WHAT ARE IRREGULAR PERVERSE SHEAVES?

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Abstract. The talk will first explain the notion of Stokes-filtered local system as introduced by Deligne and generalized in higher dimension by Mochizuki, together with the Riemann-Hilbert correspondence for good meromorphic connections (over complex numbers). It will then focus on the recent approach of Kuwagaki, relying on the work of d'Agnolo and Kashiwara, which describes a category of irregular constructible complexes and of irregular perverse sheaves, and proves functorial properties and Riemann-Hilbert correspondence for arbitrary holonomic D-modules (over complex numbers).

1. Introduction

Setting.

- X: cplx mfld or smooth alg. var. $/\mathbf{C}$,
- M: a holonomic $\mathscr{D}_X$-module, e.g. M a free $\mathscr{O}_X$-module with integrable conn.

Question. Find the right category of "irregular constructible complexes" resp. "irregular perverse sheaves" in order to have a Riemann-Hilbert correspondence (equiv. of categories)

$$\begin{array}{ccc} \mathsf{Mod}_{\mathrm{hol}}(\mathscr{D}_X) & \xrightarrow{\ \mathrm{DR}^{\mathrm{irr}}\ } & \mathrm{Perv}^{\mathrm{irr}}(\mathbf{C}_X) \\ & {\scriptstyle\mathrm{DR}}\searrow \quad \swarrow & \\ & \mathrm{Perv}(\mathbf{C}_X) & \end{array}$$

and similarly with $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}}$ and $\mathsf{D}^{\mathrm{b}}_{\mathrm{irr}}$.

For example, this would lead to a natural notion of a Betti structure, by extending the definition to $\mathrm{Perv}^{\mathrm{irr}}$ for any base field. [In fact, the notion of a Betti structure has already been introduced by Mochizuki.]

2. Stokes-filtered local systems in dim. one (Deligne)

X is a complex disc, f a meromorphic function with pole at the origin only. $\mathscr{E}^f := (\mathscr{O}_X(*0), \mathrm{d} + \mathrm{d}f)$. $\mathrm{DR}\,\mathscr{E}^f_{|X^*} = \mathbf{C}_{X^*} = L$. How to enrich the data of L in order to recover $\mathscr{E}^f$?

Deligne's approach, based on asymptotic theory of differential equations: L comes equipped with a richer structure.

- Extend L to the constant sheaf $\widetilde{L}$ on the real blow up $\varpi : \widetilde{X} \to X$ (polar coordinates).
- For any (multivalued) meromorphic function g, set $L^f_{\leqslant g} = \mathrm{DR}^{\mathrm{mod}}\,\mathscr{E}^{f+g}$. A local section is $e^{-(f+g)}$, hence $L^f_{\leqslant g|S^1} = \mathbf{C}_{I_g}$, where I_g is a union of open intervals of S^1 depending on g (and on f, but f is fixed).
- This is a filtration by $\mathbf{R}$-constructible subsheaves indexed by the set of meromorphic functions: if $g \leqslant g'$ near $\theta \in S^1$, i.e., $e^{g-g'}$ has moderate growth near θ, then $e^{g-g'}\cdot$ is an injective morphism $L^f_{\leqslant g} \to L^f_{\leqslant g'}$.
- If f and f' give rise to the same filtration, then choosing $g = -f'$, we find that $\mathrm{DR}^{\mathrm{mod}}\,\mathscr{E}^{f-f'} = \mathrm{DR}^{\mathrm{mod}}(\mathscr{O}_X, \mathrm{d}) = \mathbf{C}_{\widetilde{X}}$, meaning that the function $e^{f'-f}$ has moderate growth all around the origin, hence $f - f'$ is holomorphic.
- In other words, the filtration gives back f up to a holomorphic function.

***Definition* (Stokes-filtered local system).** A Stokes filtration on a local system L on X^* is a filtration $\widetilde{L}_\bullet$ of $\widetilde{L}$ indexed by the set of ramified meromorphic functions with pole at 0 such that, near each $\theta \in S^1$, $\widetilde{L}_\bullet \simeq \bigoplus_f \widetilde{L}^f_\bullet$.

***Theorem* (Deligne).** *RH correspondence: free $\mathscr{O}_{X,0}(*0)$-modules of finite rank with connection $\longleftrightarrow$ Stokes-filtered local systems.*

What have we learned?

- We expect to have objects with a filtration of some kind as irregular constructible sheaves.
- The local splitting condition is important since it ensures that the category is abelian.
- The growth conditions impose us to introduce semi-analytic sets (intervals on S^1), and more generally subanalytic sets.

Drawback. The index set of the filtration is very complicated and much linked with the geometry. Not clear how to extend in higher dimension when the geometry is complicated.

3. The work of Kashiwara-Schapira and Kashiwara-D'Agnolo

Kashiwara and Schapira have managed to input the notion of moderate growth within the framework of sheaf theory. They have used the formalism of Ind sheaves for that purpose (or the formalism of sheaves on the subanalytic site of a complex manifold). From the point of view of D-module theory, this enables them to recover objects like $L^f_{\leqslant 0}$ above. This is however not enough to obtain a Riemann-Hilbert correspondence. D'Agnolo had the idea to add to these data a filtration parameter in $\mathbf{R}$. This led D'Agnolo and Kashiwara to the theory of enhanced Ind sheaves and the category $\mathsf{E}^{\mathrm{b}}_{\mathbf{R}\text{-c}}(\mathrm{I}k_X)$ of $\mathbf{R}$-constructible enhanced Ind sheaves (where k is any field).

***Theorem* (Enhanced RH correspondence of D'A-K).** *There exists a functor* DR^{E} : $\mathsf{D}^{\mathrm{b}}_{\mathrm{hol}} \mapsto \mathsf{E}^{\mathrm{b}}_{\mathbf{R}\text{-}c}(\mathrm{I}\mathbf{C}_X)$ *compatible with the usual operations which is fully faithful.*

***Definition* (First definition of an irregular perverse sheaf).** The essential image of this functor is the category $\mathsf{D}^{\mathrm{b}}_{\mathrm{irr}\mathbf{C}\text{-c}}(\mathbf{C}_X)$ we look for.

Good point. There is a t-structure on $\mathsf{E}^{\mathrm{b}}_{\mathbf{R}\text{-c}}(\mathrm{I}\mathbf{C}_X)$ and the enhanced RH correspondence is t-exact.

Drawback. How to define $\mathsf{D}^{\mathrm{b}}_{\mathrm{irr}\mathbf{C}\text{-c}}(k_X)$ when k not isomorphic to a subfield of $\mathbf{C}$?

***Theorem* (Valuative criterion, T. Mochizuki).** *An object* $K \in \mathsf{E}^{\mathrm{b}}_{\mathbf{R}\text{-}c}(\mathrm{I}\mathbf{C}_X)$ *is in* $\mathsf{D}^{\mathrm{b}}_{\mathrm{irr}\mathbf{C}\text{-c}}(\mathbf{C}_X)$ *iff for any holomorphic germ* $\gamma : (\Delta, 0) \to (X, x)$, γ^*K *is in* $\mathsf{D}^{\mathrm{b}}_{\mathrm{irr}\mathbf{C}\text{-c}}(\mathbf{C}_{\Delta,0})$.

One understands better the category $\mathsf{D}^{\mathrm{b}}_{\mathrm{irr}\mathbf{C}\text{-c}}(\mathbf{C}_X)$ in the case of curves. Kashiwara has given an explicit relation with the notion of Stokes filtered local system of Deligne. Hence, one can take the valuative criterion as a *definition* for $\mathsf{D}^{\mathrm{b}}_{\mathrm{irr}\mathbf{C}\text{-c}}(k_X)$ for any k. The analogous valuative criterion holds for the "regular" setting: an object K of $\mathsf{D}^{\mathrm{b}}_{\mathbf{R}\text{-c}}(k_X)$ belongs to $\mathsf{D}^{\mathrm{b}}_{\mathbf{C}\text{-c}}(k_X)$ iff γ^*K belongs to $\mathsf{D}^{\mathrm{b}}_{\mathbf{C}\text{-c}}(k_{\Delta,0})$ for every γ. But we would like to have a definition looking like the usual definition of a constructible complex (or perverse sheaf).

4. Filtered sheaves

Idea due to d'Agnolo, and then developed by d'Agnolo-Kashiwara, and used by Kuwagaki: Take $\mathbf{R}$ as the index set, independent of the geometry. Let us consider $\mathscr{E}^f$ as before. We define the filtration of the constant sheaf $k_{X^*} = L$ by the *exponential-growth locus*, indexed by $\mathbf{R}$.
Filter L by the subsheaves $L^f_{\leqslant a}$ such that $L^f_{\leqslant a,x} = k_{\{\mathrm{Re}(f)\geqslant -a\},x}$ (for x general), i.e., $L^f_{\leqslant a,x} \neq 0$ on the locus where the horizontal section e^{-f} of $\mathscr{E}^f$ has absolute value bounded by e^a. The precise definition of $L^f_{\leqslant a}$ is,

$$L^f_{\leqslant a} = p_*\Gamma_{X^*\times[-a,\infty)}k_{\{t\geqslant \mathrm{Re}(f)\}}.$$

- Need to consider filtrations up to a shift, because one can add a holomorphic function to f without changing $\mathscr{E}^f$ up to isomorphism.

• Need to work in an abelian ambient category to develop the formalism of derived functors. By the Rees trick, one replaces an $\mathbf{R}$-filtered sheaf by an $\mathbf{R}$-graded sheaf. Base ring: $\Lambda = k[\mathbf{R}_+]$.

$\rightsquigarrow$ Ambient abelian category $\mathsf{Mod}^{\mathfrak{I}}(\Lambda_{(S,\overline{S})})$.

• Category of topological pairs $(S,\overline{S})$: $\overline{S}$ topological space, S open subset, $\partial S := \overline{S} \smallsetminus S$; morphisms $(S,\overline{S}) \to (S',\overline{S}')$: morphisms $\overline{S} \to \overline{S}'$ sending S to S'. E. g.: $(S,\overline{S}) \to (\overline{S},\overline{S})$.

• One has to restrict the notion of open covering and site when working with sheaves on $(S,\overline{S})$, but I will neglect this in the talk (think of the subanalytic site).

• Objects of $\mathsf{Mod}^{\mathfrak{I}}(\Lambda_{(S,\overline{S})})$ are direct systems of sheaves on $(S,\overline{S})$ indexed by $\mathbf{R}$. An $\mathbf{R}$-filtration is such a direct system where all transition morphisms are monomorphisms. Denote by T^b ($b \in \mathbf{R}_+$) the transition morphism between F_a and F_{a+b}. One thus regards such direct systems as graded modules over $\Lambda := \mathbf{C}[\mathbf{R}_+]$.

• Example: $\varphi : S \to \mathbf{C}$ a continuous function. The sheaf $\Lambda^\varphi \in \mathsf{Mod}^{\mathfrak{I}}(\Lambda_{(S,\overline{S})})$ is defined by

$$\Lambda^\varphi_a = p_*\Gamma_{S\times[-a,\infty)}\mathbf{C}_{\{t\geqslant \mathrm{Re}(\varphi)\}}.$$

We note that Λ^φ is strict, i.e., $\Lambda^\varphi_a \to \Lambda^\varphi_{a+b}$ is injective for all $b \geqslant 0$.

• Morphisms in $\mathsf{Mod}^{\mathfrak{I}}(\Lambda_{(S,\overline{S})})$: in order to work with filtrations up to a shift, one modifies morphisms, not objects. Set $\mathrm{Hom}_{\mathbf{R}}$ be the set of graded morphisms of some degree in $\mathbf{R}$. Then $\mathrm{Hom}^{\mathfrak{I}}$ is defined as $\mathbf{C} \otimes_{\mathbf{C}[\mathbf{R}_+]} \mathrm{Hom}_{\mathbf{R}}$, i.e., "one sets the action of T^b to 1 for any b". It means that the shift by b acts as the identity.

• Kuwagaki shows that the formalism of six functors holds for this category.

5. Irregular constructible sheaves

Let $\mathscr{S}$ be an analytic stratification of X. For each (locally closed) stratum S, consider the inclusion $\iota_S : (S,\overline{S}) \hookrightarrow (X,X)$.

• $F \in \mathsf{Mod}^{\mathfrak{I}}(\Lambda_{(S,\overline{S})})$ is *irregular locally constant on* $(S,\overline{S})$ if

(1) $\forall x \in S$, $F_x \simeq \Lambda_x$,

(2) $\forall x \in \partial S$, $\exists\, \pi : (S = S',\overline{S}')_{\pi^{-1}(x)} \to (S,\overline{S})_x$ projective modification with $(S',\overline{S}')$ a log-smooth pair, and $\forall x' \in \pi^{-1}(x)$, $\exists$ finite family $\Phi_{x'}$ of multivalued meromorphic functions with poles in $\partial S'$ such that, on small open sets (multisectors) $\pi^*F \simeq \bigoplus_{\varphi\in\Phi_{x'}} \Lambda^\varphi$.

This mimics the asymptotic theorem for good meromorphic connections on $(S',\overline{S}')$.

Lemma. *The category of irregular locally constant sheaves on* $(S,\overline{S})$ *is abelian.*

This leads to the abelian category of irregular constructible sheaves and the category $\mathsf{D}^{\mathrm{b}}_{\mathrm{irrc}}(\mathsf{Mod}^{\mathfrak{I}}(\Lambda_X))$. Mimicking the notion of perverse sheaf, one defines the notion of irregular perverse sheaf.

6. Two main theorems

***Theorem* (Kuwagaki).** *If* $k=\mathbf{C}$, $\mathsf{D}^{\mathrm{b}}_{\mathrm{irrc}}(\mathsf{Mod}^{\mathfrak{I}}(\Lambda))$ *is stable by proper pushforward.*

The proof is analytic (and not topological), since it uses the Riemann-Hilbert correspondence below. One can interpret it as the "degeneration at E_1 of the spectral sequence attached to the filtration by the exponential-growth locus". This difficulty already occurs with Deligne's approach of Stokes-filtered local systems.

***Theorem* (RH correspondence, Kuwagaki).** *There is a natural functor*

$$\mathsf{Mod}_{\mathrm{hol}}(\mathscr{D}_X) \longrightarrow \mathrm{Perv}^{\mathrm{irr}}(\mathbf{C}_X)$$

which is an equivalence of categories.

The RH correspondence strongly relies on the RH correspondence of D'Agnolo-Kashiwara, since there is (at the moment) no de Rham functor from $\mathsf{Mod}_{\mathrm{hol}}(\mathscr{D}_X)$ to $\mathrm{Perv}^{\mathrm{irr}}(\mathbf{C}_X)$. The functor in the theorem is obtained by factoring through that of D'Agnolo-Kashiwara.

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