CHAPTER 14

POLARIZABLE HODGE MODULES AND THEIR DIRECT IMAGES

Summary. This chapter contains the definition of polarizable Hodge modules. The actual presentation justifies the introduction of the language of triples. The main properties are abelianity and semi-simplicity of the category of polarizable pure Hodge modules of weight \( w \). It is convenient to also introduce polarizable Hodge-Lefschetz modules, as they appear in many intermediate steps of various proofs, due to the very definition of a polarizable Hodge module. We also give the proof of one of the two main important results concerning polarizable Hodge modules, namely, the decomposition theorem. The proof of the structure theorem will be given in Chapter 15. Here, we will use the machinery of filtered \( \mathcal{D} \)-module theory and sesquilinear pairings to reduce the proof to the case of the map from a compact Riemann surface to a point, that we have analyzed in Chapter 7, according to the results of Schmid and Zucker developed in Chapter 6. This strategy justifies the somewhat complicated and recursive definition of the category \( \mathbf{pHM}(X, w) \) of polarizable Hodge modules.

14.1. Introduction

Polarizable Hodge modules on a complex analytic manifold \( X \) are supposed to play the role of polarizable Hodge structures with a multi-dimensional parameter. These objects can acquire singularities. The way each characteristic property of a Hodge structure is translated in higher dimension of the parameter space is given by the table below.

<table>
<thead>
<tr>
<th>dimension 0</th>
<th>dimension ( n \geq 1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathcal{K} ) a ( \mathbb{C} )-vector space</td>
<td>( \mathcal{M} ) a holonomic ( \mathcal{D} )-module</td>
</tr>
<tr>
<td>( F^* \mathcal{K} ) a filtration</td>
<td>( F^* \mathcal{M} ) a coherent filtration</td>
</tr>
<tr>
<td>( \mathcal{K} = R_f \mathcal{H} )</td>
<td>( \mathcal{M} = R_f \mathcal{M} )</td>
</tr>
<tr>
<td>( H = (\mathcal{K}, \mathcal{K}^\sigma, s) ) a graded triple of ( \mathbb{C}[z] )-vector spaces</td>
<td>( M = (\mathcal{M}', \mathcal{M}'', s) ) a graded triple of ( R_f \mathcal{D} )-modules</td>
</tr>
<tr>
<td>( S : H \to H^*(-w) ) a polarization</td>
<td>( S : M \to M^*(-w) ) a polarization</td>
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Why choosing holonomic \( \mathcal{D} \)-modules as analogues of \( \mathbb{C} \)-vector space? The reason is that the category of holonomic \( \mathcal{D} \)-modules is Artinian, that is, any holonomic
A \( \mathcal{D} \)-module has finite length (locally on the underlying manifold). A related reason is that its deRham complex has constructible cohomology, generalizing the notion of local system attached to a flat bundle. Moreover, the property of holonomicity is preserved by various operations (proper pushforward, pullback by a holomorphic map), and the nearby/vanishing cycle theory (the \( V \)-filtration) is well-defined for holonomic \( \mathcal{D} \)-modules without any other assumption, so that the issue concerning nearby/vanishing cycles of filtered holonomic \( \mathcal{D} \)-modules only comes from the filtration.

In order to define the Hodge properties, we use the same method as in dimension one (see Chapter 7):

- as in Section 7.4.a, we work in the ambient abelian category \( \mathcal{D} \)-Triples(\( X \)), which has been defined in Section 12.7;
- the definition of the category \( \text{pHM}(X, w) \) is obtained by induction on the dimension of the support of the triples entering the definition; contrary to dimension one, many steps may be needed before reaching the case of polarizable Hodge structures.

The definition of a polarizable Hodge module can look frightening: in order to check that an object \( M = (\tilde{M}', \tilde{M}'', s) \) belongs to \( \text{HM}(X, w) \), we have to consider in an inductive way nearby cycles with respect to all germs of holomorphic functions.

That the category of polarizable Hodge modules is non-empty is a non-trivial fact. Already, it is not obvious at all that \( \mathcal{O}_X \) underlies a polarizable Hodge module when \( \dim X \geq 2 \). The reason is that the definition involves considering nearby and vanishing cycles along any germ of holomorphic function, whose singularities can be arbitrarily complicated. In dimension one, holomorphic functions are simply powers of coordinates, and this explains why the property is easier to check. The higher-dimensional case will be proved in Corollary 14.3.5.

The question should however be considered the other way round. Once we know at least one polarizable Hodge module, we automatically know an infinity of them, by considering (monodromy-graded) nearby or vanishing cycles with respect to any holomorphic function. Moreover, once we have proved that a polarizable variation of Hodge structure is a pure Hodge module, we obtain many such objects by applying the pushforward by any projective morphism, by the Hodge-Saito theorem 14.3.1.

In the same vein, due to this inductive definition, the proof of many properties of polarizable Hodge modules can be done by induction on the dimension of the support, and this reduces to checking the property for polarizable Hodge structures.

### 14.2. Definition and properties of polarizable Hodge modules

The notion of a polarizable Hodge module will be defined by induction on the dimension of the support, and we will make extensive use of the properties of the abelian category \( \mathcal{D} \)-Triples(\( X \)) introduced in Section 12.7, in particular the definitions relative to nearby/vanishing cycles (Section 12.7.11). We mimic the definitions in dimension one.
14.2.1. Definition (of a polarizable Hodge module of weight $w$)

The category $\text{pHM}(X, w)$ of polarizable Hodge modules of weight $w$ on $X$ is the full subcategory of $\mathcal{D}_X$-Triples($X$) whose objects $\mathcal{T}$ are coherent holonomic and for which there exists a morphism $S: \mathcal{T} \to \mathcal{T}^*(-w)$ such that $(\mathcal{T}, S)$ is a polarized Hodge module of weight $w$ on $X$.

We will denote by $M$ a triple which is a polarizable Hodge module and by $\text{pHM}(X, w)$ the full subcategory of the category of holonomic coherent filtered $\mathcal{D}_X$-triples whose objects are polarizable Hodge modules of weight $w$. The definition below has to be understood in an inductive way, with respect to the dimension of the support of a triple.

14.2.2. Definition (of a polarized Hodge module of weight $w$)

Let $\mathcal{T}$ be an object of $\mathcal{D}_X$-Triples($X$) which is coherent holonomic, and let $S: \mathcal{T} \to \mathcal{T}^*(-w)$ be a morphism ($w \in \mathbb{Z}$).

(0) If $\dim \text{Supp} \mathcal{T} = 0$, we say that $(\mathcal{T}, S)$ is a polarized Hodge module of weight $w$ on $X$ if

$$(\mathcal{T}, S) \simeq \bigoplus_{x \in \text{Supp} \mathcal{T}} \omega_x(H_x, S_x),$$

where each $(H_x, S_x)$ is a polarized Hodge structure of weight $w$.

(>0) For $d \geq 1$, assume we have defined polarized Hodge module of weight $w$ having support of dimension $< d$. and let $(\mathcal{J}, S)$ be such that $\dim \text{Supp} \mathcal{J} = d$. We say that $(\mathcal{J}, S)$ is a polarized Hodge module of weight $w$ on $X$ if $\mathcal{J}$ is strict and for any open set $U \subset X$ and any holomorphic function $g: U \to \mathbb{C}$,

(1) $\mathcal{J}$ is strictly $R$-specializable along $(g)$

(2) if moreover $g^{-1}(0) \cap \text{Supp} \mathcal{J}$ has everywhere codimension one in $\text{Supp} \mathcal{J}$, then for every $\ell \geq 0$ and every $\lambda \in S^1$,

(a) $\text{Pr}_{\psi, \lambda}(\mathcal{J}, S)$ is a polarized Hodge module of weight $w - 1$ on $U$,

(b) $\text{Pr}_{\phi, \lambda}(\mathcal{J}, S)$ is a polarized Hodge module of weight $w$ on $U$.

(See (12.7.14*) for the objects considered in (2).) Objects of $\text{pHM}(X, w)$ can be represented either by left or right triples, by using the corresponding definition for the functors in the left or right case.

Note that, by the strictness assumption, $\mathcal{M}', \mathcal{M}''$ correspond to coherent filtered holonomic $\mathcal{D}_X$-modules $(\mathcal{M}', \mathcal{F}_\ast \mathcal{M}')$ and $(\mathcal{M}'', \mathcal{F}_\ast \mathcal{M}'')$.

14.2.3. Remarks. Let us emphasize already some properties that will be proved in Theorem 14.2.17 below, or be a consequence of this theorem.

(1) The restriction on $g$ in (2) can be relaxed, and in fact (2) holds for any $g$.

(2) The morphism $S$, that we call a polarization of $\mathcal{T}$ is in fact a pre-polarization of weight $w$ of the triple $\mathcal{T}$, that is, is a Hermitian isomorphism.

(3) If Properties 14.2.2(1) and (2) are satisfied, then so are 14.2.2(1) and (2) for any $r \geq 2$. This follows from Section 12.7.19.
(4) If \((\mathcal{T}, S)\) satisfies (1)_g, (2a)_g and is a middle extension along \((g)\), then it also satisfies (2b)_g. This follows from the vanishing cycle theorem 14.2.22.

14.2.4. Hermitian duality. Hermitian duality in \(\mathcal{D}\)-Triples(X) exchanges \(\mathrm{pHM}(X, w)\) with \(\mathrm{pHM}(X, -w)^{op}\).

14.2.5. Tate twist. The Tate twist \((\ell)\) in \(\mathcal{D}\)-Triples(X) sends the category \(\mathrm{pHM}(X, w)\) to \(\mathrm{pHM}(X, w + 2\ell)\). More precisely, if \(S\) is a polarization of \(M\), then \((-1)^{\ell}S\) is a polarization of \(M(\ell)\).

14.2.6. Strictness of \(N\). We also note that, for an object \(M\) of \(\mathrm{pHM}(X, w)\) and for any function \(g : U \to \mathbb{C}\) such that \(g^{-1}(0) \cap \text{Supp } M\) has everywhere codimension one in \(\text{Supp } M\), the morphism \(N\) is strict on \(\psi_{g, \lambda} M\) and \(\phi_{g, 1} M\): this follows from Proposition 9.4.9. We will relax below the restriction on \(g\).

14.2.7. Stability by direct sums and isomorphisms. The category \(\mathrm{pHM}(X, w)\) is stable by direct sums in \(\mathcal{D}\)-Triples(X): this is clear for polarizable Hodge structures of weight \(w\) in the category \(\mathcal{C}\)-Triples (see Section 5.2), and the general case follows by induction on the dimension of the support. Similarly, we obtain that any object of \(\mathcal{D}\)-Triples(X) which is isomorphic of an object of \(\mathrm{pHM}(X, w)\) is an object of \(\mathrm{pHM}(X, w)\).

14.2.8. Stability by direct summands. The category \(\mathrm{pHM}(X, w)\) is stable by direct summand in \(\mathcal{D}\)-Triples(X). More precisely, if \(M = \mathcal{T}_1 \oplus \mathcal{T}_2\) is in \(\mathrm{pHM}(X, w)\) and if \(S\) is a polarization of \(M\), then \(\mathcal{T}_1, \mathcal{T}_2\) are in \(\mathrm{pHM}(X, w)\) and \(S\) induces a polarization on each of them. Indeed, the property of coherence and holonomicity restricts to direct summands, as well as strictness and the property of strict \(\mathbb{R}\)-specializability along any \(g\) (see Exercise 9.22(1)). We then argue by induction on the dimension of the support, the case of dimension zero reducing to Lemma 5.2.8 and Exercise 2.11(1). If the support has dimension \(\geq 1\), let \(S_1\) the morphism \(\mathcal{T}_1 \to \mathcal{T}_1(-w)\) induced by \(S\). Then, for any \(g\) such that \(g^{-1}(0) \cap \text{Supp } M\) has everywhere codimension one in \(\text{Supp } M\), \(\text{Pr}_!\psi_{g, \lambda} S\) induces \(\text{Pr}_!\psi_{g, \lambda} S_{1}\) on \(\text{Pr}_!\psi_{g, \lambda} M_{1}\), and this is a polarization by the induction hypothesis. A similar property holds for \(\phi_{g, 1}\), showing that \((\mathcal{T}_1, S_1)\) satisfies \((2)_g\).

14.2.9. Proposition (Kashiwara’s equivalence). Let \(Z \hookrightarrow X\) be a closed analytic submanifold of the analytic manifold \(X\). The functor \(\iota_*\), induces an equivalence between \(\mathrm{pHM}(Z, w)\) and \(\mathrm{pHM}_Z(X, w)\) (objects supported on \(Z\)).

\textbf{Proof.} Full faithfulness follows from Section 12.7.24. It follows that essential surjectivity is a local question, and more precisely, if essential surjectivity holds locally for polarized objects \((M, S)\), it holds globally. In the local setting, we can argue by induction and assume that \(Z = H\) is a smooth hypersurface. Then Proposition 9.6.6 (and its obvious variant for sesquilinear pairings, hence for objects in \(\mathcal{D}\)-Triples(X)), implies the assertion by induction on \(\dim X\). \(\Box\)
14.2.10. Proposition (Generic structure of polarizable Hodge modules)

Let $M$ be an object of $\text{pHM}(X, w)$ with support on the irreducible closed analytic set $Z \hookrightarrow X$. Then there exists an open dense set $Z^0 \subset Z$ and a smooth Hodge triple $H$ of weight $w$ on $Z^0$, such that $M|_{Z^0} = +_{t=0} H$. In particular, if $Z = X$, then $M_{|X^o}$ is a smooth Hodge triple of weight $w$.

Note that we use Definition 5.4.7 for a smooth Hodge triple, in order to have an object similar to the object $M$. By definition, it corresponds to a polarizable variation of Hodge structure of weight $w - \dim Z$ on $Z^0$.

**Proof.** Set $M = (\tilde{M}', \tilde{M}'', s)$ and let $S$ be a polarization. We first restrict to the smooth locus of $Z$ and apply Kashiwara’s equivalence (14.2.9) to reduce to the case when $Z = X$. By Corollary 9.7.13 (that we can apply since $M$ is strict), there exists a dense open subset $X^0$ of $X$ such that $\tilde{M}'_{|X^0}$ and $\tilde{M}''_{|X^0}$ are $\mathcal{O}_X$-locally free of finite rank. Then $s_{X^0}$ takes values in $\tilde{\mathcal{E}}_{X^0}$ (see Lemma 12.3.6). We now restrict to $X^0$ and argue by induction on $\dim X$. It will be convenient to use the left setting.

Let $t$ be a local coordinate and set $H = \{t = 0\}$. We have seen in the proof of Proposition 9.7.10 that $\gr^0_v \tilde{M} = \tilde{M}/t\tilde{M}$ for $\tilde{M} = \tilde{M}', \tilde{M}''$. After Remark 12.5.14 and Example 12.5.12, $\gr^0_v s$ is the restriction of $s$ to $t = 0$ as a $C^\infty$ function. We conclude that $\tilde{\psi}_{t, 1}M$ is the pushforward $+_{t=0}^{-1}M_{|t=0}$. It is also pure of weight $w - 1$ since $N$ is easily seen to be zero. Therefore, $M_{|t=0}$ is pure of weight $w - 1$ and, by induction on $\dim X$, is a smooth Hodge triple of weight $w - 1$. Since this holds for any $H$ and since $\tilde{M}', \tilde{M}''$ are $\mathcal{O}_X$-locally free, it is clear that $M$ is a smooth Hodge triple of weight $w$.

A similar argument shows that $S$ is a polarization of this smooth Hodge triple. \(\Box\)

14.2.11. Caveat. At this point, we do not know the converse property that a polarizable smooth Hodge triple of weight $w$ on $X$ is an object of $\text{pHM}(X, w)$, since we have not checked that $(2)_g$ holds for any nonzero $g$ for such a triple. This will be done in Corollary 14.3.5.

14.2.b. Abelianity and the $S$-decomposition theorem

Before proving the main properties of $\text{pHM}(X, w)$, we introduce other categories which will prove useful at some intermediate steps.

14.2.12. The category of $W$-filtered Hodge modules. As a first approximation of the category of mixed Hodge modules, we consider the category $\text{WHM}(X)$: this is the full subcategory of $\mathcal{W}\mathcal{D}$-Triples$(X)$ (see Section 2.6.b) such that, for each object $(\tilde{T}, W, \tilde{T})$, the graded object $\gr^W_v \tilde{T}$ belongs to $\text{pHM}(X, t)$. We can regard each $\text{pHM}(X, t)$ as a full subcategory of $\text{WHM}(X)$ by considering on $M$ the filtration $W$, which jumps at $w$ only.

14.2.13. The category of polarizable Hodge-Lefschetz modules. We also consider the category $\text{pHLM}(X, w)$ of polarizable Hodge-Lefschetz modules centered at $w$. An object in this category consists of a pair $(\tilde{T}, N)$, where $\tilde{T}$ is an object of $\mathcal{D}$-Triples$(X)$ and $N$
is a nilpotent endomorphism of $\tilde{T}$, such that there exists a morphism $S : (\tilde{T}, N) \to (\tilde{T}, N)^*(-w)$ satisfying

- $(P_t \tilde{T}, P_t S)$ is a polarized Hodge module of weight $w + \ell$ for every $\ell \geq 0$, where $P_t S$ is the morphism $P_t : P_t \tilde{T} \to (P_t \tilde{T})^*(-(w + \ell))$.

We also say that $(\tilde{T}, N, S)$ is a polarized Hodge-Lefschetz module centered at $w$. From the Lefschetz decomposition, we deduce that, setting $W_{\gamma} \tilde{T} := M_{(k-w)} \tilde{T}$ (i.e., $M_\ell = W_{w+\ell}$), $(\tilde{T}, W_\bullet)$ is an object of WHM($X$) (but pHLM($X, w$) is not a full subcategory of WHM($X$), since morphisms have to commute with $N$).

14.2.14. Caveat. We do not claim that objects and morphisms in WHM($X$) or pHLM($X, w$) are strictly specializable along any $(g)$. On the other hand, objects and morphisms in the graded category pHLM($X, w; \varepsilon$) defined below are so, since they are graded with respect to the weight or monodromy filtration.

14.2.15. The category of polarizable Hodge-Lefschetz quivers. In a way similar to that of Definition 3.2.25, we also define the notion of polarized/polarizable Hodge-Lefschetz quiver centered at $w$, starting from a Lefschetz quiver in $\tilde{D}$-Triples($X$): such an object consists of the data $((M, N, S), (M_1, N_1, S_1), c, v)$, where the first terms are polarized Hodge-Lefschetz modules centered at $w - 1$ and $w$ respectively, and $c : M_1 \to M$ and $v : M_1 \to M(-1)$ are morphisms in $\tilde{D}$-Triples($X$) such that $v \circ c = N$, $c \circ v = N_1$ and the following diagram commutes (see Definition 3.2.25(1)):

$$
\begin{array}{ccc}
M_1 & \xrightarrow{S_1} & M_1^*(-w) \\
\downarrow v & & \downarrow c^* \\
M(-1) & \xrightarrow{S} & M'^*(-w)
\end{array}
$$

The corresponding category is denoted by pHLQ($X, w$).

We can rephrase Condition (2)$'_g$ of Theorem 14.2.2 as follows:

(2)$'_g$ if moreover $g^{-1}(0) \cap \text{Supp} \tilde{T}$ has everywhere codimension one in $\text{Supp} \tilde{T}$, then for every $\ell \geq 0$ and every $\lambda \in S^1$,

(a) for each $\lambda \in S^1 \setminus \{1\}$, $(\psi_{g, \lambda}, N, -\psi_{g, \lambda} S)$ is an object of pHLM($X, w - 1$),

(b) the set of data $((\psi_{g, 1}, -\psi_{g, 1} S), (\phi_{g, 1} T, \phi_{g, 1} S)$, can, var) is a polarized object of pHLQ($X, w$).

Indeed, the only properties which need a check are those for can and var, and they have been proved in 12.7.13.

14.2.16. The category of polarizable graded Hodge-Lefschetz modules of type $\varepsilon$

Given $\varepsilon = \pm 1$, we can define the category pHLM($X, w; \varepsilon$) of polarizable graded Hodge-Lefschetz modules centered at $w$ of type $\varepsilon$, as in Section 3.2: the objects are pairs $(M, N)$, with $M = \oplus M_\ell$, and $M_\ell$ are objects of $\tilde{D}$-Triples($X$); $N$ is a graded morphism $M \to M[2](-\varepsilon)$ of degree $-2$, that is, for every $\ell$, $N$ induces a morphism $M_\ell \to M_{\ell-2}(-\varepsilon)$, such that, for $\ell \geq 0$, $N^\ell : M_\ell \to M_{\ell-2}(-\varepsilon)$ is an isomorphism;
the Hodge condition is that, for every $\ell \geq 0$, $P_\ell M$ is an object of $pHM(X, w + \varepsilon \ell)$. By considering the Lefschetz decomposition, we see that each $M_\ell$ is an object of $pHM(X, w + \varepsilon \ell)$.

Morphisms in $pHLM(X, w; \varepsilon)$ are supposed to be graded with respect to the given grading. By definition, there is a functor $gr^M$ from $pHLM(X, w)$ to $pHLM(X, w; 1)$.

We set $(M^*)_\ell = (M_{-\ell})^*$. Then $M^*(-w)$ is also an object of $pHLM(X, w; \varepsilon)$. A polarization $S$ of $M$ is a (graded, by definition) morphism $S : M \to M^*(-w)$ such that $P_\ell S$ is a polarization of $P_\ell M$ for every $\ell \geq 0$.

14.2.17. Theorem (Main properties of polarizable Hodge modules)

1. Any object $M = (M^0, M^\prime, s)$ of $pHM(X, w)$ is $S$-decomposable in $pHM(X, w)$, and the pure components of $M^0$ and $M^\prime$ are the same.

2. There is no nonzero morphism (in $\widetilde{D}$-$\text{Triples}(X)$) from an object in $pHM(X, w_1)$ to an object in $pHM(X, w_2)$ if $w_1 > w_2$.

3. Property 14.2.2(2)$_g$ holds without any restriction on $g$.

4. The category $pHM(X, w)$ is abelian. Any morphism is strict and strictly specializable along any $g$.

5. Any polarization of an object of $pHM(X, w)$ or $pHLM(X, w)$ is a Hermitian isomorphism (i.e., a pre-polarization of weight $w$ of the corresponding triple).

6. If $M_1$ is a subobject of $M$ in $pHM(X, w)$, then it is a direct summand and a polarization $S$ of $M$ induces a polarization on each summand.

7. The category $pHLM(X, w; \varepsilon)$ is abelian. Any morphism is strict and strictly specializable along any $g$. Any sub-object of an object $(M, N)$ in $pHLM(X, w; \varepsilon)$ is a direct summand and a polarization of $(M, N)$ induces a polarization on it.

8. The category $WHM(X)$ is abelian, and any morphism is strict and strictly compatible with $W_s$.

9. The category $pHLM(X, w)$ is abelian. Any morphism is strict and strictly compatible with the monodromy filtration $M_s$.

10. For any polarizable Hodge-Lefschetz quiver $(M, M_1, c, v)$ centered at $w$, we have $(M_1, N_1) = \text{Im} c \oplus \text{Ker} v$ in $pHLM(X, w)$.

Let us emphasize some direct consequences of the theorem.

14.2.18. Notation. If $Z \subset X$ is a closed irreducible analytic subset, we denote by $pHM_Z(X, w)$ the full sub-category of $pHM(X, w)$ whose objects have pure support $Z$. By the S-decomposition property 14.2.17(1), Any object of $pHM(X, w)$ resp. any morphism between objects of $pHM(X, w)$ decomposes as the direct sum of objects resp. morphisms in of $pHM_Z(X, w)$ for a suitable locally finite family of closed irreducible analytic subsets $Z_i \subset X$.

14.2.19. Corollary. Given two objects $M_1, M_2$ in $pHM(X, w)$, any morphism between them (as objects of $\widetilde{D}$-$\text{Triples}(X)$) has kernel, image and cokernel in $pHM(X, w)$; and a corresponding statement for $pHLM(X, w)$ and $pHLM(X, w; \varepsilon)$.
14.2.20. Corollary (S-decomposition theorem and semi-simplicity for pHM(X, w))

1. Each object $M$ decomposes uniquely into the direct sum of objects in pHM(X, w) having pure support a closed irreducible analytic subset of $X$.

2. The category pHM(X, w) is semi-simple (all objects are semi-simple and morphisms between simple objects are zero or isomorphisms).

3. The category pHLM(X, w; $\varepsilon$) is semi-simple.

14.2.21. Corollary. If $M$ is an object of pHM(X, w) with polarization $S$, then for every open subset $U \subseteq X$ and every holomorphic function $g : U \to \mathbb{C}$,

1. for every $\ell > 1$, $N^\ell : \psi_{g,\lambda}M \to \psi_{g,\lambda}M(-\ell)$ and $\phi_{g,1}M \to \phi_{g,1}M(-\ell)$ are strict and strictly shift $M_*(N)$ by $2\ell$, and a similar property holds for $\text{gr}N^\ell$.

2. $\text{can} : \psi_{g,1}M \to \phi_{g,1}M$ and $\text{var} : \phi_{g,1}M \to \psi_{g,1}M(-1)$ are strict and strictly shift $M_*$ by 1.

14.2.22. Corollary (The vanishing cycle theorem). Let $(M \hookrightarrow N \hookrightarrow S)$ be a polarized object of pHLM(X, w - 1). Let us endow $(\text{Im} N \hookrightarrow N | \text{Im} N)$ with the morphism $S_1 : (\text{Im} N, N|_{\text{Im} N}) \to (\text{Im} N, N|_{\text{Im} N})(-w)$ such that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Im} N & \xrightarrow{S_1} & (\text{Im} N)^*(-w) \\
\downarrow\text{incl.} & & \downarrow\text{N}^* \\
M(-1) & \xrightarrow{S} & M^*(-w)
\end{array}
\]

Then : $(\text{Im} N, N|_{\text{Im} N}, S)$ a polarized object of pHLM(X, w).

Proof. We use the same argument as in the proof of Proposition 3.2.24. Strictness of $\text{can}, \text{var}, \text{can}^*, S$ (according to 14.2.17(9)) enables us to reduce the problem to the graded case. We note that the equalities in (3.2.24*) also hold without $C_n$, and this shows that, for $\ell \geq 0$, the isomorphism $\text{can} : P_{\ell+1}M \to P_{\ell}M_1$ transports the polarization $P_{\ell+1}S$ to $P_{\ell}S_1$.

14.2.23. Corollary. Given any morphism $\varphi : M_1 \to M_2$ between objects of pHM(X, w) and any germ $g$ of holomorphic function on $X$, then, the specialized morphisms $\psi_{g,\lambda}\varphi$ ($\lambda \in S^1$) and $\phi_{g,1}\varphi$ are strictly compatible with the monodromy filtration $M_*$ and, for every $\ell \in \mathbb{Z}$, $\text{gr}_\ell^M \psi_{g,\lambda}\varphi$ (and similarly $\text{gr}_\ell^M \phi_{g,1}\varphi$) decomposes with respect to the Lefschetz decomposition, i.e.,

\[
\text{gr}_\ell^M \psi_{g,\lambda}\varphi = \begin{cases} 
\bigoplus_{k \geq 0} N^k P_{\ell+2k} \psi_{g,\lambda}\varphi & (\ell \geq 0), \\
\bigoplus_{k \geq 0} N^k P_{-\ell+2k} \psi_{g,\lambda}\varphi & (\ell \leq 0).
\end{cases}
\]

In particular we have

\[
\text{gr}_\ell^M \psi_{g,\lambda} \text{Ker} \varphi = \text{Ker} \text{gr}_\ell^M \psi_{g,\lambda} \varphi
\]
and similarly for \( \text{Coker} \), where, on the left side, the filtration \( M_\bullet \) is that induced naturally by \( M_\bullet \psi_{g,\lambda} M_1 \) or, equivalently, the monodromy filtration of \( N \) acting on \( \psi_{g,\lambda} \ker \varphi = \ker \psi_{g,\lambda} \varphi \).

14.2.24. Corollary. If \( M \) is in \( pHM(X, w) \), then the Lefschetz decomposition for \( \text{gr}_\ell^M \psi_{g,\lambda} M (\lambda \in S^1) \) resp. \( \text{gr}_\ell^M \phi_{g,\lambda} M \) holds in \( pHM(X, w - 1 + \ell) \) resp. \( pHM(X, w + \ell) \).

Proof. Indeed, \( N : \text{gr}_\ell^M \psi_{g,\lambda} M \to \text{gr}_{\ell-\lambda}^M \psi_{g,\lambda} M (-1) \) is a morphism in the category \( HM(X, w - 1 + \ell) \), which is abelian, so the primitive part is an object of this category, and therefore each term of the Lefschetz decomposition is also an object of this category.

Similarly to Proposition 7.4.9, we can simplify the data of a polarizable Hodge module.

14.2.25. Proposition (Simplified form for an object of \( pHM(X, w) \) or \( pHLM(X, w) \))

Any object \( M \) of \( pHM(X, w) \) resp. \( pHLM(X, w) \), resp. \( (M_\bullet, N) \) of \( pHLM(X, w; \varepsilon) \), is isomorphic to an object of the form

\[
(\mathcal{M}, F^* N), (\mathcal{M}, F^* M)(w), \mathcal{S}
\]

(resp. ...) such that \( \mathcal{S}^* = \mathcal{S} \) and with polarization \( (\text{Id}, \text{Id}) : M \to M^*(-w) \).

We call the data \( (\mathcal{M}, F^* M, S) \) a Hodge-Hermitian pair of weight \( w \) (resp. Hodge-Lefschetz Hermitian pair centered at \( w \), resp. graded Hodge-Lefschetz Hermitian pair centered at \( w \) of type \( \varepsilon \)) if the corresponding triple \( (\mathcal{M}, F^* M), (\mathcal{M}, F^* M)(w), \mathcal{S} \) with polarization \( (\text{Id}, \text{Id}) \) is polarized Hodge module of weight \( w \) (resp. ...).

14.2.26. Example (of filtered Hermitian pairs). We consider the following corresponding filtered Hermitian pairs (see Example 12.3.5)

\[
n^\mathcal{O}_X := (\mathcal{O}_X, F_\omega \mathcal{O}_X), \quad n^\omega_X := (\omega_X, F_\omega \omega_X),
\]

We will prove in Corollary 14.3.5 that they are Hodge-Hermitian pairs of weight \( n \).

The case where \( n = 1 \) is a consequence of the results in Chapter 7 (see Exercise 14.1).

Proof of Theorem 14.2.17. It is done by induction on the dimension of the support. By the point (0) in Definition 14.2.2, the categories of objects with support equal to a point as considered in the theorem are equivalent to the corresponding categories for \( X = \text{pt} \). In such a case, they are proved in Chapters 2 and 3.

We will thus fix \( d \geq 1 \) and assume that the assertions are proved for the subcategories consisting of objects having support of dimension \( < d \), in order to prove them when the dimension of the support of \( M \) is \( d \).

To prove (10) \( \Rightarrow (1)_d \) Let \( x_0 \in \text{Supp} \ M \) and let \( g \) be a germ of holomorphic function at \( x_0 \) such that \( g^{-1}(0) \cap \text{Supp} \ M \) has everywhere codimension one in \( M \). By Condition 14.2.13(2)' \( g \), the nearby vanishing quiver of \( M \) along \( (g) \) satisfies the assumption of (10) \( \Rightarrow (1)_d \), hence its conclusion, so \( M \) is \( S \)-decomposable along \( (g) \) in \( D\text{-Triples}(X) \).
By (14.2.8), the summands also belong to pHM(X, w). This proves S-decomposability in pHM(X, w).

We assume that there is a pure component $Z'$ of $\tilde{M}'$ which is not a pure component of $\tilde{M}''$. Then we have an summand $(\tilde{M}'_{Z}, 0, 0)$ of $M$ in pHM(X, w), according to the previous argument. We wish to show that $\tilde{M}'_{Z} = 0$, and it is enough, by the condition of the pure support, to show the vanishing on the smooth locus of $Z'$. We can thus reduce to the case where $Z' = X$, according to Proposition 9.7.10.

We now argue by induction on dim $X$, the case dim $X = 0$ reducing to the case of Hodge structures, which is easy. Let $t$ be a local coordinate on $X$. Arguing as in Corollary 9.7.11, one checks that $\tilde{M}'_{X}/t\tilde{M}'_{X} = \psi_{t,1}M_{X}$, and that $\psi_{t,1}\tilde{M}'_{X} = 0$ for $\lambda \in S^{1} \setminus \{1\}$, as well as $\phi_{t,1}\tilde{M}'_{X} = 0$. It follows that $N = 0$, so $\psi_{t,1}M_{X}$ is S-decomposable, according to Condition 14.2.2(2). By induction, the object $\psi_{t,1}(M_{X}, 0, 0)$ is zero. Hence $\tilde{M}'_{X}/\tilde{M}'_{X} = 0$, and by applying Nakayama's lemma as in Corollary 9.7.11, we obtain $\tilde{M}'_{X} = 0$.

$(1)_{d} \Rightarrow (2)_{d}$ Since any morphism between S-decomposable objects decomposes correspondingly, it is enough to consider a morphism $\varphi : M_{1} \rightarrow M_{2}$ between polarizable Hodge modules of respective weights $w_{1}, w_{2}$ having pure support. Since the result is clear for polarizable variations of Hodge structure (see Proposition 2.5.5(2)), it follows from Proposition 14.2.10 that the support of $\text{Im} \varphi$ is strictly smaller than $Z$.

By definition of the pure support (see Definition 9.7.9), this implies that $\text{Im} \varphi = 0$.

$(1)_{d} \Rightarrow (3)_{d}$ The question is local at $x_{0}$ and by assumption we can assume that $M_{x_{0}}$ has pure support a closed irreducible subset $Z_{x_{0}} \subset X_{x_{0}}$. Let $g : X_{x_{0}} \rightarrow \mathbb{C}$ be a germ of holomorphic function. If $g$ is non-constant on $Z_{x_{0}}$, it satisfies the constraint in Definition 14.2.2(2)$_{g}$. Otherwise, $\text{Supp} M_{x_{0}} \subset |g^{-1}(0)|$ and Proposition 12.7.12 implies that $M_{x_{0}} = \phi_{g,1}M_{x_{0}}$ (and similarly $S = \phi_{g,1}S$). Moreover, $\psi_{g,\lambda}M_{x_{0}} = 0$ for any $\lambda \in S^{1}$, and $N = 0$. Hence, if $M_{x_{0}}$ is an object of pHM((X, x_{0}), w), 14.2.2(2)$_{g}$ obviously holds.

$(4)_{<d} \& (6)_{<d} \Rightarrow (4)_{d}$ The question is local. Let $\varphi : M_{1} \rightarrow M_{2}$ be a morphism in pHM(X, w) between objects having support in dimension $d$. Then, by $(8)_{<d}$ applied to $\psi_{g,\lambda}\varphi, \phi_{g,1}\varphi$, for any germ $g$ satisfying the constraint of Definition 14.2.2(2)$_{g}$, $\varphi : M_{1} \rightarrow M_{2}$ is strictly $R$-specializable along $(g)$ and Corollary 10.8.5 implies that it is strict. Moreover, $\psi_{g,\lambda}\varphi$ and $\phi_{g,1}\varphi$ are strict with respect to the monodromy filtrations, since these are weight filtrations up to a shift.

It remains to check that $\text{Ker} \varphi, \text{Im} \varphi, \text{Coker} \varphi$ belong to pHM(X, w). Let us check this for $\text{Ker} \varphi$ for example. It follows from the M-strictness above that

$$\text{gr}_{T}^{M} \psi_{g,\lambda} \text{Ker} \varphi = \text{Ker} \text{gr}_{T}^{M} \psi_{g,\lambda} \varphi$$

and thus, for any $\ell \geq 0$, $P_{\ell} \psi_{g,\lambda} \text{Ker} \varphi = \text{Ker} P_{\ell} \psi_{g,\lambda} \varphi$. Since $P_{\ell} \psi_{g,\lambda} \varphi$ is a morphism in pHM(X, w − 1 + $\ell$) between objects having support in dimension < $d$, $(4)_{<d}$ implies that $\text{Ker} P_{\ell} \psi_{g,\lambda} \varphi$ is an object of pHM(X, w − 1 + $\ell$) and, according to $(6)_{<d}$, is a direct summand of $P_{\ell} \psi_{g,\lambda} M_{1}$. If $S$ is a polarization of $M_{1}$, let $S_{\varphi}$ denote the morphism
induced by $S$ on $\text{Ker} \varphi$. On the one hand, the morphism induced by $-P_\ell \psi_{g, \lambda} S$ on $\text{Ker} P_\ell \psi_{g, \lambda} \varphi$ is a polarization, according to 14.2.8. On the other hand, it is equal to $-P_\ell P_\ell \psi_{g, \lambda} S \varphi$. We can argue similarly with $\phi_{g, 1}$, by assumption on $g$. This shows that $(\text{Ker} \varphi, S)$ satisfies 14.2.2(2)\,.

$(4)_d \& (5)_{\leq d} \Rightarrow (5)_d$ A polarization $S$ of $M$ is a morphism $M \rightarrow M^*(-w)$, hence it is strict and strictly specializable along any $(g)$. Let $g$ be a holomorphic function such that $g^{-1}(0) \cap \text{Supp} \, M$ has everywhere codimension one in $\text{Supp} \, M$. $(5)_{\leq d}$ implies that $P_\ell \psi_{g, \lambda} S$ and $P_\ell \phi_{g, 1} S$ are isomorphisms for every $\ell \geq 0$, which implies the same property for $\text{gr}^{\mathbb{M}} \psi_{g, \lambda} S$ and $\text{gr}^{\mathbb{M}} \phi_{g, 1} S$ and thus for $\psi_{g, \lambda} S$ and $\phi_{g, 1} S$. By strict $\mathbb{R}$-specializability, $\psi_{g, \lambda}$ and $\phi_{g, 1}$ commute with taking $\text{Ker}$ and $\text{Coker}$ on $S$. We conclude that $\psi_{g, \lambda} \text{Ker} S = 0$ and $\phi_{g, 1} \text{Ker} S = 0$, and similarly with $\text{Coker}$. Since $\text{Ker} S$ and $\text{Coker} S$ are in $\text{pHM}(X, w)$ by $(4)_d$, we can apply to them the regularity property along $(g)$ of Corollary 10.8.4, which implies they both are zero.

That $S$ is Hermitian is obtained similarly by applying the argument to $\text{Im}(S - S^*)$.

$(1)_d \Rightarrow (6)_d$ A polarization of $M$ decomposes with respect to the $S$-decomposition of $M$, and it is clear that it induces a polarization on each summand. We can thus restrict to considering objects $M$ with pure support a closed irreducible analytic subset $Z$ of $X$.

If $\dim Z = 0$, we apply Exercise 2.11. If $\dim Z \geq 1$, we consider the exact sequences (defining $S_1$)

\[
\begin{array}{cccccc}
0 & \rightarrow & M^*_1(-w) & \stackrel{i^*}{\rightarrow} & M^*(-w) & \stackrel{i}{\rightarrow} & M^*_2(-w) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & M_1 & \stackrel{i}{\rightarrow} & M & \rightarrow & M_2 & \rightarrow & 0
\end{array}
\]

where $M_2$ is the cokernel, in the abelian category $\text{pHM}(X, w)$, of $M_1 \hookrightarrow M$. We first show that $S_1$ is an isomorphism. It is enough to prove it on an open dense subset $Z^0$ of $Z$. By Kashiwara’s equivalence (14.2.9) and the generic structure 14.2.10, we are reduced to considering the case of polarizable variations of Hodge structure, which is follows from Exercise 4.2. We conclude that we have a projection $p = S_1^{-1} \circ i^* \circ S : M \rightarrow M_1$ such that $p \circ i = \text{Id}$, and a decomposition $M = M_1 \oplus S^{-1} M_2^*(-w)$. By construction, $S$ splits correspondingly, and it is then clear that each summand is a polarization.

$(4)_d \& (6)_d \Rightarrow (7)_d$ Abelianity and strictness resp. strict $\mathbb{R}$-specializability of morphisms follow from $(4)_d$ in a straightforward way by the grading property. In the same way, $(6)_d$ implies the similar property for $\text{pHM}(X, w; \varepsilon)$.

$(4)_d \& (2)_d \Rightarrow (8)_d$ We note first that, since objects of $\text{pHM}(X, w)$ are strict, Lemma 5.1.9(1) implies that the $D_X$-modules which are components of an object in $\text{WHM}_{\leq d}(X)$ are strict. According to $(2)_d$ and Proposition 2.6.3, $(4)_d$ implies that the category $\text{WHM}_{\leq d}(X)$ is abelian and that morphisms are strictly compatible with $W$. Using Lemma 5.1.9(2), we conclude that all morphisms are strict.
(8)\textsubscript{d} \Rightarrow (9)\textsubscript{d} Since $pHLM(X, w)$ is a subcategory of $WHM(X)$ with the weight filtration given by the shifted monodromy filtration, strictness of morphisms and strict compatibility with $W$, follow from (8)\textsubscript{d}.

(9)\textsubscript{d} \& (10)\textsubscript{\textless}d \Rightarrow (10)\textsubscript{d} It follows from weight properties (see also Remark 3.1.25) that $c, v$ strictly shift by $-1$ the monodromy filtrations. We then denote by $\text{gr} c, \text{gr} v$ the corresponding morphisms, graded of degree $-1$ with respect to $M$. We then have $gr^M \text{Im} c = \text{Im} gr^M c$ and $gr^M \text{Ker} v = \text{Ker} gr^M v$. Moreover, the natural morphism $\text{Im} c \oplus \text{Ker} v \to M_1$ is strict with respect to the weight filtration, hence to the monodromy filtrations. It follows that, if the graded morphism $\text{Im} gr c \oplus \text{Ker} gr v \to gr^M M_1$ is an isomorphism, then $M_1 = \text{Im} c \oplus \text{Ker} v$, as wanted. We are therefore reduced to proving the assertion in the category of polarizable graded Hodge-Lefschetz quivers.

In such a case, $M, M_1, c, v$ are strict and strictly $R$-specializable along any $(g)$, according to (7)\textsubscript{d}, and by the regularity property (Corollary 10.8.4), it is enough to prove locally, for any holomorphic germ $g$, the decompositions

\[ \psi_{g, \lambda} M_1 = \text{Im} \psi_{g, \lambda} c \oplus \text{Ker} \psi_{g, \lambda} v, \quad \forall \lambda \in S^1, \]
\[ \phi_{g, 1} M_1 = \text{Im} \phi_{g, 1} c \oplus \text{Ker} \phi_{g, 1} v. \]

Let us argue with $\phi_{g, 1}$ for example. Recall that $M = \bigoplus_{\ell} M_{\ell}$ and $M_1 = \bigoplus_{\ell} M_{1, \ell}$, with $M_\ell \in pHM(X, w - 1 + \ell)$ and $M_{1, \ell - 1} \in pHM(X, w + \ell - 1)$, and that $\phi_{g, 1} c$ is a morphism $\phi_{g, 1} M_\ell \to \phi_{g, 1} M_{1, \ell - 1}$. It is strictly compatible with the weight filtration on these spaces, which is nothing but $M_{w + \ell - 1 + \bullet}(N_g)$, hence with the monodromy filtration of $N_g$. The same argument holds for $v$. It is thus enough to prove

\[ gr^M_j \phi_{g, 1} M_{1, \ell - 1} = \text{Im} gr^M_j \phi_{g, 1} c \oplus \text{Ker} gr^M_j \phi_{g, 1} v. \]

We can therefore apply (10)\textsubscript{\textless}d to the quiver

\[ (gr^M_j \phi_{g, 1} M_\ell, gr^M_j \phi_{g, 1} M_{1, \ell - 1}, gr^M_j \phi_{g, 1} c, gr^M_j \phi_{g, 1} v), \]

which is centered at $w + \ell - 1 + j$.

\[ \square \]

14.3. Introduction to the direct image theorem

The theory of polarizable Hodge modules was developed in order to give an analytic proof, relying on Hodge theory, of the decomposition theorem of the pushforward by a projective morphism of the intersection complex attached to a local system underlying a polarizable variation of Hodge structure. Two questions arise in this context:

- to relate polarizable variations of Hodge structure on a smooth analytic Zariski open subset of a complex analytic set with a polarizable Hodge module on a complex manifold containing this analytic set as a closed analytic subset (the structure theorem),

- to prove the Hodge-Saito (i.e., direct image) theorem for the pushforward by a projective morphism of a polarizable Hodge module.
The Hodge-Saito theorem describes the behaviour by projective pushforward of an object of $\text{pHM}(X, w)$. The case of the constant map $X \to \text{pt}$ and of the Hodge module $\mathcal{O}_X$ corresponds to the results of Section 2.4.

The proof of the Hodge-Saito theorem is obtained by reducing to the case of a constant map, by using the nearby cycle functor and its compatibility with pushforward. In the case of the constant map, one can reduce to the case where the Hodge module is a polarizable variation of Hodge structure on the complement of a normal crossing divisor in a complex manifold by using Hironaka’s theorem on resolution of singularities, and the decomposition theorem already proved (by induction) for the resolution morphism. One can use a Lefschetz pencil to apply an inductive process, after having blown up the base locus of the pencil. In such a way, one is reduced to the case of the constant map on a smooth projective curve, where one can apply the Hodge-Saito theorem.

Another approach in the case of a constant map would make full use of the higher dimensional analogues of the results proved in Chapter 6 for polarized variations of Hodge structure, but this would need to include in the inductive process the structure theorem for polarizable Hodge modules in the normal crossing case.

The Hodge-Saito theorem enables us to give a proof of a simple case of the structure theorem, namely, that a variation of Hodge structure of weight $w$ on a complex manifold $X$ is a polarizable Hodge module of weight $w + \dim X$. It is indeed difficult to check the behaviour along an arbitrary holomorphic function $g$ (e.g. strict $\mathbb{R}$-specializability), but the case where the function is a monomial can be reduced to the case where the function is a product of coordinates, and in that case Example 12.7.21 provides the result by induction on the dimension. The pushforward theorem 12.7.26 enables us to obtain the result for an arbitrary holomorphic function, according to Hironaka’s resolution of singularities of holomorphic functions.

**14.3.1. Theorem (Hodge-Saito theorem).** Let $f : X \to Y$ be a projective morphism between complex analytic manifolds and let $M$ be a polarizable right Hodge module of weight $w$ on $X$. Let $\mathcal{L}$ be an ample line bundle on $X$ and let $\mathcal{N}_\mathcal{L}$ be the corresponding Lefschetz operator. Then $(\bigoplus_k f_*^{(k)}M, \mathcal{N}_\mathcal{L})$ (where the $k$-th term is regarded in degree $-k$) is an object of $\text{pHLM}(Y, w; -1)$.

Let us make explicit this statement. Let us choose a polarization $S$ on $M = (\tilde{M}', \tilde{M}'', \mathfrak{s})$. It induces an isomorphism $\tilde{M}' \simeq \tilde{M}'(w)$ and we can assume that $M$ corresponds to a Hodge-Hermitian pair $(\tilde{M}, \mathfrak{s})$, i.e., $M = (\tilde{M}, \tilde{M}(w), \mathfrak{s})$ with polarization $S = (\text{Id}, \text{Id})$.

(a) $\nu_f \tilde{M}$, regarded as an object of $\text{D}_\text{hol}^b(\mathcal{D}_Y)$, is strict, that is, for every $k$, $\nu f_*^{(k)} \tilde{M}$ is a strict $\mathcal{D}_Y$-module. Moreover, $\nu f_*^{(k)} \tilde{M}$ is $S$-decomposable.

(b) Each $\tau f_*^{(k)} M$ is a polarizable Hodge module of weight $w + k$ on $Y$. 
For every $k > 0$, the Lefschetz operator $N_L$ induces isomorphisms in $\text{pHLM}(Y \hookrightarrow w + k)$ (this is known as the relative hard Lefschetz theorem):

\[
N_L^k : \tau f_*(k) M \longrightarrow \tau f_*(-k) M,
\]

so that $(\bigoplus_k \tau f_*(k) M, N_L)$ (where the $k$-th term is regarded in degree $-k$) is an object of $\text{pHLM}(Y, w; -1)$, that is, a graded Hodge-Lefschetz Hermitian pair centered at $w$ of type $\varepsilon = -1$.

One of the most notable consequences of the Hodge-Saito theorem is the decomposition theorem.

14.3.2. Theorem (Decomposition theorem). Let $f : X \to Y$ be a projective morphism of complex manifolds. Let $M$ be a $\mathcal{D}_X$-module underlying a polarizable Hodge module. Then the complex $\varpi f_! \tilde{M}$ in $\mathcal{D}_Y$ decomposes (in a non-canonical way) as $\bigoplus_k \varpi f_!(k) \tilde{M}[-k]$. Similarly, if $M = M/(z - 1)\tilde{M}$ is the underlying $\mathcal{D}_X$-module, then there exists a (non canonical) decomposition $\varpi f_! M \simeq \bigoplus_k \varpi f_!(k) M[-k]$ in $\mathcal{D}_Y$.

Proof. This is a direct consequence of Deligne’s criterion 3.1.7 for a spectral sequence to degenerate at $E_2$. We apply this theorem to $\varpi f_! \tilde{M}$ as an object of $\mathcal{D}_Y$, by using the Hard Lefschetz theorem furnished by the Hodge-Saito theorem.

When both $X$ and $Y$ are projective, we can combine Theorems 14.3.2, 14.3.1 and 15.3.2 to obtain:

14.3.3. Corollary. Let $f : X \to Y$ be a morphism between projective complex manifolds and let $M$ be a $\mathcal{D}_X$-module underlying a polarizable Hodge module. Then $\varpi f_! M$ decomposes non-canonically as $\bigoplus_k \varpi f_!(k) M[-k]$, and each $\varpi f_!(k) M$ is itself a semi-simple holonomic $\mathcal{D}_Y$-module.

14.3.4. Remark. Due to the work of T. Mochizuki, it is known that, in the corollary, one can relax the condition that $M$ underlies a polarizable Hodge module.

Let us already emphasize a particular case of the structure theorem of Chapter 15, which will be proved in Section 15.3.

14.3.5. Corollary. Let $(H, S)$ be a polarized smooth Hodge triple of weight $w$ (as in Definition 5.4.3). Then $(H, S)$ is a polarized Hodge module of weight $w$.

From the decomposition theorem we deduce:

14.3.6. Corollary. Let $(\mathcal{H}, \nabla)$ be vector bundle with connection on $X$ underlying a variation of polarizable Hodge structure of weight $w$. Then its direct image (in the category of $\mathcal{D}$-modules) by a projective morphism $f : X \to Y$ decomposes non-canonically in $\mathcal{D}_Y$

\[
\varpi f_!(\mathcal{H}, \nabla) \simeq \bigoplus_k \varpi f_!(k)(\mathcal{H}, \nabla),
\]
and each $df^{(k)}(\mathcal{H}, \nabla)$ underlies a polarizable Hodge module of weight $w + \dim X + k$. \hfill \Box

It follows from the semi-simplicity theorem 15.3.2 that, if $Y$ is projective, each $df^{(k)}(\mathcal{H}, \nabla)$ is also semi-simple as a holonomic $\mathcal{D}_Y$-module.

14.3.7. Sketch of the proof of Theorem 14.3.1. That holonomy is preserved by proper pushforward is recalled in Remark 8.8.25. We will now focus on the other properties defining a polarizable Hodge module. The proof of Theorem 14.3.1 is by induction on the pair

$$(\dim \text{Supp} M, \dim \text{Supp} f_* M)$$

ordered lexicographically, in the following way.

(a) The case where $X$ is a compact Riemann surface and $f : X \to \text{pt}$ is the constant map has been treated in Chapter 7 for both theorems 15.3.1, and 14.3.1 (see Corollary 7.4.19, i.e., the Hodge-Zucker theorem 6.1.1). This is $(1,0)$ with $\text{Supp} M$ smooth.

(b) $(14.3.1)_{(n,m)} \Rightarrow (14.3.1)_{(n+1,m+1)}$ is proved in Section 14.5.

(c) $(14.3.1)_{(n-1,0)}$ & $[(14.3.1)_{(1,0)}$ with $\text{Supp} M$ smooth] $\Rightarrow (14.3.1)_{(n,0)}$ for $n \geq 1$ is proved in Section 14.6 by using the method of Lefschetz pencils.

Therefore, given a pair $(n, m) \in \mathbb{N}^2$ with $m \leq n$, let us assume that the theorem is proved for every pair $(n', m') < (n, m)$. If $m \geq 1$, and since $(n-1, m-1) < (n, m)$, the second point gives the theorem by induction. We can thus assume that $m = 0$. Then, the third point, together with the first one, reduce the proof that of $(14.3.1)_{(n-1,0)}$ which is also true by induction.

14.4. Proof that a polarizable smooth Hodge triple is a polarizable Hodge module

In this section, we prove Corollary 14.3.5, assuming the validity of Hodge-Saito’s theorem 14.3.1. The proof is by induction on $\dim X$. Let $\mathcal{F}$ be a triple corresponding to a polarized variation of Hodge structure of weight $w$ on $X$, with polarization $S = (\text{Id}, \text{Id})$. Since locally a holomorphic function $g$ on a curve is a power of a local coordinate, we only need to check Properties 14.2.2(1)$g$ and (2)$g$ when $g$ is a local coordinate, according to Remark 14.2.3(3). In such a case, these properties obviously hold for a polarizable variation of Hodge structure. We thus assume that the corollary holds for $\dim X < n$, and we assume $\dim X = n$.

**Step one.** Assume first that $g$ is a product of distinct coordinates of a local coordinate system. We are thus in the setting of Example 12.7.21. We then know that $\mathcal{F}$ is strict $\mathbb{R}$-specializability and a middle extension along $(g)$, so we only need to check 14.2.2(2)$g$, according to Remark 14.2.3(4). We can apply induction to each variation $\tau^*_{\mathcal{F}}\mathcal{F}$ for $J = \mathcal{F} \in \mathcal{F}_{\ell+1}$, since its support has dimension $\dim X - (\ell + 1)$ with $\ell \leq 0$. 

Hence $(\tau^*_j T, (\text{Id}, \text{Id}))$ is a polarized Hodge module of weight $w + \dim X - (\ell + 1)$, and $(\tau^*_j \bar{T}, (\text{Id}, \text{Id}))(\ell)$ is a polarized Hodge module of weight $w + \dim X + \ell - 1$.

14.5. Behaviour of the Hodge module properties by projective pushforward

Let $f : X \to Y$ be a projective morphism between complex manifolds, let $h$ be a holomorphic function on $Y$ and set $g = h \circ f : X \to \mathbb{C}$. Let $\mathcal{L}$ be a relatively ample line bundle on $X$. In other words, we choose a relative embedding

\[
X \hookrightarrow Y \times \mathbb{P}^N
\]

so that $\mathcal{L}$ comes by pullback from an ample line bundle on $\mathbb{P}^N$.

Let $M = (\mathcal{M}, \mathcal{M}', \mathcal{B}, s)$ be an object of $\mathcal{D}(-\text{Triples}(X))$. We assume that $M$ is strictly $\mathbb{R}$-specializable along $(g)$ and is a middle extension along $(g)$, that is, can is onto and var is injective (see Definition 9.7.3).

Let $S : M \to M^*(-w)$ be a Hermitian morphism.

14.5.1. Proposition. Together with these assumptions, let us moreover assume that

(a) $\dim(\text{Supp} M \cap g^{-1}(0)) \leq d$,

(b) $(M, S)$ satisfies 14.2.2(2)g, that is, for any $\lambda \in S^1$ and any integer $\ell \geq 0$, the morphism $P_{\ell} \psi_{g, \lambda}(M, S)$ is a polarized object of $\text{pHM}(X, w - 1 + \ell)$ and $P_{\ell} \phi_{g, 1}(M, S)$ is an object of $\text{pHM}_{\ell, d}(X, w + \ell)$.

In other words, we assume that $(\text{gr}^M_{\ell} \psi_{g, \lambda} M, N, -\text{gr}^M_{\ell} \psi_{g, \lambda} S)$ is a polarized graded Hodge-Lefschetz triple of type $\varepsilon = 1$, centered at $w - 1$ and $(\text{gr}^M_{\varepsilon} \phi_{g, 1} M, N, \text{gr}^M_{\varepsilon} \phi_{g, 1} S)$ is a polarized graded Hodge-Lefschetz triple of type $\varepsilon = 1$, centered at $w$.

Then, if Theorem 14.3.1 holds in dimension $\leq d$, the following holds.

(1) $\tau^*_j (k) M$ is strictly $\mathbb{R}$-specializable and $S$-decomposable along $(h)$ for every $k \in \mathbb{Z}$.

(2) $(\bigoplus_{\ell, \varepsilon} \text{gr}^M_{\varepsilon} \phi_{h, \lambda}(\tau^*_j (k) M), (N, \text{gr} N), -\text{gr}^M_{\varepsilon} \psi_{h, \lambda}(\tau^*_j (k) S))$ is a polarized bi-graded Hodge-Lefschetz triple of type $\varepsilon = (-1, 1)$, centered at $w - 1$.

(3) $(\bigoplus_{\ell, \varepsilon} \text{gr}^M_{\varepsilon} \phi_{h, 1}(\tau^*_j (k) M), (N, \text{gr} N), \text{gr}^M_{\varepsilon} \phi_{h, 1}(\tau^*_j (k) S))$ is a polarized bi-graded Hodge-Lefschetz triple of type $\varepsilon = (-1, 1)$, centered at $w$.

Before giving the proof of this proposition, we will introduce the technical tools that are needed for it.

14.5a. Multi-graded polarizable Hodge-Lefschetz modules. For every $k \geq 1$ and $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_k) = (\pm 1, \ldots, \pm 1)$, we can define the category $\text{pHM}(X, w; \varepsilon)$ of $k$-graded polarizable Hodge-Lefschetz modules centered at $w$ of type $\varepsilon$: the objects are tuples $(M, N)$, with $N = (N_1, \ldots, N_k)$, $M = \oplus_{j \in \mathbb{Z}^k} M_j$, each $M_j$ is an object in $\text{HM}(X, w + \varepsilon \cdot j)$ with $\varepsilon \cdot j := \sum_i \varepsilon_i j_i$, the morphisms $N_i$ should pairwise commute, be of $k$-degree $(0, \ldots, -2, \ldots, 0)$ and for every $j$ with $j_i \geq 0$, $N_i^j$ should induce an
isomorphism from $M_j$ to the component where $j_i$ is replaced with $-j_i$; the primitive part $P M_j$, for $j_1, \ldots, j_k \geq 0$, is the intersection of the $\text{Ker} N_j^{1+1}$ and we have a Lefschetz multi-decomposition. Morphisms in $\text{pHLM}(X, w; \varepsilon)$ should be $k$-graded and commute with each $N_i$. As a consequence, they are also multi-graded with respect to the Lefschetz multi-decomposition (argue as in Exercise 3.2). The category is abelian, and any morphism is strict and strictly $\mathbb{R}$-specializable (this is proved as 14.2.17(7)).

14.5.2. Lemma. Let $(M, N)$ be an object of the category $\text{pHLM}(X, w; \varepsilon)$ and let $g$ be a germ of holomorphic function. Then, for every $\lambda \in S^1$, the specialized object $(\text{gr}^M \psi_{g, \lambda} M, (\text{gr}^M \psi_{g, \lambda} N, N_g))$ is an object of $\text{pHLM}(X, w - 1; (\varepsilon, 1))$ and $P_N \text{gr}^M \psi_{g, \lambda} M_j = \text{gr}^M \psi_{g, \lambda} P_N M_j$, where $P_N$ denotes the multi-primitive part with respect to $N$. A similar statement holds with $\phi_{g, \lambda}$ and $\text{pHLM}(X, w; (\varepsilon, 1))$.

Proof. The lemma is a direct consequence of the strict compatibility of the $\psi_{1, \lambda} N_i$ with the monodromy filtration $M(N_g)$, as follows from 14.2.17(9) applied to the morphisms $N_i$.

Let $(M, N)$ be an object of $\text{HLM}(X, w; \varepsilon)$ and $(M, N)^* := (M^*, -N^*)$ denote its Hermitian dual object. A polarization $S$ is a (multi) graded morphism $S : M \to M^*(-w)$ (i.e., $S$ sends $M_\ell$ to $M^*_\ell(-w) = (M_\ell)^*(-w)$), such that each $N_i$ is skew-adjoint with respect to $S$ (i.e., $S$ is a morphism $(M, N) \to (M, N)^*(-w)$) and that, for every $\ell = \ell_1, \ldots, \ell_k$ with non-negative components, the induced morphism (see Section 3.2.b)

$$N_1^{\ell_1} \cdots N_k^{\ell_k} \circ S : M_\ell \longrightarrow (M_\ell)^*(-w - \varepsilon \cdot \ell)$$

induces a polarization of the object $P_N M_\ell$ of $\text{HM}(X, w + \varepsilon \cdot \ell)$.

14.5.3. Lemma. The category $\text{pHLM}(X, w; \varepsilon)$ has an inductive definition as in Definition 14.2.2. Properties 14.2.17(5)–(7) hold for this category.

Proof. This directly follows from the commutativity of $P_N$ and $\text{gr}^M \psi_{g, \lambda}$ and $\text{gr}^M \phi_{g, \lambda}$ shown in Lemma 14.5.2.

14.5.4. Proposition. The conclusions of Propositions 3.2.34 and 3.2.35 remain valid for graded polarizable Hodge-Lefschetz modules.

Proof.

(1) Let us start with Proposition 3.2.35. Let $((M_{j_1, j_2})_{j \in \mathbb{Z}^2}, N_1, N_2)$ be an object of $\text{pHLM}(X, w; \varepsilon_1, \varepsilon_2)$ with a polarization $S$. We assume that it comes equipped with a bi-graded differential, which is a morphism $d : M_{j_1, j_2} \to M_{j_1-1, j_2-1}(-\varepsilon)$ ($\varepsilon := (\varepsilon_1 + \varepsilon_2)/2$) in $\text{HM}(X, w + \varepsilon_1 j_1 + \varepsilon_2 j_2)$, of bi-degree $(1, -1)$, which commutes with $N_1$ and $N_2$ and is self-adjoint with respect to $S$. In particular, $d$ is strict and strictly specializable and we have, for any germ $g$ of holomorphic function, any $\lambda \in S^1$ and any $\ell \geq 0$,

$$P_\ell \psi_{g, \lambda}(\text{Ker} d/\text{Im} d) = \text{Ker}(P_\ell \psi_{g, \lambda} d)/\text{Im}(P_\ell \psi_{g, \lambda} d)$$
(see Corollary 14.2.23). By induction on the dimension of the support, we can assert that \((P_{\ell^1} \psi_{\rho, \lambda}(\text{Ker} d/\text{Im} d), P_{\ell^2} \psi_{\rho, \lambda} N)\) is an object of \(\text{pHLM}(X, w - 1 + \ell, \varepsilon)\) with polarization \(P_{\ell^1} \psi_{\rho, \lambda} S\), and we conclude with Lemma 14.5.3.

(2) The analogue of Proposition 3.2.34 is proved similarly. \(\square\)

14.5.5. Corollary (Degeneration of a spectral sequence). Let \((M^*, d)\) be a bounded complex in \(\mathcal{D}\)-Triples\((X)\), with \(d : M^j \to M^{j+1}\) and \(d \circ d = 0\). Let us assume that it is equipped with the following data:

(a) a morphism of complexes \(S : (M^*, d) \to (M^*, d)^*(-w)\) which is \((-1)^w\)-Hermitian, that is, for every \(k\), a morphism \(S : M^k \to (M^{-k})^*(-w)\) which is compatible with \(d\) and \(d^*\), and such that \(S^* = (-1)^w S\),

(b) a morphism \(N' : (M^*, d) \to (M^{*+1}, d)\) which is skew-adjoint with respect to \(E\),

(c) a morphism \(N : (M^*, d) \to (M^*(-1), d)\) which is nilpotent, commutes with \(N'\), and skew-adjoint with respect to \(S\), with monodromy filtration of \(M_*(N)\).

Let us consider the spectral sequence associated to the filtered complex \((M_{-\ell}M^*, d)\) with \(E_1^{j, -\ell} = H^j(\text{gr}_{\ell}^M M^*)\). We assume that

\[
\bigoplus_{j, \ell} \left( E_1^{j, -\ell} = H^j(\text{gr}_{\ell}^M M^*), (H^j\text{gr}_{\ell}^M N', H^j\text{gr} M) \right)
\]

is a polarized object of \(\text{pHLM}(X, w; -1, 1)\). Then,

1. the spectral sequence degenerates at \(E_2\),
2. the filtration \(W_j H^j(M^*)\) naturally induced by \(M_*(M^*)\) is the monodromy filtration \(M_*\) associated to \(H^j N : H^j(M^*) \to H^j(M^*)\),
3. the object

\[
\bigoplus_{j, \ell} \left( \text{gr}_{\ell}^M H^j(M^*), (\text{gr}_{\ell}^M H^j N', \text{gr} H^j M) \right)
\]

is a polarized object of \(\text{pHLM}(X, w; -1, 1)\).

**Proof.** Let us first make clear the statement. Note that we use the bi-grading as in Remark 3.1.18. Since \(d\) and \(N'\) commute with \(N\), \(d\) and \(N'\) are compatible with the monodromy filtration \(M_*(N)\), hence we have a graded complex \((\text{gr}_{\ell}^M M^*, d)\), and \(N'\) induces for every \(\ell\) a morphism \(\text{gr}_{\ell}^M N' : (\text{gr}_{\ell}^M M^*, d) \to (\text{gr}_{\ell}^M M^{*+2}, d)\), and thus a morphism \(H^j(\text{gr}_{\ell}^M N') : E_1^{j, -\ell} \to E_1^{j+2, \ell-2}\). Similarly, \(H^j\text{gr} N\) is a morphism \(E_1^{j, -\ell} \to E_1^{j+2, \ell-2}\). We consider the bi-grading such that \(E_1^{j, -\ell}\) is in bi-degree \((j, \ell)\).

The differential \(d_1 : H^j(\text{gr}_{\ell-1}^M M^*) \to H^{j+1}(\text{gr}_{\ell-1}^M M^*)\) is a morphism of bi-degree \((1, 1)\) in \(H^j(X, w + j - \ell)\). We will check below that \(d_1\) is self-adjoint with respect to \(H^j\text{gr}_{\ell-1}^M M\). From the analogue of Proposition 3.2.35 (see Proposition 14.5.4), we deduce that \(\bigoplus_{j, \ell} E_2^{j, -\ell}\) is part of an object of \(\text{pHLM}(X, w; -1, 1)\). Now, one shows inductively that, for \(r \geq 2\), \(d_r : E_2^{j, -\ell} \to E_2^{j+r, \ell-\ell+1}\) is a morphism of pure Hodge modules, the source having weight \(w + j - \ell\) and the target \(w + j - \ell - r + 1 < w + j - \ell\) and thus, by applying 14.2.17(2), that \(d_r = 0\). This gives the result. \(\square\)
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Proof that $d_1$ is self-adjoint. We regard $\text{gr}_-^M S$ as a morphism $\text{gr}_-^M M^k \to (\text{gr}_-^M M^{-k})^*$. It is compatible with $d$ and $d^*$ on these complexes, since $\mathcal{N}$ commutes with $d$. Then, $H^j \text{gr}_-^M S$ is a morphism $H^j \text{gr}_-^M M^* \to (H^{-j} \text{gr}_-^M M^{-*})^*$. Since $d_1$ is obtained by a standard formula from $d$ on the filtered complex, the equality $S \circ d = d^* \circ S$ implies $H^j \text{gr}_-^M S \circ d_1 = (d_1)^* \circ H^j \text{gr}_-^M S$. □

14.5.b. Proof of Proposition 14.5.1 and of 14.3.7(c).

Proof of Proposition 14.5.1. One of the points to understand is the way to pass from properties of $\tau f_*(k) \text{gr}_-^M \psi_{g,\lambda} M$ to properties of $\text{gr}_-^M \psi_{h,\lambda}(\tau f_*(k) M)$, and similarly with $\phi_{g,1}$. Although we know that $\psi_{t,\lambda}(\tau f_*(k) M) \xrightarrow{\sim} \tau f_*(k) \psi_{g,\lambda} M$ if the latter is strict, according to 12.7.26, we have to check the strictness property. Moreover, we are left with the question of passing from $\tau f_*(k) \text{gr}_-^M$ to $\text{gr}_-^M \tau f_*(k)$. Here, we do not have a commutation property, but we will use Corollary 14.5.5 to analyze the corresponding spectral sequence. At this point, the existence of a polarization is essential. The $S$-decomposability is not obvious either, and the polarization also plays an essential role for proving it.

Since we assume that Theorem 14.3.1 holds for objects in $\text{pHM}_{\infty,d}(X)$ and since $\dim(\text{Supp} M \cap g^{-1}(0)) \leq d$, we deduce that, for every $\lambda \in S^1$,

$$\left( \bigoplus_{k, \ell} \tau f_*(k) \text{gr}_-^M \psi_{g,\lambda} M, (\mathcal{N}_\ell, \tau f_*(k) \text{gr} \mathcal{N}), \tau f_*(k) \text{gr}_-^M \psi_{g,\lambda} S \right)$$

is a polarized object of $\text{pHM}(Y, w - 1; -1, 1)$ if we keep here the grading convention used in Corollary 14.5.5. This corollary implies that

$$\left( \bigoplus_{k, \ell} \text{gr}_-^M \tau f_*(k) \psi_{g,\lambda} M, (\mathcal{N}_\ell, \text{gr}_-^M \tau f_*(k) \mathcal{N}), \text{gr}_-^M \tau f_*(k) \psi_{g,\lambda} S \right)$$

is a polarized object of $\text{pHM}(Y, w - 1; -1, 1)$. In particular, each $\text{gr}_-^M \tau f_*(k) \psi_{g,\lambda} M$ is strict, and therefore so is $\tau f_*(k) \psi_{g,\lambda} M$. We argue similarly for $\phi_{g,1}$.

We can now apply Corollary 9.8.9 to conclude that $\tau f_*(k) M$ is strictly $\mathbb{R}$-specializable along $(g)$ for every $k$. We also conclude from 12.7.26 that

$$(\psi_{h,\lambda} \tau f_*(k) M, N) = \tau f_*(k) (\psi_{g,\lambda} M, N), \quad (\phi_{h,1} \tau f_*(k) M, N) = \tau f_*(k) (\phi_{g,1} M, N).$$

We have thus proved that

$$\left( \bigoplus_{k, \ell} \text{gr}_-^M \psi_{h,\lambda} \tau f_*(k) M, (\mathcal{N}_\ell, \text{gr} \mathcal{N}), \text{gr}_-^M \psi_{h,\lambda} \tau f_*(k) S \right)$$

is a polarized object of $\text{pHM}(Y, w - 1; -1, 1)$, and a corresponding assertion for $\phi_{h,1}$.

Proof of $(14.3.1)_{(n,m)} \Rightarrow (14.3.1)_{(n+1,m+1)}$. Let $f : X \to Y$ be a projective morphism and let $(M, S)$ be a polarized object of $\text{pHM}_Z(X, w)$, where $Z$ is an irreducible analytic subset of $X$ of dimension $n + 1$. We can assume that $(M, S)$ is a Hodge-Hermitian pair $(\tilde{M}, \mathfrak{s})$ of weight $w$, and we will omit $S$ in the notation. Assume that $f(Z)$ has dimension $m + 1$ and that $(14.3.1)_{(n,m)}$ holds. Since Theorem 14.3.1 is a local
statement on $Y$, we can work in an open neighbourhood of a point $y_o \in f(Z)$, that we can take as small as needed. By the S-decomposability of $(\mathcal{M}, g)$ on $X$, we can therefore assume that $Z$ and $f(Z)$ are irreducible when restricted to a fundamental basis of neighborhoods of $f^{-1}(y_o)$ and $y_o$ respectively.

Let $h$ be a holomorphic function on some nb$(y_o)$ and set $g = h \circ f$. We distinguish two cases. We note that strictness of $\tau f_*^{(k)} M$ on nb$(y_o)$ is obtained by choosing any $h$ as in Case $(1)$ below.

$(1)$ $h^{-1}(0) \cap f(Z)$ has codimension one in $f(Z)$. Then $g^{-1}(0) \cap Z$ has codimension one in $Z$. We can thus apply Proposition 14.5.1 with $d = n$. It follows that each $\tau f_*^{(k)} M$ is strict and satisfies $14.2.2(1)_h$ and $(2)_h$.

$(2)$ $h$ vanish identically on the closed irreducible subset $f(Z) \cap \text{nb}(y_o)$ of nb$(y_o)$. We now omit referring to nb$(y_o)$. We denote by $\iota_g : X \hookrightarrow X \times \mathbb{C}_t$ and $\iota : X \times \{0\} \hookrightarrow X \times \mathbb{C}_t$ the inclusions, and similarly on $Y$. The only property to be checked relative to $h$ is that $\partial f_*^{(k)} \mathcal{M}$ is strictly $\mathbb{R}$-specializable along $(h)$, that is, 14.2.2(1)$_h$: indeed, in such a case, Proposition 12.7.12 implies $\phi_{g,1}(\partial f_*^{(k)} M) = \partial f_*^{(k)} M$ and $\psi_{g,\lambda}(\partial f_*^{(k)} M) = 0$ for any $\lambda \in S^1$, so 14.2.2(2)$_h$ is trivially satisfied. Since $\partial f_*^{(k)} \mathcal{M}$ is strict, $\iota\iota_*(\partial f_*^{(k)} \mathcal{M})$ is strictly $\mathbb{R}$-specializable along $(t)$ and it is enough to prove

$$\iota\iota_*(\partial f_*^{(k)} \mathcal{M}) = \iota\iota_*(\partial f_*^{(k)} \mathcal{M}) \quad \forall \, k.$$ 

The left-hand term is equal to $\iota\iota_*(\partial f_*^{(k)} \mathcal{M})$, if we still denote by $f$ the map $f \times \text{Id}_{\mathbb{C}}$. Similarly the right-hand term is equal to $\iota\iota_*(\partial f_*^{(k)} \mathcal{M})$, with obvious abuse of notation. Since $g \equiv 0$ on $Z$ and $\mathcal{M}$ is assumed to be strictly $\mathbb{R}$-specializable along $(g)$, we have $\iota\iota_* \mathcal{M} = \iota\iota_* \mathcal{M}$, hence the desired assertion.

$\square$

### 14.6. End of the proof of the Hodge-Saito theorem

Recall that we wish to prove

$$(14.3.1)_{(\leq n-1, 0)} \& [(14.3.1)_{(1, 0)} \text{ with Supp } \mathcal{M} \text{ smooth}] \Rightarrow (14.3.1)_{(n, 0)} \text{ for } n \geq 1,$$

and by the results of Section 14.5, $(14.3.1)_{(\leq n, m)}$ is true for any $m \geq 1$. We will thus assume that the properties on the left-hand side are true. Let us denote by $a_X$ the constant map on the projective manifold $X$ of dimension $n$. We wish to decompose it as $X \to \mathbb{P}^1 \to \text{pt}$ and argue by considering the Leray spectral sequence for this decomposition (a topological analogue of the Fubini theorem). Our induction process shows that Theorem 14.3.1 holds for both maps, and we will argue that it then holds for their composition by analyzing the Leray spectral sequence. However, we do not have such a decomposition of the constant map in general, and the usual trick is to consider a Lefschetz pencil instead, a procedure which introduces a supplementary complication due to the base locus of the pencil, that we can choose as generic as we want nevertheless.
Let \((M, S)\) be a polarized Hodge module of weight \(w\) on a smooth complex projective variety and let \(\mathcal{L}\) be an ample line bundle on \(X\). We can assume that \(S = (\text{Id}, \text{Id})\) and consider the Hodge-Hermitian pair \((\tilde{\mathcal{M}}, s)\) for \(M\) as in Proposition 14.2.25. We can also assume that \(M\) has pure support \(Z\), which is an irreducible closed \(n\)-dimensional algebraic subset of \(X\) \((n \geq 1)\). It is not restrictive to assume that \(\mathcal{L}\) is very ample, so that, by Kashiwara’s equivalence (Proposition 14.2.9), we can further assume that \(X = \mathbb{P}^N\) and \(\mathcal{L} = \mathcal{O}_{\mathbb{P}^N}(1)\). Let us choose a pencil of hyperplanes in \(X\) with axis \(A \cong \mathbb{P}^{N-2}\). It defines a map \(X \smallsetminus A \to \mathbb{P}^1\), whose graph is contained in \((X \smallsetminus A) \times \mathbb{P}^1\). Let \(X_A\) be the closure of this graph in \(X \times \mathbb{P}^1\) and by \(A_A\) the pullback \(\pi^{-1}(A)\). By definition, \(X_A\) is the blow-up space of \(X\) along the axis \(A\) of the pencil, and \(A_A\) is a smooth divisor in it. We have the following diagram:

\[
A \times \mathbb{P}^1 = A_A \quad \xrightarrow{\pi} \quad X_A \quad \xrightarrow{\iota} \quad X \times \mathbb{P}^1 \quad \xrightarrow{f} \quad \mathbb{P}^1
\]

(14.6.1)

The restriction of \(\pi\) to any fiber \(f^{-1}(t)\) is an isomorphism onto the corresponding hyperplane in \(X\) and, conversely, the pullback by \(\pi\) of this hyperplane is the union of \(f^{-1}(t)\) and \(A_A = A \times \mathbb{P}^1\), whose intersection \(f^{-1}(t) \cap A_A = A \times \{t\}\) is transversal. Similarly, \(\pi^{-1}Z\) consists of the union of the strict transform \(Z_A\) of \(Z\) by \(\pi\), i.e., the blow-up space of \(Z\) along the ideal \(\mathcal{I}_A\mathcal{O}_Z\), and \((A \cap Z) \times \mathbb{P}^1\).

We set \(\mathcal{L}' = \mathcal{O}_{\mathbb{P}^1}(1)\), and we consider the ample line bundle \(\mathcal{L} \otimes \mathcal{L}'\) on \(X \times \mathbb{P}^1\). We will simply denote by \(N, N'\) the Lefschetz operators \(N_\mathcal{L}, N_\mathcal{L}'\), so that \(N + N'\) is the Lefschetz operator that is to be considered on \(X \times \mathbb{P}^1\).

The proof will take five steps.

**Step one.** We show that, under a non-characteristic condition, the pullback \(\tau \pi^*(M, S)\) is well-defined, is strict and satisfies 14.2.2(1)_{\pi^{-1}} and (2)_{\pi^{-1}}, for every \(t \in \mathbb{P}^1\). We will define \(\tau \pi^*\) as the composition \(\tau \iota^* \circ \tau p^*\).

The smooth pullback \(\tau p^* M\) is well-defined as an object of \(\tilde{\mathcal{D}}\)-Triples\((X \times \mathbb{P}^1)\) (see Section 12.7.10). In order to define \(\tau \iota^* (\tau p^* M)\), we will prove strict \(\mathbb{R}\)-specializability of \(\tau p^* M\) along the graph \(\iota(X_A)\). Note however that we do not know that the pullback \(\tau p^* M\) satisfies Hodge properties along every germ of holomorphic function on \(X \times \mathbb{P}^1\). Non-characteristic properties obtained by choosing the axis of the pencil generic enough will help us to overcome this difficulty.

More precisely, let us choose the pencil generic enough so that the axis \(A\) of the pencil is non-characteristic with respect to \(\tilde{\mathcal{M}}\) (see Section 9.5.b). If the characteristic variety of \(\tilde{\mathcal{M}}\) is equal to \(\Lambda \times \mathcal{C}_{\Lambda}\), with \(\Lambda\) Lagrangian in \(T^*X\), there exists a stratification of the support of \(\tilde{\mathcal{M}}\) by locally closed sub-manifolds \(Z_i^o\) with analytic closure \(Z_i\), such that \(\Lambda \subset \bigsqcup_i T^*_Z X\). Then \(A\) is chosen to be transversal to every \(Z_i^o\). In particular, since \(A\) has codimension two, it does not intersect any zero- and one-dimensional
stratum. Moreover, for every \( i \), the blow-up \( Z_{iA} \) of \( Z \) contains \( (A \cap Z_i) \times \mathbb{P}^1 \). This implies that \( Z_{iA} = \pi^{-1}(Z_i) \).

**Claim.** The inclusion \( \iota : X_A \hookrightarrow X \times \mathbb{P}^1 \) is non-characteristic for \( \nu_{tp}^*\tilde{M} \).

According to this claim, the pullback functor \( \gamma^* \) is defined as in Section 12.7.18.

**Proof of the claim.** Since \( p \) is a projection, the characteristic variety of \( \nu_{tp}^*\tilde{M} \) is contained in the union of the sets \( T_{Z(t)}^*(X \times \mathbb{P}^1) \times \mathbb{C} \).

- Away from \( A_1 = A \times \mathbb{P}^1 \), \( \iota \) is the graph inclusion of a map to \( \mathbb{P}^1 \) and, in a local setting, we are reduced to proving the claim for the inclusion \( \iota_U : U = U \times \{0\} \hookrightarrow U \times \mathbb{C} \) and the projection \( p : U \times \mathbb{C} \to \mathbb{C} \), where the claim is obviously true.

- Let us now consider the neighbourhood of a point of \( A_1 = A \times \mathbb{P}^1 \) in \( X \times \mathbb{P}^1 \). Since \( A \) is non-characteristic with respect to each \( Z_i \), so is \( A_1 \) with respect to each \( Z_i \times \mathbb{P}^1 \) — and therefore so is \( X_A \) near any point of \( A_1 \), since in such a point the space \( T_{X_A}^*(X \times \mathbb{P}^1) \) is contained in \( T_{X_A}^*(X \times \mathbb{P}^1) \). The claim is then also true along \( A_1 \). \( \square \)

As a consequence of Theorem 9.5.5, the characteristic variety of \( \nu_{tp}^*\tilde{M} = \nu_{tp}^*\nu_{tp}^*\tilde{M} \) is contained in the union of the sets \( T_{Z(t)}^*X_A \times \mathbb{C} \).

We also claim that, for every \( t \in \mathbb{P}^1 \), the inclusion \( A \times \{t\} \hookrightarrow X_A \) is non-characteristic with respect to \( \nu_{tp}^*\tilde{M} \). Indeed, by the choice of \( A \), for every \( Z \) as above, the intersection of \( T_{A \times \{t\}}^*(X \times \mathbb{P}^1) \) with \( T_{Z(t)}^*(X \times \mathbb{P}^1) \) is contained in the zero-section of \( T^*(X \times \mathbb{P}^1) \). As we have \( T_{A \times \{t\}}^*(X \times \mathbb{P}^1) = (T^*_{\{t\}})^{-1}(T_{A \times \{t\}}^*X_A) \), it follows that \( T_{A \times \{t\}}^*X_A \cap T_{Z(t)}^*X_A \subset T_{Z(t)}^*X_A \).

This implies that, for every \( t \in \mathbb{P}^1 \), the inclusion \( f^{-1}(t) \hookrightarrow X_A \) is non-characteristic for \( \nu_{tp}^*\tilde{M} \) near any point \( (x_o, t) \in A \times \{t\} \) since \( A \times \{t\} \) is contained in \( f^{-1}(t) \).

Let us fix a point \( x_o \in A \) and let \( g = 0 \) be a local equation of the hyperplane \( f = t \) near \( x_o \). We will prove strict \( \mathbb{R} \)-specializability of \( \gamma^* \pi^*M \) along \( (f - t) \) and we will identify \( \psi_{f^{-1}(\gamma^*\pi^*M)} \) near \( (x_o, t) \in A \times \{t\} \) with \( \psi_gM \).

Since \( f \) is smooth, we can locally consider good \( V \)-filtrations along \( (f - t) \) to compute \( \psi_{f^{-1}(\nu_{tp}^*\tilde{M})} \) (provided that strictness holds). Arguing as in the beginning of the proof of Proposition 9.5.8, one obtains that \( \nu_{tp}^*\tilde{M} \) is specializable along \( f = t \) and that there exists a good \( V \)-filtration for which \( \text{gr}_{f^{-1}(t)}^*\tilde{M} = \nu_{f^{-1}(t)}^*\nu_{f^{-1}(t)}(\nu_{tp}^*\tilde{M}) \). The latter module is equal to \( \nu_{f^{-1}(t)}(\nu_{tp}^*\tilde{M}) \), which itself is equal to \( \psi_g\tilde{M} \), as \( \tilde{M} \) is assumed to be strictly non-characteristic with respect to \( \gamma \); it follows that \( \nu_{tp}^*\tilde{M} \) is so with respect to \( (f - t) \). A similar argument is used to identify the sesquilinear pairings. The identification of the sesquilinear dualities \( S \) is straightforward, as they all both equal to \( (\text{Id}, \text{Id}) \).

Using the identification above near the axis, and the properties assumed for \( (M, S) \) on and out of the axis, we get all properties asserted for \( \gamma^*\pi^*(M, S) \) along any fiber \( (f^{-1}(t)) \). This concludes the first step.
Step two. We now prove that the pushforward $\tau\pi_*^{(0)}(\tau\pi^*M)$ decomposes as a direct sum in $\mathcal{D}$-Triples($X$), one summand being $M$.

Let us first check that this is a local statement on $X$. If such a decomposition exists locally, then $\tau\pi_*^{(0)}(\tau\pi^*M) = M \oplus M_1$ locally, with $M_1$ supported on $A$. We need to prove that this decomposition is unique, in order to glue it along $X$ (along $A$ in fact, since $\pi$ is an isomorphism away from $A$). Let $g$ be a local equation for the hyperplane $f = t$ near a point $x_0 \in A$. We claim that $\pi_*^{(0)}(\pi^*(\tilde{M}))$ is strictly $\mathbb{R}$-specializable along $(g)$. Indeed, we have seen in Step two that $\pi_*^*(\tilde{M})$ is strictly $\mathbb{R}$-specializable along $(f - t)$ and we have identified locally $\psi_{f - t}(\pi^*(\tilde{M}))$ with $\psi_t\tilde{M}$ (and we have a strict non-characteristic property, so that $\phi_{f - t,1}(\pi^*(\tilde{M}))$ is zero). We have also used that $\pi : \{f = t\} \to \{g = 0\}$ is an isomorphism. By the pushforward theorem 9.8.8 or 10.6.3, we conclude that $\pi_*^{(0)}(\pi^*(\tilde{M}))$ is strictly $\mathbb{R}$-specializable along $(g)$. Since $\tilde{M}$ has pure support $Z$, if $\pi_*^{(0)}(\pi^*(\tilde{M}))$ decomposes locally as $M \oplus M_1$ with $M_1$ supported in $A$, hence in $\{g = 0\}$, then we can apply Proposition 9.7.2 to conclude that there is no non-zero morphism $M \to M_1$ and $M_1 \to \tilde{M}$, and thus the local decomposition of $\tilde{M}$ is unique. Similarly, according to Lemma 12.3.9, any sesquilinear pairing between $M$ and $M_1$ is zero, hence $\pi_*^{(0)}(\pi^*(\tilde{M}))$ decomposes uniquely as $\mathfrak{s} \oplus \mathfrak{s}_1$.

Let us then consider the local statement near $(x_0, t_0)$, that we can assume to belong to $A \times \mathbb{P}^1$, as $\pi$ is an isomorphism outside of $A$. Let $g$ be a local equation of a hyperplane containing $A$.

We claim that $\pi_*^{(0)}(\pi^*(\tilde{M}))$ is strictly non-characteristic along both components of $g \circ \pi = 0$ and their intersections. The components consist of
- the germ at $x_0$ of the hyperplane $f = t_0$ containing $A$, for which the assertion has been proved in Step one,
- the germ at $(x_0, t_0)$ of $A \times \mathbb{P}^1$; by considering the right square in (14.6.1), the assertion follows from the property that $\tilde{M}$ is strictly non-characteristic along $A$, since $A_1 \to A$ is smooth;
- the germ at $(x_0, t_0)$ of $A \times \{t_0\}$, for which we apply the same argument as the previous one.

We can therefore apply the results of Section 12.7.20 together with Remark 14.2.3(4). They show that $\pi_*^{(0)}(\pi^*(\tilde{M}))$ satisfies $14.2.2(1)_{g \circ \pi}$ and $2(2)_{g \circ \pi}$.

Arguing as in Proposition 14.5.1 (this is permissible due to the inductive assumption (14.3.1) with $Z_A \to Z$ having dimension $\leq n - 1$), we conclude that $\pi_*^{(0)}(\pi^*(\tilde{M}), N')$ is strict and satisfies $14.2.2(1)_{g}$ and $2(2)$ in the sense of Lemma 14.5.5. We can cover $A$ by finitely many open sets where we can apply the previous argument.

Let us set $M_0 := \pi_*^{(0)}(\pi^*M)$. We note that, as $N'^2 = 0$, $M_0 = P_{\pi_*^{(0)}(\pi^*M)}$ is strict and satisfies $14.2.2(1)_{g}$ and $2(2)_{g}$. By applying 14.2.17(10) to the quiver $(\psi_{g,1} M_0, \phi_{g,1} M_0, c, v)$ and arguing as in the proof of $(10)_c \Rightarrow (1)_g$, we find that $M_0$ is $S$-decomposable along $(g)$. We will identify $M$ with a direct summand of it.

Let us set $M_0 = (M_0, s_0)$. It decomposes therefore as $M_1 \oplus M_2$, with $M_2$ supported on $g^{-1}(0)$ and $M_1$ being a middle extension along $(g)$. By Proposition 8.7.24, there
is an adjunction morphism $\tilde{M} \rightarrow \tilde{M}_0$. This morphism is an isomorphism away from $A$, hence from $g^{-1}(0)$, and is injective, as $\tilde{M}$ has no coherent submodule supported on $g^{-1}(0)$. Its image is thus contained in $\tilde{M}_1$.

At this point, we cannot assert that the image is equal to $\tilde{M}_1$, since the middle extension property 9.7.2(2) of $\tilde{M}_1$ only implies the vanishing of some quotient modules, and not all of them a priori. Nevertheless, the morphism $M \rightarrow M_1$ between the underlying $\mathcal{D}_X$-modules is an isomorphism (since no restriction occurs in 9.7.2(2) for $\mathcal{D}_X$-modules). It follows then from Proposition 12.3.7 applied to any germ of hyperplane containing $A$ that $s = s_1$. It also follows that the cokernel of $\tilde{M} \rightarrow \tilde{M}_1$ is of $z$-torsion.

We thus have a monomorphism of Hermitian pairs $M \rightarrow M_1$. It is strictly $R$-specializable along $(g)$, since the associated nearby and vanishing cycle morphisms are morphisms in $\text{pHLM}(X, w - 1)$ or $\text{pHLM}(X, w)$. Therefore, this morphism is strict, according to Corollary 10.8.5. The cokernel, being strict and of $z$-torsion, must then vanish, and $M \cong M_1$, as wanted.

**Step three.** We will consider some properties of the theorem for the map $a_{X_A} : X_A \rightarrow \text{pt}$ and the object $\gamma\pi^*(M, S)$, that we consider as a pre-polarized object of $\mathcal{D}$-Triples $(X_A)$ since we do not know that it is a polarized Hodge module of weight $w$. We aim at showing that

$$\left( \bigoplus_k \tau a_{X_A}^k (\gamma \pi^* M), N + N' \right)$$

is a graded Hodge-Lefschetz structure of weight $w$ and type $-1$.

The support of $\gamma \pi^* \tilde{M}$ is $\pi^{-1}Z$, which is equal to the blow-up $Z_A$ of $Z$ as we have seen above, and the fibers of $f|_{Z_A}$ all have dimension $n - 1$ ($n = \dim Z$). According to Step one and to Assumption (14.3.1) $(n - 1, 0)$, we can argue as in Section 14.5 to obtain that $(\bigoplus_{r \geq 0} f^*_r (\gamma \pi^*(M, S)), N)$ is a polarized object of the category $\text{pHLM}(\mathbb{P}^1, w; -1)$. Recall (see Remark 8.7.23 and Remark ??) that there exists a Leray spectral sequence for the composition $a_{X_A} = a_{\mathbb{P}^1} \circ f$, with $E_2$ terms

$$E^{i,j}_2 = \tau a_{X_A}^i (\gamma \pi^* M) \implies \bigoplus_k \tau a_{X_A}^k (\gamma \pi^* M).$$

Note that, since $\dim \mathbb{P}^1 = 1$, we have $E_2^{i,j} = 0$ for $j \neq -1, 0, 1$. We claim that this spectral sequence degenerates at $E_2$. By induction on $r \geq 2$, we assume that $E_r = E_2$ and we will prove that

$$d_r : E^{i,j}_2 \rightarrow E^{i+r,j-r+1}_2$$

under work
is zero. By the vanishing properties, this is clear if \( j = -1 \), as well as if \( j = 0 \) and \( r \geq 3 \), and if \( j = 1 \) and \( r \geq 4 \). From the commutative diagrams

\[
\begin{array}{c}
\begin{array}{c}
E_2^{i,0} \xrightarrow{d_2} E_2^{i+2,-1} \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
0 \xrightarrow{d_2} E_2^{i+2,1}(1) \\
\end{array}
\end{array}
\begin{array}{c}
\begin{array}{c}
E_2^{i,-1} \xrightarrow{d_2} 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
E_2^{i,1}(1) \xrightarrow{d_2} E_2^{i+3,-1}(1) \\
\end{array}
\end{array}
\]

we obtain \( d_2 = 0 \). Therefore, \( E_3 = E_2 \). It remains to be proved that \( d_3 : E_2^{i,1} \rightarrow E_2^{i+3,-1} \) is zero, a property which follows as above from the commutative diagram

\[
\begin{array}{c}
\begin{array}{c}
E_2^{i,-1} \xrightarrow{d_3} 0 \\
\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
E_2^{i,1}(1) \xrightarrow{d_3} E_2^{i+3,-1}(1) \\
\end{array}
\end{array}
\]

Therefore, the Leray filtration \( \text{Ler}^*, a^k_{\tau A_*} \cap \pi^* M \) attached to this spectral sequence satisfies in particular the following properties:

(a) \( \text{gr}_j^1 \text{Ler}^* a^k_{\tau A_*} \cap \pi^* M = \tau^j a^k_{\tau A_*} \cap \pi^* M \) for every \( k \geq 0 \) (because \( \text{Ler}^1 = \text{gr}_1^1 \text{Ler} \));

(b) \( N_{k+1} : \text{Ler}^* a^k_{\tau A_*} \cap \pi^* M \rightarrow \text{Ler}^* a^{k+2}_{\tau A_*} \cap \pi^* M \) is an isomorphism for every \( k \geq 0 \) (because \( \text{Ler}^1 = \text{gr}_1^1 \text{Ler} \));

(c) \( \text{Ker}(N : \tau^k a^k_{\tau A_*} \cap \pi^* M) \rightarrow \tau^k a^{k+2}_{\tau A_*} \cap \pi^* M) \) is contained in \( \text{Ler}^0 \tau^k a^k_{\tau A_*} \cap \pi^* M \) for every \( k \) (because \( N : \text{gr}_1^1 \tau^k a^k_{\tau A_*} \cap \pi^* M \rightarrow \text{gr}_1^1 \tau^{k+2} a^{k+2}_{\tau A_*} \cap \pi^* M \) for every \( k \)).

On the other hand, by Assumption (14.3.1) \((1,0) \) (i.e., by Theorem 7.4.19 applied to every primitive part relative to \( N \)), \( \bigoplus_j \bigoplus_i \tau^j a^i_{\tau A_*} \cap f^* i \cap \pi^* (M, Q), N, N' \) is a polarized bigraded Hodge-Lefschetz structure of weight \( w \). It follows from Proposition 3.2.34 that

\[
\left( \bigoplus_k \left( \bigoplus_{i+j=k} \tau^j a^i_{\tau A_*} \cap f^* i \cap \pi^* (M, Q) \right), N + N' \right)
\]

is a polarized graded Hodge-Lefschetz structure of weight \( w \). We conclude from (a) above that the object

\[
\left( \bigoplus_k a^k_{\tau A_*} \cap \pi^* M, N + N' \right)
\]

is an extension of graded Hodge-Lefschetz structures of weight \( w \) and, by Remark 5.3.2, is itself such an object.

**Step four.** As \( N' \) vanishes on \( M \), we conclude from Step (??) that \( (a_{\tau A_*} + M, N) \) is a direct summand of \((\tau a_{\tau A_*} \cap \pi^*, N + N')\). From Step (??) and \([\text{Del68}]\) we have a (non canonical) decomposition \( a_{\tau A_*} \cap \pi^* \simeq \oplus_k a^k_{\tau A_*} \cap \pi^*[-k] \). Therefore, this decomposition can be chosen to induce a decomposition \( a_{\tau A_*} \cap \pi^* \simeq \oplus_k a^k_{\tau A_*} \cap \pi^*[-k] \). In particular, \( \bigoplus_k a^k_{\tau A_*} \cap \pi^* M, N \) is a graded Hodge-Lefschetz module of weight \( w \), being a direct summand of the graded Hodge-Lefschetz module \( (\oplus_k a^k_{\tau A_*} \cap \pi^*, N + N') \).
Step five. It remains to show the polarization property. In order to do so, we will use the Fact ?? in its graded Lefschetz form given by Remark ???. Let us denote by \( P a(X,+,\mathbb{C}) \) the \( N \)-primitive part of \( a^0(X,+,\mathbb{C}) \), that is, the kernel of \( N \) acting on the previous space. Then \( P^0_0 := (P a(X,+,\mathbb{C}), N) \) remains a (simply) graded Hodge-Lefschetz module of weight \( w = 0 \), polarized by the family of sesquilinear dualities \( a^0(X,+,\mathbb{C}) \).

According to Remark ??, we get the desired property if we show that

1. \( (\oplus_k a^k_{\mathcal{X},+}, M, N) \) is a sub graded Hodge-Lefschetz module of \( P^0_0 \),
2. the polarization of \( P^0_0 \) induces the family \( (a^k_{\mathcal{X},+}, S)_k \).

By definition, for \( k \geq 0 \), \( P a(X,+,\mathbb{C}) \) is the kernel of \( N^{k+1} \) acting on \( a^k_{\mathcal{X},+} \). It follows from (b) that \( P a(X,+,\mathbb{C}) \cap \text{Ler}^0 a^k_{\mathcal{X},+} = \{0\} \). On the other hand, \( a^k_{\mathcal{X},+} \) is contained in \( \text{Ler}^0 a^k_{\mathcal{X},+} \) as \( N^k \) vanishes on \( a^k_{\mathcal{X},+} \) and according to (c). Therefore, \( P a(X,+,\mathbb{C}) \) is contained in \( \text{Ler}^0 a^k_{\mathcal{X},+} \), and more precisely in the bimodule part \( \text{Ker} N^k \cap \text{Ker} N^{k+1} \subset a^0(X,+,\mathbb{C}) \). This gives (1).

As we assume that \( S = (\text{Id}, \text{Id}) \) and \( M = M^* \), the sesquilinear duality \( a^k_{\mathcal{X},+} S \) is nothing but the identification \( (a^k_{\mathcal{X},+} M)^* = a^k_{\mathcal{X},+} M \) deduced from (?). Similarly, the sesquilinear duality on \( P^0_0 \) is induced from the identification \( (a^0(X,+,\mathbb{C}) = a^0(X,+,\mathbb{C}) \).

14.7. Exercises

Exercise 14.1. Show that if the conditions in Definition 14.2.2 hold for a function \( g \), they hold for \( g^r \) for any \( r \in \mathbb{N}_0 \). [Hint: Use the example of Section 9.9.a.] Conclude that, if \( n = 1 \), Definition 14.2.2 reduces to Definition 7.4.8.

14.8. Comments

The relation between Hodge theory and the theory of nearby or vanishing cycles in dimension bigger than one starts with the work of Steenbrink [Ste76, Ste77]. It concerns one-parameter families of projective varieties, regarded as proper functions from a complex manifold to a disc. A canonical Hodge structure is constructed on the cohomology of the nearby fiber of a singular fiber of the family by means of replacing the special fiber with a divisor with normal crossings and by computing the nearby or vanishing cohomology in terms of a logarithmic de Rham complex, in order to apply Deligne’s method in [Del71b]. This gives a geometric construction of Schmid’s limit mixed Hodge structure in the case of a variation of geometric origin. The need of passing from the assumption of unipotent monodromy, as used in the work of Schmid [Sch73] to the assumption of quasi-unipotent monodromy is justified by this geometric setting. This leads Steenbrink [Ste77] to developing the notion
of logarithmic de Rham complex in the setting of V-manifolds. Steenbrink also obtains, as a consequence of this construction, the local invariant cycle theorem and the Clemens-Schmid exact sequence. We can regard this work as the localization of Hodge theory in the analytic neighbourhood of a projective variety.

The work of Varchenko [Var82] and others on asymptotic Hodge theory has localized even more Hodge theory. This work is concerned with an isolated singularity of a germ of holomorphic function and it constructs a Hodge-Lefschetz structure on the space of vanishing cycles of this function, by taking advantage that the vanishing cycles are supported at the isolated singularity, which is trivially a projective variety. The construction of Varchenko has been later analyzed in terms of $D$-modules by Pham [Pha83], Saito [Sai83b, Sai83a, Sai84, Sai85] and Scherk-Steenbrink [SS85]. It is then natural to consider the cohomology of the vanishing cycle sheaf of a holomorphic function on a complex manifold whose critical locus is projective, but possibly not the special fiber of the function, and to ask for a mixed Hodge structure on it.

The theory of polarizable Hodge modules, as developed by Saito in [Sai88], emphasizes the local aspect of Hodge theory, by constructing a category defined by local properties in a way similar, but much more complicated, to the definition of the category of variations of Hodge structure. It can then answer the question above. This idea has proved very efficient, eventually allowing to use the formalism of Grothendieck’s six operations in Hodge theory. Many standard cohomological results, like the Clemens-Schmid exact sequence and the local invariant cycle theorem, can be read in this functorial way.

The definition of complex Hodge modules as developed here, not relying on a $Q$-structure and on the notion of a perverse sheaf, is inspired by the extension of the notion of polarizable Hodge module to twistor theory, as envisioned by Simpson [Sim97], and achieved by Sabbah [Sab05] and Mochizuki [Moc07, Moc15], although the way the sesquilinear pairing is used on both theories is not exactly the same. As already mentioned in the comments of Chapter 12, the idea of using sesquilinear pairings in the framework of germs of holomorphic functions was developed by Barlet [Bar85] with the perspective of making the link between asymptotic Hodge theory on the vanishing cycles of germs of functions with isolated singularities and the classical notion of polarization. On the other hand, the idea of using sesquilinear pairings in the framework of holonomic $D$-modules is due to Kashiwara [Kas87].