(\frac{\mathfrak{N}'_k}{\pi^m}\Big)^{s/2} \prod_{j=1}^m \Gamma\Big(\frac{s-j}{2}\Big) \prod_{p\neq 2} L_k(p;s).$$

Theorem 1.3. *Assume k is even. The function $L_k(s)$ meromorphically extends to the complex plane. Moreover, there exists a sign ε_k, an integer $r_k \geqslant 0$, and a reciprocal of a polynomial with rational coefficients $L_k(2;T)$ such that, setting $\Lambda_k(s) = 2^{r_k s/2} L_k(2; 2^{-s})\Lambda'_k(s)$, the following functional equation holds:*

$$\Lambda_k(s) = \varepsilon_k \Lambda_k(k+2-s).$$

Broadhurst and Roberts conjecture that the elusive invariants take the values $r_k = \lfloor k/6 \rfloor$ and $\varepsilon_k = (-1)^{t_k}$, with t_k given by the formula

$$t_k = \lfloor k/8 \rfloor + \sum_{p\equiv 1 \,(\mathrm{mod}\, 4)} \lfloor k/2p \rfloor + \sum_{p\equiv 3 \,(\mathrm{mod}\, 4)} \lfloor k/4p \rfloor + \delta_{8\mathbf{Z}}(k),$$

where $\delta_{8\mathbf{Z}}$ is the characteristic function of multiples of 8. We explain in 5.3.1 *infra* how the last three terms above fit with the local computations at odd primes in S and at infinity.

Theorems 1.2 and 1.3 were previously known only for $k \leqslant 8$. The first four cases are straightforward, as the L-function is trivial for $k = 1, 2, 4$ and agrees, for $k = 3$, with the shifted Dirichlet L-function $L(\chi_3, s-2)$ of the non-trivial quadratic character modulo 3. The next four cases can all be related to holomorphic Hecke cusp forms on the upper half plane, as indicated in the table below.

k	$L_k(s)$	modular form	references
5	$L(f_3, s-2)$	$f_3 \in S_3(\Gamma_0(15), (\cdot/15))$	Livné [33], Peters *et al.* [41]
6	$L(f_4, s-2)$	$f_4 \in S_4(\Gamma_0(6))$	Hulek *et al.* [26]
7	$L(\mathrm{Ad}(g), s-2)$	$g \in S_3(\Gamma_0(125), (\cdot/21)\chi_5)$, χ_5 Dirichlet character modulo 5 with $\chi_5(2) = -i$	conjectured by Evans [19]; proved by Yun [56]
8	$L(f_6, s-2)$	$f_6 \in S_6(\Gamma_0(6))$	conjectured by Evans [18]; proved by Vincent and Yun [56]

1.2. **Cohomological interpretation.** After Deligne [12], Kloosterman sums arise as traces of Frobenius acting on an étale local system on the torus $\mathbb{G}_{\mathrm{m},\mathbf{F}_p}$. Let ℓ be a prime number distinct from p and let $\overline{\mathbf{Q}}_\ell$ be an algebraic closure of the field of ℓ-adic numbers. Once we view the character ψ as taking values in $\overline{\mathbf{Q}}_\ell$ by choosing a primitive p-th root of unity, there is a rank one ℓ-adic local system $\mathscr{L}_\psi$ on the affine line $\mathbb{A}^1_{\mathbf{F}_p}$ with trace function $z \mapsto \psi(\mathrm{tr}_{\mathbf{F}_q/\mathbf{F}_p}(z))$, the so-called Artin–Schreier sheaf. The *Kloosterman sheaf* Kl_2 is then defined by pulling back and pushing out $\mathscr{L}_\psi$ through the diagram

$$\begin{array}{ccccc} & & \mathbb{G}^2_{\mathrm{m},\mathbf{F}_p} & & \\ & \overset{\pi}{\swarrow} & & \overset{f}{\searrow} & \\ \mathbb{G}_{\mathrm{m},\mathbf{F}_p} & & & & \mathbb{A}^1_{\mathbf{F}_p} \end{array} \tag{1.4}$$

where, if x and z are coordinates on $\mathbb{G}^2_{\mathrm{m},\mathbf{F}_p}$, the function f is given by $x + z/x$ and π stands for the projection to the z-coordinate. Namely, we set

$$\mathrm{Kl}_2 = \mathrm{R}\pi_! f^* \mathscr{L}_\psi[1].$$

Deligne shows that the object Kl_2 is concentrated in degree zero, and that it is a rank two lisse sheaf on $\mathbb{G}_{\mathrm{m},\mathbf{F}_p}$ which is tamely ramified at zero, wildly ramified at infinity, and pure of weight one. Indeed, the "forget supports" map $\mathrm{R}\pi_! f^*\mathscr{L}_\psi \to \mathrm{R}\pi_* f^*\mathscr{L}_\psi$ is an isomorphism. Grothendieck's trace formula and base change yield the equalities

$$\mathrm{Kl}_2(a,q) = -\operatorname{tr}(\mathrm{Frob}_a | \mathrm{Kl}_2) = -(\alpha_a + \beta_a),$$

where α_a and β_a are the eigenvalues of Frobenius acting on a geometric fibre of Kl_2 above a. In the same vein, symmetric powers of Kloosterman sums are local traces of Frobenius on the symmetric powers $\mathrm{Sym}^k \mathrm{Kl}_2$. To obtain the moments we consider the action of geometric Frobenius F_p on the compactly supported étale cohomology of $\mathrm{Sym}^k \mathrm{Kl}_2$. Since it is concentrated in degree one, invoking the trace formula again we get

$$Z_k(p;T) = \det\Big(1 - F_p T \mid \mathrm{H}^1_{\text{ét,c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)\Big). \tag{1.5}$$

It follows that $Z_k(p;T)$ is a polynomial with integer coefficients of degree minus the Euler characteristic of the sheaf $\mathrm{Sym}^k \mathrm{Kl}_2$ which, by the Grothendieck–Ogg–Shafarevich formula, is equal to its Swan conductor at infinity. Fu and Wan compute it for odd primes p in [21], completing partial results by Robba [42]:

$$\deg Z_k(p;T) = \mathrm{Sw}_\infty(\mathrm{Sym}^k \mathrm{Kl}_2) = \begin{cases} \dfrac{k+1}{2} - \left\lfloor \dfrac{k}{2p} + \dfrac{1}{2} \right\rfloor & \text{if } k \text{ odd}, \\ \dfrac{k}{2} - \left\lfloor \dfrac{k}{2p} \right\rfloor & \text{if } k \text{ even}. \end{cases}$$

The remaining case $p = 2$ is treated by Yun, who proves that the Swan conductor is equal to $(k+1)/2$ if k is odd and to $\lfloor (k+2)/4 \rfloor$ if k is even [56]. Observe that, when p is large compared with k, the degree takes the uniform value $\lfloor (k+1)/2 \rfloor$. The sets S from Section 1.1 consist exactly of those prime numbers p at which the degree drops.

From this perspective, the trivial factors of the polynomial $Z_k(p;T)$ are accounted for the invariants and the coinvariants of the inertia action at zero and infinity. By "removing them", we mean replacing compactly supported étale cohomology in the right-hand side of (1.5) with *middle extension* cohomology, defined as

$$\mathrm{H}^1_{\text{ét,mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2) = \mathrm{im}\big[\mathrm{H}^1_{\text{ét,c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \longrightarrow \mathrm{H}^1_{\text{ét}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})\big].$$

The terminology is coherent with the fact that, letting $j\colon \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{P}^1$ denote the inclusion, the above image agrees with the cohomology of the middle extension sheaf $j_{!*}\operatorname{Sym}^k \mathrm{Kl}_2$ on $\mathbb{P}^1$. By definition, $M_k(p;T)$ is the characteristic polynomial of Frobenius

$$M_k(p;T) = \det\Big(1 - F_p T \mid \mathrm{H}^1_{\text{ét},\mathrm{mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \operatorname{Sym}^k \mathrm{Kl}_2)\Big).$$

Since the étale (resp. compactly supported étale) cohomology of $\operatorname{Sym}^k \mathrm{Kl}_2$ has weights $\geqslant k+1$ (resp. $\leqslant k+1$) by Weil II [14], the middle extension cohomology is pure of weight $k+1$. What was called m in Section 1.1 *supra* is the degree of $M_k(p;T)$ or, equivalently, the dimension of the middle extension cohomology for all $p \notin S$.

1.3. **Exponential motives and irregular Hodge filtration.** Theorems 1.2 and 1.3 are proved by constructing a compatible system of potentially automorphic Galois representations

$$r_{k,\ell}\colon \operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}_m(\mathbf{Q}_\ell),$$

with ℓ running over all prime numbers, such that $r_{k,\ell}$ is unramified at primes $p \neq \ell$ outside S and has traces of Frobenius

$$\operatorname{tr}(r_{k,\ell}(\mathrm{Frob}_p)) = \begin{cases} -m_2^k(p) - 1 & \text{if } 4 \nmid k \\ -m_2^k(p) - 1 - p^{k/2} & \text{if } 4 \mid k \end{cases} \qquad (p \notin S \cup \{\ell\}). \tag{1.6}$$

The search for such Galois representations was initiated by Fu and Wan, who showed in [22] that L-functions of symmetric power moments of Kloosterman sums can be realized as Hasse–Weil zeta functions of *virtual* schemes over $\operatorname{Spec}\mathbf{Z}$. An actual Galois representation with traces (1.6) was first constructed by Yun [56] as a subquotient of the étale cohomology of a smooth projective variety over $\mathbf{Q}$ cut off the affine Grassmannian of GL_2.

Our construction is instead inspired by, although it does not rely on, the theory of exponential motives, as developed by the first author and Jossen [20]. In a nutshell, this is a theory of motives for pairs (X,f) consisting of a smooth variety X over $\mathbf{Q}$ and a regular function f on X that enriches the de Rham cohomology of the flat vector bundle $E^f = (\mathscr{O}_X, \mathrm{d} + \mathrm{d}f)$, i.e.,

$$\mathrm{H}^n_{\mathrm{dR}}(X, E^f) = \boldsymbol{H}^n\Big(X, \mathscr{O}_X \xrightarrow{\ \mathrm{d}+\mathrm{d}f\ } \Omega^1_X \xrightarrow{\ \mathrm{d}+\mathrm{d}f\ } \Omega^2_X \longrightarrow \cdots\Big).$$

When f is the zero function, one recovers the usual de Rham cohomology of X and it is indeed shown in loc. cit. that Nori motives, one of the candidates for the abelian category of mixed motives, form a full subcategory of exponential motives. However, the function does not need to be identically zero for an a priori exponential motive to be classical; it will be important for us to know that the exponential motive $\mathrm{H}^n(X \times \mathbb{A}^1, tf)$ is always a classical motive. For instance, if the zero locus $Z = \{f = 0\}$ is smooth, then one finds the motive $\mathrm{H}^{n-2}(Z)(-1)$. This should be thought of as a cohomological shadow of the identity

$$\int_0^\infty \int_{T(\gamma)} e^{-tf}\omega \,\mathrm{d}t = 2\pi \mathrm{i} \int_\gamma \operatorname{Res}_Z(\omega/f),$$

where ω is a differential form on the complement of Z and $T(\gamma)$ is the tube of a chain γ in Z. In general, the existence of square roots of the Tate motive $\mathbf{Q}(-1)$ prevents exponential motives from having realizations in mixed Hodge structures, but they do realize into certain mixed Hodge modules over the affine line that Kontsevich and Soibelman call *exponential* *mixed Hodge structures* [31]. We refer the reader to Appendix B for a summary of the relevant properties of this category.

In analogy with the ℓ-adic setting, the *Kloosterman connection* Kl_2 on $\mathbb{G}_\mathrm{m}$ over a field of characteristic zero is defined by keeping the same diagram (1.4) but replacing the Artin–Schreier sheaf with the differential equation of the exponential. The pullback $f^*\mathscr{L}_\psi$ then becomes $E^f = (\mathscr{O}_{\mathbb{G}_\mathrm{m}^2}, \mathrm{d} + \mathrm{d}f)$ and one sets

$$\mathrm{Kl}_2 = \pi_+ E^f,$$

which can be thought of as the family of exponential motives $\mathrm{H}^1(\mathbb{G}_\mathrm{m}, x + z/x)$ parametrized by $z \in \mathbb{G}_\mathrm{m}$. Over the complex numbers, Kl_2 is the rank two vector bundle with connection associated with the modified Bessel differential equation $\mathrm{d}^2y/\mathrm{d}z^2 + (1/z)\mathrm{d}y/\mathrm{d}z - y = 0$, which is indeed the one the periods of $\mathrm{H}^1(\mathbb{G}_\mathrm{m}, x + z/x)$ satisfy. Note that this equation has a regular singularity at zero and an irregular singularity at infinity. We can then form the symmetric powers $\mathrm{Sym}^k \mathrm{Kl}_2$ and consider the various flavours of de Rham cohomology

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2), \qquad \mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2), \qquad \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2),$$

where the last one is again defined as the image of compactly supported de Rham cohomology on usual de Rham cohomology and agrees with $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}^1, j_{\dagger+} \mathrm{Sym}^k \mathrm{Kl}_2)$.

These vector spaces have an exponential-motivic interpretation as de Rham fibres of exponential mixed Hodge structures attached to suitable exponential motives. As such, they naturally carry a filtration, called the *irregular Hodge filtration*, by a construction extending an idea of Deligne [15] (see [45, §6]). Besides, we shall endow these spaces with a mixed Hodge structure (so that the natural morphisms between them respect these structures) by showing that they are the de Rham fibres of exponential mixed Hodge structures that are in fact classical mixed Hodge structures. For example, in the case of middle extension cohomology, the corresponding exponential motive is already classical. As a general result, we show in Appendix B that, whenever an exponential mixed Hodge structure is a mixed Hodge structure, the Hodge filtration is identified with the irregular Hodge filtration. The results of [55, 17] lead to a geometric interpretation of the irregular Hodge filtrations on these spaces that enables us (Proposition 4.13) to apply toric methods previously explored by Adolphson and Sperber [2] to compute, according to a comparison theorem in [55], the jumps of the irregular Hodge filtrations, hence of the Hodge filtrations.

Theorem 1.7. *The mixed Hodge structure on* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)$ *has weights* $\geqslant k+1$ *and the following numerical data:*

(1) *For k odd,* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)$ *is mixed of weights* $k+1$ *and* $2k+2$*, with*

$$\dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)^{p,q} = \begin{cases} 1, & p+q = k+1,\ p \in \{2, 4, \ldots, k-1\}, \\ 1, & p = q = k+1, \\ 0, & \textit{otherwise.} \end{cases}$$

(2) *For k even,* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ *is mixed of weights* $k+1$ *and* $2k+2$ *if* $k \equiv 2 \bmod 4$ *and of weights* $k+1, k+2,$ *and* $2k+2$ *if* $k \equiv 0 \bmod 4$*, with*

$$\dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)^{p,q} = \begin{cases} 1, & p+q = k+1,\ \min\{p,q\} \in \big\{2, 4, \dots, 2\lfloor (k-1)/4 \rfloor\big\}, \\ 1, & p = q \in \{k/2+1, k+1\} \cap (1+2\mathbf{Z}), \\ 0, & \textit{otherwise.} \end{cases}$$

Furthermore, the mixed Hodge structure $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ *is pure of weight* $k+1$ *and is equal to* $W_{k+1}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$.

Roughly, the argument to show that $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ carries a mixed Hodge structure goes as follows: seeing cohomology of symmetric powers as the alternating part of the cohomology of tensor powers and using a refined form of the Künneth formula, we get

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \cong \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Kl}_2^{\otimes k})^{\mathrm{sign}} \cong \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k})^{\mathrm{sign}}, \tag{1.8}$$

where f_k is the function $x_1+\cdots+x_k+z(1/x_1+\cdots+1/x_k)$ and “sign” denotes the eigenspace on which the symmetric group $\mathfrak{S}_k$ acts through the sign. The rightmost term is now the de Rham realization of an exponential motive. After pullback by the cover $t \mapsto z = t^2$, the function f_k takes the form $\widetilde{f}_k = tg^{\boxplus k}$, where $g^{\boxplus k}$ is the k-fold Thom–Sebastiani sum $g(x_1)+\cdots+g(x_k)$ of the function $g(x) = x+1/x$ with itself, and the cohomology group (1.8) becomes the invariants of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, tg^{\boxplus k})^{\mathrm{sign}}$ under the action of μ_2. The last step consists in replacing the copy of $\mathbb{G}_{\mathrm{m}}$ with coordinate t with $\mathbb{A}^1$. Playing with the localization exact sequence, we arrive at the following description: let $\mathscr{K} \subset \mathbb{G}_{\mathrm{m}}^k$ be the hypersurface defined by the equation $g^{\boxplus k} = 0$ on which the group $\mathfrak{S}_k \times \mu_2$ acts by permuting the coordinates and sending x_i to $-x_i$. Then:

$$\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \cong W_{k-1}\mathrm{H}^{k-1}_{\mathrm{dR}}(\mathscr{K})^{\mathrm{sign}\times\mu_2}(-1).$$

That a paper seemingly about L-functions bears the title “Hodge theory of Kloosterman connections” may come as a surprise. The reason for this choice is that we see Theorem 1.7 as the crux of our contribution. Once we know that all Hodge numbers are either zero or one (equivalently, that the Galois representations $r_{k,\ell}$ are *regular*), a recent theorem of Patrikis and Taylor [40], building on previous work of Barnet-Lamb, Gee, Geraghty, and Taylor [4], implies that the $r_{k,\ell}$ are potentially automorphic and hence that their L-functions meromorphically extend to the complex plane and satisfy a functional equation.

1.4. **Overview.** Briefly, the paper is organized as follows. After a few preliminaries on Kloosterman connections, the motives $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ are constructed in Section 2, where we also exhibit their avatars over finite fields. In the preparatory Section 3, we make some of the properties of $\mathrm{Sym}^k \mathrm{Kl}_2$ more precise. Section 4 is devoted to the proof of Theorem 1.7. Finally, in Section 5 we compute the étale realizations of the motives and pull everything together to prove the main theorems. The paper is supplemented by two appendices concerning exponential motives, exponential mixed Hodge structures, and the irregular Hodge filtration.

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2. Motives of symmetric power moments of Kloosterman connections

2.1. **Basic properties of Kloosterman connections.** We refer to Appendix B.1 for the notation and results from $\mathscr{D}$-module theory. In particular, the notation E^φ is defined in (B.1). Let $n \geqslant 1$ be an integer. In this section, we define Kloosterman connections Kl_{n+1} generalizing Kl_2 from the introduction. For simplicity, we work over the base field $\mathbf{C}$, although all results remain valid over a field of characteristic zero. Let us denote by $\mathbb{G}_{\mathrm{m}}$ the one-dimensional torus and consider the following diagram:

$$\begin{array}{ccccc} & & \mathbb{G}_{\mathrm{m}}^{n+1} & & \\ & \stackrel{\text{prod}}{\swarrow} & & \stackrel{\text{sum}}{\searrow} & \\ \mathbb{G}_{\mathrm{m}} & & & & \mathbb{A}^1 \end{array} \tag{2.1}$$

where the map "sum" is the sum of coordinates and "prod" is their product. We endow the down-right $\mathbb{A}^1$ with the coordinate y and the down-left $\mathbb{G}_{\mathrm{m}}$ with the coordinate z, and we identify $\Gamma(\mathbb{G}_{\mathrm{m},z}, \mathscr{O}_{\mathbb{G}_{\mathrm{m}}})$ with $\mathbf{C}[z, z^{-1}]$. We define Kl_{n+1} as the bounded complex of $\mathscr{D}_{\mathbb{G}_{\mathrm{m}}}$-modules (equivalently, $\mathbf{C}[z, z^{-1}]\langle z\partial_z\rangle$-modules)

$$\mathrm{Kl}_{n+1} = \mathrm{prod}_+ \,\mathrm{sum}^+ E^y.$$

Besides, we consider the cyclic Galois covering

$$[n+1] : \mathbb{G}_{\mathrm{m},t} \longrightarrow \mathbb{G}_{\mathrm{m},z}$$

induced by $z \mapsto t^{n+1}$ with Galois group μ_{n+1} and we set

$$\widetilde{\mathrm{Kl}}_{n+1} = [n+1]^+ \,\mathrm{Kl}_{n+1}\,.$$

Let $x_0, \dots, x_n$ denote the coordinates on $\mathbb{G}_{\mathrm{m}}^{n+1}$ and set $x = (x_1, \dots, x_n)$. By means of the isomorphism $(z, x) \mapsto (x_0, x)$ given by $x_0 = z/x_1 \cdots x_n$, we can replace the map "prod" in diagram (2.1) with the projection

$$\begin{aligned} \pi \colon \mathbb{G}_{\mathrm{m}}^{n+1} &\longrightarrow \mathbb{G}_{\mathrm{m}} \\ (z, x) := (z, x_1, \dots, x_n) &\longmapsto z. \end{aligned}$$

Denoting by $f \colon \mathbb{G}_{\mathrm{m}}^{n+1} \to \mathbb{A}^1$ the function

$$f(z, x) = x_1 + \cdots + x_n + z/x_1 \cdots x_n,$$

we can then rewrite Kl_{n+1} as

$$\mathrm{Kl}_{n+1} = \pi_+ E^f. \tag{2.2}$$

The following proposition gathers the main properties of Kl_{n+1}, which are the analogues in characteristic zero of Katz's characterisation of ℓ-adic Kloosterman sheaves [29, §4.1]:

Proposition 2.3. *The complex* Kl_{n+1} *is concentrated in degree zero, i.e.,*

$$\mathrm{Kl}_{n+1} = \mathscr{H}^0(\mathrm{prod}_+ \mathrm{sum}^+ E^y).$$

Moreover, Kl_{n+1} *has the following properties:*

(1) Kl_{n+1} *is a free* $\mathscr{O}_{\mathbb{G}_\mathrm{m}}$*-module of rank* $n+1$ *with connection.*
(2) Kl_{n+1} *has a regular singularity at* $z=0$ *with unipotent monodromy and a single Jordan block, and an irregular singularity of pure slope* $1/(n+1)$ *at* $z=\infty$.
(3) Kl_{n+1} *is irreducible and rigid as a free* $\mathscr{O}_{\mathbb{G}_\mathrm{m}}$*-module with connection.*
(4) *Let* $\mathrm{Kl}_{n+1}^\vee$ *be the* $\mathbf{C}[z,z^{-1}]$*-module dual to* Kl_{n+1} *endowed with the dual connection, and let* ι_r *denote the involution* $z \mapsto (-1)^r z$. *Then*

$$\mathrm{Kl}_{n+1}^\vee \simeq \iota_{n+1}^+ \mathrm{Kl}_{n+1}.$$

(5) $\widetilde{\mathrm{Kl}}_{n+1}$ *is the restriction to* $\mathbb{G}_\mathrm{m}$ *of the Fourier transform of a regular holonomic module on* $\mathbb{A}^1$.

Proof. Let $g\colon \mathbb{G}_\mathrm{m}^n \to \mathbb{A}^1$ be the function defined by

$$g(y_1,\dots,y_n) = f(1,y_1,\dots,y_n) = y_1+\cdots+y_n+\frac{1}{y_1\cdots y_n}, \tag{2.4}$$

and let us consider the $\mathscr{D}_{\mathbb{A}^1}$-module $M_{n+1} = \mathscr{H}^0 g_+\mathscr{O}_{\mathbb{G}_\mathrm{m}^n}$. Letting $j_0\colon \mathbb{G}_\mathrm{m} \hookrightarrow \mathbb{A}^1$ denote the inclusion, we have an identification of $\mathbf{C}[t,t^{-1}]\langle t\partial_t\rangle$-modules:

$$\widetilde{\mathrm{Kl}}_{n+1} \simeq j_0^+ \,\mathrm{FT}\, M_{n+1}. \tag{2.5}$$

Indeed, by definition, $\widetilde{\mathrm{Kl}}_{n+1} = \widetilde{\pi}_+ E^{f(t^{n+1},x)}$, where $\widetilde{\pi}$ is the projection $(t,x)\mapsto t$. Since the change of variables $\mathbb{G}_\mathrm{m}^{n+1} \ni (t,x) \mapsto (t,y) \in \mathbb{G}_\mathrm{m}^{n+1}$ given by $(t,x)\mapsto(t,y)=(t,x/t)$ is compatible with the first projection, we also have

$$\widetilde{\mathrm{Kl}}_{n+1} = \widetilde{\pi}_+ E^{f(t^{n+1},x/t)} = \widetilde{\pi}_+ E^{tg(y)}.$$

By standard base change theorems for $\mathscr{D}$-modules, we conclude that $\widetilde{\mathrm{Kl}}_{n+1} = j_0^+\,\mathrm{FT}(g_+\mathscr{O}_{\mathbb{G}_\mathrm{m}^n})$. Properties (1) and (2) (with slope 1 at ∞) for $\widetilde{\mathrm{Kl}}_{n+1}$ are obtained from [16, Prop. 3.2] as the special case $w_0=\cdots=w_n=1$ (see also (3.10) & sq. in loc. cit.). This concludes the proof of (5), and it is then standard to obtain the properties in (1) and (2) for $\widetilde{\mathrm{Kl}}_{n+1}$.

An iteration formula. One can give an iterative description of Kl_{n+1}. Let inv be the involution $z\mapsto 1/z$ on $\mathbb{G}_\mathrm{m}$ and let $\mathscr{F}\colon \mathsf{D}^\mathrm{b}_\mathrm{hol}(\mathscr{D}_{\mathbb{G}_\mathrm{m}}) \mapsto \mathsf{D}^\mathrm{b}_\mathrm{hol}(\mathscr{D}_{\mathbb{G}_\mathrm{m}})$ be the localized Fourier transform:

$$\mathscr{F} = j_0^+\,\mathrm{FT}\, j_{0+}.$$

In what follows, it will be convenient to set $\mathrm{Kl}_1 = j_0^+ E^y$ (for $n=0$).

Lemma 2.6. *For* $n \geqslant 0$, *we have*

$$\mathrm{Kl}_{n+2} \simeq \mathscr{F}\,\mathrm{inv}^+\,\mathrm{Kl}_{n+1}.$$

Proof. We set $z'=1/z$, so that $z'=1/x_0\cdots x_n$, and $p_{z'}=1/\operatorname{prod}$. We have

$$\operatorname{inv}^+\mathrm{Kl}_{n+1}=p_{z'+}E^{x_0+\cdots+x_n},$$

and thus, denoting by ζ a coordinate on a new factor $\mathbb{G}_{\mathrm{m}}$,

$$\mathscr{F}\operatorname{inv}^+\mathrm{Kl}_{n+1}=\pi_{\zeta+}E^{x_0+\cdots+x_n+\zeta z'}=\pi_{\zeta+}E^{x_0+\cdots+x_n+\zeta/x_0\cdots x_n}\simeq\mathrm{Kl}_{n+2}\,.\qquad\square$$

We note that properties (2) and (3) hold for $n=0$. Assuming they hold for Kl_{n+1}, we will prove that they also hold for Kl_{n+2}. Since $\operatorname{inv}^+\mathrm{Kl}_{n+1}$ has slope $1/(n+1)$ at 0, we have $j_{0+}\operatorname{inv}^+\mathrm{Kl}_{n+1}=j_{0\dagger+}\operatorname{inv}^+\mathrm{Kl}_{n+1}$, which is thus an irreducible $\mathscr{D}_{\mathbb{A}^1}$-module. Its Fourier transform is then also irreducible, and it is rigid according to [6], hence (3) holds. Moreover, Fourier transform commutes with taking duals of $\mathscr{D}_{\mathbb{A}^1}$-modules up to ι_1, as follows from [34, Lem. V.3.6 p. 86], and the latter duality corresponds to the duality of free $\mathscr{O}_{\mathbb{G}_{\mathrm{m}}}$-modules with connection through the pair of functors j_0^+ and $j_{0\dagger+}$. This gives (4), thus finishing the proof of Proposition 2.3. $\square$

2.2. **Basic properties of** $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$**.** We let $\mathrm{Sym}^k\mathrm{Kl}_{n+1}$ (resp. $\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_{n+1}$) be the $\mathfrak{S}_k$-invariant part of $\mathrm{Kl}_{n+1}^{\otimes k}$ (resp. $\widetilde{\mathrm{Kl}}_{n+1}^{\otimes k}$). We have

$$\begin{aligned}
&\widetilde{\mathrm{Kl}}_{n+1}^{\otimes k}\simeq[n+1]^+\mathrm{Kl}_{n+1}^{\otimes k}, && \mathrm{Sym}^k\widetilde{\mathrm{Kl}}_{n+1}\simeq[n+1]^+\mathrm{Sym}^k\mathrm{Kl}_{n+1},\\
&\mathrm{Kl}_{n+1}^{\otimes k}\simeq([n+1]_+\widetilde{\mathrm{Kl}}_{n+1}^{\otimes k})^{\mu_{n+1}}, && \mathrm{Sym}^k\mathrm{Kl}_{n+1}\simeq([n+1]_+\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_{n+1})^{\mu_{n+1}},\\
&[n+1]_+\widetilde{\mathrm{Kl}}_{n+1}^{\otimes k}\simeq\bigoplus_{i=0}^{k-1}z^{i/k}\,\mathrm{Kl}_{n+1}^{\otimes k}, && [n+1]_+\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_{n+1}\simeq\bigoplus_{i=0}^{k-1}z^{i/k}\,\mathrm{Sym}^k\mathrm{Kl}_{n+1},
\end{aligned}\tag{2.7}$$

where $z^{i/k}\,\mathrm{Sym}^k\mathrm{Kl}_{n+1}$ denotes the twist $(\mathscr{O},\mathrm{d}+\frac{i}{k}\mathrm{d}z/z)\otimes\mathrm{Sym}^k\mathrm{Kl}_{n+1}$. This follows from the decomposition $[n+1]_+\mathscr{O}=\bigoplus_{i=0}^{k-1}(\mathscr{O},\mathrm{d}+\frac{i}{k}\mathrm{d}z/z)$ into invariant spaces of different characters with respect to the action of the Galois group μ_{n+1} of the covering $[n+1]$. Notice also that

$$\operatorname{rk}\mathrm{Sym}^k\mathrm{Kl}_{n+1}=\operatorname{rk}\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_{n+1}=\binom{n+k}{k}.$$

Let us consider the Laurent polynomial

$$f_k=\sum_{j=1}^k\Big(\sum_{i=1}^n x_{ji}+z\prod_{i=1}^n\frac{1}{x_{ji}}\Big)\colon\mathbb{G}_{\mathrm{m}}^{kn+1}\longrightarrow\mathbb{A}^1\tag{2.8}$$

and let $\pi_k\colon\mathbb{G}_{\mathrm{m}}^{kn+1}\to\mathbb{G}_{\mathrm{m},z}$ denote the projection to the coordinate z. With the above notation, we have $f_1=f$ and $\pi_1=\pi$. Let us consider the Cartesian square

$$\begin{array}{ccc}
\mathbb{G}_{\mathrm{m}}^{kn+1} & \xrightarrow{[n+1]} & \mathbb{G}_{\mathrm{m}}^{kn+1}\\
{\scriptstyle\widetilde{\pi}_k}\big\downarrow & & \big\downarrow{\scriptstyle\pi_k}\\
\mathbb{G}_{\mathrm{m},t} & \xrightarrow{[n+1]} & \mathbb{G}_{\mathrm{m},z}
\end{array}$$

and let us set $\widetilde{f}_k=[n+1]^*f_k$. There is a natural action of μ_{n+1} on $\mathscr{H}^0[n+1]_+E^{\widetilde{f}_k}$ and $E^{f_k}=(\mathscr{H}^0[n+1]_+E^{\widetilde{f}_k})^{\mu_{n+1}}$.

Proposition 2.9. *We have*

$$\mathrm{Kl}_{n+1}^{\otimes k} \simeq \mathscr{H}^0\pi_{k+}E^{f_k} = \pi_{k+}E^{f_k} \quad \textit{and} \quad \widetilde{\mathrm{Kl}}_{n+1}^{\otimes k} \simeq \mathscr{H}^0\widetilde{\pi}_{k+}E^{\widetilde{f}_k} = \widetilde{\pi}_{k+}E^{\widetilde{f}_k}.$$

Proof. Let us start with $\widetilde{\mathrm{Kl}}_{n+1}^{\otimes k}$. Recall the function g from (2.4). By changing variables $(t,x)\mapsto(t,y)$ as in the proof of Proposition 2.3, we write $\widetilde{f}_k = t\cdot g^{\boxplus k}$, where $g^{\boxplus k}$ is the k-fold Thom–Sebastiani sum of g with itself.

Let us set $M_{n+1}^{(k)} = \mathscr{H}^0 f_{k+}\mathscr{O}_{\mathbb{G}_{\mathrm{m}}^{kn+1}}$. This is a regular holonomic module on the affine line $\mathbb{A}^1$ and we interpret $\mathscr{H}^0\widetilde{\pi}_{k+}E^{\widetilde{f}_k}$ as $j_0^+\,\mathrm{FT}\,M_{n+1}^{(k)}$ in a way similar to (2.5). The desired formula amounts to

$$j_0^+\,\mathrm{FT}\,M_{n+1}^{(k)} \simeq \bigotimes^k j_0^+\,\mathrm{FT}\,M_{n+1}, \tag{2.10}$$

where the tensor product is taken over the coordinate ring $\mathbf{C}[t,t^{-1}]$ of $\mathbb{G}_{\mathrm{m},t}$ and is understood as an isomorphism of $\mathbf{C}[t,t^{-1}]$-modules with connection.

By applying the reasoning of [9, Prop. 3.4] done for polynomials to the Laurent polynomial g, we obtain that g satisfies the tameness assumption called "M-tameness" and by [39, Prop. 2], so does $g^{\boxplus\ell}$ for any ℓ. In such a case, we can apply the results of [39] by induction on k. For example, if $k=2$, [39] considers the Brieskorn lattice $G_0(g)$, resp. $G_0(g\boxplus g)$, which is a $\mathbf{C}[t^{-1}]$-lattice of $j_0^+\,\mathrm{FT}\,M_{n+1}$, resp. $j_0^+\,\mathrm{FT}\,M_{n+1}^{(2)}$, with a connection having a pole of order $\leqslant 2$ at ∞, and shows (Lemma 1 in loc. cit.) that

$$G_0(g\boxplus g)\simeq G_0(g)\otimes_{\mathbf{C}[t]} G_0(g),$$

as $\mathbf{C}[t^{-1}]$-modules with meromorphic connection. Tensoring with $\mathbf{C}[t,t^{-1}]$ gives (2.10) if $k=2$.

Let us now prove the first isomorphism. By functoriality, we have

$$\mathscr{H}^0\pi_{k+}E^{f_k} = (\mathscr{H}^0\widetilde{\pi}_{k+}E^{\widetilde{f}_k})^{\mu_{n+1}},$$

and the first isomorphism is obtained by taking the μ_{n+1}-invariant part of the second one. □

Let ι be the involution on $\mathbb{G}_{\mathrm{m}}^{kn+1}$ defined by $(z,x_{ji})\mapsto((-1)^{n+1}z,-x_{ji})$. We have the Poincaré pairing

$$\mathrm{H}^{kn+1}_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}^{kn+1},E^{f_k})\otimes\mathrm{H}^{kn+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{kn+1},E^{-f_k})\longrightarrow\mathrm{H}^{2kn+2}_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}^{kn+1})=\mathbf{C},$$

which is compatible with the induced actions of ι. Since $\iota_+E^{f_k}=\iota^+E^{f_k}=E^{-f_k}$, the action ι induces a self-duality

$$\mathrm{H}^{kn+1}_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}^{kn+1},E^{f_k})\otimes\mathrm{H}^{kn+1}_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}^{kn+1},E^{f_k})\longrightarrow\mathbf{C}. \tag{2.11}$$

Furthermore, as ι acts trivially on $\mathrm{H}^{2kn+2}_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}^{kn+1})$, the last pairing is $(-1)^{kn+1}$-symmetric. We have a similar pairing for $\mathrm{H}^{kn+1}_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}^{kn+1},E^{\widetilde{f}_k})$ which induces (2.11) by taking μ_{n+1}-invariants.

In what follows, we set $\chi\colon\mathfrak{S}_k,\mathfrak{S}_k\times\mu_{n+1}\to\{\pm1\}$ to be the sign character on the symmetric group and $\chi_n=\chi^n$. For a representation V of $G=\mathfrak{S}_k$ or $\mathfrak{S}_k\times\mu_{n+1}$ over a field K of characteristic zero, V^{G,χ_n} denotes the χ_n-isotypic part of V, i.e., the image of the idempotent

$$\frac{1}{|G|}\sum_{\sigma\in G}\chi_n(\sigma)\sigma$$

in the group ring $K[G]$ acting on V.

Proposition 2.12. *The de Rham cohomology of* $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ *and* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$ *is concentrated in degree one. Moreover,*

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \simeq \mathrm{H}^{kn+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{f_k})^{\mathfrak{S}_k,\chi_n} \simeq \mathrm{H}^{kn+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{\widetilde{f}_k})^{\mathfrak{S}_k\times\mu_{n+1},\chi_n},$$
$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}) \simeq \mathrm{H}^{kn+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{kn+1}, E^{\widetilde{f}_k})^{\mathfrak{S}_k,\chi_n}.$$

Proof. Since $\mathbb{G}_{\mathrm{m}}$ is affine of dimension one, there is no cohomology in degree two. In order to prove that $\mathrm{H}^0_{\mathrm{dR}} = 0$, it suffices to show that $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ and $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$ have no global flat sections. We will argue for $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$, the other case being similar. Since the nilpotent part N of the monodromy around the origin for Kl_{n+1} has only one Jordan block, there is only one flat holomorphic section of $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ annihilated by N acting on flat sections of $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$, and it takes the form v^k, where v is a flat holomorphic section of Kl_{n+1} annihilated by N. Would v^k be algebraic, so would be v, but this contradicts the irreducibility of Kl_{n+1}.

If q resp. $\widetilde{q}$ denote the structure morphism of $\mathbb{G}_{\mathrm{m},z}$ resp. $\mathbb{G}_{\mathrm{m},t}$, then we have an isomorphism of functors $(q\circ\pi)_+ \simeq q_+ \circ \pi_+$ resp. $(\widetilde{q}\circ\widetilde{\pi})_+ \simeq \widetilde{q}_+ \circ \widetilde{\pi}_+$, and we deduce from Proposition 2.9 and the first part that there are isomorphisms

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Kl}_{n+1}^{\otimes k}) \simeq \mathrm{H}^{kn+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^k, E^{f_k}), \quad \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \widetilde{\mathrm{Kl}}_{n+1}^{\otimes k}) \simeq \mathrm{H}^{kn+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^k, E^{\widetilde{f}_k}).$$

Notice however that the actions of $\sigma \in \mathfrak{S}_k$ on the left and right sides of each of the isomorphisms in Proposition 2.9 commute up to the sign $\chi_n(\sigma)$ which can be seen e.g. by representing elements as differential forms. (Compare the discussion in [20, §12.3.1] for taking $\det \mathrm{H}^1$.) The desired isomorphisms are then clear. ☐

Let $j\colon \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{P}^1$ be the inclusion. Together with the meromorphic extension $j_+ \mathrm{Sym}^k \mathrm{Kl}_{n+1}$, we consider the following $\mathscr{D}_{\mathbb{P}^1}$-modules which also extend on $\mathbb{P}^1$ the $\mathscr{D}_{\mathbb{G}_{\mathrm{m}}}$-module $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$:

$$j_\dagger \mathrm{Sym}^k \mathrm{Kl}_{n+1} := \boldsymbol{D} j_+ \mathrm{Sym}^k \mathrm{Kl}_{n+1}^\vee \simeq \iota_{n+1}^+ \boldsymbol{D} j_+ \mathrm{Sym}^k \mathrm{Kl}_{n+1} \text{ (after 2.3(4))},$$
$$j_{\dagger+} \mathrm{Sym}^k \mathrm{Kl}_{n+1} := \mathrm{im}\big[j_\dagger \mathrm{Sym}^k \mathrm{Kl}_{n+1} \longrightarrow j_+ \mathrm{Sym}^k \mathrm{Kl}_{n+1}\big],$$

where $\boldsymbol{D}$ denotes the duality of $\mathscr{D}$-modules. By definition, we have

$$\mathrm{H}^r_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) = \mathrm{H}^r_{\mathrm{dR}}(\mathbb{P}^1, j_+ \mathrm{Sym}^k \mathrm{Kl}_{n+1}),$$

and we set

$$\mathrm{H}^r_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) := \mathrm{H}^r_{\mathrm{dR}}(\mathbb{P}^1, j_\dagger \mathrm{Sym}^k \mathrm{Kl}_{n+1}),$$
$$\mathrm{H}^r_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) := \mathrm{im}\big[\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})\big].$$

Corollary 2.13. *The cohomology* $\mathrm{H}^r_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ *and* $\mathrm{H}^r_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ *vanish for* $r \neq 1$ *and*

$$\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) = \mathrm{H}^r_{\mathrm{dR}}(\mathbb{P}^1, j_{\dagger+} \mathrm{Sym}^k \mathrm{Kl}_{n+1}).$$

Moreover there is the perfect $(-1)^{kn+1}$*-symmetric pairing*

$$\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \otimes \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \longrightarrow \mathbf{C}$$

induced by Poincaré duality. We have similar properties for $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$.

Proof. By Poincaré duality for $\mathscr{D}_{\mathbb{P}^1}$-modules and since ι_{n+1}^+ does not affect de Rham cohomology on $\mathbb{G}_{\mathrm{m}}$ or $\mathbb{P}^1$, we have

$$\mathrm{H}^r_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \simeq \mathrm{H}^{2-r}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})^\vee,$$

hence its vanishing for $r \neq 1$.

To proceed, let $H^r = \mathrm{H}^r_{\mathrm{dR}}(\mathbb{P}^1, j_{\dagger+} \mathrm{Sym}^k \mathrm{Kl}_{n+1})$. Similarly, we have $H^r \simeq (H^{2-r})^\vee$. Since $H^0 \subset \mathrm{H}^0_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$, it vanishes, and thus so does H^2. On the other hand, since $\ker[j_\dagger \to j_{\dagger+}]$ is supported in dimension zero [30, Prop. 2.9.8], the morphism $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \to H^1$ is onto, and dually $H^1 \to \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ is injective.

Notice that the pairing (2.11) is compatible with the induced actions of $\mathfrak{S}_k$ and it acts trivially on the target $\mathbf{C}$ Taking the χ_n-isotypic parts yields the desired self-duality for $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$.

Same proof for $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$. □

2.3. **The exponential motives attached to** $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$**.** We refer to Appendix A for the definitions and notation used below. In particular, see Footnote 2 for the relation with [20]. The goal of this section is to construct a motive

$$\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \in \mathsf{Mot}^{\mathrm{cl}} \subset \mathsf{Mot}^{\mathrm{exp}},$$

pure of weight $kn+1$, whose de Rham realization is the space $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ with its irregular Hodge filtration. The theory of exponential motives provides a convenient and natural ground field to treat various cohomological invariants associated with varieties with potential and it unifies the flow of thoughts. However, the reader will find that our main theorems concerning Hodge numbers and L-functions are independent of that theory.

The construction of the object $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ in $\mathsf{Mot}^{\mathrm{cl}}$ goes as follows: we

(i) pass the coefficient from the symmetric power $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ to tensor power $\mathrm{Kl}_{n+1}^{\otimes k}$ which enables us to define $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ in $\mathsf{Mot}^{\mathrm{exp}}$ as a direct summand of the motive of a variety with potential $(\mathbb{G}_{\mathrm{m}}^{kn+1}, f_k)$,
(ii) take a cyclic cover $(\mathbb{G}_{\mathrm{m}}^{kn+1}, tg^{\boxplus k})$ of $(\mathbb{G}_{\mathrm{m}}^{kn+1}, f_k)$,
(iii) extend the potential $tg^{\boxplus k}$ over $\mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}}^{kn}$, and
(iv) pass to the zero locus of the potential $g^{\boxplus k}$ without the scaling parameter t.

(i) Recall the Laurent polynomial f_k on the torus $\mathbb{G}_{\mathrm{m}}^{kn+1}$ over $\mathbf{Q}$ defined in (2.8). For $? = \varnothing, \mathrm{c}, \mathrm{mid}$, we consider the object

$$\mathrm{H}^{kn+1}_{?}(\mathbb{G}_{\mathrm{m}}^{kn+1}, f_k) \tag{2.14}$$

of the category $\mathsf{Mot}^{\mathrm{exp}}$. By Proposition 2.9, its de Rham realization equals the de Rham cohomology $\mathrm{H}^1_{\mathrm{dR},?}(\mathbb{G}_{\mathrm{m}}, \mathrm{Kl}_{n+1}^{\otimes k})$ of the connection $\mathrm{Kl}_{n+1}^{\otimes k}$ on $\mathbb{G}_{\mathrm{m}}$. The pair $(\mathbb{G}_{\mathrm{m}}^{kn+1}, f_k)$ is equipped with an action of the symmetric group $\mathfrak{S}_k$ by exchanging the copies $\mathbb{G}_{\mathrm{m}}^n$ of the base $\mathbb{G}_{\mathrm{m}}^{kn+1} = \mathbb{G}_{\mathrm{m},z} \times (\mathbb{G}_{\mathrm{m}}^n)^k$. Thus the motive (2.14) is in the category $\mathsf{Mot}^{\mathrm{exp}}_{\mathfrak{S}_k}$ of exponential motives with $\mathfrak{S}_k$-actions.

Definition 2.15. For $? = \varnothing, \mathrm{c}, \mathrm{mid}$, let

$$\mathrm{H}^1_?(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) = \mathrm{H}^{kn+1}_?(\mathbb{G}_\mathrm{m}^{kn+1}, f_k)^{\mathfrak{S}_k, \chi_n}$$

in the category $\mathsf{Mot}^{\mathrm{exp}}_{\mathfrak{S}_k}$.

Since the symmetric power $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ is defined to be the $\mathfrak{S}_k$-invariant part of $\mathrm{Kl}_{n+1}^{\otimes k}$, the de Rham realization of $\mathrm{H}^1_?(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ equals $\mathrm{H}^1_{\mathrm{dR},?}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ as in the discussion of the de Rham cohomology above.

Remark 2.16. When $k = n = 1$, one has $\mathrm{H}^1(\mathbb{G}_\mathrm{m}, \mathrm{Kl}_2) = \mathbf{Q}(-2)$ and $\mathrm{H}^1_\mathrm{c}(\mathbb{G}_\mathrm{m}, \mathrm{Kl}_2) = \mathbf{Q}(0)$, hence $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Kl}_2)$ vanishes. Indeed, the change of variables $(x, z) \mapsto (x, z/x)$ and the Künneth formula yield an isomorphism $\mathrm{H}^1(\mathbb{G}_\mathrm{m}, \mathrm{Kl}_2) = \mathrm{H}^1(\mathbb{G}_\mathrm{m}, x)^{\otimes 2}$, and $\mathrm{H}^1(\mathbb{G}_\mathrm{m}, x)$ is isomorphic to $\mathbf{Q}(-1)$ by the Gysin long exact sequence for the inclusion $(\mathbb{G}_m, x) \hookrightarrow (\mathbb{A}^1, x)$ and the vanishing of $\mathrm{H}^1(\mathbb{A}^1, x)$. The statement for the compactly supported motive follows by duality.

Theorem 2.17. *Assume $kn \geqslant 2$. The motive $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1})$ is an object of $\mathsf{Mot}^{\mathrm{cl}}$; it is pure of weight $(kn+1)$ and equipped with a $(-1)^{kn+1}$-symmetric perfect pairing*

$$\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \otimes \mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \longrightarrow \mathbf{Q}(-kn-1). \tag{2.18}$$

More explicitly, consider the hypersurface $\mathscr{K} \subset \mathbb{G}^{kn}_{\mathrm{m},\mathbf{Q}}$ defined as the zero locus of the Laurent polynomial

$$g^{\boxplus k}(y) = \sum_{j=1}^{k} \biggl(\sum_{i=1}^{n} y_{ji} + \prod_{i=1}^{n} 1/y_{ji} \biggr) \tag{2.19}$$

on which $\mathfrak{S}_k \times \mu_{n+1}$ acts by permuting the j-index in y_{ji} and by $y_{ji} \mapsto \zeta^{-1} y_{ji}$ for each $\zeta \in \mu_{n+1}$. There is an equivariant morphism of motives $\mathrm{H}^{kn-1}_\mathrm{c}(\mathscr{K})(-1) \to \mathrm{H}^{kn-1}_\mathrm{c}(\mathscr{K})^\vee(-kn)$ such that

$$\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \cong \mathrm{im}\Bigl\{ \mathrm{H}^{kn-1}_\mathrm{c}(\mathscr{K})(-1) \longrightarrow \mathrm{H}^{kn-1}_\mathrm{c}(\mathscr{K})^\vee(-kn) \Bigr\}^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n}$$

as objects in $\mathsf{Mot}^{\mathrm{cl}}$.

Proof. The self-duality is induced by the Poincaré pairing and the involution ι as in Corollary 2.13 for de Rham realization.

(ii) Recall that $\widetilde{f}_k = [n+1]^* f_k$. We can thus regard $\mathrm{H}^{kn+1}_{\mathrm{mid}}(\mathbb{G}_\mathrm{m}^{kn+1}, \widetilde{f}_k)$ as an object of $\mathsf{Mot}^{\mathrm{exp}}_{\mathfrak{S}_k \times \mu_{n+1}}$, which contains $\mathrm{H}^{kn+1}_{\mathrm{mid}}(\mathbb{G}_\mathrm{m}^{kn+1}, f_k)$ as a direct summand of μ_{n+1}-invariants (see (2.7)). Thus

$$\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) = \mathrm{H}^{kn+1}_{\mathrm{mid}}(\mathbb{G}_\mathrm{m}^{kn+1}, \widetilde{f}_k)^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n}.$$

(iii) Consider the open embedding $(\mathbb{G}_\mathrm{m}^{kn+1}, \widetilde{f}_k) \subset (\mathbb{A}^1_t \times \mathbb{G}_\mathrm{m}^{kn}, \widetilde{f}_k)$ of pairs compatible with the action of $\mathfrak{S}_k \times \mu_{n+1}$. Noting that

$$(\mathbb{A}^1_t \times \mathbb{G}_\mathrm{m}^{kn}, \widetilde{f}_k) \smallsetminus (\mathbb{G}_\mathrm{m}^{kn+1}, \widetilde{f}_k) = (\mathbb{G}_\mathrm{m}^{kn}, 0),$$

it induces two localization exact sequences

$$\begin{array}{ccccccccc} 0 \to \mathrm{H}^{kn-1}(\mathbb{G}_{\mathrm{m}}^{kn})(-1) & \to & \mathrm{H}^{kn+1}(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f}_k) & \longrightarrow & \mathrm{H}^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f}_k) & \to & \mathrm{H}^{kn}(\mathbb{G}_{\mathrm{m}}^{kn})(-1) \to 0 \\ & & \uparrow & & \uparrow & & \\ 0 \longleftarrow \mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn}) & \longleftarrow & \mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f}_k) & \xleftarrow{\ \gamma\ } & \mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f}_k) & \longleftarrow & \mathrm{H}_{\mathrm{c}}^{kn}(\mathbb{G}_{\mathrm{m}}^{kn}) \longleftarrow 0 \end{array}$$

with a commutative square indicated in the diagram. Exactness at the lower left part comes from the vanishing of $\mathrm{H}_{\mathrm{c}}^{kn+2}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f}_k)$, which follows from Proposition 2.9, and that in the lower right comes from the vanishing of $\mathrm{H}_{\mathrm{c}}^{kn}(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f}_k)$ since $\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}$ is affine. Exactness at the upper corners is obtained similarly.

Lemma 2.20. *Let $W_\bullet$ denote the weight filtration. We have natural isomorphisms*

$$\begin{aligned} W_{2kn-1}\mathrm{H}^{kn+1}(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f}_k) &\xrightarrow{\sim} W_{2kn-1}\mathrm{H}^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f}_k), \\ \mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f}_k)/W_2 &\xrightarrow{\sim} \mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f}_k)/W_2, \\ \mathrm{H}_{\mathrm{mid}}^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f}_k) &\xrightarrow{\sim} \mathrm{H}_{\mathrm{mid}}^{kn+1}(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f}_k) \end{aligned}$$

in the category $\mathsf{Mot}^{\mathrm{exp}}_{\mathfrak{S}_k \times \mu_{n+1}}$ where the last map is induced by γ.

Proof. The four terms involving motives of $\mathbb{G}_{\mathrm{m}}^{kn}$ in the corners are pure of weights $2kn+2$, $2kn$, 2 and 0, respectively. The first two isomorphisms follows. Since $kn \geqslant 2$, the two long exact sequences yield isomorphisms

$$\begin{aligned} \mathrm{gr}^W_{kn+1}\,\mathrm{H}^{kn+1}(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f}_k) &\xrightarrow{\sim} \mathrm{gr}^W_{kn+1}\,\mathrm{H}^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f}_k), \\ \mathrm{gr}^W_{kn+1}\,\mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{G}_{\mathrm{m}}^{kn+1}, \widetilde{f}_k) &\xrightarrow{\sim} \mathrm{gr}^W_{kn+1}\,\mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f}_k) \end{aligned}$$

of motives of weight $kn+1$. The last isomorphism follows. ☐

(iv) Setting $x_{ji} = ty_{ji}$, we obtain $\widetilde{f}_k(t,x) = tg^{\boxplus k}(y)$, and the action of $\zeta \in \mu_{n+1}$ in the variables (t,y) is given by

$$t \longmapsto \zeta t, \quad y_{ji} \longmapsto \zeta^{-1} y_{ji},$$

while $\mathfrak{S}_k$ permutes the indices j of the variables y_{ji} and leaves t invariant. We have another open embedding

$$(\mathbb{A}^1 \times (\mathbb{G}_{\mathrm{m}}^{kn} \smallsetminus \mathscr{K}), tg^{\boxplus k}) \subset (\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f}_k)$$

in $\mathsf{Mot}^{\mathrm{exp}}_{\mathfrak{S}_k \times \mu_{n+1}}$, and we note that

$$(\mathbb{A}^1_t \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f}_k) \smallsetminus (\mathbb{A}^1_t \times (\mathbb{G}_{\mathrm{m}}^{kn} \smallsetminus \mathscr{K}), tg^{\boxplus k}) = (\mathbb{A}^1_t \times \mathscr{K}, 0).$$

Since any cohomology realization of $(\mathbb{A}^1 \times (\mathbb{G}_{\mathrm{m}}^{kn} \smallsetminus \mathscr{K}), tg^{\boxplus k})$ is zero (as the potential $g^{\boxplus k}$ on the second component $(\mathbb{G}_{\mathrm{m}}^{kn} \smallsetminus \mathscr{K})$ is invertible, see e.g. [55, Lem. 3.2]), we have

$$\mathrm{H}^r(\mathbb{A}^1 \times (\mathbb{G}_{\mathrm{m}}^{kn} \smallsetminus \mathscr{K}), tg^{\boxplus k}) = \mathrm{H}_{\mathrm{c}}^r(\mathbb{A}^1 \times (\mathbb{G}_{\mathrm{m}}^{kn} \smallsetminus \mathscr{K}), tg^{\boxplus k}) = 0$$

in $\mathsf{Mot}^{\mathrm{exp}}$ for all r. The localization sequence for compactly supported motives yields

$$\mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f}_k) \xrightarrow{\sim} \mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{A}^1 \times \mathscr{K}).$$

Furthermore, by the Künneth formula, we have

$$\mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{A}^1 \times \mathscr{K}) = \mathrm{H}_{\mathrm{c}}^2(\mathbb{A}^1) \otimes \mathrm{H}_{\mathrm{c}}^{kn-1}(\mathscr{K}) = \mathrm{H}_{\mathrm{c}}^{kn-1}(\mathscr{K})(-1).$$

The first equality respects the action of μ_{n+1} and it acts trivially on $\mathrm{H}_{\mathrm{c}}^2(\mathbb{A}^1)$.

On the other hand, since $\mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}}^{kn}$ is smooth, one has

$$\begin{aligned}\mathrm{H}^{kn+1}(\mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f}_k) &\cong \mathrm{H}_{\mathrm{c}}^{kn+1}(\mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}}^{kn}, -\tilde{f}_k)^\vee(-kn-1) \quad \text{(Poincaré duality)}\\ &\cong \mathrm{H}_{\mathrm{c}}^{kn-1}(\mathscr{K})^\vee(-kn).\end{aligned}$$

Therefore

$$\mathrm{H}_{\mathrm{mid}}^{kn+1}(\mathbb{A}^1 \times \mathbb{G}_{\mathrm{m}}^{kn}, \widetilde{f}_k) \cong \mathrm{im}\left\{\mathrm{H}_{\mathrm{c}}^{kn-1}(\mathscr{K})(-1) \longrightarrow \mathrm{H}_{\mathrm{c}}^{kn-1}(\mathscr{K})^\vee(-kn)\right\}.$$

The last object is now in the full sub-category $\mathsf{Mot}^{\mathrm{cl}}_{\mathfrak{S}_k\times\mu_{n+1}}$ of $\mathsf{Mot}^{\mathrm{exp}}_{\mathfrak{S}_k\times\mu_{n+1}}$ ([20, Th. 5.1.1]).

One ends the proof of Theorem 2.17 by taking the χ_n-isotypic components for the action of $\mathfrak{S}_k \times \mu_{n+1}$. □

Remark 2.21. The hypersurface $\mathscr{K}$ has at worst isolated singularities; its $\overline{\mathbf{Q}}$-singular points are $y_i = \zeta_i$ with $\zeta_i \in \mu_{n+1}$ satisfying $\sum_{i=1}^k \zeta_i = 0$. (See the proof of [21, Th. 3.1] for a similarity in the structure of $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ at ∞ in characteristic p.) For example, if $(n+1)$ is a prime, then $\mathscr{K}$ is singular if and only if k is a multiple of $(n+1)$; this includes the case $n=1$, in which $\mathscr{K}$ is singular if and only k is even. If $\mathscr{K}$ is smooth, one has

$$\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) \cong \mathrm{im}\left\{\mathrm{H}_{\mathrm{c}}^{kn-1}(\mathscr{K})(-1) \longrightarrow \mathrm{H}^{kn-1}(\mathscr{K})(-1)\right\}^{\mathfrak{S}_k\times\mu_{n+1},\chi_n}$$

as an object in $\mathsf{Mot}^{\mathrm{cl}}_{\mathfrak{S}_k\times\mu_{n+1}}$ by Poincaré duality. We do not know if the arrow inside the curly brackets is the canonical map from the compactly supported motive to the usual one.

2.4. **Other analogies.**

2.4.1. *The étale case in characteristic p.* We recall the construction of the ℓ-adic Kloosterman sheaf in characteristic p. Fix two distinct prime numbers p and ℓ. Let ζ be a primitive p-th root of unity in $\overline{\mathbf{Q}}_\ell$ and let $\psi\colon \mathbf{F}_p \to \mathbf{Q}_\ell(\zeta)^\times$ be a non-trivial additive character. The Artin–Schreier sheaf $\mathscr{L}_\psi$ is the rank one lisse étale sheaf with coefficients in $\mathbf{Q}_\ell(\zeta)$ on the affine line $\mathbb{A}^1_{\mathbf{F}_p}$ such that, at each point $z \in \mathbb{A}^1(\mathbf{F}_q) = \mathbf{F}_q$, geometric Frobenius acts as multiplication by $\psi(\mathrm{tr}_{\mathbf{F}_q/\mathbf{F}_p}(z))$. This sheaf is the ℓ-adic counterpart of the connection E^z on $\mathbb{A}^1_{\mathbf{C}}$.

On $\mathbb{G}_{\mathrm{m}}$ over the finite field $\mathbf{F}_p$, there is the Kloosterman sheaf which we also denote by Kl_{n+1}; it is the lisse étale sheaf with coefficients in $\mathbf{Q}_\ell(\zeta)$ of rank $(n+1)$ such that at each $z \in \mathbb{G}_{\mathrm{m}}(\mathbf{F}_q)$, the geometric Frobenius action has trace $(-1)^n \mathrm{Kl}_{n+1}(z;q)$ where

$$\mathrm{Kl}_{n+1}(z;q) = \sum_{x_1,\cdots,x_n\in\mathbf{F}_q^\times} \psi(\mathrm{tr}_{\mathbf{F}_q/\mathbf{F}_p}(x_1+\cdots+x_n+z/x_1\cdots x_n))$$

generalizing (1.1). The corresponding properties in §§2.1 and 2.2 hold true for Kloosterman sheaf and its symmetric powers after replacing the connection E^h by the pullback $\mathscr{L}_h := h^*\mathscr{L}_\psi$ for a morphism $h\colon X \to \mathbb{A}^1$ over $\mathbf{F}_p$. In particular, one has

$$\mathrm{H}^1_{\text{ét},?}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}) = \mathrm{H}^{kn+1}_{\text{ét},?}(\mathbb{G}^{kn+1}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathscr{L}_{\widetilde{f}_k})^{\mathfrak{S}_k\times\mu_{n+1},\chi_n} \tag{2.22}$$

for $? = \varnothing, \mathrm{c}, \mathrm{mid}$. (If $n+1 = p^r m$ with $(p, m) = 1$, the covering $[n+1]$ on $\mathbb{G}_\mathrm{m}$ factors through an étale cyclic covering $[m]$ and a purely inseparable covering $[p^r]$. The latter induces an equivalence of categories of étale covers and hence does not change the cohomology. In this case the action of μ_{n+1} factors through μ_m.)

Let $\mathscr{K} \subset \mathbb{G}_\mathrm{m}^{k+1}$ over $\mathbf{F}_p$ be defined by the same equation as in (2.19). Following the same localization exact sequence argument as in the construction of the motives in characteristic zero in the previous subsection, we obtain the isomorphisms

$$\mathrm{H}^{k+1}_{\text{ét,c}}(\mathbb{G}^{kn+1}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathscr{L}_{\widetilde{f}_k})^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n}/W_2 \xrightarrow{\sim} \mathrm{H}^{k+1}_{\text{ét,c}}((\mathbb{A}^1 \times \mathbb{G}^k_\mathrm{m})_{\overline{\mathbf{F}}_p}, \mathscr{L}_{\widetilde{f}_k})^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n}/W_2, \tag{2.23}$$

$$\mathrm{H}^{k+1}_{\text{ét,c}}((\mathbb{A}^1 \times \mathbb{G}^k_\mathrm{m})_{\overline{\mathbf{F}}_p}, \mathscr{L}_{\widetilde{f}_k}) \xrightarrow{\sim} \mathrm{H}^{k+1}_{\text{ét,c}}((\mathbb{A}^1 \times \mathscr{K})_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell)[\zeta] = \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}_p}, \mathbf{Q}_\ell)(-1)[\zeta]. \tag{2.24}$$

2.4.2. *The rigid case in characteristic p.* Let us fix an algebraic closure $\overline{\mathbf{Q}}_p$ of $\mathbf{Q}_p$ and an element $\pi \in \overline{\mathbf{Q}}_p$ such that $\pi^{p-1} = -p$. Let now $\mathscr{L}_\pi$ be the overconvergent Dwork F-isocrystal on $\mathbb{A}^1$; it is the rank one connection $\mathrm{d} + \pi \mathrm{d}z$ with the Frobenius $\exp(\pi(z^p - z))$ on the overconvergent structure sheaf of $\mathbb{G}_\mathrm{m}$ over $\mathbf{Q}_p(\pi)$. The Kloosterman F-isocrystal on $\mathbb{G}_\mathrm{m}$, again denoted by Kl_{n+1}, is the overconvergent F-isocrystal obtained by taking the n-th pushforward of $\mathscr{L}_f := f^*\mathscr{L}_\pi$ on $\mathbb{G}_\mathrm{m}^{n+1}$ to $\mathbb{G}_{\mathrm{m},z}$. See [10, §1] for a detailed discussion. Similarly to the ℓ-adic case above, we then have

$$\mathrm{H}^1_{\mathrm{rig},?}(\mathbb{G}_\mathrm{m}/\mathbf{Q}_p(\pi), \mathrm{Sym}^k \mathrm{Kl}_{n+1}) = \mathrm{H}^{kn+1}_{\mathrm{rig},?}(\mathbb{G}_\mathrm{m}^{kn+1}/\mathbf{Q}_p(\pi), \mathscr{L}_{\widetilde{f}_k})^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n}$$

for $? = \varnothing, \mathrm{c}, \mathrm{mid}$, and

$$\mathrm{H}^{k+1}_{\mathrm{rig,c}}(\mathbb{G}_\mathrm{m}^{k+1}/\mathbf{Q}_p(\pi), \mathscr{L}_{\widetilde{f}_k})^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n}/W_2 \xrightarrow{\sim} \mathrm{H}^{k+1}_{\mathrm{rig,c}}(\mathbb{A}^1 \times \mathbb{G}^k_\mathrm{m}/\mathbf{Q}_p(\pi), \mathscr{L}_{\widetilde{f}_k})^{\mathfrak{S}_k \times \mu_{n+1}, \chi_n}/W_2,$$
$$\mathrm{H}^{k+1}_{\mathrm{rig,c}}(\mathbb{A}^1 \times \mathbb{G}^k_\mathrm{m}/\mathbf{Q}_p(\pi), \mathscr{L}_{\widetilde{f}_k}) \xrightarrow{\sim} \mathrm{H}^{k+1}_{\mathrm{rig,c}}(\mathbb{A}^1 \times \mathscr{K}/\mathbf{Q}_p)[\pi] = \mathrm{H}^{k-1}_{\mathrm{rig,c}}(\mathscr{K}/\mathbf{Q}_p)(-1)[\pi]$$

via the localization sequence ([32, Prop. 8.2.18(ii)]).

2.4.3. *Mixed Hodge structures for the de Rham cohomologies of* $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$. In this section, we prove an analogue of Theorem 2.17 for the de Rham realisation of the motives, endowed with their irregular Hodge filtration and their weight filtration $W_\bullet$. We will use the results of Appendix B.

The exponential motives of Definition 2.15 have a realization in the category EMHS of exponential mixed Hodge structure (see [20, §11.5]). The corresponding exponential mixed Hodge structures give rise to bi-filtered vector spaces (see Proposition B.3)

$$\begin{gathered}(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}), F^{\mathrm{irr}}_\bullet, W_\bullet), \quad (\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}), F^{\mathrm{irr}}_\bullet, W_\bullet), \\ (\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_{n+1}), F^{\mathrm{irr}}_\bullet, W_\bullet).\end{gathered} \tag{2.25}$$

A similar definition applies to $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$.

Theorem 2.26. *The bi-filtered vector spaces* (2.25) *underlie mixed Hodge structures of weights* $\geqslant kn+1$*, resp.* $\leqslant kn+1$*, resp.* $kn+1$ *and the natural morphisms between them are morphisms of mixed Hodge structures. A similar result holds for* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$.

Proof. It is enough to treat the case of $\bigotimes^k \widetilde{\mathrm{Kl}}_{n+1}$, since those of $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_{n+1}$ and $\mathrm{Sym}^k \mathrm{Kl}_{n+1}$ are obtained by taking invariants under the action $\mathfrak{S}_k$ and $\mathfrak{S}_k \times \mu_{n+1}$, respectively.

For $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \bigotimes^k \widetilde{\mathrm{Kl}}_{n+1})$, we regard it as the Hodge realization of $\mathrm{H}^{kn+1}_{\mathrm{mid}}(\mathbb{A}^1_t \times \mathbb{G}^{kn}_{\mathrm{m}}, \widetilde{f}_k)$, and apply Theorem 2.17. (It also follows from the argument below by using Theorem B.10(1).)

For $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \bigotimes^k \widetilde{\mathrm{Kl}}_{n+1})$ resp. $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \bigotimes^k \widetilde{\mathrm{Kl}}_{n+1})$, we apply Theorem B.10(2) respectively B.10(3) with $V = \mathbb{G}^{kn}_{\mathrm{m}}$, $M^{\mathrm{H}}_V = \mathscr{O}^{\mathrm{H}}_V$, $f = \widetilde{f}_k$ and $r = kn+1$.

It remains to check the weights. We will do it for $\bigotimes^k \mathrm{Kl}_{n+1}$ for example. For that purpose, recall that the mixed Hodge modules giving rise via the projector Π to the (exponential) mixed Hodge structures above are $\mathscr{H}^0{}_{\mathrm{H}}f_{k*}\mathscr{O}^{\mathrm{H}}_{\mathbb{G}^{kn+1}_{\mathrm{m}}}$ and $\mathscr{H}^0{}_{\mathrm{H}}f_{k!}\mathscr{O}^{\mathrm{H}}_{\mathbb{G}^{kn+1}_{\mathrm{m}}}$ (see Notation B.2). Let $j\colon \mathbb{G}^{kn+1}_{\mathrm{m}} \hookrightarrow \overline{X}$ be the open inclusion in a good compactification $(\overline{X}, \overline{f_k})$ of $(\mathbb{G}^{kn+1}_{\mathrm{m}}, f_k)$ (see Appendix A). Since $\mathscr{O}^{\mathrm{H}}_{\mathbb{G}^{kn+1}_{\mathrm{m}}}$ is pure of weight $kn+1$, ${}_{\mathrm{H}}j_*\mathscr{O}^{\mathrm{H}}_{\mathbb{G}^{kn+1}_{\mathrm{m}}}$ resp. ${}_{\mathrm{H}}j_!\mathscr{O}^{\mathrm{H}}_{\mathbb{G}^{kn+1}_{\mathrm{m}}}$ is mixed of weights $\geqslant kn+1$ resp. $\leqslant kn+1$, and so is

$$\mathscr{H}^0{}_{\mathrm{H}}f_{k*}\mathscr{O}^{\mathrm{H}}_{\mathbb{G}^{kn+1}_{\mathrm{m}}} \simeq \mathscr{H}^0{}_{\mathrm{H}}\overline{f_k}_*({}_{\mathrm{H}}j_*\mathscr{O}^{\mathrm{H}}_{\mathbb{G}^{kn+1}_{\mathrm{m}}}), \quad \text{resp. } \mathscr{H}^0{}_{\mathrm{H}}f_{k!}\mathscr{O}^{\mathrm{H}}_{\mathbb{G}^{kn+1}_{\mathrm{m}}} \simeq \mathscr{H}^0{}_{\mathrm{H}}\overline{f_k}_!({}_{\mathrm{H}}j_!\mathscr{O}^{\mathrm{H}}_{\mathbb{G}^{kn+1}_{\mathrm{m}}}). \quad \square$$

3. Basic properties of $\mathrm{Sym}^k \mathrm{Kl}_2$

3.1. **Structure of Kl_2 and $\widetilde{\mathrm{Kl}}_2$.** We make the assertions of Proposition 2.3 more explicit in the case $n = 1$.

Explicit bases of Kl_2 *and* $\widetilde{\mathrm{Kl}}_2$. We have $\mathrm{Kl}_2 = \mathscr{F}(E^{1/y})$ (see Lemma 2.6). It has rank two, a regular singularity at $z = 0$, and an irregular singularity at $z = \infty$. Recall that $E^{1/y}$ is the free $\mathbf{C}[y, y^{-1}]$-module with connection generated by one element, denoted by $\mathrm{e}^{1/y}$, which satisfies the differential equation $(y^2\partial_y + 1)\mathrm{e}^{1/y} = 0$. It is better to consider the generator $v_0 = \mathrm{e}^{1/y}/y$, which in turn satisfies $(y\partial_y y + 1)v_0 = 0$. Regarding v_0 as an element of $j_0^+ \mathrm{FT}\, E^{1/y}$, with the usual rule $z = -\partial_y$ and $\partial_z = y$, we see that v_0 satisfies

$$(z\partial_z)^2 v_0 = z v_0.$$

We set $v_1 = z\partial_z v_0$, i.e., $v_1 = \mathrm{e}^{1/y}/y^2$. Then $\{v_0, v_1\}$ is a $\mathbf{C}[z, z^{-1}]$-basis of Kl_2 and the matrix of the connection in this basis is given by

$$z\partial_z(v_0, v_1) = (v_0, v_1)\begin{pmatrix} 0 & z \\ 1 & 0 \end{pmatrix}.$$

It follows that the monodromy around the regular singularity is unipotent with one Jordan block of size two.

Let us consider the interpretation (2.2) as $\mathrm{Kl}_2 = \mathscr{H}^0\pi_+ E^f$ with $f(z, x) = x + z/x$ and $\pi\colon \mathbb{G}^2_{\mathrm{m}} \to \mathbb{G}_{\mathrm{m},z}$ the projection. As a $\mathbf{C}[z, z^{-1}]$-module, Kl_2 is equal to the cokernel of the relative de Rham complex

$$\mathbf{C}[x^{\pm1}, z^{\pm1}] \xrightarrow{\mathrm{d}+\partial_x f \mathrm{d}x} \mathbf{C}[x^{\pm1}, z^{\pm1}]\mathrm{d}x.$$

Then v_0 and v_1 are respectively given by the classes of $\mathrm{d}x/x$ and $x\mathrm{d}x/x$ in this twisted de Rham cohomology group.

The basis $\{v_0, v_1\}$ lifts as a $\mathbf{C}[t,t^{-1}]$-basis $\{\widetilde{v}_0, \widetilde{v}_1\}$ of $\widetilde{\mathrm{Kl}}_2$. We thus have

$$\tfrac{1}{2}t\partial_t\widetilde{v}_0 = \widetilde{v}_1, \quad \tfrac{1}{2}t\partial_t\widetilde{v}_1 = t^2\widetilde{v}_0.$$

Duality. Let $\{v_0^\vee, v_1^\vee\}$ denote the basis of $\mathrm{Kl}_2^\vee$ dual to $\{v_0, v_1\}$. The matrix of the dual connection in this basis is equal to $-\left(\begin{smallmatrix}0&1\\t&0\end{smallmatrix}\right)$. We thus have an isomorphism $\mathrm{Kl}_2 \xrightarrow{\sim} \mathrm{Kl}_2^\vee$ given by $(v_0, v_1) \mapsto (v_1^\vee, -v_0^\vee)$, that is, a non-degenerate skew symmetric pairing $\mathrm{Kl}_2 \otimes \mathrm{Kl}_2 \to \mathbf{C}[z, z^{-1}]$ compatible with the connection, defined by $\langle v_0, v_1\rangle = -\langle v_1, v_0\rangle = 1$. There is a similar formula for $\widetilde{\mathrm{Kl}}_2$.

Structure at infinity. The irregular singularity at $z = \infty$ has slope $1/2$. The formal stationary phase formula gives (see [44, Th. 5.1])

$$\widehat{\mathrm{Kl}}_2 := \mathbf{C}(\!(z^{-1})\!) \otimes_{\mathbf{C}[z,z^{-1}]\langle z\partial_z\rangle} \mathrm{Kl}_2 \simeq [2]_+(E^{2t} \otimes L_{-1}), \quad [2]\colon t \longmapsto z = t^2,$$

where L_{-1} is the rank-one $\mathbf{C}(\!(t^{-1})\!)$-connection $(\mathbf{C}(\!(t^{-1})\!), \mathrm{d} + \frac{1}{2}\mathrm{d}(t^{-1})/t^{-1})$, so that the monodromy is equal to -1. Set $L_{\mathsf{i}} = (\mathbf{C}(\!(z^{-1})\!), \mathrm{d} + \frac{1}{4}\mathrm{d}(z^{-1})/z^{-1})$, so that $L_{-1} = [2]^+L_{\mathsf{i}}$. Then we also have

$$\widehat{\mathrm{Kl}}_2 \simeq ([2]_+E^{2t}) \otimes L_{\mathsf{i}}. \tag{3.1}$$

The pullback $\widetilde{\mathrm{Kl}}_2 = [2]^+\,\mathrm{Kl}_2$ has slope one at $t = \infty$ and unipotent monodromy with one Jordan block at the origin. On the other hand,

$$[2]^+([2]_+E^{2t}) \simeq E^{2t} \oplus E^{-2t}.$$

It has a μ_2-action induced by $t \mapsto -t$ which exchanges both summands, whose invariant submodule is identified with $[2]_+E^{2t}$ as a $\mathbf{C}(\!(z^{-1})\!)$-module.

$\widetilde{\mathrm{Kl}}_2$ *as a localized Fourier transform.* Let $g\colon \mathbb{G}_{\mathrm{m}} \to \mathbb{A}^1_\tau$ be the degree-two morphism defined by $y \mapsto y + 1/y$. The critical points are $y = \pm 1$ and the critical values are $\tau = \pm 2$. We have a decomposition $g_+\mathscr{O}_{\mathbb{G}_{\mathrm{m}}} = \mathscr{O}_{\mathbb{A}^1} \oplus M_2$, where M_2 is an irreducible $\mathbf{C}[\tau]\langle\partial_\tau\rangle$-module with regular singularities at $\tau = \pm 2$ and $\tau = \infty$. In fact, M_2 is a free $\mathbf{C}[\tau, (\tau \pm 2)^{-1}]$-module of rank one with a connection having monodromy $-\mathrm{Id}$ around each point $\tau = \pm 2$ and monodromy Id around ∞. Then $\mathrm{FT}(M_2)$ is an irreducible $\mathbf{C}[t]\langle\partial_t\rangle$-module with a regular singularity at $t = 0$ and irregular singularity of slope one at $t = \infty$. Lemma 2.5 gives in particular:

$$\widetilde{\mathrm{Kl}}_2 = j_0^+\,\mathrm{FT}(g_+\mathscr{O}_{\mathbb{G}_{\mathrm{m}}}) = j_0^+\,\mathrm{FT}(M_2).$$

Therefore, by [6], $\widetilde{\mathrm{Kl}}_2$ is a rigid irreducible $\mathbf{C}[t, t^{-1}]$-module with connection and $\mathrm{FT}(M_2)$ is equal to the intermediate extension $j_{0\dagger+}\widetilde{\mathrm{Kl}}_2$. Since the monodromy of M_2 at infinity is equal to Id, the inverse Fourier stationary phase formula implies that the monodromy on the vanishing cycles $\phi_t\widetilde{\mathrm{Kl}}_2$ (a one-dimensional vector space) is also equal to Id. By the intermediate extension property, it follows that the monodromy on the nearby cycles $\psi_t\widetilde{\mathrm{Kl}}_2$ (a two-dimensional vector space) is the unipotent automorphism with one Jordan block of size two. It follows that the same property holds for Kl_2.

3.2. Structure of $\mathrm{Sym}^k \mathrm{Kl}_2$ and $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$

Bases of $\mathrm{Sym}^k \mathrm{Kl}_2$ *and* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$. We obtain a $\mathbf{C}[z,z^{-1}]$-basis (resp. $\mathbf{C}[t,t^{-1}]$-basis) of $\mathrm{Sym}^k \mathrm{Kl}_2$ (resp. $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$) by considering the monomials

$$u_a = v_0^{k-a} v_1^a, \quad \text{resp. } \widetilde{u}_a = \widetilde{v}_0{}^{k-a} \widetilde{v}_1{}^a, \quad 0 \leqslant a \leqslant k. \tag{3.2}$$

In the basis $\{u_a\}$ resp. $\{\widetilde{u}_a\}$, the connection of $\mathrm{Sym}^k \mathrm{Kl}_2$ resp. $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ is given by

$$\begin{aligned} z\partial_z u_a &= (k-a)u_{a+1} + azu_{a-1}, \\ \tfrac{1}{2} t\partial_t \widetilde{u}_a &= (k-a)\widetilde{u}_{a+1} + at^2 \widetilde{u}_{a-1}, \end{aligned} \quad \text{with } u_b, \widetilde{u}_b := 0 \text{ if } b < 0 \text{ or } b > k. \tag{3.3}$$

We gather in the following proposition the properties of $\mathrm{Sym}^k \mathrm{Kl}_2$ and $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$.

Proposition 3.4. *The free* $\mathbf{C}[z,z^{-1}]$*-module resp.* $\mathbf{C}[t,t^{-1}]$*-module with connection* $\mathrm{Sym}^k \mathrm{Kl}_2$ *resp.* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ *satisfies the following properties:*

1. $\mathrm{rk}\,\mathrm{Sym}^k \mathrm{Kl}_2 = \mathrm{rk}\,\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2 = k+1$ *and the monodromy of* $\mathrm{Sym}^k \mathrm{Kl}_2$ *around* $z=0$ *(resp. of* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ *around* $t=0$*) is unipotent with only one Jordan block of size* $k+1$.
2. $\mathrm{Sym}^k \mathrm{Kl}_2$ *(resp.* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$*) is endowed with a* $(-1)^k$*-symmetric non-degenerate pairing.*
3. *Let* i *be a square root of* -1 *and let* L_{i} *denote the rank-one meromorphic connection with monodromy* i *around* ∞*. Then*

$$\mathrm{Sym}^k \widehat{\mathrm{Kl}}_2 \simeq \begin{cases} \bigoplus_{j=0}^{(k-1)/2} ([2]_+ E^{2(2j-k)t}) \otimes L_{\mathsf{i}}^{\otimes k} & \textit{if } k \textit{ is odd,} \\ L_{\mathsf{i}}^{\otimes k} \oplus \bigoplus_{j=0}^{k/2-1} ([2]_+ E^{2(2j-k)t}) \otimes L_{\mathsf{i}}^{\otimes k} & \textit{if } k \textit{ is even.} \end{cases}$$

In particular, $\mathrm{irr}_\infty(\mathrm{Sym}^k \mathrm{Kl}_2) = \lfloor (k+1)/2 \rfloor$ *and, if* k *is odd,* $\mathrm{Sym}^k \widehat{\mathrm{Kl}}_2$ *is purely irregular.*

Proof. (1) It follows from the property that Sym^k of the standard representation of $\mathfrak{sl}_2$ is an irreducible representation of size $k+1$, hence (1).

(2) $(-1)^k$-symmetric self-duality of $\mathrm{Sym}^k \mathrm{Kl}_2$ resp. $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ follows from skew-symmetric self-duality of Kl_2 resp. $\widetilde{\mathrm{Kl}}_2$.

(3) The formal structure of $\widehat{\mathrm{Kl}}_2$ at infinity given by (3.1) implies the following.

Lemma 3.5. *We have* $\mathrm{Sym}^k \widehat{\mathrm{Kl}}_2 \simeq \mathrm{Sym}^k([2]_+ E^{2t}) \otimes L_{\mathsf{i}}^{\otimes k}$*. On the other hand,* $\mathrm{Sym}^k([2]_+ E^{2t})$ *is the* μ_2*-invariant submodule (by* $t \mapsto -t$*) of* $[2]^+ \mathrm{Sym}^k([2]_+ E^{2t}) \simeq \mathrm{Sym}^k([2]^+[2]_+ E^{2t})$*. The latter* $\mathbf{C}(\!(t^{-1})\!)$*-module with connection decomposes as* $\bigoplus_{j=0}^k E^{2(2j-k)t}$*.* □

We know that $\mathrm{Sym}^k([2]_+ E^{2t})$ is the invariant part by $t \mapsto -t$ of $\bigoplus_{j=0}^k E^{2(2j-k)t}$. On the one hand, the invariant part of $E^{at} \oplus E^{-at}$ $(a \neq 0)$ is $[2]_+ E^{at} = [2]_+ E^{-at}$. On the other hand, the invariant part of $(\mathbf{C}(\!(t^{-1})\!), \mathrm{d})$ is $(\mathbf{C}(\!(z^{-1})\!), \mathrm{d})$. We conclude that

$$\mathrm{Sym}^k([2]_+ E^{2t}) \simeq \begin{cases} \bigoplus_{j=0}^{(k-1)/2} [2]_+ E^{2(2j-k)t} & \text{if } k \text{ is odd,} \\ (\mathbf{C}(\!(z^{-1})\!), \mathrm{d}) \oplus \bigoplus_{j=0}^{k/2-1} [2]_+ E^{2(2j-k)t} & \text{if } k \text{ is even,} \end{cases}$$

from which 3.4(3) follows. As a consequence, the formal regular component $(\mathrm{Sym}^k \widehat{\mathrm{Kl}}_2)_{\mathrm{reg}}$ has rank zero (k odd) or rank one (k even), and in the latter case the formal monodromy has eigenvalue one if and only if $k \equiv 0 \bmod 4$. □

We also obtain from the above proof:

$$\mathrm{irr}_\infty(\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) = \begin{cases} k+1 & \text{if } k \text{ is odd,} \\ k & \text{if } k \text{ is even.} \end{cases}$$

The formal behaviour of $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ at infinity is similarly given by

$$\mathrm{Sym}^k \widehat{\widetilde{\mathrm{Kl}}_2} \simeq \bigoplus_{j=0}^{k} E^{2(2j-k)t} \otimes L_{-1}^{\otimes k}. \tag{3.6}$$

Corollary 3.7.

(1) *The natural morphism*

$$j_{\infty\dagger+} \mathrm{Sym}^k \mathrm{Kl}_2 \longrightarrow j_{\infty+} \mathrm{Sym}^k \mathrm{Kl}_2$$

is an isomorphism if $k \not\equiv 0 \bmod 4$. *A similar result holds for* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ *if* $k \not\equiv 0 \bmod 2$.

(2) *Let N be a proper $\mathbf{C}[z]\langle\partial_z\rangle$-submodule of $j_{0+} \mathrm{Sym}^k \mathrm{Kl}_2$ such that $j_0^+ N = \mathrm{Sym}^k \mathrm{Kl}_2$. Then $N = j_{0\dagger+} \mathrm{Sym}^k \mathrm{Kl}_2$. A similar property holds for* $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$.

Proof. (1) follows from 3.4(3). For (2), the question is local analytic at $z = 0$. If we set $E = \psi_{z,1} \mathrm{Sym}^k \mathrm{Kl}_2$ with nilpotent endomorphism N, then giving N is equivalent to giving a subspace F of E stable by N, together with two morphisms $E \xrightarrow{\mathrm{can}} F \xrightarrow{\mathrm{var}} E$ commuting with N, such that $\mathrm{var} \circ \mathrm{can} = \mathrm{N}$ and var is the natural inclusion. We then have $F \supset \mathrm{im}\,\mathrm{can} = \mathrm{im}\,\mathrm{N}$. Since N has only one Jordan block, $\mathrm{im}\,\mathrm{N}$ has codimension one in E and since $F \neq E$ by the properness assumption, this implies $F = \mathrm{im}\,\mathrm{N}$, as wanted. □

3.3. **The inverse Fourier transform of $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$.** We consider the exact sequence

$$0 \longrightarrow j_{0\dagger+} \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2 \longrightarrow j_{0+} \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2 \longrightarrow \widetilde{C}_k \longrightarrow 0, \tag{3.8}$$

where $\widetilde{C}_k$ is supported at the origin (recall that $j_0 \colon \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1$ denotes the inclusion). Applying the vanishing cycle functor at $t = 0$ we find the exact sequence

$$0 \longrightarrow \phi_{t,1} j_{0\dagger+}(\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) \longrightarrow \phi_{t,1} j_{0+}(\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) \longrightarrow \phi_{t,1}\widetilde{C}_k \longrightarrow 0.$$

By definition of j_{0+} and $j_{0\dagger+}$, the middle term $\phi_{t,1}(j_{0+} \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ is canonically identified with the nearby cycle module $\psi_{t,1}(j_{0+} \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$, and $\phi_{t,1}(j_{0\dagger+} \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ is then identified with the subspace $\mathrm{im}\,\widetilde{\mathrm{N}}$, where $\widetilde{\mathrm{N}}$ is the nilpotent endomorphism induced by $t\partial_t$. Since $\widetilde{\mathrm{N}}$ has only one Jordan block of size $k+1$, we conclude that $\dim \phi_{t,1}\widetilde{C}_k = 1$, and $\widetilde{\mathrm{N}}$ on $\phi_{t,1}(j_{0\dagger+} \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ has only one Jordan block of size k.

Let us apply inverse Fourier transformation to (3.8). One can define an endofunctor Π of the category of regular holonomic $\mathscr{D}_{\mathbb{A}^1}$-modules, which is a projector, whose image consists of regular holonomic $\mathscr{D}_{\mathbb{A}^1}$-modules with zero global de Rham cohomology (see also §B.3). We get the following exact sequence of regular holonomic $\mathscr{D}_{\mathbb{A}^1_\tau}$-modules:

$$0 \longrightarrow \widetilde{M} \longrightarrow \Pi(\widetilde{M}) \longrightarrow \widetilde{M}' \longrightarrow 0, \tag{3.9}$$

where $\widetilde{M}'$ is constant of rank one on $\mathbb{A}^1_\tau$. The origin $\tau = 0$ is a singular point for $\widetilde{M}$ and $\Pi(\widetilde{M})$ if and only if the formal regular component of $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2$ at infinity is non-zero, and then $\dim \phi_\tau \widetilde{M} = \dim \phi_\tau \Pi(\widetilde{M})$ is equal to the rank of this formal component. By Lemma 3.5, this rank is equal to zero if k is odd and one if k is even, and in the latter case the eigenvalue of the corresponding formal monodromy is one.

Let us summarize the properties of $\widetilde{M}$ and $\Pi(\widetilde{M})$.

Corollary 3.10.

(1) $\Pi(\widetilde{M})$ *is a regular holonomic* $\mathbf{C}[\tau]\langle\partial_\tau\rangle$*-module, generically of rank* $k+1$ *with singularities at the points* $\tau = 2(2j-k)$ $(j = 0, \dots, k)$*. For each singularity, the vanishing cycle space has rank one with local monodromy equal to identity. At* $\tau = \infty$*, the monodromy is unipotent, with only one Jordan block of size* $k+1$*.*

(2) $\widetilde{M}$ *is a regular holonomic* $\mathbf{C}[\tau]\langle\partial_\tau\rangle$*-module, generically of rank* k*, with singularities at the points* $\tau = 2(2j-k)$ $(j = 0, \dots, k)$*. For each singularity, the vanishing cycle space has rank one with local monodromy equal to identity. At* $\tau = \infty$*, the monodromy is unipotent, with only one Jordan block of size* k*.* □

3.4. Irreducibility and non-rigidity.

Proposition 3.11. *The* $\mathbf{C}[z, z^{-1}]$*-module with connection* $\operatorname{Sym}^k \mathrm{Kl}_2$ *(resp. the* $\mathbf{C}[t, t^{-1}]$*-module with connection* $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2$*) is irreducible. Moreover,* $\operatorname{Sym}^k \mathrm{Kl}_2$ *is rigid if and only if* $k = 1$ *or* $k = 2$*, and* $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2$ *is rigid if and only if* $k = 1$*.*

Let us first give a direct consequence of the irreducibility statement.

Corollary 3.12.

(1) *The intermediate extensions*

$$j_{0\dagger+} \operatorname{Sym}^k \mathrm{Kl}_2, \quad \textit{resp. } j_{0\dagger+} \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2, \quad \textit{resp. } j_{\dagger+} \operatorname{Sym}^k \mathrm{Kl}_2, \quad \textit{resp. } j_{\dagger+} \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2,$$

that we consider as $\mathscr{D}_{\mathbb{A}^1_z}$*- (resp.* $\mathscr{D}_{\mathbb{A}^1_t}$*-, resp.* $\mathscr{D}_{\mathbb{P}^1_z}$*-, resp.* $\mathscr{D}_{\mathbb{P}^1_t}$*-) modules, are irreducible and self-dual.*

(2) *We have*

$$[2]_+ j_{\dagger+} \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2 \simeq j_{\dagger+} \operatorname{Sym}^k \mathrm{Kl}_2 \oplus j_{\dagger+}(\operatorname{Sym}^k \mathrm{Kl}_2 \otimes L_{-1})$$

and

$$\begin{aligned} j_{0\dagger+}(\operatorname{Sym}^k \mathrm{Kl}_2 \otimes L_{-1}) &\simeq j_{0+}(\operatorname{Sym}^k \mathrm{Kl}_2 \otimes L_{-1}), \\ j_{\infty\dagger+}(\operatorname{Sym}^k \mathrm{Kl}_2 \otimes L_{-1}) &\simeq j_{\infty+}(\operatorname{Sym}^k \mathrm{Kl}_2 \otimes L_{-1}) \quad \textit{if } k \not\equiv 2 \bmod 4. \end{aligned}$$

Proof of the irreducibility property in Proposition 3.11. It is enough to prove that $\operatorname{Sym}^k \mathrm{Kl}_2$ resp. $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2$ is semi-simple, since the structure of its monodromy at $z = 0$ implies that it has no direct summand (and the same argument for $\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_2$). Being a direct summand of $\bigotimes^k\mathrm{Kl}_2$, it is enough to prove that the latter is semi-simple, and it is enough to prove the same property for $[2]^+(\bigotimes^k\mathrm{Kl}_2)\simeq\bigotimes^k\widetilde{\mathrm{Kl}}_2$. For that purpose, we will use a geometric interpretation of $\bigotimes^k\widetilde{\mathrm{Kl}}_2$.

Let $g^{\boxplus k}\colon U=\mathbb{G}_{\mathrm{m}}^k\to\mathbb{A}^1_\tau$ be the k-fold Thom–Sebastiani sum of g with itself. We have $g^{\boxplus k}(y_1,\dots,y_k)=\sum_{i=1}^k(y_i+1/y_i)$. This is a convenient non-degenerate Laurent polynomial in the sense of Kouchnirenko. We consider the Gauss–Manin system $\mathscr{H}^0g_+^{\boxplus k}\mathscr{O}_U$ and its localized Fourier transform $j_0^+\mathrm{FT}(\mathscr{H}^0g_+^{\boxplus k}\mathscr{O}_U)$. The second isomorphism of Proposition 2.9 reads

$$\bigotimes^k\widetilde{\mathrm{Kl}}_2\simeq j_0^+\mathrm{FT}(\mathscr{H}^0g_+^{\boxplus k}\mathscr{O}_U).$$

Since $g^{\boxplus k}$ is M-tame (see proof of Proposition 2.9), the cone of the natural morphism $g_\dagger^{\boxplus k}\mathscr{O}_U\to g_+^{\boxplus k}\mathscr{O}_U$ has constant cohomology, according to [39, §1], hence so does the kernel and cokernel of the induced morphism $\mathscr{H}^0g_\dagger^{\boxplus k}\mathscr{O}_U\to\mathscr{H}^0g_+^{\boxplus k}\mathscr{O}_U$. We denote by $\mathscr{H}^0g_{\dagger+}^{\boxplus k}\mathscr{O}_U$ its image. It follows that the induced morphisms (after inverting ∂_τ)

$$\bigl(\mathscr{H}^0g_\dagger^{\boxplus k}\mathscr{O}_U\bigr)[\partial_\tau^{-1}]\longrightarrow\bigl(\mathscr{H}^0g_{\dagger+}^{\boxplus k}\mathscr{O}_U\bigr)[\partial_\tau^{-1}]\longrightarrow(\mathscr{H}^0g_+^{\boxplus k}\mathscr{O}_U)[\partial_\tau^{-1}]$$

are isomorphisms. We finally obtain a morphism

$$\mathscr{H}^0g_{\dagger+}^{\boxplus k}\mathscr{O}_U\longrightarrow\mathrm{FT}^{-1}(j_{0+}(\bigotimes^k\widetilde{\mathrm{Kl}}_2))$$

whose kernel and cokernel are constant $\mathscr{D}_{\mathbb{A}^1}$-modules. Semi-simplicity of the left-hand side as a $\mathbf{C}[\tau]\langle\partial_\tau\rangle$-module implies semi-simplicity of $j_{0+}(\bigotimes^k\widetilde{\mathrm{Kl}}_2)$ as a $\mathbf{C}[t]\langle\partial_t\rangle$-module, hence that of $\bigotimes^k\widetilde{\mathrm{Kl}}_2$ as a $\mathbf{C}[t,t^{-1}]$-module with connection, as wanted.

We refer to Appendix B for a reminder on mixed Hodge modules. Let $U=\mathbb{G}_{\mathrm{m}}^k$. Recall that the perverse sheaf associated with the mixed Hodge module $\mathbf{Q}_U^{\mathrm{H}}$ is $\mathbf{Q}_U[k]$, the filtered $\mathscr{D}_U$-module is $\mathscr{O}_U$ with its trivial filtration jumping at the index 0, and it is pure of weight $k=\dim U$. The proper pushforward (in MHM) $\mathscr{H}^0{}_{\mathrm{H}}g_!^{\boxplus k}\mathbf{Q}_U^{\mathrm{H}}$ has weights $\leqslant k$, the pushforward $\mathscr{H}^0{}_{\mathrm{H}}g_*^{\boxplus k}\mathbf{Q}_U^{\mathrm{H}}$ has weights $\geqslant k$ and $\mathscr{H}^0{}_{\mathrm{H}}g_{!*}^{\boxplus k}\mathbf{Q}_U^{\mathrm{H}}$ is pure of weight k, as Hodge modules on $\mathbb{A}^1_\tau$, according to [48]. Away from the singularities, $\mathscr{H}^0{}_{\mathrm{H}}g_{!*}^{\boxplus k}\mathbf{Q}_U^{\mathrm{H}}$ corresponds to a polarizable variation of Hodge structure of weight $k-1$, hence its underlying local system is semi-simple. From [47] we know that $\mathscr{H}^0g_{\dagger+}^{\boxplus k}\mathscr{O}_U$ Riemann–Hilbert corresponds to the intermediate extension of this local system by the inclusion into $\mathbb{A}^1_\tau$. It follows that $\mathscr{H}^0g_{\dagger+}^{\boxplus k}\mathscr{O}_U$ is semi-simple, as wanted. □

Proof of the non-rigidity property in Proposition 3.11. Since both are irreducible, according to the first part of the statement, it is enough to compute the index of rigidity by using the formula of [3, Prop. 3.1] applied to $V:=\mathrm{End}(\mathrm{Sym}^k\mathrm{Kl}_2)$ and $\widetilde{V}:=\mathrm{End}(\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_2)$. A straightforward computation gives

$$\mathrm{rig}(\mathrm{Sym}^k\widetilde{\mathrm{Kl}}_2)=(k+1)(2-k),$$

which is $\leqslant 0$ for $k\geqslant 2$, and

$$\mathrm{rig}(\mathrm{Sym}^k\mathrm{Kl}_2)=\begin{cases}2(1-m^2)&\text{if } k=2m+1,\ m\geqslant 0,\\-2(m^2-m-1)&\text{if } k=2m,\ m\geqslant 1,\end{cases}$$

so that $\mathrm{rig}(\mathrm{Sym}^k \mathrm{Kl}_2) = 2$ if and only if $k = 1$ or $k = 2$. □

The group $\mathfrak{S}_k$ acts on $\mathbb{G}_\mathrm{m}^k$ by $\sigma\cdot(y_1, \dots, y_k) = (y_{\sigma(1)}, \dots, y_{\sigma(k)})$. It preserves $g^{\boxplus k}$. Therefore, $\mathfrak{S}_k$ acts on $\mathscr{H}^0 g_+^{\boxplus k}\mathscr{O}_U$ and thus on its Fourier transform $\mathrm{FT}\,\mathscr{H}^0 g_+^{\boxplus k}\mathscr{O}_U$ and its localized Fourier transform $j_0^+\,\mathrm{FT}\,\mathscr{H}^0 g_+^{\boxplus k}\mathscr{O}_U$. We identify

$$\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2 \simeq j_0^+\,\mathrm{FT}(\mathscr{H}^0 g_+^{\boxplus k}\mathscr{O}_U)^{\mathfrak{S}_k,\chi_n}.$$

We obtain a morphism

$$(\mathscr{H}^0 g_{\dagger+}^{\boxplus k}\mathscr{O}_U)^{\mathfrak{S}_k,\chi_n} \longrightarrow \mathrm{FT}^{-1}(j_{0+}\,\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) = \Pi(\widetilde{M}) \tag{3.13}$$

inducing an isomorphism after inverting ∂_τ, a property which implies

$$\Pi(\widetilde{M}) \simeq \Pi((\mathscr{H}^0 g_{\dagger+}^{\boxplus k}\mathscr{O}_U)^{\mathfrak{S}_k,\chi_n}) \simeq \Pi(\mathscr{H}^0 g_{\dagger+}^{\boxplus k}\mathscr{O}_U)^{\mathfrak{S}_k,\chi_n}.$$

Corollary 3.14. *$\widetilde{M}$ (see (3.9)) is irreducible and equal to the image of (3.13).*

Proof. Corollary 3.12 gives the irreducibility of $j_{0\dagger+}\,\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$, hence that of $\widetilde{M}$ by inverse Fourier transformation. On the other hand, $j_{0+}\,\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ is not semi-simple as a $\mathbf{C}[t]\langle\partial_t\rangle$-module (see (3.8)), hence (3.13) is not onto. By Corollary 3.7(2), its image is equal to $j_{0\dagger+}\,\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$, hence the statement after applying FT^{-1}. □

As recalled in Appendix B, the projector Π lifts as a projector on $\mathsf{MHM}(\mathbb{A}^1)$.

Proposition 3.15. *The exact sequence (3.9) underlies an exact sequence*

$$0 \longrightarrow \widetilde{M}^{\mathrm{H}} \longrightarrow \Pi(\widetilde{M}^{\mathrm{H}}) \longrightarrow \widetilde{M}'^{\mathrm{H}} \longrightarrow 0,$$

in $\mathsf{MHM}(\mathbb{A}^1)$. More precisely, $\widetilde{M}^{\mathrm{H}}$ is pure of weight k and is identified with $W_k\Pi(\widetilde{M}^{\mathrm{H}})$, and $\widetilde{M}'^{\mathrm{H}}$ is pure of weight $2k+1$. Moreover, the Hodge bundles $F^\bullet\widetilde{M}^{\mathrm{H}}$ satisfy

$$\mathrm{rk}\,\mathrm{gr}_F^p \widetilde{M}^{\mathrm{H}} = \begin{cases} 1 & \text{if } p = 0, \dots, k-1, \\ 0 & \text{otherwise,} \end{cases}$$

and the constant variation $\widetilde{M}'^{\mathrm{H}}$ has $\mathrm{gr}_F^p \widetilde{M}'^{\mathrm{H}} = 0$ unless $p = k$.

Proof. As seen in the proof of Proposition 3.11 (after taking the χ_n-isotypic component under $\mathfrak{S}_k$), $(\mathscr{H}^0{}_{\mathrm{H}}g_{!*}^{\boxplus k}\mathscr{O}_U)^{\mathfrak{S}_k,\chi_n}$ is pure of weight k in $\mathsf{MHM}(\mathbb{A}^1_\tau)$, and from Section B.2 we conclude that $\Pi(\mathscr{H}^0{}_{\mathrm{H}}g_{!*}^{\boxplus k}\mathbf{Q}_U^{\mathrm{H}})^{\mathfrak{S}_k,\chi_n}$ is in $\mathsf{MHM}(\mathbb{A}^1)$ (in EMHS in fact) and its underlying $\mathbf{C}[t]\langle\partial_t\rangle$-module is $\Pi(\widetilde{M})$. The image of the MHM lift of (3.13) is a pure Hodge module of weight k which lifts $\widetilde{M}$ to MHM and is denoted by $\widetilde{M}^{\mathrm{H}}$. Therefore, $\Pi(\mathscr{H}^0{}_{\mathrm{H}}g_{!*}^{\boxplus k}\mathbf{Q}_U^{\mathrm{H}})^{\mathfrak{S}_k,\chi_n} = \Pi(\widetilde{M}^{\mathrm{H}})$. We denote by $\widetilde{M}'^{\mathrm{H}}$ the quotient object in MHM.

Let us check the weight properties. It is enough to show that $\widetilde{M}'^{\mathrm{H}}$ is pure of weight $2k+1$, and it is equivalent to showing that $\psi_{1/\tau}\widetilde{M}'^{\mathrm{H}}$ has weight $2k$ (note that $\widetilde{M}'^{\mathrm{H}}$ extends smoothly at infinity). By [35, App.], although $\Pi(\widetilde{M}^{\mathrm{H}})$ is mixed, the weight filtration W on $\psi_{1/\tau,1}\Pi(\widetilde{M}^{\mathrm{H}})$ is nevertheless identified with a shifted monodromy filtration and more precisely it is the monodromy filtration centered at k. Moreover, $\psi_{1/\tau}\widetilde{M}'^{\mathrm{H}}$ is identified with the primitive part $\mathrm{P}_k\,\mathrm{gr}_{2k}^W\,\psi_{1/\tau,1}\Pi(\widetilde{M}^{\mathrm{H}})$, hence is pure of weight $2k$, as wanted.

Let us end with the Hodge filtration. Because the monodromy around infinity has only one Jordan block (Corollary 3.10), the non-zero graded pieces of the weight filtration of $\psi_{1/\tau}\Pi(\widetilde{M}^{\mathrm{H}})$ are the $\widetilde{\mathrm{N}}^\ell \mathrm{P}_k$, hence are of the form $\mathrm{gr}^W_{2j}\,\psi_{1/\tau}\Pi(\widetilde{M}^{\mathrm{H}})$ and of dimension one. It follows that the mixed Hodge structure $\psi_{1/\tau}\Pi(\widetilde{M}^{\mathrm{H}})$ is Hodge-Tate, and thus so is $\psi_{1/\tau}\widetilde{M}^{\mathrm{H}}$. In other words,

$$\mathrm{gr}^p_F\,\psi_{1/\tau}\widetilde{M}^{\mathrm{H}} = \mathrm{gr}^W_{2p}\,\psi_{1/\tau}\widetilde{M}^{\mathrm{H}}, \quad p = 0, \dots, k-1,$$

has dimension one. By the standard properties of Hodge bundles, this implies that $\mathrm{gr}^p_F\,\widetilde{M}^{\mathrm{H}}$ is a rank-one bundle for $p = 0, \dots, k-1$. □

4. Computation of the Hodge filtration

4.1. **De Rham cohomology on $\mathbb{G}_{\mathrm{m}}$.** We have seen in Proposition 2.12 that the de Rham cohomology of $\mathrm{Sym}^k \mathrm{Kl}_2$ is concentrated in degree one. By the analogue of the Grothendieck–Ogg–Shafarevich formula, $\dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ is equal to the irregularity number of $\mathrm{Sym}^k \mathrm{Kl}_2$ at infinity. From Proposition 3.4 we obtain

$$\dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) = \mathrm{irr}_\infty(\mathrm{Sym}^k \mathrm{Kl}_2) = \left\lfloor \frac{k+1}{2} \right\rfloor. \tag{4.1}$$

A similar computation gives

$$\dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) = \mathrm{irr}_\infty(\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) = \begin{cases} k+1 & \text{if } k \text{ is odd,} \\ k & \text{if } k \text{ is even.} \end{cases}$$

Note that, by self-duality (Proposition 3.4(2)) and Poincaré duality, we have

$$\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \simeq \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)^\vee,$$
$$\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) \simeq \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^\vee.$$

We consider the middle extensions $\mathscr{D}_{\mathbb{P}^1}$-modules $j_{\dagger+}\mathrm{Sym}^k \mathrm{Kl}_2$ and $j_{\dagger+}\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$ with respect to the inclusion $j\colon \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{P}^1$, which according to Corollary 2.13 compute the middle extension de Rham cohomology. Recall from loc. cit. that it is concentrated in degree one.

Proposition 4.2. *We have*

$$\dim \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) = \left\lfloor \frac{k-1}{2} \right\rfloor - \delta_{4\mathbf{Z}}(k) = \begin{cases} \dfrac{k-1}{2} & \textit{if } k \textit{ is odd,} \\ 2\left\lfloor \dfrac{k-1}{4} \right\rfloor & \textit{if } k \textit{ is even,} \end{cases}$$

$$\dim \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2) = \begin{cases} k & \textit{if } k \textit{ is odd,} \\ k-2 & \textit{if } k \textit{ is even.} \end{cases}$$

Proof. We first consider the middle extension by $j_0\colon \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1_z$. Corollary 3.7(2) and its proof imply that the cokernel of the injective morphism of $\mathbf{C}[z]\langle\partial_z\rangle$-modules

$$j_{0\dagger+}\mathrm{Sym}^k \mathrm{Kl}_2 \longrightarrow j_{0+}\mathrm{Sym}^k \mathrm{Kl}_2$$

takes the form $i_{0+}\mathbf{C}$, where $i_0\colon \{0\} \hookrightarrow \mathbb{A}^1_z$ is the closed inclusion. Besides, for the middle extension by $j_\infty\colon \mathbb{G}_\mathrm{m} \hookrightarrow \mathbb{A}^1_{1/z}$, we note that $j_{\infty\dagger+}\operatorname{Sym}^k \mathrm{Kl}_2 \to j_{\infty+}\operatorname{Sym}^k \mathrm{Kl}_2$ is an isomorphism if the formal completion of $\operatorname{Sym}^k \mathrm{Kl}_2$ at ∞ is purely irregular or has no monodromy invariants, that is (Lemma 3.5), if $k \not\equiv 0 \bmod 4$. Otherwise, since the formal regular component has rank one and monodromy identity, it is injective with cokernel isomorphic to $i_{\infty+}\mathbf{C}$. Therefore,

$$\dim \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \mathrm{Kl}_2) = \begin{cases} \dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \mathrm{Kl}_2) - 1 & \text{if } 4 \nmid k \\ \dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \mathrm{Kl}_2) - 2 & \text{if } 4 \mid k \end{cases} \tag{4.3}$$

and we apply (4.1). The proof for $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2$ is similar. □

We now give explicit bases of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \mathrm{Kl}_2)$ and $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$ in terms of (3.2).

Proposition 4.4. *The space* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \mathrm{Kl}_2)$ *has a basis consisting of the classes*

$$z^j v_0^k \frac{\mathrm{d}z}{z}, \quad 0 \leqslant j < \left\lfloor \frac{k+1}{2} \right\rfloor,$$

and the space $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)$ *has a basis consisting of the classes*

$$t^j \widetilde{v}_0{}^k \frac{\mathrm{d}t}{t}, \quad 0 \leqslant j < 2\left\lfloor \frac{k+1}{2} \right\rfloor.$$

Proof. We will only consider the case of $\operatorname{Sym}^k \mathrm{Kl}_2$, that of $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2$ being similar by taking below $\deg t = 1$. The de Rham cohomology $\mathrm{H}^r_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \operatorname{Sym}^k \mathrm{Kl}_2)$ is identified with the cohomology of the two-term complex

$$G \xrightarrow{z\partial_z} G, \quad G := \text{the } \mathbf{C}[z, z^{-1}]\langle z\partial_z\rangle\text{-module } \operatorname{Sym}^k \mathrm{Kl}_2.$$

Let $G^+ = \bigoplus_{a=0}^k \mathbf{C}[z]u_a$, regarded as a $\mathbf{C}[z]$-submodule of G. Formula (3.3) shows that G^+ is stable under $z\partial_z$. In fact, the coherent sheaf on $\mathbb{A}^1$ associated with G^+ is the Deligne canonical extension of the connection $\operatorname{Sym}^k \mathrm{Kl}_2$, regularly singular at the origin, where the residue action on the singular fibre has all eigenvalues equal to 0.

Lemma 4.5. *The inclusion* $(G^+, z\partial_z) \to (G, z\partial_z)$ *is a quasi-isomorphism.*

Proof. One has $G = \bigcup_{r \geqslant 0} z^{-r} G^+$ and, for $r \geqslant 0$, $z\partial_z$ is invertible on $z^{-(r+1)}G^+/z^{-r}G^+$ (with eigenvalue $-(r+1)$ and one Jordan block). □

Define the multiplicative degree map $\deg : G^+ \to (\mathbf{Z}_{\geqslant 0}, +)$ by

$$\deg z = 2, \quad \deg u_a = a. \tag{4.6}$$

(This degree is the one induced from the Newton degree associated with the Laurent polynomial $f_k = \sum_{j=1}^k x_j + z \sum_{j=1}^k 1/x_j$, computing the tensor power $\mathrm{Kl}_2^{\otimes k}$. See the proof of Lemma 4.14.) Then $z\partial_z$ is (non-homogeneous) of degree one. Let $\operatorname{gr} G^+$ be the associated graded module. The induced graded $\mathbf{C}$-linear map $\overline{z\partial_z} : \operatorname{gr} G^+ \to \operatorname{gr} G^+[1]$ is $\mathbf{C}[t]$-linear and we regard it as a two-term complex $(\operatorname{gr} G^+, \overline{z\partial_z})$.

Lemma 4.7. *If k is odd, $\mathrm{H}^0(\mathrm{gr}\, G^+, \overline{z\partial_z}) = 0$ and the vector space $\mathrm{H}^1(\mathrm{gr}\, G^+, \overline{z\partial_z})$ is generated by classes of $z^j u_0$, $0 \leqslant j \leqslant (k-1)/2$. If k is even, $\mathrm{H}^0(\mathrm{gr}\, G^+, \overline{z\partial_z})$ and $\mathrm{H}^1(\mathrm{gr}\, G^+, \overline{z\partial_z})$ are free of rank one over the graded ring $\mathbf{C}[t]$ and are generated by $\sum_{i=0}^{k/2} c_i z^i u_{k-2i}$ and the class of u_0 respectively, where $c_i = (-1)^i \binom{k/2}{i}$.*

Proof. We are dealing with finitely generated modules over the principal ideal domain $\mathbf{C}[z]$. Notice that the induced map

$$\overline{z\partial_z} : \bigoplus_{i=0}^{k-1} \mathbf{C}[z]u_i \longrightarrow \mathrm{gr}\, G^+/\mathbf{C}[z]u_0 \tag{4.8}$$

from the submodule of $\mathrm{gr}\, G^+$ to the quotient is clearly an isomorphism. Assume k is odd. With respect to the basis $\{u_a\}$, the action $\overline{z\partial_z}$ has determinant $(k!!)^2(-z)^{(k+1)/2}$. On the other hand, the cokernel has dimension $(k+1)/2$ and the isomorphism (4.8) says that it coincides with the image of $\mathbf{C}[z]u_0$. Therefore the elements $z^j u_0$, $0 \leqslant j \leqslant (k-1)/2$, form a basis of $\mathrm{H}^1(\overline{G}^+, \overline{z\partial_z})$.

Suppose k is even. In this case we have $\det \overline{z\partial_z} = 0$. Then (4.8) implies that $\mathbf{C}[z]u_0 \xrightarrow{\sim} \mathrm{H}^1(\overline{G}^+, \overline{z\partial_z})$. On the other hand, notice that the map $\overline{z\partial_z}$ splits as a direct sum $\overline{z\partial_z}' \oplus \overline{z\partial_z}''$ where

$$\overline{z\partial_z}' : \bigoplus_{j=0}^{k/2} \mathbf{C}[z]u_{2j} \longrightarrow \bigoplus_{j=1}^{k/2} \mathbf{C}[z]u_{2j-1}, \quad \overline{z\partial_z}'' : \bigoplus_{j=1}^{k/2} \mathbf{C}[z]u_{2j-1} \longrightarrow \bigoplus_{j=0}^{k/2} \mathbf{C}[z]u_{2j}. \tag{4.9}$$

Moreover $\overline{z\partial_z}'$ is surjective and $\overline{z\partial_z}''$ injective. Thus $\mathrm{H}^0(\overline{G}^+, \overline{z\partial_z})$ is contained in the submodule $\bigoplus_{j=0}^{k/2} \mathbf{C}[z]u_{2j}$. An inspection of the coefficients reveals the result. □

To conclude Proposition 4.4, we use the spectral sequence

$$E_1^{p,q} = \mathrm{H}^p\Big(\mathrm{gr}_{q-p}\, G^+ \xrightarrow{\overline{z\partial_z}} \mathrm{gr}_{q-p+1}\, G^+\Big) \Longrightarrow \mathrm{H}^p(G^+, z\partial_z) \quad (p \in \{0,1\},\ q \geqslant 0)$$

associated with the grading (4.6). The spectral sequence must degenerate at E_2-page. The first part of the above lemma gives the statement for k odd. For k even, $\mathrm{gr}\, \mathrm{H}^0_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl})$ and $\mathrm{gr}\, \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl})$ are respectively the kernel and cokernel of the induced map

$$z\partial_z : \mathrm{H}^0(\overline{G}^+, \overline{z\partial_z}) \longrightarrow \mathrm{H}^1(\overline{G}^+, \overline{z\partial_z}).$$

For $r \geqslant 0$, we have in $\mathrm{H}^1(\overline{G}^+, \overline{z\partial_z})$ that

$$z\partial_z\Big(z^r \sum_{i=0}^{k/2} a_i z^i u_{k-2i}\Big) = \sum_{i=0}^{k/2} (r+i) a_i t^{r+i} u_{k-2i} \tag{4.10}$$

$$\equiv c_r t^{r+k/2} u_0 \tag{4.11}$$

for some $c_r \in \mathbf{C}$. We have to show that the element (4.10) is non-zero in $\mathrm{H}^1(\overline{G}^+, \overline{z\partial_z})$. Recall the maps in (4.9). It is straightforward to check that the image of $\overline{z\partial_z}''$ consists of $\mathbf{C}$-linear combinations of all homogeneous elements $\sum_{i=0}^{k/2} b_i z^{r+i} u_{k-2i}$ of degree $k + 2r \geqslant 2$ satisfying

$b_i = 0$ if $i < -r$ and

$$\sum_{i=0}^{k/2} (-1)^i \binom{k/2}{i} \binom{k}{2i}^{-1} b_i = 0.$$

However for the element (4.10), we have

$$\sum_{i=0}^{k/2} (-1)^i \binom{k/2}{i} \binom{k}{2i}^{-1} (r+i) a_i = \sum_{i=0}^{k/2} \binom{k/2}{i}^2 \binom{k}{2i}^{-1} (r+i).$$

Since each summand in the above expression is strictly positive, (4.10) is non-trivial and hence $c_r \neq 0$ for all $r \geqslant 0$ in (4.11). The desired statement follows. □

4.2. **The Hodge filtration.** In this section, we start the proof of Theorem 1.7. We will also prove its analogue for $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$, which is stated as follows:

Proposition 4.12. *The mixed Hodge structure* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mathrm{H}}$ *has weights* $\geqslant k+1$ *and the following numerical data:*

(1) *For k odd,* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mathrm{H}}$ *is mixed of weights $k+1$ and $2k+2$, with*

$$\dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{p,q} = \begin{cases} 1, & p+q = k+1,\ p \in \{1, \dots, k\}, \\ 1, & p = q = k+1, \\ 0, & \textit{otherwise.} \end{cases}$$

(2) *For k even,* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mathrm{H}}$ *is mixed of weights $k+1$, $k+2$ and $2k+2$, with*

$$\dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{p,q} = \begin{cases} 1, & p+q = k+1,\ p \in \{1, \dots, k\} \textit{ and } p \neq k/2,\ k/2+1, \\ 1, & p = q = k/2+1, \\ 1, & p = q = k+1, \\ 0, & \textit{otherwise.} \end{cases}$$

Furthermore, the mixed Hodge structure $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mathrm{H}}$ *is pure of weight $k+1$ and is isomorphic to* $W_{k+1}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mathrm{H}}$.

We start with giving precise information on the Hodge filtration. We compute here the irregular Hodge filtration, which is the same as the Hodge filtration of the mixed Hodge structure $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mathrm{H}}$ (and similarly for $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)^{\mathrm{H}}$), according to Theorem B.10. We still denote by $F^\bullet$ the irregular Hodge filtration.

Proposition 4.13. *With respect to the bases of Proposition* 4.4*,*

(1) *the Hodge filtration of* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mathrm{H}}$ *is given by*

$$F^p \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mathrm{H}} = \Big\langle t^j \widetilde{v}_0^k \frac{\mathrm{d}t}{t} \;\Big|\; 0 \leqslant j \leqslant k+1-p \Big\rangle$$

if k is odd, or if k is even and $p > k/2$.

(2) *the Hodge filtration of* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)^{\mathrm{H}}$ *is given by*

$$F^p\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)^{\mathrm{H}} = \Big\langle z^j v_0^k \frac{\mathrm{d}z}{z} \;\Big|\; 0 \leqslant j \leqslant \Big\lfloor \frac{k+1-p}{2} \Big\rfloor \Big\rangle$$

if k *is odd, or if* k *is even and* $p > k/2$.

4.2.1. *Proof of Proposition* 4.13 *for odd* k. We start with (2). Under the inclusion

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Kl}_2^{\otimes k}) \simeq \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k})$$

the basis element $z^j v_0^k \,\mathrm{d}z/z \in \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ is mapped to

$$w_j := z^j \frac{\mathrm{d}z}{z} \frac{\mathrm{d}x_1}{x_1} \cdots \frac{\mathrm{d}x_k}{x_k} \in \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k}).$$

Lemma 4.14. *Assume* k *is odd. Then* $w_j \in F^p\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k})^{\mathrm{H}}$ *if and only if* $p \leqslant k+1-2j$.

Proof. For any even or odd k, the exponents of the monomials appearing in f_k all lie in the affine hyperplane $\xi = 1$ in the lattice of monomials where

$$\xi\Big(z^a \prod_{i=1}^k x_i^{b_i}\Big) = 2a + \sum_{i=1}^k b_i.$$

Thus the Newton polytope $\Delta \subset \mathbf{R}^{k+1}$ of f_k has only one facet that does not contain the origin; it lies on the plane $\xi = 1$. The cone $\mathbf{R}_{\geqslant 0}\Delta$ is given by the 2^k inequalities

$$s + \sum_{i=1}^k \varepsilon_i y_i \geqslant 0, \quad \varepsilon_i \in \{0, 1\}.$$

(Here s and y_i are the dual coordinates of z and x_i and indicate the exponents of z and x_i, respectively. E.g., $\xi = 2s + \sum_{i=1}^k y_i$.) It is straightforward to check that f_k is non-degenerate with respect to Δ if (and only if) k is odd. In this case with $\dim \Delta = k+1$, the irregular Hodge filtration on $\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{f_k})$ can be computed via the Newton filtration on monomials in $\mathbf{R}_{\geqslant 0}\Delta$ (see [2, Th. 1.4], [55, Th. 4.6]).

Now return to the assumption that k is odd so that f_k is non-degenerate. It is clear that $z^j \in \mathbf{R}_{\geqslant 0}\Delta$ with $\xi(z^j) = 2j$. We remark that a top form $m\frac{\mathrm{d}z}{z}\frac{\mathrm{d}x_1}{x_1}\cdots\frac{\mathrm{d}x_k}{x_k}$ with monomial $m \in \mathbf{R}_{\geqslant 0}\Delta$ and Newton degree $\xi(m)$ then lies in the irregular Hodge filtration $F^p\mathrm{H}^{k+1}_{\mathrm{dR}}$ with $p \leqslant k+1-\xi(m)$. □

For Proposition 4.13(1), we consider the function

$$\widetilde{\xi}\Big(t^a \prod_{i=1}^k y_i^{b_i}\Big) = a$$

on the monomials in the cone generated by the Newton polytope $\widetilde{\Delta}$ of $\widetilde{f}_k$. In the case k odd, $\widetilde{f}_k$ is non-degenerate. Furthermore, for $m \in \mathbf{R}_{\geqslant 0}\widetilde{\Delta}$, we have $m\frac{\mathrm{d}t}{t}\frac{\mathrm{d}y_1}{y_1}\cdots\frac{\mathrm{d}y_k}{y_k} \in F^p\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{\widetilde{f}_k})^{\mathrm{H}}$ if and only if $p \leqslant k+1-\widetilde{\xi}(m)$. We omit the details. □

4.2.2. *A toric compactification.* Before giving the proof of Proposition 4.13 for even k, we describe an explicit compactification of $(\mathbb{G}_{\mathrm{m}}^{k+1}, \widetilde{f}_k)$ which will allow us to understand the Hodge filtration on the cohomology of $E^{\widetilde{f}_k}$. Since the construction is also used in the study of the étale realizations in Section 5.1, we take the base field to be $\mathbf{Q}$ before dealing with Hodge filtrations in the second half of this subsection.

Recall the isomorphism of Proposition 2.12 for $n = 1$, with $\widetilde{f}_k = tg^{\boxplus k} = t\sum_{i=1}^k (y_i + 1/y_i)$. We now compactify $(\mathbb{G}_{\mathrm{m}}^k, g^{\boxplus k})$. Let $M = \bigoplus_{i=1}^k \mathbf{Z}y_i$ be the lattice of monomials on $\mathbb{G}_{\mathrm{m}}^k$ and $N = \bigoplus_{i=1}^k \mathbf{Z}e_i$ the dual lattice with basis e_i dual to y_i. Consider the toric compactification X of $\mathbb{G}_{\mathrm{m}}^k$ attached to the simplicial fan F in $N_{\mathbf{R}} := \bigoplus_{i=1}^k \mathbf{R}e_i$ generated by the rays

$$\mathbf{R}_{\geqslant 0} \cdot \sum_{i=1}^k \varepsilon_i e_i \quad \text{with } \varepsilon_i \in \{0, \pm 1\} \text{ and } (\varepsilon_1, \dots, \varepsilon_k) \neq 0.$$

There are $2^k k!$ simplicial cones of maximal dimension k in F and each provides an affine chart of X isomorphic to $\mathbb{A}^k$ over which the function $g^{\boxplus k}$ has the same structure. Explicitly as an example, consider the maximal cone of F generated by the k vectors

$$\sum_{i=1}^r e_i, \quad 1 \leqslant r \leqslant k.$$

The affine ring associated with the dual cone in $M_{\mathbf{R}}$ is the polynomial ring $\mathbf{Q}[z_i]_{i=1}^k$ where

$$z_r := y_r/y_{r+1}, \quad 1 \leqslant r < k, \quad \text{and} \quad z_k := y_k.$$

On this chart $X_1 = \mathrm{Spec}(\mathbf{Q}[z_i]_{i=1}^k) \cong \mathbb{A}^k$, we have $g^{\boxplus k} = g_1/z_1 \cdots z_k$ with

$$g_1 = \sum_{r=1}^k z_1 \cdots z_{r-1} + z_1 \cdots z_k \sum_{r=1}^k z_r \cdots z_k \in \Gamma(X_1, \mathscr{O}). \tag{4.15}$$

The toric variety X provides an example of a non-degenerate compactification of $(\mathbb{G}_{\mathrm{m}}^k, g^{\boxplus k})$ as in a neighborhood of $X \smallsetminus \mathbb{G}_{\mathrm{m}}^k$, the closure of the zero locus of $g^{\boxplus k}$ and $X \smallsetminus \mathbb{G}_{\mathrm{m}}^k$ form a strict normal crossing divisor. See also the paragraph before §5.1.3.

Consider the product $\mathbb{P}_t^1 \times X$ as a compactification of $(\mathbb{G}_{\mathrm{m}}^{k+1}, tg^{\boxplus k}) \cong (\mathbb{G}_{\mathrm{m}}^{k+1}, \widetilde{f}_k)$. It is a non-degenerate compactification of $(\mathbb{G}_{\mathrm{m}}^{k+1}, \widetilde{f}_k)$ if k is odd. When k is even, $\mathbb{P}_t^1 \times X$ is non-degenerate away from the $\binom{k}{k/2}$ points $(t, y_i) = (\infty, \varepsilon_i)$ defined over $\mathbf{Q}$, where

$$\varepsilon_i = \pm 1, \quad \sum_{i=1}^k \varepsilon_i = 0.$$

With suitable choice of (analytic or étale) local coordinates on X, at each x of these points, we have $g = z_1^2 + \cdots + z_k^2$, i.e., x is an ordinary quadratic point on $g^{\boxplus k} = 0$. Perform two blowups on $\mathbb{P}_t^1 \times X$: first at each x and then along the intersection of the exceptional divisor and the proper transform of $\infty \times X$. Call the resulting variety $\widetilde{X}$ and the exceptional divisors E_1, E_2 from the two steps, respectively.

A direct computation reveals that $\mathrm{ord}_{E_1} \widetilde{f}_k = 1$, $\mathrm{ord}_{E_2} \widetilde{f}_k = 0$ and $\widetilde{X}$ is a non-degenerate compactification of $(\mathbb{G}_{\mathrm{m}}^{k+1}, \widetilde{f}_k)$.

4.2.3. *Proof of Proposition 4.13 when k is even.*

(1) The cohomology $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ is the χ-isotypic part of $\mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{\widetilde{f}_k})$ under $\mathfrak{S}_k$. Consider the top form

$$\widetilde{w}_j = t^j \,\frac{\mathrm{d}t}{t}\frac{\mathrm{d}y_1}{y_i}\cdots\frac{\mathrm{d}y_k}{y_k}$$

on $\mathbb{G}_{\mathrm{m}}^{k+1}$ corresponding to the basis $t^j\widetilde{v}_0^k\frac{\mathrm{d}t}{t}$ in Proposition 4.4. Take the non-degenerate compactification $\widetilde{X}$ constructed above. Let $D = \widetilde{X} \smallsetminus \mathbb{G}_{\mathrm{m}}^{k+1}$ and P the pole divisor of $\widetilde{f}_k\colon \widetilde{X} \dashrightarrow \mathbb{P}^1$. Then $E_1, E_2 \not\subset |P|$, the support of P, and furthermore one has

$$\widetilde{w}_j \in \Gamma\Big(\widetilde{X}, \Omega^{k+1}(\log D)(jP - (k-j)E_1 - (k-2j)E_2)\Big) \tag{4.16}$$

by a direct computation. Accordingly

$$\widetilde{w}_j \in \Gamma\Big(\widetilde{X}, \Omega^{k+1}(\log D)(\lfloor k+1-\eta\rfloor\, P)\Big)$$

if $j \leqslant k+1-\eta$ and $k-2j \geqslant 0$ (so that $\widetilde{w}_j$ is holomorphic generically on the divisors E_i). There is the natural map (see §A.2.1)

$$\Omega^{k+1}_{\widetilde{X}}(\log D)(\lfloor k+1-\eta\rfloor\, P)[-(k+1)] \longrightarrow F^{\log,\mathrm{dR}}_{-\eta}(E^{\widetilde{f}_k}) \quad (k+1-\eta \geqslant 0),$$

which induces

$$\Gamma\Big(\widetilde{X}, \Omega^{k+1}(\log D)(\lfloor k+1-\eta\rfloor\, P)\Big) \longrightarrow \boldsymbol{H}^{k+1}\big(\widetilde{X}, F^{\log,\mathrm{dR}}_{-\eta}(E^{\widetilde{f}_k})\big) = F^\eta \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}^{k+1}, E^{\widetilde{f}_k}).$$

This gives the desired statement.

(2) We have

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) = \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mu_2}$$

compatible with Hodge filtrations. Therefore with notations as in Proposition 4.4 one has

$$z^j v_0^k \,\frac{\mathrm{d}z}{z} \in F^p\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2) \quad \text{if } 0 \leqslant j \leqslant \Big\lfloor \frac{k+1-p}{2} \Big\rfloor \text{ and } p > \frac{k}{2}$$

as the pullback $[2]^* z^j v_0^k \frac{\mathrm{d}z}{z} = 2t^j\widetilde{v}_0^k\frac{\mathrm{d}t}{t}$ by the double covering is in $F^p\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$. □

Remark 4.17. Notice that by (4.16), we also have

$$\widetilde{w}_j \in \Gamma\left(\widetilde{X}, \Omega^{k+1}(\log D)(*P - E_1 - E_2)\right) \quad \text{if } 0 < j \text{ and } k-2j > 0,$$

and hence $\widetilde{w}_j$ represents a class in $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)$ for k even.

The following lemma will be instrumental in solving the case when k is not a multiple of 4 in Theorem 1.7 and Proposition 4.12. For the notation of the Tate objects, see Notation A.1.

Lemma 4.18. *There are epimorphisms of mixed Hodge structures:*

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)^{\mathrm{H}} \longrightarrow \mathbf{Q}(-k-1)_{\mathbf{C}}, \qquad \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mathrm{H}} \longrightarrow \mathbf{Q}(-k-1)_{\mathbf{C}}.$$

Proof. Notice that f_k can be extended as a regular function on $\mathbb{A}^1_z \times \mathbb{G}_{\mathrm{m}}^k$. Let us consider the closed-open decomposition

$$\{0\} \times \mathbb{G}_{\mathrm{m}}^k \hookrightarrow \mathbb{A}^1_z \times \mathbb{G}_{\mathrm{m}}^k \hookleftarrow \mathbb{G}_{\mathrm{m}}^{k+1}$$

and note that $f_k|_{z=0} = h_k := \sum_i x_i$. Since $\mathrm{H}^k(\mathbb{G}_\mathrm{m}^k, h_k) = \mathrm{H}^1(\mathbb{G}_\mathrm{m}, x)^{\otimes k} = \mathbf{Q}(-k)$ in $\mathsf{Mot}^{\exp}$, it is a classical motive of pure weight k. The localization sequence in $\mathsf{Mot}^{\exp}$ (see (iii) in the proof of Theorem 2.17)

$$\cdots \longrightarrow \mathrm{H}^{k+1}(\mathbb{A}^1 \times \mathbb{G}_\mathrm{m}^k, f_k) \longrightarrow \mathrm{H}^{k+1}(\mathbb{G}_\mathrm{m}^{k+1}, f_k) \longrightarrow \mathrm{H}^k(\mathbb{G}_\mathrm{m}^k, h_k)(-1) \longrightarrow \cdots$$

gives thus rise to an exact sequence in MHS:

$$\cdots \longrightarrow \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{A}^1 \times \mathbb{G}_\mathrm{m}^k, E^{f_k})^{\mathrm{H}} \longrightarrow \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}^{k+1}, E^{f_k})^{\mathrm{H}} \xrightarrow{(*)} \mathrm{H}^k_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}^k, E^h)^{\mathrm{H}}(-1) \longrightarrow \cdots$$

The arrow $(*)$ can be computed by taking residues. On the other hand, one has the maps

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Kl}_2^{\otimes k}) \xrightarrow{\sim} \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}^{k+1}, E^{f_k}). \tag{4.19}$$

and the desired morphism is obtained by composing (4.19) and $(*)$. It is non-trivial (hence surjective) since one has

$$\begin{array}{ccccc} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) & \longrightarrow & \mathrm{H}^{k+1}_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}^{k+1}, E^{f_k}) & \xrightarrow{(*)} & \mathrm{H}^k_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}^k, E^h)(-1) \\ v_0^k \dfrac{\mathrm{d}z}{z} & \longmapsto & \dfrac{\mathrm{d}z}{z}\dfrac{\mathrm{d}x_1}{x_1}\cdots\dfrac{\mathrm{d}x_k}{x_k} & \longmapsto & \dfrac{\mathrm{d}x_1}{x_1}\cdots\dfrac{\mathrm{d}x_k}{x_k} \end{array}$$

and the last form is a generator of $\mathrm{H}^k_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}^k, E^h) = \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, E^x)^{\otimes k}$.

The proof is similar for $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$, where we have $h \equiv 0$. □

We will also need the following lemma when $k \equiv 0 \bmod 4$ (for $\mathrm{Sym}^k \mathrm{Kl}_2$) or k even (for $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$). Its proof will be given at the end of Section 4.3.

Lemma 4.20. *We have*

(1) $\mathrm{gr}_F^{k+2} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)^{\mathrm{H}} \neq 0$ *if* $k \equiv 0 \bmod 4$,
(2) $\mathrm{gr}_F^{k+2} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mathrm{H}} \neq 0$ *if* k *is even.*

Proof of Theorem 1.7 *when* $k \not\equiv 0 \bmod 4$. By Theorem 2.26 we know that

- $W_k \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) = 0$ and
- $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) \subset W_{k+1} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)$.

On the other hand, (4.3) shows that $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)$ has codimension one in $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)$. Together with Lemma 4.18, this implies that

$$\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2) = W_{k+1} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2),$$

$$\dim \mathrm{gr}_\ell^W \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)^{\mathrm{H}} = \begin{cases} 0 & \text{if } \ell > k+1 \;\&\; \ell \neq 2k+2, \\ 1 & \text{if } \ell = 2k+2. \end{cases}$$

If k is odd, Proposition 4.13(2) together with Proposition 4.4 give

$$\dim \mathrm{gr}_F^p \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)^{\mathrm{H}} = \begin{cases} 1 & \text{for } p = 2, 4, \ldots, k+1, \\ 0 & \text{otherwise.} \end{cases}$$

The term gr_F^{k+1} must correspond to $H^{k+1,k+1}$, according to Lemma 4.18. Therefore, the spaces $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}, \mathrm{Sym}^k \mathrm{Kl}_2)^{p,q}$ are as asserted in the theorem.

If $k=4m+2$, Proposition 4.13(2) together with Proposition 4.4 give

$$\dim \operatorname{gr}_F^p \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2) = 1 \quad \text{for } p = k+1, k-1, k-3, \dots \text{ and } p \geqslant k/2+1 = 2m+2.$$

Forgetting $p=k+1$ which contributes to $\operatorname{gr}^W_{2k+2} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$, this gives the m terms

$$\dim \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)^{p,k+1-p} = 1, \quad p = 2m+3, 2m+5, \dots, 4m+1.$$

Hodge symmetry together with $\dim \operatorname{gr}^W_{k+1} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2) = 2m$ implies that the spaces $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)^{p,q}$ are as asserted in the theorem. ☐

Proof of Theorem 1.7 *when* $k \equiv 0 \bmod 4$. We set $k=4m$. Lemma 4.20, together with Lemma 4.18, implies that $W_{k+1}\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ has codimension $\geqslant 2$. On the other hand, since it contains $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ which has codimension two (after (4.1) and Proposition 4.2), the two spaces coincide. Hence, $\dim \operatorname{gr}_F^{k+2} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2) = 1$, so it has type $(2m+1, 2m+1)$.

Arguing as above, we find

$$\dim \operatorname{gr}_F^p \mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2) = 1 \quad \text{for } p = k+1, k-1, k-3, \dots \text{ and } p \geqslant k/2+1 = 2m+1.$$

On the one hand, $\operatorname{gr}_F^{k+1}$ corresponds to $\operatorname{gr}^W_{2k+2}$, while $\operatorname{gr}_F^{k/2+1}$ corresponds to $\operatorname{gr}^W_{k+2}$. The remaining values of p are $4m-1, 4m-3, \dots, 2m+3$, and $q = k+1-p$ takes values $2, 4, \dots, 2(m-1)$. By Hodge symmetry in weight $k+1$, one finds that the spaces $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)^{p,q}$ are as asserted in the theorem. ☐

Proof of Proposition 4.12 *when* k *is odd.* The proof is completely similar to that for Theorem 1.7, by using Proposition 4.13(1). ☐

Proof of Proposition 4.12 *when* k *is even.* Proposition 4.13(1) gives $\dim \operatorname{gr}_F^p = 1$ if $k/2 < p \leqslant k+1$. The case $p=k+1$ corresponds to $\operatorname{gr}^W_{2k+2}$ according to Lemma 4.18, while the case $p = k/2+1$ corresponds to $\operatorname{gr}^W_{k+1}$, according to Lemma 4.20. We remain with $p = k/2+2, \dots, k$ in weight $k+1$, so that $q = 1, \dots, k/2-1$. Hodge symmetry implies that the spaces $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)^{p,q}$ are as asserted. ☐

4.3. **Another proof of Proposition 4.12.** In this section, we give another approach to Proposition 4.12 by using the theory of mixed Hodge modules. Although a shorter proof could be given by using the more general theory of mixed twistor $\mathscr{D}$-modules of T. Mochizuki [37], the present approach remains more related to motivic questions and leads as well to a proof of Lemma 4.20.

We use notation and results of Sections 3.3, 3.4 and B.5. We have $\operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2 = \mathrm{FT}\,\Pi(\widetilde{M})$ and the assumptions of Section B.5 are fulfilled for $\widetilde{M}$. The mixed Hodge structure $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \widetilde{\mathrm{Kl}}_2)^{\mathrm{H}}$ is the extension of $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M}')^{\mathrm{H}}$ by $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M})^{\mathrm{H}}$. We can apply Proposition B.12 with $w=k$ for $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M}')^{\mathrm{H}}$. According to Proposition 3.15, we find that $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M}')^{\mathrm{H}}$ is pure of weight $2k+2$ and

$$\dim \operatorname{gr}_F^p \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M}')^{\mathrm{H}} = \begin{cases} 1 & \text{if } p = k+1, \\ 0 & \text{otherwise.} \end{cases}$$

If k is odd, 0 is not a singular point of $\widetilde{M}$, so we can apply Proposition B.12 with $w = k$ for $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M})^{\mathrm{H}}$. According to Proposition 3.15, we find that $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M})^{\mathrm{H}}$ is pure of weight $k+1$, and

$$\dim \mathrm{gr}^p_F \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M})^{\mathrm{H}} = \begin{cases} 1 & \text{if } p = 1, \dots, k, \\ 0 & \text{otherwise.} \end{cases}$$

If k is even, 0 is a singular point of $\widetilde{M}$ and, according to Theorem B.15, we identify the mixed Hodge structure $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M})^{\mathrm{H}}$ with $\operatorname{coker} \widetilde{\mathrm{N}} \colon \psi_{\tau,1}\widetilde{M}^{\mathrm{H}} \to \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}(-1)$. Since $\widetilde{M}$ is an intermediate extension at $\tau = 0$ and $\dim \phi_{\tau,1}\widetilde{M} = 1$, we have $\widetilde{\mathrm{N}}^2 = 0$. Recall also that the weight filtration on $\psi_{\tau,1}\widetilde{M}^{\mathrm{H}}$ is the monodromy filtration of $\widetilde{\mathrm{N}}$ centered at $k-1$ (k is the pure weight of $\widetilde{M}$). The primitive parts of the Lefschetz decomposition of $\mathrm{gr}^W \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}$ are thus

- $\mathrm{gr}^W_k \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}$ of rank one,
- $\mathrm{gr}^W_{k-1} \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}$ of rank $k-1$.

It follows that

$$\mathrm{gr}^W \operatorname{coker} \widetilde{\mathrm{N}} = \mathrm{gr}^W_{k-1} \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}(-1) \oplus \mathrm{gr}^W_k \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}(-1).$$

In particular, $\mathrm{gr}^W_{k+2} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\widetilde{M})^{\mathrm{H}} = \mathrm{gr}^W_{k+2} \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}\,\Pi(\widetilde{M}))^{\mathrm{H}}$ corresponds to the summand $\mathrm{gr}^W_k \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}(-1)$ and has dimension one, proving thereby Lemma 4.20(2) for $\mathrm{Sym}^k \widetilde{\mathrm{Kl}}_2$. The computation of the dimension of gr^p_F is obtained with the formulas of Proposition 3.15. □

Proof of Lemma 4.20. The proof of Lemma 4.20(2) has been given above. Let us prove (1). It amounts to proving that the μ_2 action on $\mathrm{gr}^W_k \psi_{\tau,1}\widetilde{M}^{\mathrm{H}}(-1)$ is trivial if $k \equiv 0 \bmod 4$ and it is enough to consider the action on the underlying vector space. According to the previous analysis, this amounts to proving that $\phi_{\tau,1}\widetilde{M}$ is invariant under μ_2 for such a k. By the stationary phase formula, this amounts to proving that the μ_2-invariant submodule of the regular part at $t = \infty$ of $\mathrm{Sym}^k \widehat{\widetilde{\mathrm{Kl}}}_2$ has formal monodromy equal to Id. Lemma 3.5 and (3.6) imply that this holds if and only if $k \equiv 0 \bmod 4$. □

5. L-functions

In this section, we compute the L-function associated with the motive $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ over $\mathbf{Q}$. We first compare, in Theorems 5.9 and 5.15, the traces of Frobenius at unramified primes of the ℓ-adic realization with symmetric power moments of Kloosterman sums. The results, which largely overlap with Yun's theorem [56, Th. 1.1.6], show that the two approaches yield the same Galois representation up to semi-simplification by Chebotarev density and Brauer–Nesbitt theorems as realizations of two different geometric models. In some sense, we replace the use of the theories of affine Grassmannians and homogeneous Fourier transforms in [56] by the theories of exponential motives and irregular Hodge filtrations, and we obtain an easier geometric model $\mathscr{K}$ which leads to the motive (compare with [56, §4.1.6]). As an advantage, one obtains the structure at the ramified primes in the case k odd by applying the Picard–Lefschetz formula. In addition, in Proposition 5.20 we show that the Galois representations are crystalline at $p > k$ when k is odd (resp. $p > k/2$ when k is even). We show the potential automorphy in the end.

5.1. **Étale realizations.**

5.1.1. *Cohomology of* $\mathrm{Sym}^k \mathrm{Kl}_2$ *over finite fields.* Recall the ℓ-adic Kloosterman sheaf Kl_2 on $\mathbb{G}_\mathrm{m}$ over a finite field from Section 2.4.1. In this paragraph, we gather the main properties about étale cohomology of its symmetric powers. All results are due to Fu–Wan [21, Th. 0.2] and Yun [56, Lem. 4.2.1, Cor. 4.2.3 and 4.3.5], who prove them by a thorough study on the structure of Kl_2 at zero and infinity. Throughout, F_p denotes the geometric Frobenius in $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ and we consider the reciprocal characteristic polynomials

$$Z_k(p;T) = \det\Big(1 - F_pT \mid \mathrm{H}^1_{\text{ét,c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)\Big)$$
$$M_k(p;T) = \det\Big(1 - F_pT \mid \mathrm{H}^1_{\text{ét,mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)\Big).$$

- If k is odd, then

$$\deg Z_k(p;T) = \begin{cases} \dfrac{k+1}{2}, & p=2 \\[2mm] \dfrac{k+1}{2} - \Big\lfloor \dfrac{k}{2p} + \dfrac{1}{2} \Big\rfloor, & p \geqslant 3. \end{cases} \tag{5.1}$$

  One has the decomposition

$$Z_k(p;T) = (1-T)M_k(p;T), \tag{5.2}$$

  where the reciprocal roots of M_k are Weil numbers of weight $k+1$. Moreover, we have

$$\mathrm{H}^1_{\text{ét,mid}}(\mathbb{G}_{m,\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2) = \mathrm{H}^1_{\text{ét,c}}(\mathbb{G}_{m,\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)/W_0, \tag{5.3}$$

  where $W_\bullet$ denotes the weight filtration (see [56, Lem. 2.5.4] for a comparison between $\mathrm{H}_{\text{ét,mid}}$ and $\mathrm{H}_{\text{ét,c}}$ in characteristic p in a more general setting).

- If k is even and p is odd, then $\deg Z_k(p;T) = k/2 - \lfloor k/2p \rfloor$ and we have a factorization

$$Z_k(p;T) = (1-T)R_k(p;T)M_k(p;T) \tag{5.4}$$

  such that the reciprocal roots of M_k are again Weil numbers of weight $k+1$. Above, the polynomial $R_k(p;T)$ is given by

$$R_k(p;T) = \big(1-(-1)^{(p-1)/2}p^{k/2}T\big)^{n_k(p)}\big(1-p^{k/2}T\big)^{m_k(p)-n_k(p)},$$

$$n_k(p) = \Big\lfloor \frac{k}{4p} + \frac{1}{2} \Big\rfloor, \quad m_k(p) = \Big\lfloor \frac{k}{2p} \Big\rfloor + \delta_{4\mathbf{Z}}(k).$$

- There is also an explicit description in the case where k is even and $p=2$ in [56], namely $Z_k(2;T)$ has degree $\lfloor (k+2)/4 \rfloor$ and factors as

$$Z_k(2;T) = (1-T)\big(1-2^{k/2}T\big)^{a_k}\big(1+2^{k/2}T\big)^{b_k}M_k(2;T) \tag{5.5}$$

where $\deg M_k = 2\lfloor (k+2)/12 \rfloor - 2\delta_{12\mathbf{Z}}(k)$ and the exponents a_k and b_k are given by

$$a_k = \begin{cases} \lfloor k/24 \rfloor + 1, & k \equiv 0, 8, 12, 16, 18, 20 \mod 24 \\ \lfloor k/24 \rfloor, & k \equiv 2, 4, 6, 10, 14, 22 \mod 24 \end{cases}$$

$$b_k = \begin{cases} \lfloor k/24 \rfloor + 1, & k \equiv 6, 12, 14, 18, 20, 22 \mod 24 \\ \lfloor k/24 \rfloor, & k \equiv 0, 2, 4, 8, 10, 16 \mod 24. \end{cases}$$

In all three cases, the reciprocal roots α of the polynomial $M_k(p;T)$ are stable under the transformation $\alpha \mapsto p^{(k+1)/2}\alpha^{-1}$.

5.1.2. *Galois representations of symmetric power moments.* Recall from (2.19) the function $g^{\boxplus k} = \sum_{j=1}^k y_j + 1/y_j$ on the torus $\mathbb{G}_{\mathrm{m}}^k$ and the hypersurface $\mathscr{K}$ defined as its vanishing locus. For a prime number ℓ, let us consider the ℓ-adic realization of the motive $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$, which is a $\mathbf{Q}_\ell$-vector space $V_{k,\ell}$ equipped with a continuous $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-action

$$r_{k,\ell}\colon\ \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \mathrm{GL}(V_{k,\ell}).$$

By Theorem 2.17, for each $k \geqslant 2$ we have

$$\begin{aligned} V_{k,\ell} &= \mathrm{im}\Big\{\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)(-1) \longrightarrow \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)^\vee(-k)\Big\}^{\mathfrak{S}_k\times\mu_2,\,\chi} \\ &= \Big(\mathrm{gr}^W_{k-1}\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}}, \mathbf{Q}_\ell)^{\mathfrak{S}_k\times\mu_2,\,\chi}\Big)(-1). \end{aligned} \tag{5.6}$$

The goal of the next two sections is to compare the traces of Frobenius at unramified primes with symmetric power moments of Kloosterman sums. For this, we shall consider the toric compactification Z of $\mathbb{G}_{\mathrm{m}}^k$ introduced in Section 4.2.2 and let $\overline{\mathscr{K}}$ be the closure of $\mathscr{K}$ in Z. On each of the $2^k k!$ affine charts $Z_1 \cong \mathbb{A}^k = \mathrm{Spec}(\mathbf{Q}[z_i]_{i=1}^k)$ of Z, we have $g^{\boxplus k} = g_1/z_1 \cdots z_k$ and $\overline{\mathscr{K}} \cap Z_1 = (g_1)$, where g_1 is given by (4.15). We also regard these varieties as defined over general rings, e.g. over $\mathbf{F}_p, \mathbf{Z}_p$, etc. Then $\overline{\mathscr{K}} \smallsetminus \mathscr{K}$ is a strict normal crossing divisor over any ring. Indeed, in the chart Z_1 the boundary $(\overline{\mathscr{K}} \smallsetminus \mathscr{K}) \cap Z_1$ is the intersection with the coordinate hyperplanes of Z_1, which extends to a relative strict normal crossing divisor of $\overline{\mathscr{K}} \cap Z_1$ over $\mathbf{Z}$ since $\overline{\mathscr{K}} \cap (z_1) = \varnothing$ and $\partial_{z_1} g_1$ is invertible on $(\overline{\mathscr{K}} \smallsetminus \mathscr{K}) \cap Z_1$. Over $\mathbf{Q}$, the variety $\overline{\mathscr{K}}$ is smooth when k is odd, while if k is even, its singular locus consists of $\binom{k}{k/2}$ ordinary quadratic points with coordinates $y_i \in \{\pm 1\}$ such that $\sum_{i=1}^k y_i = 0$.

5.1.3. *The ℓ-adic case for odd symmetric powers.* Let $k \geqslant 1$ be an odd integer and p an odd prime number. The singular locus Σ of $\mathscr{K}$ in characteristic p consists of $\lfloor k/2p + 1/2 \rfloor$ orbits of ordinary quadratic points under the action of $\mathfrak{S}_k \times \mu_2$. Indeed, the orbits are indexed by odd positive integers a with $ap \leqslant k$, each of them being represented by the point with coordinates $y_i = 1$ (resp. -1) for $1 \leqslant i \leqslant (ap+k)/2$ (resp. $i > (ap+k)/2$). Locally around this point, writing $y_i = z_i + 1$ (resp. $y_i = z_i - 1$), the defining equation of $\mathscr{K}$ in $\mathbf{Z}_p[\![z_1, \ldots, z_k]\!]$ is given by

$$g^{\boxplus k}(z_1, \ldots, z_k) = 2ap + Q_{ap} + \text{higher order terms},$$

where Q_{ap} is the non-degenerate quadratic form

$$Q_{ap} = \sum_{i \leqslant (ap+k)/2} z_i^2 - \sum_{j > (ap+k)/2} z_j^2. \tag{5.7}$$

Write $k=2m+1$. Each $x\in\Sigma$ creates a *vanishing cycle class* δ_x in $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}},\mathbf{Q}_\ell)(m)$ that is well defined up to sign. These classes satisfy

$$\langle\delta_x,\delta_y\rangle=\begin{cases}(-1)^m2 & \text{if } x=y,\\ 0 & \text{if } x\neq y\end{cases}$$

under the intersection form $\langle\,,\rangle$ and the identification $\mathrm{H}^{2k-2}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}})(2m)\cong\mathbf{Q}_\ell$ given by the trace. By the Picard–Lefschetz formula [1, Exp. XV, Th. 3.4], there is an exact sequence

$$0\longrightarrow\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_p})\longrightarrow\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}})\xrightarrow{\ \gamma\ }\bigoplus_{x\in\Sigma}\mathbf{Q}_\ell(-m)\longrightarrow 0,$$

where the map γ is obtained by taking pairings with δ_x.

In what follows, we keep the notation ζ for a primitive p-th root of unity in $\overline{\mathbf{Q}}_\ell$ and $[\zeta]$ for the scalar extension to $\mathbf{Q}_\ell(\zeta)$, and we set

$$\begin{aligned}\Theta_p^+&=\{a\geqslant 1 \text{ odd integer}\mid ap\leqslant k \text{ with } \mathrm{ord}_p(a) \text{ odd}\},\\ \Theta_p^-&=\{a\geqslant 1 \text{ odd integer}\mid ap\leqslant k \text{ with } \mathrm{ord}_p(a) \text{ even}\},\end{aligned}$$

so that one has

$$\#\Theta_p^+ + \#\Theta_p^- = \Big\lfloor\frac{k}{2p}+\frac{1}{2}\Big\rfloor. \tag{5.8}$$

Theorem 5.9. *Let $k=2m+1$ be a positive odd integer and let p and ℓ be distinct prime numbers. Let $V_{k,\ell}$ denote the ℓ-adic realization of the motive $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}},\mathrm{Sym}^k\mathrm{Kl}_2)$ over $\mathbf{Q}$. Fix a place of $\overline{\mathbf{Q}}$ above p, let I_p be the corresponding inertia group in $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)\subset\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, and identify the quotient with $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$.*

1. *The representation $V_{k,\ell}$ is unramified at 2 and the $\mathrm{Gal}(\overline{\mathbf{F}}_2/\mathbf{F}_2)$-module $V_{k,\ell}[\zeta]$ is isomorphic to $\mathrm{H}^1_{\text{ét},\mathrm{mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_2},\mathrm{Sym}^k\mathrm{Kl}_2)$.*
2. *If p is an odd prime, then $V_{k,\ell}$ is at most tamely ramified at p. More precisely, the restriction of $V_{k,\ell}$ to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ decomposes into an orthogonal sum $M\oplus E$, where*
   - *$M[\zeta]=\mathrm{H}^1_{\text{ét},\mathrm{mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p},\mathrm{Sym}^k\mathrm{Kl}_2)$,*
   - *E is generated by vanishing cycles, one for each $a\in\Theta_p^+\cup\Theta_p^-$, on which the Galois group acts through the character $\varepsilon_a\otimes\chi_{\mathrm{cyc}}^{-m-1}$, where $\varepsilon_a\colon\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)\to\{\pm1\}$ stands for the non-trivial character associated with the quadratic extension $\mathbf{Q}_p(\sqrt{(-1)^{(1+ap)/2}2ap})$ of $\mathbf{Q}_p$.*

   In particular, decomposing $E=E^+\oplus E^-$ according to whether a belongs to Θ_p^+ or Θ_p^-, the invariants under inertia are $V_{k,\ell}^{I_p}=M\oplus E^+$ and E^+ is a semi-simple $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$-module with reciprocal characteristic polynomial of Frobenius

   $$\det(1-F_pT\mid E^+)=\prod_{a\in\Theta_p^+}\Big(1-\Big(\frac{(-1)^{(1+ap)/2}2a'}{p}\Big)p^{m+1}T\Big),$$

   where $a'=ap^{-\mathrm{ord}_p(a)}$ denotes the prime-to-p part of a and $(\cdot/p)$ the Legendre symbol.

Proof. There is nothing to prove if $k = 1$, so we assume $k \geqslant 3$. To shorten notation, we omit the coefficients $\mathbf{Q}_\ell$ from the étale cohomology groups and write $G = \mathfrak{S}_k \times \mu_2$, so that $V_{k,\ell} = \mathrm{gr}^W_{k-1} \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}})^{G,\chi}(-1)$. Set $\overline{\mathscr{K}}^{(0)} = \overline{\mathscr{K}}$ and let $\overline{\mathscr{K}}^{(i)}$ be the disjoint union of all i-fold intersections of distinct irreducible components of $\overline{\mathscr{K}} \smallsetminus \mathscr{K}$ for $i \geqslant 1$. The weight filtration on the compactly supported étale cohomology of $\mathscr{K}$ is the abutment of the spectral sequence

$$E_1^{i,j} = \mathrm{H}^j_{\text{ét}}(\overline{\mathscr{K}}^{(i)}_{\overline{\mathbf{Q}}}) \Longrightarrow \mathrm{H}^{i+j}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}}),$$

which degenerates at E_2 by weight considerations, thus yielding

$$\begin{aligned}\mathrm{gr}^W_{k-1} \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}}) &= \ker\Big\{\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}}) \longrightarrow \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}^{(1)}_{\overline{\mathbf{Q}}})\Big\} \\ &= \mathrm{im}\Big\{\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}}) \longrightarrow \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}})\Big\}.\end{aligned}$$

By the discussion in Section 2.4.1, namely identities (2.22), (2.23), and (2.24), we have

$$\Big(\mathrm{gr}^W_{>0} \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}_p})^{G,\chi}\Big)(-1)[\zeta] \cong \mathrm{gr}^W_{>2} \mathrm{H}^1_{\text{ét,c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)$$

and by (5.3) the right-hand side is equal to $\mathrm{H}^1_{\text{ét,mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)$. It is pure of weight $k+1$, and has dimension m if $p = 2$ and $m - \lfloor k/2p + 1/2 \rfloor$ if $p \geqslant 3$ by (5.1). Since $\overline{\mathscr{K}}^{(i)}$ is smooth and proper for $i \geqslant 1$, a similar spectral sequence argument gives

$$\mathrm{gr}^W_{k-1} \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}_p})^{G,\chi} = \mathrm{im}\Big\{\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}_p}) \longrightarrow \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_p})\Big\}^{G,\chi}.$$

Assume $p = 2$. The proper variety $\overline{\mathscr{K}}_{\overline{\mathbf{F}}_2}$ has quadratic non-ordinary isolated singularities consisting of the single point with coordinates $y_i = 1$. In this case, the cohomology sheaf $\mathrm{R}^n\Phi$ of the vanishing cycle complex on $\overline{\mathscr{K}}_{\overline{\mathbf{F}}_2}$ is non-zero only in degree $n = k-1$ by [27, Cor. 2.10], which implies that the cospecialization morphism $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_2}) \to \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}})$ is injective. By considering the G-action on the images from the open part $\mathscr{K}$ and counting dimensions, one obtains $V_{k,\ell} = \mathrm{gr}^W_{k-1} \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}_2})^{G,\chi}(-1)$ as representations of $\mathrm{Gal}(\overline{\mathbf{F}}_2/\mathbf{F}_2)$. This proves (1).

Now suppose $p \geqslant 3$ and consider the G-equivariant commutative diagram with exact rows and columns

$$\begin{array}{ccccccccc}
 & & \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{F}}_p}) & \longrightarrow & \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}}) & & & & \\
 & & \beta\downarrow & & \downarrow\alpha & & & & \\
0 & \longrightarrow & \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_p}) & \longrightarrow & \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}}) & \xrightarrow{\gamma} & \bigoplus_{x\in\Sigma} \mathbf{Q}_\ell(-m) & \longrightarrow & 0, \\
 & & \downarrow & & \downarrow & & & & \\
 & & \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}^{(1)}_{\overline{\mathbf{F}}_p}) & \xrightarrow{\sim} & \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}^{(1)}_{\overline{\mathbf{Q}}}) & & & &
\end{array}$$

where the middle row is given by the Picard–Lefschetz formula. Let $\Delta = \bigoplus_{x\in\Sigma} \mathbf{Q}_\ell(-m)\delta_x$ be the subspace of $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}})$ generated by vanishing cycle classes, which is the orthogonal complement of the image of $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_p})$ by cospecialization. Let $\mathrm{R}\Psi$ be the complex of nearby cycles on $\overline{\mathscr{K}}_{\overline{\mathbf{F}}_p}$. Since δ_x is a generator of the local cohomology $\mathrm{H}^{k-1}_{\{x\}}(\overline{\mathscr{K}}_{\overline{\mathbf{F}}_p}, \mathrm{R}\Psi(m))$

with support $\{x\}$ contained in $\mathscr{K}_{\overline{\mathbf{F}}_p}$, the subspace Δ lies in the image of α. The image of β and Δ being orthogonal as subspaces of $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}_{\overline{\mathbf{Q}}})$, we then have

$$\operatorname{im}(\alpha)^{G,\chi} = \operatorname{im}(\beta)^{G,\chi} \oplus \Delta^{G,\chi}$$

with $\dim \Delta^{G,\chi} = \lfloor k/2p + 1/2 \rfloor$ by a dimension count. These are the factors M and E in (2).

To compute the Galois action on E, recall the quadratic form Q_{ap} from (5.7) and consider the projective quadric $D = (2apw^2 + Q_{ap})$ in $\mathbb{P}^k_{\mathbf{Q}_p}$ as well as the hyperplane section $C = D \cap (w)$. The space $\mathbf{Q}_\ell(-m)\delta_x$ is described in [1, Exp. XV, Prop. 2.2.3] as $\mathrm{H}^{k-1}_{\text{ét,c}}((D \smallsetminus C)_{\overline{\mathbf{Q}}_p})$, which is equal to the primitive part of $\mathrm{H}^{k-1}_{\text{ét}}(D_{\overline{\mathbf{Q}}_p})$ by the localization sequence for compactly supported étale cohomology. As a non-degenerate quadratic form over $\mathbf{Q}_p$, the defining equation of D has discriminant $d = (-1)^{(k-ap)/2} 2ap$, hence $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ acts on

$$\det \mathrm{H}^{k-1}_{\text{ét}}(D_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_\ell(m)) = \mathrm{H}^{k-1}_{\text{ét,prim}}(D_{\overline{\mathbf{Q}}_p})(m)$$

via the character ε_a corresponding to the quadratic extension $\mathbf{Q}_p(\sqrt{(-1)^{(k+1)/2} d})$ by [50, §5.2]. Noting the extra twist in the expression (5.6) of $V_{k,\ell}$ in terms of $\mathrm{H}^{k-1}_{\text{ét}}(\mathscr{K}_{\overline{\mathbf{Q}}})$, this proves the first statement about E. This extension is unramified if and only if $\operatorname{ord}_p(a)$ is odd, in which case it is equal to $\mathbf{Q}_p(\sqrt{(-1)^{(1+ap)/2} 2a'})$ and the last assertion in (2) follows. □

Remark 5.10. In the case at hand, instead of invoking [50] one can directly see the action of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ on the primitive cohomology of the quadric by regarding it as defined over $\mathbf{Z}$. Indeed, D has good reduction at all primes r with even $\operatorname{ord}_r(d)$ and, e.g. by point counting over $\mathbf{F}_r$, Frobenius acts as multiplication by $\left(\frac{(-1)^{(k+1)/2} d}{r}\right) r^m$. Chebotarev's density theorem then implies that $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$ acts on $\mathrm{H}^{k-1}_{\text{ét,prim}}(D_{\overline{\mathbf{Q}}})(m)$ via the character corresponding to the extension $\mathbf{Q}(\sqrt{(-1)^{(1+ap)/2} 2ap})$. Note that, if $\operatorname{ord}_p(a)$ is odd, then D has good reduction at p.

From Theorem 5.9 and Serre's recipe [51], we immediately derive the local L-factors and the conductor of the system of Galois representations $\{V_{k,\ell}\}_\ell$ associated with the motive $\mathrm{H}^1_{\text{mid}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ over $\mathbf{Q}$. For each prime p, define $L_k(p; T)$ as the reciprocal of the polynomial with integer coefficients

$$\det(1 - F_p T \mid V_{k,\ell}^{I_p}) = \begin{cases} M_k(2;T) & \text{if } p = 2, \\ M_k(p;T) \prod_{a \in \Theta_p^+} \left(1 - \left(\frac{(-1)^{(1+ap)/2} 2a'}{p}\right) p^{m+1} T\right) & \text{if } p \geqslant 3. \end{cases} \tag{5.11}$$

The L-function of $\mathrm{H}^1_{\text{mid}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ is the Euler product

$$L_k(s) = \prod_p L_k(p; p^{-s}),$$

which converges absolutely on the half plane $\operatorname{Re}(s) > 1 + (k+1)/2$.

Recall from [51, eq. (11) and (29)] that the exponent of p in the global conductor of $\{V_{k,\ell}\}$ is given by the sum of the Swan conductor of $V_{k,\ell}$ restricted to $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and the codimension of $V_{k,\ell}^{I_p}$. Since $V_{k,\ell}$ is at most tamely ramified at all primes $p \neq \ell$, the Swan conductor vanishes and we are left with $(k-1)/2 - \dim V_{k,\ell}^{I_p}$, which in view of (5.1), (5.2), and (5.8) is equal to $\#\Theta_p^-$ if p is odd and to zero if $p = 2$. Thus the value of the conductor is

$$\mathfrak{N}_k = \prod_{p \text{ odd}} p^{\#\Theta_p^-} = 1_{\mathrm{s}} 3_{\mathrm{s}} 5_{\mathrm{s}} \cdots k_{\mathrm{s}}, \tag{5.12}$$

where n_{s} denotes the product of all primes p such that $\mathrm{ord}_p(n)$ is odd.

Remark 5.13. By Theorem 5.9, it is clear that both the L-factor $L_k(p;T)$ and the conductor $\mathfrak{N}_k$ remain unchanged if one replaces the input $\{V_{k,\ell}\}_\ell$ with its semi-simplification $\{V_{k,\ell}^{\mathrm{ss}}\}_\ell$.

5.1.4. *The ℓ-adic case for even symmetric powers.* Let $k \geqslant 2$ be an even integer and p an odd prime number. In this case, the singular locus of $\mathscr{K}_{\overline{\mathbf{F}}_p}$ consists of $1 + \lfloor k/2p \rfloor$ orbits of ordinary quadratic points under the action of $\mathfrak{S}_k \times \mu_2$. They are indexed by non-negative even integers b satisfying $bp \leqslant k$ with points with coordinates $y_i = 1$ (resp. -1) for $1 \leqslant i \leqslant (bp+k)/2$ (resp. $i > (bp+k)/2$) as representatives. Writing $y_i = z_i + 1$ (resp. $y_i = z_i - 1$), locally around each singularity the defining equation of $\mathscr{K}$ in $\mathbf{Z}_p[\![z_1, \cdots, z_k]\!]$ has the shape

$$2bp + Q_{bp} + \text{higher order terms}, \quad Q_{bp} = \sum_{i \leqslant (bp+k)/2} z_i^2 - \sum_{j > (bp+k)/2} z_j^2. \tag{5.14}$$

Theorem 5.15. *Suppose k is even. Let p and ℓ be distinct prime integers with $p \geqslant 3$ and let $V_{k,\ell}$ be the ℓ-adic realization of $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ over $\mathbf{Q}$. Fix a place of $\overline{\mathbf{Q}}$ above p and let I_p be the corresponding inertia group in $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \subset \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$.*

The representation $V_{k,\ell}$ is at most tamely ramified at p. More precisely, there is an isotropic subspace $U \subset V_{k,\ell}$ of dimension $\lfloor k/2p \rfloor$ generated by vanishing cycles such that $V_{k,\ell}^{I_p} = U^\perp$ and that, for each $\sigma \in I_p$, the induced map $\sigma - 1\colon V_{k,\ell} \to V_{k,\ell}/U$ is zero. As $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$-modules,

$$V_{k,\ell}^{I_p}[\zeta] = \mathrm{H}^1_{\text{ét,c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)/E[\zeta],$$

where E is equal to $\mathbf{Q}_\ell(0)$ if 4 does not divide k and an extension of $\mathbf{Q}_\ell(-k/2)$ by $\mathbf{Q}_\ell$ otherwise.

Proof. There is nothing to prove for $k = 2$, so we assume $k \geqslant 4$. Again, we set $G = \mathfrak{S}_k \times \mu_2$ and omit the coefficients $\mathbf{Q}_\ell$ from the étale cohomology.

In characteristic zero, $\mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K})^{G,\chi}/W_0$ is pure of weight $k-1$ and has dimension $(k-2)/2$ if $k \equiv 2 \bmod 4$. If $k \equiv 0 \bmod 4$, it is mixed of weights $k-2$ and $k-1$ with graded pieces of dimension 1 and $(k-4)/2$, respectively

Let S be the singular locus of $\mathscr{K}$ in characteristic zero consisting of the $\binom{k}{k/2}$ ordinary quadratic points. Let $\mathscr{K}'$ be the strict transform of $\mathscr{K}$ inside the blowup of $\mathbb{G}_{\mathrm{m}}^k$ at S. The preimage of S in $\mathscr{K}'$ is a disjoint union T of quadrics. We have the commutative diagram

$$\begin{array}{ccc} T & \subset & \mathscr{K}' \\ \downarrow & & \downarrow \\ S & \subset & \mathscr{K} \end{array}$$

and the commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{H}^{k-2}(T) & \longrightarrow & \mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}'\smallsetminus T) & \longrightarrow & \mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}') & \longrightarrow & \mathrm{H}^{k-1}(T)\\
\downarrow & & \downarrow\cong & & \downarrow & & \downarrow\\
\mathrm{H}^{k-2}(S) & \longrightarrow & \mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}\smallsetminus S) & \longrightarrow & \mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}) & \longrightarrow & \mathrm{H}^{k-1}(S).
\end{array}$$

Since k is even and $\geqslant 4$, one has $\mathrm{H}^{k-2}(S)=\mathrm{H}^{k-1}(S)=\mathrm{H}^{k-1}(T)=0$ and hence

$$\mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}')\xrightarrow{\ \sim\ }\mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}). \tag{5.16}$$

The above isomorphism remains true if one replaces $\mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K})$ and $\mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}')$ with $\mathrm{H}^{k-1}_{\text{ét},\mathrm{c}}(\mathscr{K}_{\mathbf{F}})$ and $\mathrm{H}^{k-1}_{\text{ét},\mathrm{c}}(\mathscr{K}'_{\mathbf{F}})$, respectively for $\mathbf{F}=\overline{\mathbf{Q}}$ or $\overline{\mathbf{F}}_p$.

Let $\overline{\mathscr{K}}'\subset Z'$ be the closure of $\mathscr{K}'$ and let $\overline{\mathscr{K}}'^{(i)}$ be the disjoint union of all i-fold intersections of distinct irreducible components of the boundary divisor $\overline{\mathscr{K}}'\smallsetminus\mathscr{K}'$. Consider the associated spectral sequence

$$(E_1^{i,j})_{\mathbf{F}}=\mathrm{H}^j_{\text{ét}}(\overline{\mathscr{K}}'^{(i)}_{\overline{\mathbf{F}}})\Longrightarrow\mathrm{H}^{i+j}_{\text{ét},\mathrm{c}}(\mathscr{K}'_{\overline{\mathbf{F}}}).$$

In characteristic zero, since all $\overline{\mathscr{K}}'^{(i)}_{\overline{\mathbf{Q}}}$ are smooth and proper, the spectral sequence degenerates at E_2 and one gets

$$\begin{aligned}\mathrm{gr}^W_{k-1}\mathrm{H}^{k-1}_{\text{ét},\mathrm{c}}(\mathscr{K}'_{\overline{\mathbf{Q}}})&=\ker\Big\{\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}})\longrightarrow\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'^{(1)}_{\overline{\mathbf{Q}}})\Big\}\\&=\mathrm{im}\Big\{\mathrm{H}^{k-1}_{\text{ét},\mathrm{c}}(\mathscr{K}'_{\overline{\mathbf{Q}}})\longrightarrow\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}})\Big\}\end{aligned}\tag{5.17}$$

exactly as in the case where k is odd. The χ-isotypic component is then equal to $V_{k,\ell}(1)$. Moreover, the $E_2^{1,k-2}$-term reads

$$\begin{aligned}&\mathrm{gr}^W_{k-2}\mathrm{H}^{k-1}_{\text{ét},\mathrm{c}}(\mathscr{K}'_{\overline{\mathbf{Q}}})=\\&\quad\ker\Big\{\mathrm{H}^{k-2}_{\text{ét}}(\overline{\mathscr{K}}'^{(1)}_{\overline{\mathbf{Q}}})\longrightarrow\mathrm{H}^{k-2}_{\text{ét}}(\overline{\mathscr{K}}'^{(2)}_{\overline{\mathbf{Q}}})\Big\}\Big/\mathrm{im}\Big\{\mathrm{H}^{k-2}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}})\longrightarrow\mathrm{H}^{k-2}_{\text{ét}}(\overline{\mathscr{K}}'^{(1)}_{\overline{\mathbf{Q}}})\Big\},\end{aligned}\tag{5.18}$$

whose χ-isotypic subspace has dimension $\delta_{4\mathbf{Z}}(k)$.

Since the singularities of $\overline{\mathscr{K}}'_{\mathbf{F}_p}$ consist only of ordinary quadratic points supported on $\mathscr{K}'_{\mathbf{F}_p}$, the Picard–Lefschetz formula and base-change yield isomorphisms

$$\begin{aligned}\mathrm{H}^n_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{F}}_p})&\xrightarrow{\ \sim\ }\mathrm{H}^n_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}}),\quad n\leqslant k-2,\\ \mathrm{H}^j_{\text{ét}}(\overline{\mathscr{K}}'^{(i)}_{\overline{\mathbf{F}}_p})&\xrightarrow{\ \sim\ }\mathrm{H}^j_{\text{ét}}(\overline{\mathscr{K}}'^{(i)}_{\overline{\mathbf{Q}}}),\quad i\geqslant 1.\end{aligned}$$

In particular, we have $(E_2^{i,j})_{\mathbf{F}_p}=(E_2^{i,j})_{\mathbf{Q}}$ for all $i+j=k-1$ with $i\geqslant 1$, from which the degeneration $(E_2^{1,k-2})^{G,\chi}_{\mathbf{F}_p}=(E_\infty^{1,k-2})^{G,\chi}_{\mathbf{F}_p}$ follows. By comparison with (5.18) and weight counting, this space vanishes if $k\equiv 2\bmod 4$ and is one-dimensional of weight $k-2$ if 4 divides k. Consider again the G-equivariant commutative diagram

$$\begin{array}{ccccccc} & & \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{F}}_p}) & \longrightarrow & \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{Q}}}) & & \\ & & \beta\downarrow & & \downarrow\alpha & & \\ 0 & \longrightarrow & \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{F}}_p}) & \longrightarrow & \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}}) & \xrightarrow{\gamma} & \bigoplus_{x\in\Sigma}\mathbf{Q}_\ell(-k/2) \end{array}$$

in which the second row is exact and the map γ is defined by pairing with vanishing cycle classes $\delta_x \in \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}})((k-2)/2)$, one for each $x \in \Sigma$. By the calculation (5.4) of the Frobenius eigenvalues of $\mathrm{H}^1_{\text{ét,c}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \operatorname{Sym}^k \mathrm{Kl}_2)$ and the shape of the E_2-terms, we have

$$\dim \operatorname{im}(\beta)^{G,\chi} = \begin{cases} \dfrac{k-2}{2} - \left\lfloor \dfrac{k}{2p} \right\rfloor, & k \equiv 2 \bmod 4 \\[2ex] \dfrac{k-4}{2} - \left\lfloor \dfrac{k}{2p} \right\rfloor, & k \equiv 0 \bmod 4. \end{cases}$$

Let $\Delta = \bigoplus_{x\in\Sigma} \mathbf{Q}_\ell((2-k)/2)\delta_x$ be the subspace of $\mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}})$ generated by the vanishing cycle classes. In this case, $\Delta \subset \Delta^\perp \cap \ker\gamma$ and we have $\dim \Delta^{G,\chi} \leqslant |G\backslash\Sigma| = \lfloor k/2p \rfloor$. As in the case where k is odd, $\Delta \subset \operatorname{im}(\alpha)$. Since $\operatorname{im}(\alpha)^{G,\chi}$ is self-dual and $\operatorname{im}(\beta) \subset \ker\gamma$, dimension counting reveals that

$$(\Delta^\perp)^{G,\chi} = \operatorname{im}(\beta)^{G,\chi} \quad \text{and} \quad \dim \Delta^{G,\chi} = \left\lfloor \frac{k}{2p} \right\rfloor.$$

By the Picard–Lefschetz formula [1, Exp. XV., Th. 3.4], an element σ of the inertia group I_p acts on $v \in \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}})$ as

$$\sigma(v) = v - (-1)^{k/2} t_\ell(\sigma) \sum_{x\in\Sigma} \langle v, \delta_x \rangle\, \delta_x,$$

where $\langle v, \delta_x \rangle \in \mathrm{H}^{2k-2}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{Q}}})((k-2)/2) \cong \mathbf{Q}_\ell(-k/2)$ and $t_\ell \colon I_p \to \varprojlim \mu_{\ell^n}(\overline{\mathbf{Q}}_\ell)$ is the fundamental tame character. One concludes that $V_{k,\ell}$ is tame at p and $V_{k,\ell}^{I_p} = \operatorname{im}(\beta)^{G,\chi}(-1)$. This completes the proof in the case $k \equiv 2 \bmod 4$.

Finally, we look at the action of Frobenius on Δ. Recall that each $V_x = \mathbf{Q}_\ell((2-k)/2)\delta_x$ corresponds to the singularity defined by the equation (5.14) for an even positive integer b. Consider the quadric $C = (Q_{bp}) \subset \mathbb{P}^{k-1}_{\mathbf{F}_p}$, whose primitive cohomology $\mathrm{H}^{k-2}_{\text{ét,prim}}(C_{\overline{\mathbf{F}}_p})$ coincides with V_x by [1, Exp. XV, Prop. 2.2.3]. The quadratic form Q_{bp} has discriminant $d = (-1)^{(k-bp)/2}$ and therefore F_p acts on the primitive cohomology as multiplication by

$$\left(\frac{(-1)^{k/2} d}{p}\right) p^{(k-2)/2} = (-1)^{bp(p-1)/4} p^{(k-2)/2}.$$

For $p \equiv 1 \bmod 4$, the sign is always positive whereas, for $p \equiv 3 \bmod 4$, there are $\lfloor k/4p + 1/2 \rfloor$ values of b such that the sign is negative. Comparing with the eigenvalues of F_p in (5.4), one concludes that in the case $k \equiv 0 \bmod 4$, the kernel of $\beta \colon \mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{F}}_p})^{G,\chi}/W_0 \to \mathrm{H}^{k-1}_{\text{ét}}(\overline{\mathscr{K}}'_{\overline{\mathbf{F}}_p})^{G,\chi}$ is a factor $\mathbf{Q}_\ell((2-k)/2)$. This completes the proof. □

Similarly to the case of odd symmetric powers, the above theorem gives the local L-factors and the conductor away from $p=2$. Indeed, defining $L_k(p;T)=\det(1-F_pT\mid V_{k,\ell}^{I_p})^{-1}$ for a prime number p, Theorem 5.15 and (5.4) imply that, for odd p, we have

$$(5.19)\quad L_k(p;T)^{-1}=\begin{cases}(1-p^{k/2}T)^{\lfloor k/2p\rfloor}M_k(p;T) & \text{if } p\equiv 1 \bmod 4,\\ (1+p^{k/2}T)^{\left\lfloor\frac{k}{4p}+\frac12\right\rfloor}(1-p^{k/2}T)^{\left\lfloor\frac{k}{2p}\right\rfloor-\left\lfloor\frac{k}{4p}+\frac12\right\rfloor}M_k(p;T) & \text{if } p\equiv 3 \bmod 4,\end{cases}$$

The L-function of $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}},\mathrm{Sym}^k\mathrm{Kl}_2)$ is the Euler product

$$L_k(s)=\prod_p L_k(p;p^{-s}),$$

which again converges absolutely for $\mathrm{Re}(s)>1+(k+1)/2$.

As for the conductor, Serre's recipe yields in this case that the exponent of an odd prime p is given by $\lfloor k/2p\rfloor$. The conductor is thus equal to

$$2^{r_k}\prod_{p\text{ odd}}p^{\lfloor k/2p\rfloor}=2^{r_k}\cdot 2_{\mathrm{u}}4_{\mathrm{u}}6_{\mathrm{u}}\cdots k_{\mathrm{u}},$$

where $r_k=\mathrm{Sw}(V_{k,\ell}|_{\mathrm{Gal}(\overline{\mathbf{Q}}_2/\mathbf{Q}_2)})+\mathrm{codim}\,V_{k,\ell}^{I_2}$ and n_{u} stands for the odd part of the radical, i.e., the product of all odd primes dividing n. Broadhurst and Roberts conjecture that $r_k=\lfloor k/6\rfloor$.

5.1.5. *The p-adic case.* We keep the setting of Section 2.4.2 but abbreviate $K=\mathbf{Q}_p(\pi)$. In what follows, $\mathbf{B}_{\mathrm{dR}}$, $\mathbf{B}_{\mathrm{crys}}$, and $\mathbf{B}_{\mathrm{st}}$ denote Fontaine's p-adic de Rham, crystalline, and semistable period rings over $\mathbf{Q}_p$. For all odd p, the equality of Frobenius modules

$$\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_{\mathrm{m}}/K,\mathrm{Sym}^k\mathrm{Kl}_2)=\mathrm{gr}^W_{k+1}\mathrm{H}^1_{\mathrm{rig,c}}(\mathbb{G}_{\mathrm{m}}/K,\mathrm{Sym}^k\mathrm{Kl}_2)$$

is proved by examining the behaviours of the Kloosterman F-isocrystal Kl_2 at zero and infinity [10, §1], exactly as in [21] and [56] using [32, Prop. 6.4.16] for the long exact sequence relating compactly supported and usual rigid cohomology. If $p>k$, the above cohomology is a K-vector space of dimension $\lfloor(k-1)/2\rfloor-\delta_{4\mathbf{Z}}(k)$.

Proposition 5.20. *Fix an integer $k\geqslant 1$, a prime number p, and a place of $\overline{\mathbf{Q}}$ above p. The p-adic representation $V_{k,p}$ of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ is de Rham. If p is odd, then $V_{k,p}$ is semistable and there is an inclusion of Frobenius modules*

$$\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_{\mathrm{m}}/K,\mathrm{Sym}^k\mathrm{Kl}_2)\longrightarrow(V_{k,p}\otimes\mathbf{B}_{\mathrm{st}})^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}\otimes K.$$

Under the extra assumption that $p>k$ if k is odd or $2p>k$ if k is even, the representation $V_{k,p}$ is crystalline and the associated Frobenius module $(V_{k,p}\otimes\mathbf{B}_{\mathrm{crys}})^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}\otimes K$ is isomorphic to $\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_{\mathrm{m}}/K,\mathrm{Sym}^k\mathrm{Kl}_2)$.

Proof. For the first assertion, we simply note that the representation $\mathrm{H}^{k-1}_{\text{ét}}(\mathscr{K}_{\overline{\mathbf{Q}}},\mathbf{Q}_p)(-1)$ coming from $\mathscr{K}$ is de Rham (see e.g. [5, §3.3(i) and §3.4]) and that any subquotient of a de Rham representation is still de Rham. For the remaining statements, assume p is odd. We only treat the case of even k, leaving to the reader the easier case of odd k.

As in the proof of Theorem 5.15, consider the resolution of singularities $\mathscr{K}'$ of $\mathscr{K}$ and its compactification $\overline{\mathscr{K}}'$ induced from the blowup of the ambient torus and the explicit toric compactification Z over $\mathbf{Z}_p$. We have

$$\Big(\mathrm{gr}^W_{k-1}\,\mathrm{H}^{k-1}_{\mathrm{rig,c}}(\mathscr{K}_{\mathbf{F}_p}/\mathbf{Q}_p)^{\mathfrak{S}_k\times\mu_2,\,\chi}\Big)(-1)[\pi] = \mathrm{gr}^W_{k+1}\,\mathrm{H}^1_{\mathrm{rig,c}}(\mathbb{G}_{\mathrm{m},\mathbf{F}_p}/K,\mathrm{Sym}^k\mathrm{Kl}_2)$$

via localization sequences. On the other hand, we have isomorphisms

$$\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{Q}}_p},\mathbf{Q}_p)\xrightarrow{\sim}\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}_{\overline{\mathbf{Q}}_p},\mathbf{Q}_p),\quad \mathrm{H}^{k-1}_{\mathrm{rig,c}}(\mathscr{K}'_{\mathbf{F}_p}/\mathbf{Q}_p)\xrightarrow{\sim}\mathrm{H}^{k-1}_{\mathrm{rig,c}}(\mathscr{K}_{\mathbf{F}_p}/\mathbf{Q}_p)$$

of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$-modules and Frobenius modules respectively, as in (5.16). For étale cohomology, equation (5.17) yields

$$\Big(\mathrm{gr}^W_{k-1}\,\mathrm{H}^{k-1}_{\text{ét,c}}(\mathscr{K}'_{\overline{\mathbf{Q}}_p},\mathbf{Q}_p)\otimes\mathbf{B}_{\mathrm{dR}}\Big)^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)} = \ker\Big\{\mathrm{H}^{k-1}_{\mathrm{dR}}(\overline{\mathscr{K}}'_{\mathbf{Q}_p})\longrightarrow\mathrm{H}^{k-1}_{\mathrm{dR}}(\overline{\mathscr{K}}'^{(1)}_{\mathbf{Q}_p})\Big\}\tag{5.21}$$

by p-adic Hodge comparison. For rigid cohomology, consider the spectral sequence

$$E_1^{i,j}=\mathrm{H}^j_{\mathrm{rig}}(\overline{\mathscr{K}}'^{(i)}_{\mathbf{F}_p}/\mathbf{Q}_p)\Longrightarrow\mathrm{H}^{i+j}_{\mathrm{rig,c}}(\mathscr{K}'_{\mathbf{F}_p}/\mathbf{Q}_p)$$

similar to the ℓ-adic case ([32, Prop. 8.2.17 & 8.2.18(ii)]).

Since the singularities of $\overline{\mathscr{K}}'$ consist only of ordinary quadratic points supported on $\mathscr{K}'_{\mathbf{F}_p}$, by the p-adic Picard–Lefschetz formula [36, Th. 2.13 & 1.1], we have the two vertical arrows in the commutative diagram

$$\begin{array}{ccccc}
\mathrm{H}^{k-1}_{\mathrm{rig,c}}(\mathscr{K}'_{\mathbf{F}_p}/\mathbf{Q}_p) & \longrightarrow & \mathrm{H}^{k-1}_{\mathrm{rig}}(\overline{\mathscr{K}}'_{\mathbf{F}_p}/\mathbf{Q}_p) & \xrightarrow{\alpha} & \mathrm{H}^{k-1}_{\mathrm{rig}}(\overline{\mathscr{K}}'^{(1)}_{\mathbf{F}_p}/\mathbf{Q}_p)\\
 & & \big\downarrow{\scriptstyle\beta} & & \big\Vert\\
 & & \mathrm{H}^{k-1}_{\mathrm{dR}}(\overline{\mathscr{K}}'_{\mathbf{Q}_p}) & \longrightarrow & \mathrm{H}^{k-1}_{\mathrm{dR}}(\overline{\mathscr{K}}'^{(1)}_{\mathbf{Q}_p})
\end{array}$$

with β injective. Since $\overline{\mathscr{K}}'^{(i)}$ is proper and smooth for all $i\geqslant 1$, by counting weights the spectral sequence yields an isomorphism

$$\mathrm{gr}^W_{k-1}\,\mathrm{H}^{k-1}_{\mathrm{rig,c}}(\mathscr{K}'_{\mathbf{F}_p}/\mathbf{Q}_p)\longrightarrow\mathrm{gr}^W_{k-1}\ker\alpha.$$

Besides, let $\overline{\mathscr{K}}''$ be the blowup of $\overline{\mathscr{K}}'$ along the ordinary quadratic points. Then $\overline{\mathscr{K}}''$ is semistable over $\mathbf{Z}_p$ (see [36, §(2.3)]). Therefore, one has the equality as in (5.21) by replacing $\mathbf{B}_{\mathrm{dR}}$ with $\mathbf{B}_{\mathrm{st}}$ [5, §3.3(ii)]. Taking χ-isotypic components and extending scalars to K in rigid cohomology, one concludes that $V_{k,p}$ is semistable.

If k is even and $2p>k$, then $\overline{\mathscr{K}}'$ is smooth. Consequently, β is an isomorphism and $\ker\alpha$ is pure of weight $k-1$. By [5, §3.3(iii)] and a similar argument in the semistable case, we obtain the identity

$$(V_{k,p}\otimes\mathbf{B}_{\mathrm{crys}})^{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}[\pi]=\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_{\mathrm{m},\mathbf{F}_p}/K,\mathrm{Sym}^k\mathrm{Kl}_2),$$

which finishes the proof. □

Corollary 5.22. *Let $k\geqslant 1$ be an integer and p an odd prime number. The Newton polygon of the Frobenius module $\mathrm{H}^1_{\mathrm{rig,c}}(\mathbb{G}_{\mathrm{m},\mathbf{F}_p}/K,\mathrm{Sym}^k\mathrm{Kl}_2)$ lies above the Hodge polygon of*

$\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$. *In case $p > k$ if k is odd or $2p > k$ if k is even, the endpoints of both polygons coincide.*

Proof. Regarded as a representation of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, the p-adic étale realization $V_{k,p}$ of $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ is semistable and (weakly) admissible. Thus by the above proposition, the Newton polygon of the Frobenius module $\mathrm{H}^1_{\mathrm{rig,mid}}(\mathbb{G}_{\mathrm{m},\mathbf{F}_p}/K, \mathrm{Sym}^k \mathrm{Kl}_2)$, which is a summand (after extending scalars to K) of the associated Frobenius module of $V_{k,p}$, lies above the Hodge polygon of $\mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$. Moreover, under the condition $p > k$ (resp. $2p > k$) if k is odd (resp. even), the two polygons have the same endpoints. Putting the trivial factor back, the statement follows. □

Remark 5.23. Writing $Z_k(p;T) = \sum c_n T^n$, the above corollary implies in particular that

$$\mathrm{ord}_p(c_n) \geqslant n(n-1)$$

for all $p \geqslant 3$. This sharpens a result of Haessig [24, Th. 1.1], who obtained the bound $(1 - 1/(p-1))\, n(n-1)$ for all $p \geqslant 5$ using p-adic analysis à la Dwork.

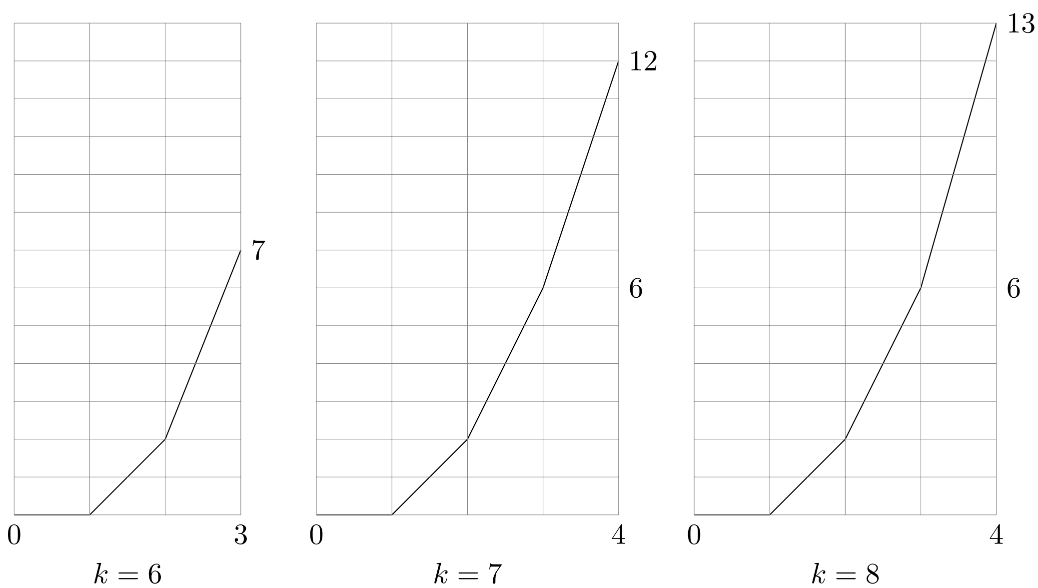

FIGURE 1. The Hodge polygons of $\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$

5.2. **The gamma factor.** We first recall Serre's recipe [51, §3] describing the conjectural shape of the gamma factor at infinity in the complete L-function of a pure motive over $\mathbf{Q}$. Let V be a finite-dimensional vector space over $\mathbf{C}$ together with an $\mathbf{R}$-Hodge decomposition of weight w, i.e., the data of a grading $V = \bigoplus_{p\in\mathbf{Z}} V^p$ and a $\mathbf{C}$-linear involution σ of V such that $\sigma(V^p) = V^{w-p}$. Given an $\mathbf{R}$-Hodge decomposition, we set $h(p) = \dim V^p$ and

$$h(w/2)^{\pm} = \Big\{ v \in V^{w/2} \mid \sigma(v) = \pm(-1)^{w/2} v \Big\}$$

if w is even and $h(w/2)^{\pm} = 0$ otherwise. Setting

$$\Gamma_{\mathbf{R}}(s) = \pi^{-s/2}\Gamma(s/2), \quad \Gamma_{\mathbf{C}}(s) = 2(2\pi)^{-s}\Gamma(s) = \Gamma_{\mathbf{R}}(s)\Gamma_{\mathbf{R}}(s+1),$$

the *gamma factor* $\Gamma_V(s)$ of V is defined as

$$\Gamma_V(s) = \Gamma_{\mathbf{R}}(s - w/2)^{h(w/2)^+}\Gamma_{\mathbf{R}}(s - w/2 + 1)^{h(w/2)^-} \prod_{p<w/2} \Gamma_{\mathbf{C}}(s-p)^{h(p)}.$$

Corollary 5.24. *For each integer $k \geqslant 1$, the gamma factor of the motive $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$ is equal to*

$$L_k(\infty, s) = \pi^{-ms/2} \prod_{j=1}^{m} \Gamma\Big(\frac{s-j}{2}\Big), \quad m = \Big\lfloor \frac{k-1}{2} \Big\rfloor - \delta_{4\mathbf{Z}}(k).$$

Proof. In our geometric setting, the grading is given by

$$V^p = \mathrm{gr}^p_F \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$$

and the $\mathbf{R}$-structure comes from the maps σ induced by complex conjugation $\mathscr{K}(\mathbf{C}) \to \mathscr{K}(\mathbf{C})$ on singular cohomology $\mathrm{H}^{k-1}(\mathscr{K}(\mathbf{C}))$ and singular cohomology with compact support $\mathrm{H}^{k-1}_{\mathrm{c}}(\mathscr{K}(\mathbf{C}))$, see [51, §3.3(b)]. These form an $\mathbf{R}$-Hodge decomposition of weight $w = k+1$.

Observe that the middle degree factor $V^{w/2}$ is non-trivial if and only if $k = 4r+3$ for some integer $r \geqslant 0$, in which case the weight is $w = 4r+4$ and $V^{w/2}$ has dimension one. Assuming this, let $\varepsilon \in \{\pm 1\}$ denote the sign of the action of σ on $V^{w/2}$. Since $\dim V = 2r+1$ and σ interchanges V^p and V^{w-p}, one has $\det \sigma = (-1)^r \varepsilon$ on $\det \mathrm{H}^1_{\mathrm{dR,mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$. Therefore, it suffices to compute $\det \sigma$. Thanks to the orthogonal pairing (2.18), the above determinant is, up to a twist, the de Rham realization of the rank one Artin motive associated with a quadratic field extension of $\mathbf{Q}$ and one only needs to decide whether this field is real or imaginary. To do so, we look at the ℓ-adic representation $r_{k,\ell}\colon \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \to \mathrm{GL}(V_{k,\ell})$. For each odd prime p, the determinant of Frobenius was computed in [23, Th. 0.1]:

$$\det(F_p | \mathrm{H}^1_{\text{ét,mid}}(\mathbb{G}_{\mathrm{m},\overline{\mathbf{F}}_p}, \mathrm{Sym}^k \mathrm{Kl}_2)) = p^{(k+1)\dim \mathrm{H}^1_{\mathrm{mid}}/2} \Big(\frac{2}{p}\Big)^{\lfloor k/2p + 1/2 \rfloor} \prod_{\substack{0 \leqslant j \leqslant (k-1)/2 \\ p \nmid 2j+1}} \Big(\frac{(-1)^j(2j+1)}{p}\Big)$$

From this we immediately derive that, for all primes $p > k$,

$$\det(r_{k,\ell}(\mathrm{Frob}_p)) = \Big(\frac{(-3)\cdot 5 \cdots (-1)^{(k-1)/2}k}{p}\Big) p^{(k^2-1)/4} = \Big(\frac{p}{k!!}\Big) p^{(k^2-1)/4}$$

where $k!! = 3 \cdot 5 \cdots k$. Chebotarev's density theorem then yields $\det r_{k,\ell} = (\cdot/k!!)\chi_{\mathrm{cyc}}^{(1-k^2)/4}$. It follows that the quadratic number field to which this character gives rises through class field theory is equal to $\mathbf{Q}(\sqrt{\pm k!!})$, with sign adjusted by the condition that the radicand is congruent to 1 modulo 4 (otherwise, 2 would be a ramified prime). Noting that $k = 4r+3$, this sign is given by $(-1)^{r+1}$ and the power of the cyclotomic character appearing in $\det r_{k,\ell}$ is even. Putting everything together, one derives $\varepsilon = -1 = -(-1)^{w/2}$, hence $h(w/2)^+ = 0$ and $h(w/2)^- = 1$, which was the missing information to compute the gamma factor. □

5.3. **Automorphy, meromorphic continuation, and functional equation.** In this final section, we pull everything together to prove Theorems 1.2 and 1.3 from the introduction. To begin with, we compute the ε-factors of the Galois representations $V_{k,\ell}$ and recall the particular case of a theorem of Patrikis and Taylor that we will use.

5.3.1. *Weil–Deligne representations and ε-factors.* For each integer $k \geqslant 1$, consider the system $\{V_{k,\ell}\}$ of ℓ-adic realizations of the motive $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \mathrm{Sym}^k \mathrm{Kl}_2)$. We investigate its global ε-factor $\varepsilon_k(s)$, an entire holomorphic function defined by Tate and Langlands, by means of the information obtained in Theorems 5.9 and 5.15.

As inputs for defining the local ε-factor of $\{V_{k,\ell}\}$ at each place p of $\mathbf{Q}$, we fix the additive character ψ and the Haar measure $\mathrm{d}x$ on $\mathbf{Q}_p$ as follows. If $p < \infty$, then ψ is the composition

$$\mathbf{Q}_p \longrightarrow \mathbf{Q}_p/\mathbf{Z}_p = \mathbf{Z}[1/p]/\mathbf{Z} \longrightarrow \mathbf{C}^\times,$$

where the first map is the quotient and the last map sends α to $\exp(2\pi\mathsf{i}\alpha)$. The Haar measure $\mathrm{d}x$ is such that $\int_{\mathbf{Z}_p} \mathrm{d}x = 1$; note that it is self-dual with respect to ψ. For $p = \infty$, we set $\psi(\alpha) = \exp(-2\pi\mathsf{i}\alpha)$ for $\alpha \in \mathbf{R}$, and we take as $\mathrm{d}x$ the usual Lebesgue measure. Letting $\mathbf{A}_\mathbf{Q}$ denote the adele ring of $\mathbf{Q}$, these local characters and Haar measures are compatible in the sense that the product of the ψ's induces a character of $\mathbf{A}_\mathbf{Q}/\mathbf{Q}$ and the compact quotient $\mathbf{A}_\mathbf{Q}/\mathbf{Q}$ has volume 1 with respect to the induced measure [11, §3.10].

For each $p < \infty$, let $W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ be the Weil group of $\mathbf{Q}_p$ i.e., the subgroup of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ consisting of those elements whose image in $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$ is an integral power of Frobenius together with the topology making I_p with its usual topology into an open subgroup, and let $F_p \in W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ be a lifting of the geometric Frobenius. Local class field theory provides an isomorphism between $\mathbf{Q}_p^\times$ and the maximal abelian quotient $W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)^{\mathrm{ab}}$ that, following the convention of [11, §2.3], we normalize in such a way that it identifies p with F_p. For $s \in \mathbf{C}$, let

$$\omega_s \colon W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p) \longrightarrow \mathbf{C}^\times$$

be the homomorphism defined by composition of the quotient map to $W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)^{\mathrm{ab}} \cong \mathbf{Q}_p^\times$ with the map from $\mathbf{Q}_p^\times$ to $\mathbf{C}^\times$ sending α to $\|\alpha\|^s$, with $\|p\| = 1/p$.

With a continuous representation ρ of $W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ on a discrete topological vector space V over a field of characteristic zero is associated a local ε-factor $\varepsilon(\rho, s) = \varepsilon(\rho \cdot \omega_s, 0)$, depending on ψ and $\mathrm{d}x$, in [11, Th. 4.1]. By (5.5.2) in loc. cit., we have

$$\varepsilon(\rho, s) = \omega_s(p^{a(\rho)}) \cdot \varepsilon(\rho, 0) = p^{-a(\rho)s} \cdot \varepsilon(\rho, 0), \tag{5.25}$$

where $a(\rho)$ denotes the conductor and we regard ω_s as a map $\mathbf{Q}_p^\times \cong W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)^{\mathrm{ab}} \to \mathbf{C}^\times$.

A *Weil–Deligne representation* (ρ, N) on V consists of a representation ρ on V as above and a nilpotent endomorphism N of V satisfying $\rho(w)N\rho(w)^{-1} = p^{v(w)}N$ for all $w \in W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, where $v(w)$ denotes the power of F_p to which w is mapped in $\mathrm{Gal}(\overline{\mathbf{F}}_p/\mathbf{F}_p)$. There is a canonical way to attach a Weil–Deligne representation to an ℓ-adic representation r of $W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$. Namely, by Grothendieck's quasi-unipotency theorem, there exists a unique nilpotent endomorphism N such that $r(\sigma) = \exp(t_\ell(\sigma)N)$ for all σ in a finite index subgroup of I_p and we set

$$\rho(\sigma F_p^n) = r(\sigma F_p^n)\exp(-t_\ell(\sigma)N) \tag{5.26}$$

for all $\sigma \in I_p$ and all $n \in \mathbf{Z}$, see [11, §8.4]. Let

$$\varepsilon((\rho,N),s) = \varepsilon(\rho,s)\cdot\det\bigl(-p^{-s}F_p \mid V^{\rho(I_p)}/\ker(N)^{\rho(I_p)}\bigr) \tag{5.27}$$

be the associated local ε-factor defined before Remarque 5.2.1 of [13].

Let ℓ be a prime number distinct from p. For $s \in \mathbf{Z}$, we also regard ω_s as a homomorphism to $\mathbf{Q}_\ell^\times$. We consider the Weil–Deligne representation (ρ, N) on $V_{k,\ell}$ corresponding to the ℓ-adic representation $V_{k,\ell}$ of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ and denote by $\varepsilon_k(p,s) = \varepsilon((\rho,N),s)$ its ε-factor.

Suppose k is odd. For $2 < p < \infty$, the representation $V_{k,\ell}$ of the inertia group I_p is tame and factors through characters of subgroups of indices at most two by Theorem 5.9. The associated Weil–Deligne representation (ρ, N) has thus $N = 0$ and ρ equals the restriction of $V_{k,\ell}$ to $W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, so that $\varepsilon_k(p,s) = \varepsilon(\rho,s)$ in this case. By definition [11, (4.5.4)], the conductor is given by

$$a(\rho) = \dim V_{k,\ell} - \dim V_{k,\ell}^{\rho(I_p)} = \#\Theta_p^-$$

and from the "formulaire" in loc. cit. we find

$$\begin{aligned} 1 &= \varepsilon(\rho,0)\cdot\varepsilon(\rho^\vee\cdot\omega_1,0)\cdot\det(\rho)(-1) && \text{by [11, (5.4), (5.7.1)]} \\ &= \varepsilon(\rho,0)\cdot\varepsilon(\rho\cdot\omega_{k+2},0)\cdot\det(\rho)(-1) && \text{since } V_{k,\ell}^\vee = V_{k,\ell}(k+1) \\ &= \varepsilon(\rho,0)^2\cdot(p^{\#\Theta_p^-})^{-(k+2)}\cdot\det(\rho)(-1) && \text{by (5.25).} \end{aligned} \tag{5.28}$$

Recall from the proof of Corollary 5.24 that $\det(\rho)$ is the non-trivial character associated with the quadratic extension $\mathbf{Q}_p(\sqrt{\pm k!!})$ with positive sign if $k \equiv 1, 7 \bmod 8$ and negative otherwise. Therefore, $\det(\rho)(-1)$ is given by the Hilbert symbol $(-1, \pm k!!)$ and there exists a fourth root of unity $w_p \in \mu_4(\mathbf{C})$ with $w_p^2 = (-1, \pm k!!)$ such that

$$\varepsilon_k(p,s) = w_p\cdot(p^{\#\Theta_p^-})^{(k+2)/2-s}.$$

Moreover, if $V_{k,\ell}$ is unramified, then $\varepsilon_k(p,s) = 1$ (recall that this includes the case $p = 2$). According to [13, §5.3], at $p = \infty$ the associated ε-factor $\varepsilon_k(\infty,s)$ is given, in the notation of Section 5.2 *supra*, by i to the power

$$\sum_{p<q}(q-p+1)h(p) + h((k+1)/2)^- = \frac{k^2-1}{8}.$$

Now the product formula for Hilbert symbols implies that $\varepsilon_k(\infty,s)\cdot\prod_{p<\infty} w_p \in \{\pm 1\}$ and putting everything together we get

$$\varepsilon_k(s) = \prod_{p\leqslant\infty}\varepsilon_k(p,s) = \pm\mathfrak{N}_k^{(k+2)/2-s},$$

where $\mathfrak{N}_k$ is the integer defined in (5.12).

Remark 5.29. It is obvious that in this case the ε-factors remain unchanged if one replaces the input $\{V_{k,\ell}\}_\ell$ with its semi-simplification $\{V_{k,\ell}^{\mathrm{ss}}\}_\ell$.

We now suppose that k is even and keep notation from Theorem 5.15. If $2 < p < \infty$, there exist a basis $\{e_i\}$ of U and elements $\{e'_i\}$ inducing a basis of $V_{k,\ell}/U^\perp$ such that $\ker(N) = U^\perp$ and $N(e'_i) = -(-1)^{k/2} e_i$. In this case, the Weil representation ρ given by (5.26) is unramified, hence $a(\rho) = 0$ and $\varepsilon(\rho, s) = 1$. Thanks to the identity $V_{k,\ell}^\vee = V_{k,\ell}(k+1)$ as representations of $\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$, we have $\det(\rho(F_p), V_{k,\ell}) = p^{m(k+1)/2}$ where $m = \dim V_{k,\ell}$ is even. By the definition (5.27) and Theorem 5.15, we obtain

$$\varepsilon_k(p,s) = (-1)^{v_p} \cdot p^{\lfloor k/2p \rfloor ((k+2)/2 - s)}, \quad v_p = \begin{cases} \left\lfloor \dfrac{k}{2p} \right\rfloor & \text{if } p \equiv 1 \bmod 4 \\ \left\lfloor \dfrac{k}{4p} \right\rfloor & \text{if } p \equiv 3 \bmod 4. \end{cases} \tag{5.30}$$

Besides, the computation of Hodge numbers yields

$$\varepsilon_k(\infty, s) = \left.\begin{cases} 1, & k \equiv 2 \bmod 4 \\ (-1)^{(k-4)/4}, & k \equiv 0 \bmod 4 \end{cases}\right\} = (-1)^{\delta_{8\mathbf{Z}}(k)},$$

from which we get the value of the global epsilon factor away from $p = 2$:

$$\prod_{2 < p \leqslant \infty} \varepsilon_k(p,s) = (-1)^{v'} \mathfrak{N}_k'^{\,(k+2)/2 - s}, \quad v' = \sum_{p \equiv 1\,(4)} \left\lfloor \frac{k}{2p} \right\rfloor + \sum_{p \equiv 3\,(4)} \left\lfloor \frac{k}{4p} \right\rfloor + \delta_{8\mathbf{Z}}(k).$$

Remark 5.31. The factor $\varepsilon_k(s)$ is of the form AB^{-s} since all its local factors are. Suppose we have the functional equation

$$\widehat{L}_k(s) = \varepsilon_k(s) \cdot \widehat{L}_k(k+2-s), \quad \widehat{L}_k(s) := L_k(\infty, s) \cdot \prod_{p < \infty} L_k(p, p^{-s}).$$

By applying it twice, we get $A^2 = B^{k+2}$. Suppose k is even and $p = 2$, and let $a = a(\rho)$ be the conductor of the associated Weil–Deligne representation. The same computation as in (5.28) gives $|\varepsilon(\rho, 0)| = 2^{a(k+2)/2}$. On the other hand, suppose the quotient $V^{\rho(I_2)}/\ker(N)^{\rho(I_2)}$ has dimension r and that $\det(F_2)$ acts as δ. Plugging into (5.27), one obtains

$$\varepsilon_k(2,s) = w'' |\delta| \, 2^{a(k+2)/2} \, 2^{-(a+r)s}$$

for some $|w''| = 1$. The assumption of the validity of the functional equation with (5.30) then yields $|\delta| = 2^{r(k+2)/2}$. Therefore, we conclude

$$\varepsilon_k(s) = w \cdot (2^{a+r} \mathfrak{N}'_k)^{(k+2)/2 - s}, \quad w = (-1)^{v'} w''.$$

In [8] based on the numerical data, it is conjectured that $a + r = \lfloor k/6 \rfloor$, the exponent of 3 in $\mathfrak{N}'_k$. It is further conjectured that $w'' = (-1)^{v''}$ with $v'' = \lfloor k/8 \rfloor$. One possible structure which fits these data would be that $V_{k,\ell}$ is tamely ramified at 2 such that the associated Weil–Deligne representation is unramified and

$$\det\Big(1 - F_2 T \mid V_{k,\ell}^{I_2}\Big) = \big(1 - 2^{k/2} T\big)^{\lfloor k/8 \rfloor} \big(1 + 2^{k/2} T\big)^{b_k} M_k(2;T)$$

of degree $\frac{k-2}{2} - \delta_{4\mathbf{Z}}(k) - \left\lfloor \frac{k}{6} \right\rfloor$ where b_k and M_k are defined in (5.5).

5.3.2. *The theorem of Patrikis–Taylor.* Let $m \geqslant 1$ be an integer and S a finite set of prime numbers. We consider a *weakly compatible* system of continuous semi-simple representations

$$r_\ell\colon \operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \operatorname{GL}_m(\overline{\mathbf{Q}}_\ell)$$

with ℓ running over all prime numbers. By "weakly compatible", a notion borrowed from [4, 5.1], we mean that the following conditions hold:

- if $p \notin S$, then for all $\ell \neq p$ the representation r_ℓ is unramified at p and the characteristic polynomial of $r_\ell(\operatorname{Frob}_p)$ lies in $\mathbf{Q}[T]$ and is independent of ℓ;
- each representation r_ℓ is de Rham and in fact crystalline if $\ell \notin S$;
- the Hodge–Tate weights of r_ℓ are independent of ℓ.

Theorem 5.32 (Patrikis–Taylor, [40]). *Suppose that the weakly compatible system $\mathscr{R} = \{r_\ell\}$ satisfies the following three properties:*

- *(Purity) There exists an integer w such that, for each prime $p \notin S$, the roots of the common characteristic polynomial of $r_\ell(\operatorname{Frob}_p)$ are Weil numbers of weight w.*
- *(Regularity) The representation r_ℓ has m distinct Hodge–Tate numbers.*
- *(Odd essential self-duality) Either each r_ℓ factors through a map to $\operatorname{GO}_m(\overline{\mathbf{Q}}_\ell)$ with even similitude character or each r_ℓ factors through a map to $\operatorname{GSp}_m(\overline{\mathbf{Q}}_\ell)$ with odd similitude character. Moreover, these characters form a weakly compatible system.*

Then there exists a finite, Galois, totally real number field over which all of the r_ℓ become automorphic. In particular, the partial L-function

$$L^S(\mathscr{R}, s) = \prod_{p \notin S} \det(1 - r_\ell(\operatorname{Frob}_p)p^{-s})^{-1}$$

admits a meromorphic continuation to the complex plane.

Let p and ℓ be distinct prime numbers and let (ρ, N) be the Weil–Deligne representation of $W(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ on $V \simeq \overline{\mathbf{Q}}_\ell^m$ associated with r_ℓ. There is a unique *monodromy filtration* $V^\bullet \subset V$ attached to the nilpotent endomorphism N such that $NV^a \subset V^{a-2}$ and that the map $V^a/V^{1+a} \to V^{-a}/V^{1-a}$ induced by N^a is an isomorphism for each integer a. Recall that (ρ, N) is called *pure* of weight w if the eigenvalues of Frobenius F_p acting on V^a/V^{1+a} are p-Weil numbers of weight $w + a$ for all a. Following vast work on constructions of Galois representations attached to automorphic representations, which is partly summarized in [4, Th, 2.1.1], we have the following consequence of the above theorem.

Corollary 5.33. *Let $\mathscr{R} = \{r_\ell\}$ be a weakly compatible system which is pure of weight w, regular, and odd essential self-dual. For any distinct primes p and ℓ, the Weil–Deligne representation $\operatorname{WD}_p(\mathscr{R})$ of $\operatorname{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)$ associated with r_ℓ is pure of weight w.*

Moreover, the completed L-function

$$\Lambda(\mathscr{R}, s) = L_\infty(\mathscr{R}, s) \cdot \prod_{p \in S} L(\operatorname{WD}_p(\mathscr{R}), s) \cdot L^S(\mathscr{R}, s)$$

satisfies the functional equation $\Lambda(\mathscr{R}, s) = \varepsilon(\mathscr{R}, s)\Lambda(\mathscr{R}^\vee, 1 - s)$.

5.3.3. *Proof of Theorems* 1.2 *and* 1.3. For each integer $k \geqslant 1$, the ℓ-adic representations

$$r_{k,\ell}\colon \operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \operatorname{GL}(V_{k,\ell} \otimes \overline{\mathbf{Q}}_\ell) \simeq \operatorname{GL}_m(\overline{\mathbf{Q}}_\ell)$$

are pure of weight $k+1$ and their semi-simplifications $r_{k,\ell}^{\mathrm{ss}}$ form a weakly compatible system. They are also regular since the condition that all their Hodge–Tate weights are distinct amounts to the Hodge numbers of $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$ being either zero or one, which is the content of Theorem 1.7. Besides, the existence of the $(-1)^{k+1}$-symmetric perfect pairing (2.18) implies that the $r_{k,\ell}$ factor through $\mathrm{GO}_m(\overline{\mathbf{Q}}_\ell)$ (resp. $\mathrm{GSp}_m(\overline{\mathbf{Q}}_\ell)$) if k is odd (resp. even) with similitude character $\chi_{\mathrm{cyc}}^{-k-1}$. Choose a basis of $\mathbf{Q}_\ell(-k-1)$ and regard the perfect pairing $V_{k,\ell} \times V_{k,\ell} \to \mathbf{Q}_\ell(-k-1)$ as a compatible non-degenerate bilinear form on the module $V_{k,\ell}$ over the group ring $\mathbf{Q}_\ell[\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})]$ with the involution $g \mapsto \chi_{\mathrm{cyc}}^{-k-1}(g)g^{-1}$. Then by [52, Th. 4.2.1], $r_{k,\ell}^{\mathrm{ss}}$ also factors through $\mathrm{GO}_m(\overline{\mathbf{Q}}_\ell)$ (resp. $\mathrm{GSp}_m(\overline{\mathbf{Q}}_\ell)$) with similitude character $\chi_{\mathrm{cyc}}^{-k-1}$. Moreover $r_{k,\ell}^{\mathrm{ss}}$ is de Rham at all primes ℓ and indeed crystalline if $\ell > k$ by Proposition 5.20). By the theorem of Patrikis and Taylor, the partial L-function $L^S(s)$ of $\{r_{k,\ell}^{\mathrm{ss}}\}$ has meromorphic continuation and satisfies the expected automorphic functional equation.

We now show that the L-function and the ε-factor of $\{r_{k,\ell}^{\mathrm{ss}}\}$ coincide with those of $\{r_{k,\ell}\}$. When k is odd, this was the content of Remarks 5.13 and 5.29. Suppose k is even and fix an odd ramified prime $3 \leqslant p \leqslant k/2$ for the motive $\mathrm{H}^1_{\mathrm{mid}}(\mathbb{G}_{\mathrm{m}}, \operatorname{Sym}^k \mathrm{Kl}_2)$. Take a prime $\ell \neq p$. Let

$$0 = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_c = V_{k,\ell} \otimes \overline{\mathbf{Q}}_\ell$$

be a maximal sequence of $\operatorname{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$-invariant subspaces and let $\overline{V} = \bigoplus V_i/V_{i-1}$ be the semi-simplification. To show that the L-factor and the ε-factor at p of $V_{k,\ell}$ are the same as those of $\overline{V}$, it suffices to prove the equality of dimensions

$$\dim \overline{V}^{I_p} = \dim V_{k,\ell}^{I_p}.$$

(For the ε-factor, this is due to the fact that the associated Weil–Deligne representation (ρ, N) is unramified and $\ker(N)^{\rho(I_p)} = V_{k,\ell}^{I_p}$.) Suppose otherwise. Since $\overline{V}^\vee \cong \overline{V}(k+1)$, there must exist an $e' \in V_{k,\ell} \smallsetminus V_{k,\ell}^{I_p}$ with $e' \in V_{i+1} \smallsetminus V_i$ such that the induced class $[e']$ in V_{i+1}/V_i is fixed by I_p and $F_p[e'] = \pm p^{\frac{k+2}{2}}[e']$ by Theorem 5.15. Since $\{\sigma(e') - e' \mid \sigma \in I_p\} \neq \{0\}$ and the former is contained in the isotropic subspace U generated by vanishing cycle classes, there exists a non-zero F_p-eigenvector $e \in U$ such that $F_p e = \pm p^{\frac{k}{2}} e$ and $e \in V_i$. We assume that the e' is chosen such that the index i is minimal for the existence of such an e'. It follows that the subrepresentation V_i is unramified at the prime p, the eigenvalues of F_p on V_i are p-Weil numbers of weight $\frac{k+1}{2}$ and $\frac{k}{2}$, and at least one of them has weight $\frac{k}{2}$ coming from e (with eigenvalue $\pm p^{\frac{k}{2}}$). This is impossible by Corollary 5.33.

The discussion of paragraph 5.3.1 *supra* then implies that this functional equation is, up to sign, precisely the one from Theorems 1.2 and 1.3. To conclude, we need to show that for odd k the sign is always positive; for this we use T. Saito's result [49] that the sign of the functional equation of the L-function of an orthogonal motive of even weight is always positive. $\square$

Remark 5.34. The above formulas of $\Lambda_k(s)$ for both k odd and even match the numerical observations made in [7, Eq. (128)] and [8][1] (see also [43]). The computations of the ℓ-adic realizations thus provide the theoretical explanations from the viewpoint of motives.

Appendix A. Exponential motives with compact support and realizations

We work over the base field $\mathbf{Q}$ for the exponential motives in Section A.1, and over $\mathbf{C}$ for the de Rham cohomology in Section A.2.

Notation A.1. We denote by $\mathsf{Mot}^{\mathrm{cl}}$ and $\mathsf{Mot}^{\exp}$ the $\mathbf{Q}$-linear neutral Tannakian categories of Nori motives and exponential motives over the base field $\mathbf{Q}$, respectively. The former is contained in the latter as a full subcategory ([20, Th. 5.1.1]). For a finite group G, $\mathsf{Mot}^{\mathrm{cl}}_G$ denotes the $\mathbf{Q}$-linear Tannakian category consisting of objects in $\mathsf{Mot}^{\mathrm{cl}}$ together with a G-action and similarly for $\mathsf{Mot}^{\exp}_G$.

For an integer r, let $\mathbf{Q}(r)$ be the r-th Tate object of weight $-2r$ in $\mathsf{Mot}^{\mathrm{cl}}$; it is identified with $\mathbf{Q}$ as a $\mathbf{Q}$-vector space. Its period realization is given by $(\mathbf{Q}(r), \mathbf{Q}(r)_{\mathbf{C}}, \cong)$, where $\mathbf{Q}(r)_{\mathbf{C}}$ is the filtered complex vector space $\mathbf{C}$ with $\mathrm{gr}^W_{-2r}\mathbf{Q}(r)_{\mathbf{C}} = \mathrm{gr}^{-r}_F\mathbf{Q}(r)_{\mathbf{C}} = \mathbf{Q}(r)_{\mathbf{C}}$ and the comparison isomorphism is multiplication by $(2\pi i)^{-r}$. The 0-th Tate object is simply denoted by $\mathbf{Q}$. We have $\mathbf{Q}(-1) \cong \mathrm{H}^1(\mathbb{G}_{\mathrm{m}}) \cong \mathrm{H}^2(\mathbb{P}^1)$ in $\mathsf{Mot}^{\mathrm{cl}}$.

A.1. Exponential motives.

Let (U, f) be a smooth quasi-projective variety with potential and (X, f_X) a good compactification, i.e., a smooth projective variety X containing U as the complement of a strict normal crossing divisor $D = X \smallsetminus U$, and a morphism $f_X\colon X \to \mathbb{P}^1$ extending f. Write the boundary as a sum of reduced divisors

$$D = D_\infty + D_{<\infty},$$

where D_∞ is the support of the pole divisor of f_X. Let

$$(Y, D^{(\infty)}_{<\infty}, f_Y) = (X \smallsetminus D_\infty, D_{<\infty} \smallsetminus D_\infty, f_X|_Y)$$

and view it as an object $[Y, D^{(\infty)}_{<\infty}, f_Y, r, 0]$ in the quiver of exponential relative variety over $\mathbf{Q}$ ([20, Def. 4.2.1]).[2] One has the canonical morphism (of type (a))

$$[Y, D^{(\infty)}_{<\infty}, f_Y, r, i] \longrightarrow [U, \varnothing, f, r, i] \tag{A.2}$$

in the quiver ([20, §4.2]). We have the r-th exponential motive $\mathrm{H}^r(U, f)$ in $\mathsf{Mot}^{\exp}$ associated with $[U, \varnothing, f, r, 0]$.

[1] The $\Lambda_k(s)$ here are obtained by replacing the variable s by $(s-2)$ of those in [8] due to the shift of weights by 4 by a Tate twist.

[2] Warning of the notation: The tuple $[Y, Z, f, r, i]$ and the motive $\mathrm{H}^r(Y, f)$ etc. here in fact correspond to the tuple $[Y, Z, -f, r, i]$ and the motive $\mathrm{H}^r(Y, -f)$ etc. in [20]. For example, the de Rham realization $\mathrm{H}^r_{\mathrm{dR}}(Y, E^f)$ of $\mathrm{H}^r(Y, f)$ is the hypercohomology of the de Rham complex of the connection $\mathrm{d} + \mathrm{d}f$ in this paper, contrary to $\mathrm{d} - \mathrm{d}f$ in [20]. We follow this convention in order to be coherent with the main references in irregular Hodge theory.

Definition A.3. With notations as above, the r-th exponential motive *with compact support* of (U, f) is the object in $\mathsf{Mot}^{\mathrm{exp}}$

$$\mathrm{H}^r_{\mathrm{c}}(U, f) = \mathrm{H}^r(Y, D^{(\infty)}_{<\infty}, f_Y)$$

associated with $[Y, D^{(\infty)}_{<\infty}, f_Y, r, 0]$. The map (A.2) induces a morphism $\mathrm{H}^r_{\mathrm{c}}(U, f) \to \mathrm{H}^r(U, f)$ in $\mathsf{Mot}^{\mathrm{exp}}$. Let the motive $\mathrm{H}^r_{\mathrm{mid}}(U, f)$ be the image

$$\mathrm{H}^r_{\mathrm{mid}}(U, f) = \mathrm{im}\,\{\mathrm{H}^r_{\mathrm{c}}(U, f) \longrightarrow \mathrm{H}^r(U, f)\}\,.$$

The subscript 'mid' corresponds to the subscript '!∗' indicating the 'balanced' version of motives and cohomology spaces in [56].

Remark A.4. We have the following properties.

(1) Suppose U is absolutely connected of dimension d. By [20, Prop. 4.8.1], there is the canonical morphism

$$\mathrm{H}^r_{\mathrm{c}}(U, f) \otimes \mathrm{H}^{2d-r}(U, -f) \longrightarrow \mathbf{Q}(-d),$$

which underlies the Poincaré pairing in various cohomological realizations. Since the exponential motive $\mathrm{H}^{2d-r}(U, -f)$ is independent of the choice of the compactification (X, f_X), so is the motive $\mathrm{H}^r_{\mathrm{c}}(U, f)$ of (U, f) with compact support, which serves canonically as the dual of $\mathrm{H}^{2d-r}(U, -f)(d)$. Consequently, $\mathrm{H}^r_{\mathrm{mid}}(U, f)$ is defined independently of (X, f_X).

(2) By definition, we have $\mathrm{H}^r_{\mathrm{c}}(Y, f_Y) = \mathrm{H}^r(Y, f_Y)$. By [20, (4.2.4.2)] with the chain $\varnothing \subset D^{(\infty)}_{<\infty} \subset Y$, we have the localization exact sequence in $\mathsf{Mot}^{\mathrm{exp}}$

$$\cdots \longrightarrow \mathrm{H}^r_{\mathrm{c}}(U, f) \longrightarrow \mathrm{H}^r_{\mathrm{c}}(Y, f_Y) \longrightarrow \mathrm{H}^r_{\mathrm{c}}(D^{(\infty)}_{<\infty}, f_Y) \longrightarrow \mathrm{H}^{r+1}_{\mathrm{c}}(U, f) \longrightarrow \cdots$$

where we define $\mathrm{H}^r_{\mathrm{c}}(D^{(\infty)}_{<\infty}, f_Y) = \mathrm{H}^r(D^{(\infty)}_{<\infty}, f_Y)$. The de Rham version of the sequence was obtained in [55, Prop. 1.9(ii)] in case $D^{(\infty)}_{<\infty}$ is smooth.

A.2. **De Rham cohomology.** In this section, U denotes a complex quasi-projective smooth variety together with a morphism $f\colon U \to \mathbb{A}^1$. We fix a compactification X of U such that the boundary $D = X \smallsetminus U$ is a strict normal crossings divisor and that f extends to a rational morphism, still denoted by f,

$$f\colon X \dashrightarrow \mathbb{P}^1$$

which is *non-degenerate* along D in the sense of Mochizuki [38, Def. 2.6]. This means that, locally for the analytic topology, there is a coordinate system $\{x_1, \ldots, x_r, y_1, \ldots, y_m, z_1, \ldots, z_l\}$ of X such that

$$D = (xy), \quad f = 1/x^e \text{ or } z_1/x^e \tag{A.5}$$

for some $e \in \mathbf{Z}^r_{>0}$.

Let P be the pole divisor of f and write

$$D = D_\infty + D_{<\infty},$$

where D_∞ is the support of P. Consider the $\mathscr{D}$-module

$$\mathscr{O}(!D_{<\infty}) := [(X \smallsetminus D_{<\infty}) \longrightarrow X]_\dagger\, \mathscr{O}.$$

It is the filtered $\mathscr{D}$-module underlying a Hodge module, dual to $\mathscr{O}(*D_{<\infty})$.

On the above local chart, the sheaf $\mathscr{O}(!D_{<\infty})$ is equal to $\mathscr{D}\big/\sum_{i=1}^m \mathscr{D}\cdot y_i\partial_{y_i}$ and the (increasing) Hodge filtration is induced by the order of differential operators. On one hand, the local description shows that there are the natural maps

$$\mathscr{O}(-D_{<\infty}) \longrightarrow \mathscr{O} \longrightarrow \mathscr{O}(!D_{<\infty}) \tag{A.6}$$

of $\mathscr{O}_X$-modules induced by the inclusion $\mathscr{O}\to\mathscr{D}$. On the other hand, the filtration has an inductive description from the exterior product structure as follows. Write the total space as $U\times V$ where U and V have coordinates

$$\{x_1,\dots,x_r,y_1,\dots,y_{m-1},z_1,\dots,z_l\} \quad\text{and}\quad \{y_m\}, \tag{A.7}$$

respectively. Let $D'_{<\infty}=(y_1\cdots y_{m-1})$ (resp. $D''_{<\infty}=(y_m)$) be the divisor on U (resp. V) and $\mathscr{O}(!D'_{<\infty})$ (resp. $\mathscr{O}(!D''_{<\infty})$) the associated $\mathscr{D}$-module. Then $F_{<0}\mathscr{O}(!D''_{<\infty})=0$ and

$$\begin{aligned}F_k\mathscr{O}(!D''_{<\infty}) &= \mathscr{O}\oplus\bigoplus_{i=1}^{k}\mathbf{C}\partial_{y_m}^{j}\\ &= \mathscr{O}(-D''_{<\infty})\oplus\bigoplus_{j=0}^{k}\mathbf{C}\partial_{y_m}^{j},\quad k\geqslant 0\end{aligned}$$

as vector spaces. We have $\mathscr{O}(!D_{<\infty})=\mathscr{O}(!D'_{<\infty})\boxtimes\mathscr{O}(!D''_{<\infty})$ as quasi-coherent $\mathscr{O}$-modules and hence the second equality in the expression

$$\begin{aligned}F_p\mathscr{O}(!D_{<\infty}) &= \sum_{i+j=p}F_i\mathscr{O}(!D'_{<\infty})\boxtimes F_j\mathscr{O}(!D''_{<\infty})\\ &= F_p\mathscr{O}(!D'_{<\infty})\boxtimes\mathscr{O}(-D''_{<\infty})\oplus\sum_{i\geqslant 0}F_{p-i}\mathscr{O}(!D'_{<\infty})\partial_{y_m}^{i}.\end{aligned}$$

Here the first summation is taken in $\mathscr{O}(!D'_{<\infty})\boxtimes\mathscr{O}(!D''_{<\infty})$ and the sums in the second line are as vector spaces; in the last term, we shorten $h\otimes\partial_{y_m}^i$ as $h\partial_{y_m}^i$ for a section h of $F_{p-i}\mathscr{O}(!D'_{<\infty})$.

Let $\mathscr{E}^f$ be the $\mathscr{D}$-module on X associated with the connection $(\mathscr{O}(*D_\infty),\mathrm{d}+\mathrm{d}f)$. Consider the $\mathscr{D}$-module

$$\mathscr{M}:=\mathscr{E}^f\otimes_{\mathscr{O}}\mathscr{O}(!D_{<\infty}).$$

It is the dual of $\mathscr{E}^{-f}(*D_{<\infty})$ and is equipped with the (increasing) irregular Hodge filtration $F_\lambda^{\mathrm{irr}}\mathscr{M}$ indexed discretely by $\lambda\in\mathbf{Q}$. In this case, the jumps λ of the filtration only occur when the denominators are among the multiplicities of the pole divisor P of f supported on D_∞.

On the local chart where (A.5) holds, consider the product decomposition (A.7). Let $\mathscr{M}'=\mathscr{E}^f\otimes\mathscr{O}(!D'_{<\infty})$ be the corresponding $\mathscr{D}$-module on U. One again has $\mathscr{M}=\mathscr{M}'\boxtimes\mathscr{O}(!D''_{<\infty})$ and the decomposition

$$\begin{aligned}F_\lambda^{\mathrm{irr}}\mathscr{M} &= \sum_{i+j=\lambda}F_i^{\mathrm{irr}}\mathscr{M}'\boxtimes F_j\mathscr{O}(!D''_{<\infty})\\ &= F_\lambda^{\mathrm{irr}}\mathscr{M}'\boxtimes\mathscr{O}(-D''_{<\infty})\oplus\sum_{i\geqslant 0}F_{\lambda-i}^{\mathrm{irr}}\mathscr{M}'\partial_{y_m}^{i}\end{aligned}$$

with the same convention for the last term as in $F_p\mathscr{O}(!D_{<\infty})$.

We have the induced filtered de Rham complex

$$F_\lambda^{\mathrm{D,dR}}(\mathscr{M}) = \bigl(F^{\mathrm{irr}}_{\lambda+\bullet}\mathscr{M} \otimes_{\mathscr{O}} \Omega^\bullet, \nabla\bigr).$$

It defines the irregular Hodge filtration on the de Rham cohomology $\mathrm{H}^\bullet_{\mathrm{dR}}(X, \mathscr{M})$, which equals the compact-supported cohomology $\mathrm{H}^\bullet_{\mathrm{dR,c}}(U, E^f)$ of the $\mathscr{D}_U$-module E^f, and the associated Hodge to de Rham spectral sequence degenerates in the initial page.

A.2.1. *Logarithmic de Rham complexes.*

Definition A.8. On X, the *filtered extended logarithmic de Rham complex* of $\mathscr{M}$ is the collection of the subcomplexes of $(\Omega(*D_\infty), \nabla)$

$$F_\lambda^{\log,\mathrm{dR}}(\mathscr{M}) = \Bigl(\mathscr{O}(\lfloor(\lambda+\bullet)P\rfloor - D_{<\infty}) \otimes_{\mathscr{O}} \Omega^\bullet(\log D), \nabla\Bigr)_{\bullet \geqslant -\lfloor\lambda\rfloor} \quad (\lambda \in \mathbf{Q}).$$

Let

$$\Omega^k_f = \ker\Bigl\{\Omega^k(\log D) \xrightarrow{\ \nabla\ } \Omega^{k+1}(*D)/\Omega^{k+1}(\log D)\Bigr\}.$$

The *filtered Kontsevich de Rham complex* of $\mathscr{M}$ is the collection of the complexes

$$F_\lambda^{\mathrm{Kont,dR}}(\mathscr{M}) = \Bigl(\mathscr{O}(\lfloor\lambda - \lfloor\lambda\rfloor)P\rfloor - D_{<\infty}) \otimes_{\mathscr{O}} \Omega^\bullet_f, \nabla\Bigr)_{\bullet \geqslant -\lfloor\lambda\rfloor} \quad (\lambda \in \mathbf{Q}).$$

Directly from the definitions, one has the natural inclusion $F_\lambda^{\mathrm{Kont,dR}}(\mathscr{M}) \to F_\lambda^{\log,\mathrm{dR}}(\mathscr{M})$ for each $\lambda \in \mathbf{Q}$. The $\{F_\lambda^{\log,\mathrm{dR}}(\mathscr{M})\}_{\lambda\in\mathbf{Q}}$ forms an increasing filtration on the complex $(\Omega(*D_\infty), \nabla)$. For each fixed $0 \leqslant \alpha < 1$, one has the finite filtration $F_{\alpha+p}^{\mathrm{Kont,dR}}(\mathscr{M}), p \in \mathbf{Z}$, with $F_{\alpha+p}^{\mathrm{Kont,dR}}(\mathscr{M}) = F_\alpha^{\mathrm{Kont,dR}}(\mathscr{M})$ for $p \geqslant 0$. Furthermore, we regard $F_\lambda^{\log,\mathrm{dR}}(\mathscr{M})$ as a subcomplex of $F_\lambda^{\mathrm{D,dR}}(\mathscr{M})$ under the natural maps (A.6) (see [17, 46]).

On the local chart (A.5), the two filtrations also respect the direct product (A.7). Indeed, on V, they reduce to the usual filtered logarithmic de Rham complex

$$\begin{aligned} 0, &\quad \lambda < -1\\ \bigl[0 \longrightarrow \Omega^1\bigr], &\quad -1 \leqslant \lambda < 0\\ \Bigl[\mathscr{O}(-D''_{<\infty}) \xrightarrow{\ \mathrm{d}\ } \Omega^1\Bigr], &\quad \lambda \geqslant 0\end{aligned}$$

for cohomology with compact support. On $U \times V$, one has

$$F_\lambda^{\log,\mathrm{dR}}(\mathscr{M}) = F_\lambda^{\log,\mathrm{dR}}(\mathscr{M}') \boxtimes \mathscr{O}(-D''_{<\infty}) \boxplus F_{\lambda+1}^{\log,\mathrm{dR}}(\mathscr{M}') \boxtimes \Omega^1_V[-1]$$

and similarly for $F_\lambda^{\mathrm{Kont,dR}}(\mathscr{M})$. (Here the direct sum means degree-wise sum, which does not respect the connection ∇.)

Theorem A.9. *For any λ, the natural arrows*

$$F_\lambda^{\mathrm{Kont,dR}}(\mathscr{M}) \longrightarrow F_\lambda^{\log,\mathrm{dR}}(\mathscr{M}) \longrightarrow F_\lambda^{\mathrm{D,dR}}(\mathscr{M})$$

are quasi-isomorphisms. In particular, each of them gives the irregular Hodge filtration of $\mathrm{H}^\bullet_{\mathrm{dR,c}}(U, E^f)$.

Proof. The assertion is local. We choose the local coordinates such that (A.5) holds and proceed by induction on the index m via the product (A.7). The case $m = 0$ is known in [17, 46].

Suppose $m \geqslant 1$. For either ? = Kont or log, consider the subcomplex

$$(A^\bullet \oplus B^\bullet, \nabla) \longrightarrow F_\lambda^{\mathrm{D,dR}}(\mathscr{M}), \tag{A.10}$$

where the degree p terms are given by

$$A^p = F_\lambda^{?,\mathrm{dR}}(\mathscr{M}')^p \boxtimes \mathscr{O}(-D''_{<\infty}) \oplus \sum_{i\geqslant 0} F_{\lambda-i}^{?,\mathrm{dR}}(\mathscr{M}')^p \partial_{y_m}^i$$
$$B^p = F_{\lambda+1}^{?,\mathrm{dR}}(\mathscr{M}')^{p-1} \boxtimes \Omega_V^1 \oplus \sum_{i\geqslant 0} F_{\lambda-i}^{?,\mathrm{dR}}(\mathscr{M}')^{p-1} \partial_{y_m}^{i+1} \mathrm{d}y_m.$$

Notice that the decomposition $A^\bullet \oplus B^\bullet$ does not respect ∇ though. Put the filtration $K^1 \subset K^0$ on $F_\lambda^{\mathrm{D,dR}}(\mathscr{M})$ given by

$$K^0 = F_\lambda^{\mathrm{D,dR}}(\mathscr{M}), \quad K^1 = F_\lambda^{\mathrm{D,dR}} \cap \left(\mathscr{M} \otimes \Omega_X^{\bullet-1} \wedge \mathrm{d}y_m, \nabla\right).$$

We have

$$K^1 = F_{\lambda+1}^{\mathrm{D,dR}}(\mathscr{M}')[-1] \boxtimes \Omega_V^1 \oplus \sum_{i\geqslant 0} F_{\lambda-i}^{\mathrm{D,dR}}(\mathscr{M}')[-1] \partial_{y_m}^{i+1} \mathrm{d}y_m$$
$$\mathrm{gr}_K^0 = F_\lambda^{\mathrm{D,dR}}(\mathscr{M}') \boxtimes \mathscr{O}(-D''_{<\infty}) \oplus \sum_{i\geqslant 0} F_{\lambda-i}^{\mathrm{D,dR}}(\mathscr{M}') \partial_{y_m}^i.$$

By induction, one has the quasi-isomorphism $F_\lambda^{?,\mathrm{dR}}(\mathscr{M}') \to F_\lambda^{\mathrm{D,dR}}(\mathscr{M}')$. Therefore the inclusion (A.10) is a quasi-isomorphism.

Consider the finite filtration $\{G^c\}_{c\in\mathbf{Z}}$ of $(A^\bullet \oplus B^\bullet, \nabla)$ whose degree p component is $G^cA^p \oplus G^cB^p$ with

$$G^cA^p = \begin{cases} F_\lambda^{?,\mathrm{dR}}(\mathscr{M}')^p \boxtimes \mathscr{O}(-D''_{<\infty}), & p < c \\ A^p, & p \geqslant c \end{cases}$$
$$G^cB^p = \begin{cases} F_{\lambda+1}^{?,\mathrm{dR}}(\mathscr{M}')^{p-1} \boxtimes \Omega_V^1, & p < c+1 \\ B^p, & p \geqslant c+1. \end{cases}$$

Then $G^\infty := \bigcup G^c$ equals $F_\lambda^{?,\mathrm{dR}}(\mathscr{M})$. For each c, the quotient gr_G^c is the two-term complex

$$\left[\sum_{i\geqslant 0} F_{\lambda-i}^{?,\mathrm{dR}}(\mathscr{M}')^c \partial_{y_m}^i \longrightarrow \sum_{i\geqslant 0} F_{\lambda-i}^{?,\mathrm{dR}}(\mathscr{M}')^c \partial_{y_m}^{i+1} \mathrm{d}y_m\right][-c]$$
$$\xi \partial_{y_m}^i \longmapsto \xi \partial_{y_m}^{i+1} \mathrm{d}y_m,$$

which is acyclic. Therefore the assertion follows. □

A.2.2. *The toric case.*

Proposition A.11. *Let f be a Laurent polynomial on the torus $\mathbb{G}_\mathrm{m}^n$. Assume f is non-degenerate with respect to its Newton polytope Δ and $\dim \Delta = n$. Let X be a non-degenerate toric compactification of $(\mathbb{G}_\mathrm{m}^n, f)$. Then $\mathrm{H}^i\big(X, \mathscr{O}(\lfloor \lambda P \rfloor - D_{<\infty})\big) = 0$ for any $\lambda \geqslant 0$ and $i \geqslant 1$.*

Proof. Since $\mathrm{H}^i_{\mathrm{dR,c}}(\mathbb{G}_\mathrm{m}^n, E^f)$ is dual to $\mathrm{H}^{2n-i}_{\mathrm{dR}}(\mathbb{G}_\mathrm{m}^n, E^{-f})$ and f is non-degenerate with $\dim\Delta = n$, the former space may be non-trivial only if $i = n$ by [2, Th. 1.4]. On the other hand, by the above theorem, the i-th hypercohomology of $F^{\log,\mathrm{dR}}_\lambda(\mathscr{M})$ maps injectively onto $F^{\mathrm{irr}}_\lambda\mathrm{H}^i_{\mathrm{dR,c}}(\mathbb{G}_\mathrm{m}^n, E^f)$. We use this to deduce the assertion by induction.

Notice that on the smooth projective toric X, the sheaves $\Omega^i(\log D)$ for all i are trivial. Fix one $\alpha\in[0,1)$. We have

$$\mathrm{H}^i\left(X, \Omega^n(\log D)(\lfloor\alpha P\rfloor - D_{<\infty})\right) = \boldsymbol{H}^{n+i}\big(X, F^{\log,\mathrm{dR}}_{\alpha-n}(\mathscr{M})\big) \hookrightarrow \mathrm{H}^{n+i}_{\mathrm{dR,c}}\big(\mathbb{G}_\mathrm{m}^n, E^f\big).$$

The assertion follows for $\lambda=\alpha$. Suppose that the vanishing is true for $\lambda\leqslant\alpha+p$, $\lambda-\alpha\in\mathbf{Z}$ for some integer $p\geqslant 0$. Consider the short exact sequence

$$0\longrightarrow\Omega^n(\log D)(\lfloor(\alpha+p+1)P\rfloor - D_{<\infty})[-n]\longrightarrow F^{\log,\mathrm{dR}}_{\alpha+p+1-n}(\mathscr{M})\longrightarrow Q\longrightarrow 0$$

where Q is obtained by deleting the last non-trivial term of $F^{\log,\mathrm{dR}}_{\alpha+p+1-n}(\mathscr{M})$. By induction, each component of Q has trivial higher cohomology. Since Q is concentrated at degrees in the range $[n-p-1, n-1]$, one has $\boldsymbol{H}^i(X,Q)=0$ for $i\geqslant n$. By taking the associated long exact sequence of the hypercohomologies of terms in the short exact sequence, one obtains the assertion for $\lambda=\alpha+p+1$. □

Corollary A.12. *With the some assumptions as in the previous proposition,* $\mathrm{H}^i_{\mathrm{dR,c}}(\mathbb{G}_\mathrm{m}^n, E^f)$ *is non-zero only if* $i=n$. *In this case the middle cohomology space equals the cokernel of*

$$\Gamma\left(X, \Omega^{n-1}(\log D)\big(\lfloor(\alpha+m-1)P\rfloor - D_{<\infty}\big)\right)\xrightarrow{\ \nabla\ }\Gamma\left(X, \Omega^{n}(\log D)\big(\lfloor(\alpha+m)P\rfloor - D_{<\infty}\big)\right)$$

for any $0\leqslant\alpha<1$ *and integer* $m\geqslant n$. *The filtration* $F^{\mathrm{irr}}_\lambda\mathrm{H}^n_{\mathrm{dR,c}}(\mathbb{G}_\mathrm{m}^n, E^f)$ *coincides with the image of the global sections*

$$\Gamma\left(X, \Omega^{n}(\log D)\big(\lfloor(\lambda+n)P\rfloor - D_{<\infty}\big)\right).$$

Appendix B. Exponential mixed Hodge structures and irregular Hodge filtration

B.1. **Notation and results from $\mathscr{D}$-module theory.** We refer e.g. to [25, 28] for this section, although the notation may be somewhat different.

Given an algebraic morphism $h\colon X\to Y$ between smooth complex algebraic varieties, we denote by h_+ the derived pushforward in the sense of $\mathscr{D}$-modules, that is, for a $\mathscr{D}_X$-module or a bounded complex of $\mathscr{D}_X$-modules M, we set

$$h_+M = \mathrm{R}h_*\left(\mathscr{D}_{Y\leftarrow X}\overset{L}{\otimes}_{\mathscr{D}_X} M\right).$$

We denote by $\mathrm{DR}\,M$ the analytic de Rham complex of M, where M^{an} sits in degree zero, and by ${}^{\mathrm{p}}\mathrm{DR}M$ the shifted complex $\mathrm{DR}\,M[\dim X]$. We also denote by $h_\dagger$ the adjoint by duality of h_+, and there is a natural morphism $h_\dagger M\to h_+M$. In particular, for a holonomic $\mathscr{D}_X$-module M which is regular at infinity on X, we have isomorphisms ${}^{\mathrm{p}}\mathrm{DR}h_+M\simeq\mathrm{R}h_*{}^{\mathrm{p}}\mathrm{DR}M$ and ${}^{\mathrm{p}}\mathrm{DR}h_\dagger M\simeq\mathrm{R}f_!{}^{\mathrm{p}}\mathrm{DR}M$, and the morphism $h_\dagger M\to h_+M$ induces, by the de Rham functor, the natural morphism $\mathrm{R}h_!{}^{\mathrm{p}}\mathrm{DR}M\to\mathrm{R}h_*{}^{\mathrm{p}}\mathrm{DR}M$.

For a $\mathscr{D}_X$-module N on a complex manifold (resp. smooth algebraic variety) X, we denote by $\mathrm{H}^k_{\mathrm{dR}}(X,N)$ the hypercohomology on X of the analytic (resp. algebraic) de Rham complex of N (here we do not shift the de Rham complex as usually done in $\mathscr{D}$-module theory). If X is affine, we will not distinguish between algebraic $\mathscr{D}_X$-modules and their global sections.

If $j\colon U\hookrightarrow X$ is the complementary open embedding of a divisor, then for a holonomic $\mathscr{D}_U$-module M, the extension j_+M is the holonomic $\mathscr{D}_X$-module on which any local equation of the divisor $X\smallsetminus U$ acts in an invertible way, and $j_{\dagger+}M$ denotes the intermediate extension, that is, the maximal $\mathscr{D}_X$-module of j_+M which has no quotient supported on $X\smallsetminus U$. For a holonomic $\mathscr{D}_U$-module M which is regular at infinity, the natural morphism $j_{\dagger+}M\to j_+M$ corresponds, via ${}^{\mathrm{p}}\mathrm{DR}$, to the natural morphism $j_{!*}{}^{\mathrm{p}}\mathrm{DR}M\to \boldsymbol{R}j_*{}^{\mathrm{p}}\mathrm{DR}M$.

For the pullback, we use the notation h^+, e.g. if h is smooth, it sends holonomic $\mathscr{D}_Y$-modules to holonomic $\mathscr{D}_X$-modules. It corresponds to the usual pullback of connections, and the same notation is also used in the derived sense. If $j\colon U\hookrightarrow X$ is an open embedding, then j^+ is the usual restriction functor to the open subset U.

If $\varphi\in\Gamma(X,\mathscr{O}_X(*P))$ is a meromorphic function on X with pole divisor P, we set

$$E^{\varphi}=(\mathscr{O}_X(*P),\mathrm{d}+\mathrm{d}\varphi), \tag{B.1}$$

and we denote by e^{φ} the generator $1\in\mathscr{O}_X(*P)$ of E^{φ}.

Consider the product $\mathbb{A}^1_t\times\mathbb{A}^1_\tau$ of affine lines and denote by p_τ and p_t the first and second projections respectively. Given a $\mathscr{D}$-module M (or a bounded complex) on the affine line $\mathbb{A}^1_\tau$, we denote by $\mathrm{FT}_\tau M$ its *Fourier transform*, defined as[3]

$$\mathrm{FT}_\tau M=p_{t+}(p_\tau^+M\otimes E^{t\tau}).$$

If M is a holonomic $\mathbf{C}[\tau]\langle\partial_\tau\rangle$-module, then $\mathrm{FT}_\tau M$ has cohomology in degree zero only, i.e., it is a holonomic $\mathbf{C}[t]\langle\partial_t\rangle$-module. This yields a functor $\mathrm{FT}_\tau\colon \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{\mathbb{A}^1_\tau})\mapsto \mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}(\mathscr{D}_{\mathbb{A}^1_t})$.

B.2. **Notation and results from the theory of mixed Hodge modules.** Let X be a smooth complex projective variety and let $M^{\mathrm{H}}=(M,F_\bullet M,\mathscr{F}_{\mathbf{Q}})$ consist of the data of a regular holonomic left $\mathscr{D}_X$-module M endowed with an increasing good filtration $F_\bullet M$ and a $\mathbf{Q}$-perverse sheaf $\mathscr{F}_{\mathbf{Q}}$ related by an isomorphism (that should be included in the notation, but that we omit) ${}^{\mathrm{p}}\mathrm{DR}M\xrightarrow{\sim}\mathscr{F}_{\mathbf{C}}$. We say that M^{H} is a pure polarizable Hodge module of weight w if the associated right filtered $\mathscr{D}_X$-module $(\omega_X\otimes_{\mathscr{O}_X}M,\omega_X\otimes_{\mathscr{O}_X}F_{\bullet-\dim X}M)$, together with $\mathscr{F}_{\mathbf{Q}}$ and the corresponding isomorphism $\mathrm{Sp}(\omega_X\otimes M)={}^{\mathrm{p}}\mathrm{DR}M\simeq\mathscr{F}_{\mathbf{C}}$ (where Sp denotes the Spencer complex of a right $\mathscr{D}_X$-module) forms a pure polarizable Hodge module of weight w in the sense of Saito [47]. We have a similar definition for a left mixed Hodge module, with the left-to-right correspondence $W_\bullet M\leftrightarrow\omega_X\otimes W_\bullet M$ of the weight filtrations, and refer to [48] for the properties of mixed Hodge modules (which are always assumed to be graded-polarizable), where the category $\mathsf{MHM}(X)$ of algebraic mixed Hodge modules is defined, together with the functors corresponding to the six operations.

[3]For the sake of clarity, we put the variable as index of various functors, like FT and Π.

The pushforward functor by a morphism for left mixed Hodge modules is obtained by composing the similar functor for mixed Hodge modules with the side-changing functor in an obvious way. We have a similar definition for other functors.

Assume for example that $(M, F_\bullet M)$ underlies a pure Hodge module of weight w which is smooth, i.e., M is $\mathscr{O}_X$-locally free of finite rank with integrable connection ∇. Let us consider the decreasing filtration $F^pM := F_{-p}M$. Then $(M, \nabla, F^\bullet M)$ is a polarizable variation of pure Hodge structure of weight $w - \dim X$.

Let U be a smooth complex quasi-projective variety of dimension $d = \dim U$. We denote by $\mathscr{O}_U^{\mathrm{H}}$ the pure Hodge module of weight d (also denoted by $\mathbf{Q}_U^{\mathrm{H}}$ in [48]; its underlying perverse sheaf is $\mathbf{Q}_U[d]$). For a morphism $f\colon U \to \mathbb{A}^1_s$, there are functors ${}_{\mathrm{H}}f_*, {}_{\mathrm{H}}f_!\colon \mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(U)) \mapsto \mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(\mathbb{A}^1_s))$. For the complementary open inclusion $j\colon U \smallsetminus D \hookrightarrow U$ of a divisor, the localization functor ${}_{\mathrm{H}}j_*\,{}_{\mathrm{H}}j^*$ is simply denoted as $[*D]$ (and the corresponding functor for $\mathscr{D}$-modules by $(*D)$), while the dual localization functor ${}_{\mathrm{H}}j_!\,{}_{\mathrm{H}}j^*$ is denoted by $[!D]$ (and $(!D)$, respectively).

In [31, §4] Kontsevich and Soibelman define the category EMHS of exponential mixed Hodge structure as the full subcategory of $\mathsf{MHM}(\mathbb{A}^1_s)$ consisting of objects whose underlying perverse sheaf has zero global cohomology. There is a projector $\Pi_s\colon \mathsf{MHM}(\mathbb{A}^1_s) \mapsto \mathsf{MHM}(\mathbb{A}^1_s)$ whose image is EMHS. Explicitly,

$$\Pi_s(M^{\mathrm{H}}) = M^{\mathrm{H}} * {}_{\mathrm{H}}j_!\mathscr{O}^{\mathrm{H}}_{\mathbb{G}_{\mathrm{m}}},$$

where $*$ denotes additive convolution and $j\colon \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1$ the inclusion.

B.3. **Fourier transform of an object in EMHS.** Let M^{H} be a mixed Hodge module on $\mathbb{A}^1_s$, with $(M, F_\bullet M, W_\bullet M)$ as associated algebraic bi-filtered $\mathscr{D}_{\mathbb{A}^1_s}$-module. There is a natural morphism $M \to \Pi_s(M)$ in $\mathsf{MHM}(\mathbb{A}^1_s)$ whose kernel and cokernel are constant mixed Hodge modules. Let us denote by $a_{\mathbb{A}^1_s}$ the structure morphism of $\mathbb{A}^1_s$. The de Rham fibre functor on EMHS is given by

$$M^{\mathrm{H}} \longmapsto \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M \otimes E^s) = \mathscr{H}^0 a_{\mathbb{A}^1_s,+}(M \otimes E^s). \tag{B.2}$$

We recall that the functor $\mathsf{Mod}_{\mathrm{holreg}}(\mathscr{D}_{\mathbb{A}^1_s}) \mapsto \mathsf{Vect}_{\mathbf{C}}$ defined by $M \mapsto \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M \otimes E^s)$ is exact and that $\mathrm{H}^k_{\mathrm{dR}}(\mathbb{A}^1_s, M \otimes E^s) = 0$ if $k \neq 1$. It follows trivially that the spectral sequence $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, (W_\bullet M) \otimes E^s) \Rightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M \otimes E^s)$ degenerates at E_1 and any morphism $M_1^{\mathrm{H}} \to M_2^{\mathrm{H}}$ induces a strictly filtered morphism

$$(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M_1 \otimes E^s), W_\bullet) \longrightarrow (\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M_2 \otimes E^s), W_\bullet).$$

On the other hand, the Hodge filtration $F_\bullet M$ of M^{H} gives rise to the irregular Hodge filtration (indexed by $\mathbf{Q}$) $F^{\mathrm{irr}}_\bullet j_{\infty,+}(M \otimes E^s)$ ($j_\infty : \mathbb{A}^1_s \hookrightarrow \mathbb{P}^1_s$), also denoted by $F^{\mathrm{Del}}_\bullet$ in [45, §6].

Proposition B.3. *The spectral sequence*

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}^1_s, F^{\mathrm{irr}}_\bullet j_{\infty,+}(M \otimes E^s)) \Longrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}^1_s, j_{\infty,+}(M \otimes E^s)) = \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M \otimes E^s)$$

degenerates at E_1 and, defining

$$F^{\mathrm{irr}}_\bullet \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M \otimes E^s) := \mathrm{im}\Big[\mathrm{H}^1_{\mathrm{dR}}(\mathbb{P}^1_s, F^{\mathrm{irr}}_\bullet j_{\infty,+}(M \otimes E^s)) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M \otimes E^s)\Big],$$

the functor

$$\mathsf{MHM}(\mathbb{A}^1_s) \ni M^{\mathrm{H}} \longmapsto \big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M \otimes E^s), F^{\mathrm{irr}}_\bullet, W_\bullet\big)$$

factors through Π. *Any morphism in* $\mathsf{MHM}(\mathbb{A}^1_s)$ *(or in* EMHS*) gives rise to a strictly bifiltered morphism.*

Proof. The degeneration of the spectral sequence for $F^{\mathrm{irr}}_\bullet$ is proved in [45, §6] for polarized Hodge modules. The case of mixed Hodge modules is deduced from it in [17, Th. 3.3.1] (in a more general setting). Moreover, any morphism $M_1^{\mathrm{H}} \to M_2^{\mathrm{H}}$ gives rise to a strictly filtered morphism $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M_1 \otimes E^s) \to \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M_2 \otimes E^s)$ independently with respect to $F^{\mathrm{irr}}_\bullet$ and $W_\bullet$ and it follows easily that $\mathrm{gr}^W_\ell \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M_1 \otimes E^s) \to \mathrm{gr}^W_\ell \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M_2 \otimes E^s)$ is strictly filtered with respect to $F^{\mathrm{irr}}_\bullet$, hence the last statement.

Lastly, since the kernel and cokernel of $M^{\mathrm{H}} \to \Pi_s(M^{\mathrm{H}})$ are constant mixed Hodge modules, the second statement is then clear. □

Let $j_0 \colon \mathbb{G}_{\mathrm{m}} \hookrightarrow \mathbb{A}^1_s$ denote the open inclusion, $i_0 \colon \{0\} \hookrightarrow \mathbb{A}^1_s$ the complementary closed inclusion, and let ${}_{\mathrm{H}}i_{0!}$ be the pushforward functor $\mathsf{MHS} \to \mathsf{MHM}(\mathbb{A}^1_s)$. In such a way, we regard MHS as full subcategory of $\mathsf{MHM}(\mathbb{A}^1_s)$, and we have $\phi_{s,1} \circ {}_{\mathrm{H}}i_{0!} \simeq \mathrm{Id}_{\mathsf{MHS}}$.

Let us now consider the composed functor $\Pi_s \circ {}_{\mathrm{H}}i_{0!} : \mathsf{MHS} \to \mathsf{EMHS}$. It realises MHS as a full subcategory of EMHS, and $\phi_{s,1} \circ (\Pi_s \circ {}_{\mathrm{H}}i_{0!}) \simeq \mathrm{Id}_{\mathsf{MHS}}$, since $\phi_{s,1} \circ \Pi_s \simeq \Pi_s$ (because $\phi_{s,1}$ of a constant object in $\mathsf{MHM}(\mathbb{A}^1_s)$ is zero).

Proposition B.4. *Let* M^{H} *be an object of* $\mathsf{MHM}(\mathbb{A}^1_s)$ *such that* $\Pi_s(M^{\mathrm{H}})$ *belongs to* MHS. *Then the bi-filtered vector space* $\big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M \otimes E^s), F^{\mathrm{irr}}_\bullet, W_\bullet\big)$ *is naturally isomorphic to that associated with the mixed Hodge structure* $\phi_{s,1} M^{\mathrm{H}}$.

Notation B.5. If M^{H} is an object of $\mathsf{MHM}(\mathbb{A}^1_s)$ such that $\Pi_s(M^{\mathrm{H}})$ belongs to MHS, we denote by $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M \otimes E^s)^{\mathrm{H}}$ the vector space $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M \otimes E^s)$ endowed with the mixed Hodge structure coming from $\phi_{s,1} M^{\mathrm{H}}$ via the previous isomorphism.

Proof. Let V^{H} be a mixed Hodge structure. The vector space $\big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, i_{0+}V \otimes E^s), F^{\mathrm{irr}}_\bullet, W_\bullet\big)$ endowed with its two filtrations is easily identified with $(V, F_\bullet, W_\bullet)$. Since

$$\big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, \Pi_s(i_{0+}V) \otimes E^s), F^{\mathrm{irr}}_\bullet, W_\bullet\big) \simeq \big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, i_{0+}V \otimes E^s), F^{\mathrm{irr}}_\bullet, W_\bullet\big),$$

we conclude

$$\big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, \Pi_s(i_{0+}V) \otimes E^s), F^{\mathrm{irr}}_\bullet, W_\bullet\big) \simeq (V, F_\bullet, W_\bullet). \tag{B.6}$$

For M^{H} as in Proposition B.4, we have $\Pi_s(M^{\mathrm{H}}) = \Pi_s({}_{\mathrm{H}}i_{0!}V^{\mathrm{H}})$ for some mixed Hodge structure $(V, F_\bullet, W_\bullet)$, and thus $(V, F_\bullet, W_\bullet) \simeq \phi_{s,1}\Pi_s(M^{\mathrm{H}}) \simeq \phi_{s,1} M^{\mathrm{H}}$. Then (B.6) gives

$$\begin{aligned}\phi_{s,1} M^{\mathrm{H}} &\simeq \big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, \Pi_s(i_{0+}\phi_{s,1} M) \otimes E^s), F^{\mathrm{irr}}_\bullet, W_\bullet\big)\\ &\simeq \big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, \Pi_s(M) \otimes E^s), F^{\mathrm{irr}}_\bullet, W_\bullet\big)\\ &\simeq \big(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, M \otimes E^s), F^{\mathrm{irr}}_\bullet, W_\bullet\big).\end{aligned}$$

□

B.4. **A criterion for an object of EMHS to belong to MHS.** Let $f\colon U\to\mathbb{A}^1_s$ be a regular function on a smooth complex quasi-projective variety U of dimension d. Let M_U^{H} be an object of $\mathsf{MHM}(U)$. For each $r\in\mathbf{Z}$, we consider the mixed Hodge modules

$$M_{r*}^{\mathrm{H}}:=\mathscr{H}^{r-d}{}_{\mathrm{H}}f_*M_U^{\mathrm{H}},\quad M_{r!}^{\mathrm{H}}:=\mathscr{H}^{r-d}{}_{\mathrm{H}}f_!M_U^{\mathrm{H}}.$$

They define objects $\Pi_s(M_{r*}^{\mathrm{H}})$ and $\Pi_s(M_{r!}^{\mathrm{H}})$ of EMHS. We denote by $M_{r*},M_{r!}$ the $\mathscr{D}_{\mathbb{A}^1_s}$-modules underlying $M_{r*}^{\mathrm{H}},M_{r!}^{\mathrm{H}}$. We have filtered isomorphisms

$$(\mathrm{H}^r_{\mathrm{dR}}(U,M_U\otimes E^f),F_\bullet^{\mathrm{irr}})\simeq(\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s,M_{r*}\otimes E^s),F_\bullet^{\mathrm{irr}}),\tag{B.7}$$

$$(\mathrm{H}^r_{\mathrm{dR,c}}(U,M_U\otimes E^f),F_\bullet^{\mathrm{irr}})\simeq(\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{A}^1_s,M_{r!}\otimes E^s),F_\bullet^{\mathrm{irr}}).\tag{B.8}$$

This follows from [46, Th. 1.3(4)] applied to a compactification of the morphism f and by taking the pushforward by the structure morphism. We can thus regard $(\mathrm{H}^r_{\mathrm{dR}}(U,M_U\otimes E^f),F_\bullet^{\mathrm{irr}})$ resp. $(\mathrm{H}^r_{\mathrm{dR,c}}(U,M_U\otimes E^f),F_\bullet^{\mathrm{irr}})$ as the filtered vector space attached to the object $\Pi_s(M_{r*}^{\mathrm{H}})$ resp. $\Pi_s(M_{r!}^{\mathrm{H}})$ of EMHS. As such, they are endowed with a filtration $W_\bullet$.

Notation B.9. When the right-hand side below is defined according to notation B.5, we set

$$\mathrm{H}^r_{\mathrm{dR}}(U,M_U\otimes E^f)^{\mathrm{H}}=\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s,M_{r*}\otimes E^s)^{\mathrm{H}},$$
$$\mathrm{H}^r_{\mathrm{dR,c}}(U,M_U\otimes E^f)^{\mathrm{H}}=\mathrm{H}^1_{\mathrm{dR,c}}(\mathbb{A}^1_s,M_{r!}\otimes E^s)^{\mathrm{H}}.$$

We now give a criterion in order that $\Pi_s(M_{r*}^{\mathrm{H}})$ resp. $\Pi_s(M_{r!}^{\mathrm{H}})$ are objects of the subcategory MHS of EMHS, in order to apply Proposition B.4.

Theorem B.10 (see also [55, Th. 3.3], [20, Lem. 6.5.3])**.** *Assume $U=\mathbb{A}^1_t\times V$, $f=tg$ for some regular function $g\colon V\to\mathbb{A}^1$ and let M_V^{H} be an object of* $\mathsf{MHM}(V)$.

(1) *If $M_U^{\mathrm{H}}=\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}\boxtimes M_V^{\mathrm{H}}$, then $\Pi_s(M_{r*}^{\mathrm{H}})$ and $\Pi_s(M_{r!}^{\mathrm{H}})$ are in* MHS *for all r, and*

$$(\mathrm{H}^r_{\mathrm{dR}}(U,M_U\otimes E^f),F_\bullet^{\mathrm{irr}},W_\bullet)\quad\text{and}\quad(\mathrm{H}^r_{\mathrm{dR,c}}(U,M_U\otimes E^f),F_\bullet^{\mathrm{irr}},W_\bullet)$$

underlie the mixed Hodge structures $\mathrm{H}^r_{\mathrm{dR}}(U,M_U\otimes E^f)^{\mathrm{H}}$ and $\mathrm{H}^r_{\mathrm{dR,c}}(U,M_U\otimes E^f)^{\mathrm{H}}$.

(2) *If $M_U^{\mathrm{H}}=\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}[*t]\boxtimes M_V^{\mathrm{H}}$, then for any r, $\Pi_s(M_{r*}^{\mathrm{H}})$ is in* MHS *and*

$$(\mathrm{H}^r_{\mathrm{dR}}(\mathbb{G}_{\mathrm{m}}\times V,(\mathscr{O}_{\mathbb{G}_{\mathrm{m}}}\boxtimes M_V)\otimes E^f),F_\bullet^{\mathrm{irr}},W_\bullet)$$

underlies the mixed Hodge structure $\mathrm{H}^r_{\mathrm{dR}}(U,M_U\otimes E^f)^{\mathrm{H}}$.

(3) *If $M_U^{\mathrm{H}}=\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}[!t]\boxtimes M_V^{\mathrm{H}}$, then for any r, $\Pi_s(M_{r!}^{\mathrm{H}})$ is in* MHS *and*

$$(\mathrm{H}^r_{\mathrm{dR,c}}(\mathbb{G}_{\mathrm{m}}\times V,(\mathscr{O}_{\mathbb{G}_{\mathrm{m}}}\boxtimes M_V)\otimes E^f),F_\bullet^{\mathrm{irr}},W_\bullet)$$

underlies the mixed Hodge structure $\mathrm{H}^r_{\mathrm{dR,c}}(U,M_U\otimes E^f)^{\mathrm{H}}$.

Proof of (1). The last statement follows from Proposition B.4, (B.7) and (B.8).

Let us consider first $\Pi_s(M_{r*}^{\mathrm{H}})$. Set $D=\mathbb{A}^1_t\times g^{-1}(0)$ and $U'=U\smallsetminus D$. The object $\mathscr{M}_{U*}^{\mathrm{H}}:=[M_U^{\mathrm{H}}\to M_U^{\mathrm{H}}[*D]]$ of $\mathsf{D}^{\mathrm{b}}(\mathsf{MHM}(U))$ is supported on $f=0$, hence $\mathscr{H}^r{}_{\mathrm{H}}f_*\mathscr{M}_{U*}^{\mathrm{H}}$, when regarded in EMHS, belongs to MHS for any r, according to Proposition B.4. It is then enough to prove that $\mathrm{H}^r_{\mathrm{dR}}(U,M_U(*D)\otimes E^f)=0$ for any r. By taking pushforward by $(t,g):\mathbb{A}^1_t\times V\to\mathbb{A}^1_t\times\mathbb{A}^1_\tau$ we can reduce the proof to the case when $V = \mathbb{A}^1_\tau$ and $g = \tau$. We then simply denote $M = M_{\mathbb{A}^1_\tau}$, and we are reduced to proving

$$a_{\mathbb{A}^1_t\times\mathbb{A}^1_\tau,+}\bigl((\mathscr{O}_{\mathbb{A}^1_t}\boxtimes M(*0))\otimes E^{t\tau}\bigr)=0.$$

Let $p_t : \mathbb{A}^1_t \times \mathbb{A}^1_\tau \to \mathbb{A}^1_t$ denote the projection. We have $a_{\mathbb{A}^1_t\times\mathbb{A}^1_\tau} = a_{\mathbb{A}^1_t} \circ p_t$. We note that the complex $p_{t,+}\bigl((\mathscr{O}_{\mathbb{A}^1_t}\boxtimes M(*0))\otimes E^{t\tau}\bigr)$ is nothing but the Fourier transform $\mathrm{FT}_\tau(M(*0))$ (in particular it is concentrated in degree zero). Then, identifying a $\mathscr{O}_{\mathbb{A}^1_t}$-module with connection with a $\mathbf{C}[t]$-module with connection, $a_{\mathbb{A}^1_t,+}\mathrm{FT}_\tau(M(*0))$ is represented by the complex

$$\Bigl[\mathrm{FT}_\tau(M(*0))\xrightarrow{\partial_t}\underset{\bullet}{\mathrm{FT}_\tau(M(*0))}\Bigr]\simeq\Bigl[M(*0)\xrightarrow{\tau}\underset{\bullet}{M(*0)}\Bigr]$$

(where the $\bullet$ indicates the degree zero term). This complex is thus quasi-isomorphic to zero.

For $\Pi_s(M^{\mathrm{H}}_{r!})$, we argue similarly. We consider $\mathscr{M}^{\mathrm{H}}_{U!} := [M^{\mathrm{H}}_U[!D] \to M^{\mathrm{H}}_U]$ and note that $\mathscr{H}^r{}_{\mathrm{H}}f_!\mathscr{M}^{\mathrm{H}}_{U!}$, when regarded in EMHS, belongs to MHS for any r. It is then enough to prove that $\mathrm{H}^r_{\mathrm{dR,c}}(U, M_U(!D)\otimes E^f) = 0$ for any r and, by taking proper pushforward by (t, g), we are reduced to proving

$$a_{\mathbb{A}^1_t\times\mathbb{A}^1_\tau,\dagger}\bigl((\mathscr{O}_{\mathbb{A}^1_t}\boxtimes M(!0))\otimes E^{t\tau}\bigr)=0.$$

It is known that the complex $p_{t,\dagger}\bigl((\mathscr{O}_{\mathbb{A}^1_t}\boxtimes M(!0))\otimes E^{t\tau}\bigr)$ is also isomorphic to the Fourier transform $\mathrm{FT}_\tau(M(!0))$ (in particular it is concentrated in degree zero). We have $\mathrm{FT}_\tau(M(!0)) \simeq \iota^+\boldsymbol{D}\,\mathrm{FT}_\tau((\boldsymbol{D}M)(*0))$, where ι is the involution $t \mapsto -t$. Lastly,

$$a_{\mathbb{A}^1_\tau,\dagger}\iota^+\boldsymbol{D}\,\mathrm{FT}_\tau((\boldsymbol{D}M)(*0))\simeq\boldsymbol{D}a_{\mathbb{A}^1_\tau,+}\mathrm{FT}_\tau((\boldsymbol{D}M)(*0))\simeq0$$

by the first part of the proof applied to $\boldsymbol{D}M$. □

Proof of (2). As in (1), we reduce to the case where $V = \mathbb{A}^1_\tau$ and $g = \tau$, so that $f = t\tau$, and we simply denote M^{H}_V by M^{H}. We consider the morphisms in $\mathsf{MHM}(\mathbb{A}^1_t \times \mathbb{A}^1_\tau)$

$$\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}[*t]\boxtimes M^{\mathrm{H}}\longrightarrow\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}[*t]\boxtimes\Pi_\tau(M^{\mathrm{H}})\longleftarrow\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}\boxtimes\Pi_\tau(M^{\mathrm{H}})$$

and their pushforwards

$$\mathscr{H}^r{}_{\mathrm{H}}f_*\bigl(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}[*t]\boxtimes M^{\mathrm{H}}\bigr)\longrightarrow\mathscr{H}^r{}_{\mathrm{H}}f_*\bigl(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}[*t]\boxtimes\Pi_\tau(M^{\mathrm{H}})\bigr)\longleftarrow\mathscr{H}^r{}_{\mathrm{H}}f_*\bigl(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}\boxtimes\Pi_\tau(M^{\mathrm{H}})\bigr).$$

We will prove that, after applying the projector Π_s, they become isomorphisms in EMHS. Since Π_s of the right-hand side belongs to MHS, according to (1), so does Π_s of the left-hand side, as wanted.

For the left morphism, we are reduced to proving that, if M^{H} is a constant mixed Hodge module on $\mathbb{A}^1_\tau$, then the mixed Hodge module $\mathscr{H}^r{}_{\mathrm{H}}f_*\bigl(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}[*t]\boxtimes M^{\mathrm{H}}\bigr)$ is constant. For that purpose, it is enough to prove that the underlying $\mathscr{D}$-module is constant (see [53, Th. 4.20]), and this amounts to proving that its de Rham fibre (B.2) is zero. This fibre is isomorphic to

$$\mathscr{H}^r a_{\mathbb{A}^1_t\times\mathbb{A}^1_\tau,+}\bigl((\mathscr{O}_{\mathbb{A}^1_t}(*t)\boxtimes M)\otimes E^{t\tau}\bigr).$$

Projecting first to $\mathbb{A}^1_t$, we find that

$$p_{t+}\bigl((\mathscr{O}_{\mathbb{A}^1_t}(*t)\boxtimes M)\otimes E^{t\tau}\bigr)\simeq\mathscr{O}_{\mathbb{A}^1_t}(*t)\otimes\mathrm{FT}_\tau(M)=0,$$

since M is constant, as wanted.

For the right morphism, with a similar argument we are reduced to proving that $\mathscr{H}^r f_+\big((\mathscr{O}_{\mathbb{A}^1_t}(*t)/\mathscr{O}_{\mathbb{A}^1_t}) \boxtimes \Pi_\tau(M)\big)$ is constant, i.e., that its de Rham fibre

$$\mathscr{H}^r a_{\mathbb{A}^1_t\times\mathbb{A}^1_\tau,+}\big(((\mathscr{O}_{\mathbb{A}^1_t}(*t)/\mathscr{O}_{\mathbb{A}^1_t}) \boxtimes \Pi_\tau(M)) \otimes E^{t\tau}\big)$$

is zero. This follows by projecting first to $\mathbb{A}^1_t$, since $(\mathscr{O}_{\mathbb{A}^1_t}(*t)/\mathscr{O}_{\mathbb{A}^1_t}) \otimes \mathrm{FT}(\Pi_\tau(M)) = 0$ (because $\mathrm{FT}_\tau(\Pi_\tau(M)) = \mathrm{FT}_\tau(\Pi_\tau(M))(*0)$).

The second part of the statement follows from Proposition B.4 and (B.7). □

Proof of (3). We argue similarly, by replacing $\Pi_\tau(M^{\mathrm{H}})$ with $\Pi_{\tau,!}(M^{\mathrm{H}}) := {}_{\mathrm{H}}\iota^* \boldsymbol{D}\Pi_\tau(\boldsymbol{D}M^{\mathrm{H}})$, whose underlying $\mathscr{D}$-module $\Pi_{\tau,!}(M)$ satisfies $\mathrm{FT}_\tau(\Pi_{\tau,!}(M)) \simeq \mathrm{FT}_\tau M(!0)$. □

B.5. **Computation of the weight and Hodge filtrations.** We now assume that M^{H} is a *pure object of* $\mathsf{MHM}(\mathbb{A}^1_\tau)$ *of weight* w *whose underlying* $\mathscr{D}_{\mathbb{A}^1_\tau}$*-module* M *has no non-zero section supported at the origin.* (In particular, M is an intermediate extension at the origin.) We have an exact sequence in $\mathsf{MHM}(\mathbb{A}^1)$:

$$0 \longrightarrow M^{\mathrm{H}} \longrightarrow \Pi_\tau(M^{\mathrm{H}}) \longrightarrow M'^{\mathrm{H}} \longrightarrow 0,$$

where M'^{H} is a constant mixed Hodge module on $\mathbb{A}^1_\tau$. Recall that, according to [35, App.], M'^{H} has weights $\geqslant w+1$. Let N^{H} be any of the mixed Hodge modules in this exact sequence. Set $f = t\tau$.

Lemma B.11. *Under these conditions, we have* $\Pi_s(\mathscr{H}^r f_+(\mathscr{O}_{\mathbb{A}^1_t} \boxtimes N)) = 0$ *for* $r \neq 0$.

Proof. Indeed, proving this vanishing property is equivalent to proving the vanishing of the de Rham fibre

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, \mathscr{H}^r f_+(\mathscr{O}_{\mathbb{A}^1_t} \boxtimes N) \otimes E^s) \simeq \mathrm{H}^{r+1}_{\mathrm{dR}}(\mathbb{A}^1_t \times \mathbb{A}^1_\tau, (\mathscr{O}_{\mathbb{A}^1_t} \boxtimes N) \otimes E^{t\tau}),$$

and we interpret it in terms of the Fourier transform $\mathrm{FT}_\tau N$ of N, as

$$\mathrm{H}^{r+1}_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau N).$$

This is the cohomology of the complex

$$[\underset{\bullet}{\mathrm{FT}_\tau} N \xrightarrow{\partial_t} \mathrm{FT}_\tau N] \simeq [N \xrightarrow{\tau} N].$$

We are thus left to proving that N has no submodule supported at the origin. This property clearly holds for $N = M$ and $N = M'$, so it also holds for $N = \Pi_\tau(M)$. □

We thus have, for N as above,

$$\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau N) \simeq \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t \times \mathbb{A}^1_\tau, (\mathscr{O}_{\mathbb{A}^1_t} \boxtimes N) \otimes E^{t\tau}) \simeq \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_s, \mathscr{H}^0 f_+(\mathscr{O}_{\mathbb{A}^1_t} \boxtimes N) \otimes E^s)$$

and, according to Proposition B.4, the mixed Hodge structure defined by Theorem B.10(1) on the second term is isomorphic to the mixed Hodge structure on the space

$$\phi_{s,1}\mathscr{H}^0{}_{\mathrm{H}}f_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes N^{\mathrm{H}}).$$

In such a way, $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau N)$ underlies a mixed Hodge structure $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau N)^{\mathrm{H}}$. We thus get an exact sequence of mixed Hodge structures

$$0 \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau M)^{\mathrm{H}} \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau \Pi_\tau(M))^{\mathrm{H}} \longrightarrow \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau M')^{\mathrm{H}} \longrightarrow 0.$$

Proposition B.12.

(1) *We have* $\dim \mathrm{gr}_F^p \mathrm{gr}_\ell^W \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau M') = \mathrm{rk}\, \mathrm{gr}_F^{p-1} \mathrm{gr}_{\ell-1}^W M'$, *for all* $\ell, p \in \mathbf{Z}$.

(2) *If* 0 *is not a singular point of* M, $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau M)^{\mathrm{H}}$ *is pure of weight* $w+1$ *and* $\dim \mathrm{gr}_F^p \mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau M) = \mathrm{rk}\, \mathrm{gr}_F^{p-1} M$.

Let us first introduce the general construction which enables us to compute the weights. We can compactify the map $f = t\tau$ by considering the blowing up $e\colon \widetilde{\mathbb{P}} \to \mathbb{P}^1_t \times \mathbb{P}^1_\tau$ of the points $(\infty, 0)$ and $(0,\infty)$ as described in Figure 2. We have set $E = e^{-1}(0,\infty)$ and $F = e^{-1}(\infty, 0)$. The thick lines represent $\widetilde{f}^{-1}(0)$, where we have set $\widetilde{f} = f \circ e$. Let $\widetilde{\jmath}\colon \mathbb{A}^1_t \times \mathbb{A}^1_\tau \hookrightarrow \widetilde{\mathbb{P}}$ denote

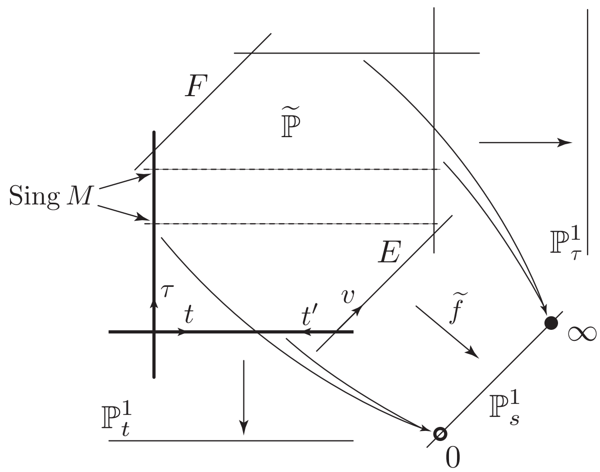

FIGURE 2. The map $\widetilde{f} := e \circ f$

the inclusion. The singularities of ${}_{\mathrm{H}}\widetilde{\jmath}_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes N^{\mathrm{H}})$ consist of all horizontal lines in Figure 2 except possibly $\tau = 0$ (if 0 is not a singularity for M) together with E, F and $\widetilde{f}^{-1}(\infty)$. This singularity set is a strict normal crossing divisor. Since e is proper, vanishing cycles commute with ${}_{\mathrm{H}}e_*$, that is,

$$\phi_{s,1} \mathscr{H}^0 {}_{\mathrm{H}}f_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes N^{\mathrm{H}}) \simeq \mathscr{H}^0 {}_{\mathrm{H}}\widetilde{f}_*\big(\phi_{\widetilde{f},1} {}_{\mathrm{H}}\widetilde{\jmath}_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes N^{\mathrm{H}})\big). \tag{B.13}$$

We will first compute $\phi_{\widetilde{f},1}\big({}_{\mathrm{H}}\widetilde{\jmath}_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes N^{\mathrm{H}})\big)$, which is a priori supported on $\widetilde{f}^{-1}(0)$. An easy computation shows that it is in fact supported on the horizontal part of $\widetilde{f}^{-1}(0)$, that we denote by $\widetilde{\mathbb{P}}^1_t \simeq \mathbb{P}^1_t$ and which is covered by two charts, denoted by $\mathbb{A}^1_t$ and $\mathbb{A}^1_{t'}$. Moreover, since M'^{H} is constant, $\phi_{\widetilde{f},1}\big({}_{\mathrm{H}}\widetilde{\jmath}_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes M'^{\mathrm{H}})\big)$ is supported at the origin $t = 0$.

Lemma B.14. *We have* $\phi_{\widetilde{f},1}\big({}_{\mathrm{H}}\widetilde{\jmath}_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes N^{\mathrm{H}})\big) = \phi_{\widetilde{f},1}\big({}_{\mathrm{H}}\widetilde{\jmath}_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes N^{\mathrm{H}})\big)[*\infty]$.

Proof. By the support property, the lemma obviously holds if $N^{\mathrm{H}} = M'^{\mathrm{H}}$. We can thus assume that $N^{\mathrm{H}} = M^{\mathrm{H}}$. Since we know that the objects are in $\mathsf{MHM}(\mathbb{P}^1_t)$,[4] it is enough to prove the assertion for the underlying $\mathscr{D}$-modules.

[4]We implicitly identify $\mathsf{MHM}_{\mathbb{P}^1_t}(\widetilde{\mathbb{P}})$ with $\mathsf{MHM}(\mathbb{P}^1_t)$.

In a small analytic chart U^{an} of $\widetilde{\mathbb{P}}$ with coordinates (t', v), the function $\widetilde{f}$ is the projection $(t', v) \mapsto v$ and t' and v act in an invertible way on $\widetilde{\jmath}_+(\mathscr{O}_{\mathbb{A}^1_t} \boxtimes M)_{|U^{\mathrm{an}}}$, which corresponds thus to a meromorphic flat bundle with poles in $t'v = 0$. The result is then clear. □

It follows that

$$\phi_{\widetilde{f},1}\big({}_{\mathrm{H}}\widetilde{\jmath}_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes N^{\mathrm{H}})\big) = {}_{\mathrm{H}}j_*\phi_{\widetilde{f},1}\big({}_{\mathrm{H}}\widetilde{\jmath}_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes N^{\mathrm{H}})\big)_{|\mathbb{A}^1_t}, \quad j\colon \mathbb{A}^1_t \hookrightarrow \mathbb{P}^1_t.$$

Moreover, the only singularity of $\phi_{\widetilde{f},1}\big({}_{\mathrm{H}}\widetilde{\jmath}_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes N^{\mathrm{H}})\big)_{|\mathbb{A}^1_t}$ occurs at $t = 0$.

Proof of Proposition B.12. Let us set $N^{\mathrm{H}} = M'^{\mathrm{H}}$ resp. M^{H}. In any case, N^{H} is smooth in the neighbourhood of $\tau = 0$: this holds because M'^{H} is constant for (1), and by our assumption for (2). Hence so is $\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes N^{\mathrm{H}}$, and its weights are $\geqslant w + 2$ resp. equal to $w + 1$. One can then check that $\phi_{\widetilde{f},1}\big({}_{\mathrm{H}}\widetilde{\jmath}_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes N^{\mathrm{H}})\big)$ is supported at $(0, 0)$. We are therefore led to computing the vanishing cycles (at $(0, 0)$) of the non-degenerate quadratic function $t\tau$ in dimension two, with coefficients in a variation of mixed Hodge structure of weights $\geqslant w + 2$ resp. a variation of pure Hodge structure of weight $w + 1$. It is standard that the weights and Hodge numbers are as described in the proposition. We now apply (B.13), which is tautological due to the support property of $\phi_{\widetilde{f},1}\big({}_{\mathrm{H}}\widetilde{\jmath}_*(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t} \boxtimes N^{\mathrm{H}})\big)$. □

We now describe the mixed Hodge structure on $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau M)^{\mathrm{H}}$ when we do not make the assumption of Proposition B.12(2). We denote by $(\psi_{\tau,1}M^{\mathrm{H}}, \mathrm{N}_\tau)$ the nearby cycle mixed Hodge structure of M^{H} at $\tau = 0$ for the eigenvalue one of the monodromy T_τ, endowed with the nilpotent operator $\mathrm{N}_\tau\colon \psi_{\tau,1}M^{\mathrm{H}} \to \psi_{\tau,1}M^{\mathrm{H}}(-1)$ such that $\exp -2\pi \mathrm{i}\mathrm{N}_\tau = \mathrm{T}_\tau$. By our assumption, the Hodge nearby/vanishing cycle quiver of M^{H} at $\tau = 0$ is isomorphic to

$$\psi_{\tau,1}M^{\mathrm{H}} \underset{\underset{(-1)\ \mathrm{incl}}{\longleftarrow}}{\overset{\mathrm{N}_\tau}{\longrightarrow}} \operatorname{im}\mathrm{N}_\tau = \phi_{\tau,1}M^{\mathrm{H}}$$

where we regard incl as a morphism $\operatorname{im}\mathrm{N}_\tau \hookrightarrow \psi_{\tau,1}M^{\mathrm{H}}(-1)$ of mixed Hodge structures.

Theorem B.15. *The mixed Hodge structure* $\mathrm{H}^1_{\mathrm{dR}}(\mathbb{A}^1_t, \mathrm{FT}_\tau M)^{\mathrm{H}}$ *is isomorphic to*

$$\operatorname{coker}\mathrm{N}_\tau : \psi_{\tau,1}M^{\mathrm{H}} \longrightarrow \psi_{\tau,1}M^{\mathrm{H}}(-1).$$

Remark B.16. If 0 is not a singular point of M, then $\mathrm{N}_\tau = 0$ and $\psi_{\tau,1}M^{\mathrm{H}}$ is pure of weight $w - 1$. We thus recover Proposition B.12(2).

Proof of Theorem B.15. Let us describe the structure of the $\mathscr{D}_{\mathbb{P}^1_t}$-module $\phi_{\widetilde{f},1}\big(\widetilde{\jmath}_+(\mathscr{O}_{\mathbb{A}^1_t} \boxtimes M)\big)$ when 0 is a possible singularity of M. According to Lemma B.14 and to the fact that the only singularity on $\mathbb{A}^1_t$ is zero, we obtain that $\phi_{\widetilde{f},1}\big(\widetilde{\jmath}_+(\mathscr{O}_{\mathbb{A}^1_t} \boxtimes M)\big)$ is completely determined by its nearby/vanishing cycle quiver at $t = 0$.

Lemma B.17. *We have* $\psi_{t,\neq 1}\phi_{\widetilde{f},1}\big(\widetilde{\jmath}_{+}(\mathscr{O}_{\mathbb{A}^1_t}\boxtimes M)\big)=0$ *and an isomorphism of quivers*

$$\psi_{t,1}\phi_{\widetilde{f},1}(\mathscr{O}_{\mathbb{A}^1_t}\boxtimes M)\ \underset{\mathrm{var}}{\overset{\mathrm{can}}{\rightleftarrows}}\ \phi_{t,1}\phi_{\widetilde{f},1}(\mathscr{O}_{\mathbb{A}^1_t}\boxtimes M)\ \simeq\ \operatorname{im}\mathrm{N}_\tau\ \underset{\mathrm{N}_\tau}{\overset{\mathrm{incl}}{\rightleftarrows}}\ \psi_{\tau,1}M$$

where the nilpotent operators N_t *on the left-hand quiver correspond to those induced by* N_τ *on the right-hand quiver.*

Proof. By our assumption on M, $\mathscr{O}_{\mathbb{A}^1_t}\boxtimes M$ is an intermediate extension along the divisor $\widetilde{f}^{-1}(0)$, and if we compute $\psi_{\widetilde{f},1}(\mathscr{O}_{\mathbb{A}^1_t}\boxtimes M)$ together with its nilpotent operator $\mathrm{N}_{\widetilde{f}}$ we will deduce $\phi_{\widetilde{f},1}(\mathscr{O}_{\mathbb{A}^1_t}\boxtimes M)$ as $\operatorname{im}(\mathrm{N}_{\widetilde{f}})$ with nilpotent operator induced by $\mathrm{N}_{\widetilde{f}}$.

The nearby/vanishing cycle quiver for $\psi_{\widetilde{f},1}(\mathscr{O}_{\mathbb{A}^1_t}\boxtimes M)$ can be computed with [48, Th. 3.4], together with the corresponding nilpotent endomorphisms, and the computation of the quiver $\operatorname{im}(\mathrm{N}_{\widetilde{f}})$ gives the right-hand quiver in the lemma. ☐

Let us continue with Theorem B.15. The results of [48, §3.18] imply then that the MHS quiver at $t=0$ for $\phi_{\widetilde{f},1}(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}\boxtimes M^{\mathrm{H}})$ is isomorphic to the quiver

$$\operatorname{im}\mathrm{N}_\tau\ \underset{\mathrm{N}_\tau\ (-1)}{\overset{\mathrm{incl}}{\rightleftarrows}}\ \psi_{\tau,1}M^{\mathrm{H}}(-1)$$

where we regard N_τ as a morphism $\psi_{\tau,1}M^{\mathrm{H}}(-1)\to\operatorname{im}\mathrm{N}_\tau(-1)$.

We will set for short $\phi^{\mathrm{H}}_{\widetilde{f},1}=\phi_{\widetilde{f},1}(\mathscr{O}^{\mathrm{H}}_{\mathbb{A}^1_t}\boxtimes M^{\mathrm{H}})$ and $\phi_{\widetilde{f},1}=\phi_{\widetilde{f},1}(\mathscr{O}_{\mathbb{A}^1_t}\boxtimes M)$ (the underlying $\mathscr{D}_{\mathbb{A}^1_t}$-module). Recall that $j_0\colon\mathbb{G}_{\mathrm{m}}\hookrightarrow\mathbb{A}^1_t$ and $i_0\colon\{0\}\hookrightarrow\mathbb{A}^1$ denote the complementary inclusions.

Lemma B.18. *We have* $\mathrm{H}^r_{\mathrm{dR}}(\mathbb{A}^1_t, j_{0\dagger}j_0^+\phi_{\widetilde{f},1})=0$ *for all* r *and the morphism* ${}_{\mathrm{H}}j_{0!\mathrm{H}}j_0^*\phi^{\mathrm{H}}_{\widetilde{f},1}\to\phi^{\mathrm{H}}_{\widetilde{f},1}$ *is injective, with cokernel isomorphic to* ${}_{\mathrm{H}}i_{0*}\operatorname{coker}\mathrm{N}^{\mathrm{H}}_\tau$.

Proof. The first statement is due to the fact that $\phi_{\widetilde{f},1}$ has a singularity at 0 at most. By the regularity property, one can reduce to the statement that, for a locally constant sheaf L on $\mathbf{C}^*$, we have $\mathrm{H}^r(\mathbf{C}, j_{0!}L)=0$ for any r, which is standard. For the second statement, since we know that the cone of the morphism is supported at the origin, it is enough to consider the morphism induced between the nearby/vanishing cycle quivers

$$\begin{array}{ccc}\operatorname{im}\mathrm{N}_\tau & = & \operatorname{im}\mathrm{N}_\tau\\ {\scriptstyle \mathrm{N}_\tau\,(-1)}\uparrow\ \big\|\ & & {\scriptstyle \mathrm{N}_\tau\,(-1)}\uparrow\ \downarrow{\scriptstyle\mathrm{incl}}\\ \operatorname{im}\mathrm{N}_\tau & \hookrightarrow & \psi_{\tau,1}M^{\mathrm{H}}(-1)\end{array}$$

whose cokernel is the quiver defining $i_{0+}\operatorname{coker}\mathrm{N}_\tau$. ☐

The theorem now directly follows from the lemma. ☐

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