ON THE FOURIER-LAPLACE TRANSFORM OF
A VARIATION OF HODGE STRUCTURE

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Abstract. Generalizing the notion of (variations of) Hodge structure is needed by various recent mathematical developments (e.g. Mirror symmetry and tt* geometry). Harmonic Higgs bundles with supplementary data are good candidates. In the talk, I will explain how this new structure is stable by the Fourier-Laplace transform, a result related to previous work of S. Szabo on the Nahm transform.

1. Variation of polarized Hodge structure

Let \( P = \{ p_1, \ldots, p_r, p_{r+1} = \infty \} \) be a non empty finite set of points on the Riemann sphere \( \mathbb{P}^1 \). We will denote by \( t \) the coordinate on the affine line \( \mathbb{A}^1 = \mathbb{P}^1 \setminus \{ \infty \} \). The main object of interest in this talk will be a \((\text{complex})\) variation of polarized Hodge structure on \( \mathbb{P}^1 \setminus P \).

Examples.

(1) Let \( \mathcal{V} \) be a unitary local system on \( \mathbb{P}^1 \setminus P \). This defines a complex variation of Hodge structure of type \((0, 0)\). Giving a local system is equivalent to giving a set of matrices (monodromy matrices) \( T_1, \ldots, T_r \in \text{GL}(\mathbb{C}^{\text{rk}V}) \), up to conjugation by the same invertible matrix. The local system is unitary if one can find \( T_1, \ldots, T_r \) in the unitary group.

(2) Let \( f : X \to \mathbb{P}^1 \) be a projective morphism from a smooth complex projective variety \( X \) to the projective line. Away from the critical values \( P \) of \( f \), each cohomology sheaf \( H^k(f^{-1}(t), \mathbb{C})_{t \in \mathbb{P}^1 \setminus P} \) forms a local system and underlies a variation of polarized Hodge structure.

(3) Let \( U \) be a smooth complex quasiprojective variety of dimension \( n \) (e.g. \( \mathbb{C}^n \) or \((\mathbb{C}^*)^n\)). Let \( f : U \to \mathbb{C} \) be a regular function on \( U \) (e.g. a polynomial or a Laurent polynomial). Let us assume that \( f \) is tame, that is, \( f \) has only isolated critical points on \( U \) (and we denote by \( P \) the set of critical values including \( \infty \)) and has no “critical point at infinity with finite critical value”. The local system \( H^n(U, f^{-1}(t), \mathbb{C}) \) underlies a variation of mixed Hodge structure, which contains, as a subquotient, a variation of polarized pure Hodge structure, all other
subquotients having no singular points at \( p_1, \ldots, p_r \). For the purpose of this talk, this will be as good as the pure case.

**Definition.** A polarized variation of Hodge structure of weight \( w \) on \( \mathbb{P}^1 \setminus P \) consists of a \( C^\infty \) vector bundle \( H \) on \( \mathbb{P}^1 \setminus P \) equipped with a flat connection \( D \), a decomposition \( H = \bigoplus_{p \in \mathbb{Z}} H^p \) (\( H^p \) is usually written as \( H^{p,w-p} \)) and a Hermitian metric \( h \) on \( H \), satisfying the following properties:

- the decomposition is orthogonal with respect to \( h \) and the nondegenerate \((-1)^w\)-Hermitian form \( k = \bigoplus_{p \in \mathbb{Z}} (-1)^p h|_{H^p} \) is \( D \)-flat,
- (Griffiths’ transversality)
  \[
  D'(H^p) \subset (H^p \oplus H^{p-1}) \otimes \mathcal{O}_{\mathbb{P}^1 \setminus P}^1,
  \]
  \[
  D''(H^p) \subset (H^p \oplus H^{p+1}) \otimes \mathcal{O}_{\mathbb{P}^1 \setminus P}^1.
  \]

The Hodge filtration \( F^*H \) is

\[
F^p H = \bigoplus_{q \geq p} H^q,
\]

so that \( D'F^p H \subset F^{p-1} H \otimes \mathcal{O}_{\mathbb{P}^1 \setminus P}^1 \).

Griffiths’ transversality (1) gives a decomposition \( D = D^+ + \theta \), where \( D^+ \) is unitary with respect to \( h \) and \( \theta \) is self-adjoint (Higgs field). Considering types and grading, the Higgs condition \((D^+)^\prime = 0\) is satisfied. Then \((H,(D^+)^\prime,\theta')\) is a holomorphic Higgs bundle.

Let \((V,\nabla)\) be the holomorphic bundle with connection \((\text{Ker } D'', D')\) and \( F^p V = F^p H \cap V \). We have \( \nabla F^p V \subset F^{p-1} V \otimes \mathcal{O}_{\mathbb{P}^1 \setminus P}^1 \). We can identify \((H,(D^+)^\prime,\theta')\) with \((\text{gr}_F V,\text{gr}_{F^{-1}} \nabla)\).

### 2. The Fourier-Laplace transform

#### 2.1. The twisted \(L^2\)-complex.

On \( H \) we consider the twisted connection \( D - 2dt \). Using a metric on \( \mathbb{P}^1 \setminus P \) which is equivalent to the Poincaré metric on the punctured disc near each puncture \( p_i \in P \) (hence a complete metric), and the Hermitian metric \( h \) on \( H \), we make the \( L^2 \) de Rham complex \( \mathcal{L}^2_2(\mathbb{P}^1 \setminus P, H, (D^+)^\prime + \theta' - dt, h) \).

**Theorem 1 (S. Szabo, CS).** This complex has cohomology in degree 1 at most, and this cohomology is a finite dimensional vector space, equipped in a natural way with a Hermitian metric. Moreover, it is canonically identified with the cohomology of the \( L^2 \) complex \( \mathcal{L}^2_2(\mathbb{P}^1 \setminus P, H, (D^+)^\prime + \theta' - dt, h) \).

In particular, as the metric on \( \mathbb{P}^1 \setminus P \) is complete, we can apply Hodge theory and compute this cohomology with \( L^2 \) harmonic forms.
2.2. Algebraic interpretation of the $L^2$ complex. The holomorphic bundle with flat connection $(V, \nabla)$ extends in a unique way as an algebraic bundle with flat algebraic connection and, according to results of Schmid, this extensions is obtained by considering holomorphic sections whose $h$-norm has moderate growth at $P$. Note that the twist of the de Rham complex is of no consequence at finite distance. The work of Zucker tells us that, near a puncture at finite distance, in order to compare the de Rham complex with the previous $L^2$ complex, we should replace the algebraic bundle with a $D$-module called the "intermediate" (or minimal) extension. Taking global sections on $A^1$ of this $D$-module gives a $\mathbb{C}[t]\langle \partial_t \rangle$-module $M$ which is holonomic and has regular singularities everywhere.

**Theorem 2.** The twisted algebraic de Rham complex $M e^{t\nabla e^{-t}} M \otimes dt$ has cohomology in degree 1 at most. This cohomology can be identified with that of the previous $L^2$ complex.

One can give the following interpretation of the dimension $\mu$ of this cohomology: let $\mathcal{V} = \text{Ker} \nabla$ be the local system of horizontal section of $(V, \nabla)$ (or equivalently, $(H, D)$); near each puncture $p_i \neq \infty$, define $\mu_i(\mathcal{V}) = \text{rk} \mathcal{V} - \dim \Gamma(\text{nb}(p_i)^*, \mathcal{V})$; then $\mu = \sum_i \mu_i(\mathcal{V})$.

**Examples.**

1. Given unitary matrices $T_1, \ldots, T_r$ of size $\text{rk} V$, $\mu = \text{rk} V - \dim(\text{Ker} T_i - \text{Id})$.
2. Let $f : X \to \mathbb{P}^1$ projective, $X$ smooth, $p_i$ the critical values at finite distance. Then $\mu_i^{(k)} = \dim H^k(f^{-1}(p_i), \phi_{f,p_i}(\mathbb{C}))$.
3. Let $f : U \to \mathbb{C}$ be a tame regular function. For any critical value $p_i$ of $f$, corresponding to critical points $x_i^{(1)}, \ldots, x_i^{(k)}$, the corresponding number $\mu_i$ is the sum of the Milnor numbers of $f$ at $x_i^{(j)}$, $j = 1, \ldots, k$, and $\mu$ is the total sum of Milnor numbers of $f$ at its critical points.

2.3. Rescaling parameter. We now rescale the variable $t$ with a nonzero complex parameter $\tau$. From the algebraic point of view, this consists in considering the twisted de Rham complex $\mathbb{C}[\tau, \tau^{-1}] \otimes_\mathbb{C} M \overset{\nabla_{\eta^{-\tau}}}{\longrightarrow} \mathbb{C}[\tau, \tau^{-1}] \otimes_\mathbb{C} M$. It has cohomology in degree 1 at most, and this cohomology $\hat{V}$ is a free $\mathbb{C}[\tau, \tau^{-1}]$-module of rank $\mu$. It comes equipped with an algebraic connection $\hat{\nabla}$.

The twisted $L^2$ de Rham complex also defines a flat $C^\infty$ bundle $(\hat{H}, \hat{D})$ on $\mathbb{C}^*$, equipped with a Hermitian metric $\hat{h}$.

**Theorem 3 (S. Szabo, CS).** The metric flat bundle $(\hat{H}, \hat{D}, \hat{h})$ is harmonic.

**Remark.** Up to now, the theory uses less than the variation of Hodge structure: it only uses the harmonicity property of the original Hermitian metric $h$. 
**Remark.** On the other hand, one cannot expect, in general, that the new metric has a tame behaviour at $\tau = \infty$. In particular, this implies that it does not correspond to a usual variation of polarized Hodge structure.

**Question.** What kind of a supplementary structure does the flat harmonic bundle $(\hat{H}, \hat{D}, \hat{h})$ underlie?

3. The supersymmetric index

The answer to this question (or a variant of it) has been given in 1991 by two physicists, Cecotti and Vafa. They give us two operators $\hat{W}$ and $\hat{D}$ on $\hat{H}$, where $\hat{D}$ is selfadjoint with respect to $\hat{h}$, $\hat{W}$ is $(\hat{D}^+)^\nu$-holomorphic, and which satisfy a series of differential equations:

$$\hat{W} = \hat{D},$$
$$[\theta', \hat{W}] = 0,$$
$$(\hat{D}^+)'(\hat{W}) - [\theta', \hat{D}] + \theta' = 0,$$
$$(\hat{D}^+)'(\hat{D}) + [\theta', \hat{W}] = 0.$$

These differential equations are better interpreted as an integrability condition, by adding a new variable $z$.

For instance, the operators associated to the variation of Hodge structure are $\hat{W} = 0$ and $\hat{D} = - \bigoplus_p p \mathrm{Id}_{H^p, w-p}$. On the other hand, the eigenvalues of $\hat{D}$ need not be constant.

3.1. The spectrum. At each $p_i \in \mathbb{P}^1$ is associated the spectrum of the variation of Hodge structure. For $p_{r+1} = \infty$, we call it the spectrum at infinity. In Example (3), this spectrum coincides with the Varchenko-Steenbrink spectrum of the critical points of $f$. In any case, at $p_i \neq \infty$, the corresponding polynomial has degree $\mu_i$.

Let me explain the definition of the spectrum at finite distance. I will set $SP_{p_i}(T) = \prod \gamma (T - \gamma)^{\nu(i)}$. For any $\alpha \in (-1, 0]$, let $V^\alpha$ be the holomorphic bundle on $\mathbb{A}^1$ with connection having a logarithmic pole at each $p_i$, extending $(V, \nabla)$, and such that the residue of the connection on $V^\alpha$ has eigenvalues in $[\alpha, \alpha + 1]$. If $\alpha \neq 0$ and $p \in \mathbb{Z}$, I set $\nu_{\alpha+p}^{(i)} = \dim(F^p \cap V^\alpha)/(F^{p+1} \cap V^\alpha + F^p \cap V^{>\alpha})$. When $\alpha = 0$, the definition has to be modified a little bit. At infinity, we have a similar definition, and there is also a small change to be done at $\alpha = 0$, but different from that done at finite distance.
Theorem 4. $\lim_{\tau \to \infty} \chi(\hat{\mathcal{D}}(\tau))(T) = \prod_{i=1}^{r} \text{SP}_{p_i}(T)$ and $\lim_{\tau \to 0} \chi(\hat{\mathcal{D}}(\tau))(T) = \text{SP}_{\infty}(T)$.