RELATIVE REGULAR RIEMANN-HILBERT CORRESPONDENCE

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Abstract. In previous works by the last named authors, the notion of regularity for a relative holonomic \(\mathcal{D}\)-module has been introduced, as well as that of relative constructible complex, and it has been proved that, if the parameter space has dimension one, the solution functor from the bounded derived category of relative modules with regular holonomic cohomology to that of relative constructible complexes is essentially surjective, by constructing a right quasi-inverse functor. In the present article, we prove that this functor satisfies the left quasi-inverse property.

Contents

Introduction 1
1. Preliminaries 3
2. Regular holonomic \(\mathcal{D}_{X\times S/S}\)-modules 6
3. The functor \(\text{RH}^S\) 9
4. Relative Riemann-Hilbert correspondence 15
References 21

Introduction

Let \(X\) and \(S\) be complex manifolds. In their previous work [13], the two last named authors have introduced the notion of relative regular holonomic \(\mathcal{D}_{X\times S/S}\)-modules. It encodes the notion of a \(\text{dim}\ S\)-parameter holomorphic family of regular holonomic \(\mathcal{D}_X\)-modules whose characteristic variety is bounded by a same Lagrangian subvariety of \(T^*X\).

For example, if \(S = \mathbb{C}^*\), a \(\mathcal{D}_{X\times S/S}\)-module underlying a regular mixed twistor \(\mathcal{D}\)-module is regular holonomic in this sense. In this article, we prove the Riemann-Hilbert correspondence for such modules when the parameter space \(S\) has dimension one, in the following form.

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Theorem 1. If $\dim S = 1$, the functors
\[
p\text{Sol} : \mathcal{D}_{\text{hol}}^b(D_X \times S/S) \to \mathcal{D}_{\text{C-c}}^b(p_X^{-1}\mathcal{O}_S) \\
\text{RH}^S : \mathcal{D}_{\text{C-c}}^b(p_X^{-1}\mathcal{O}_S) \to \mathcal{D}_{\text{hol}}^b(D_X \times S/S)
\]
are quasi-inverse equivalences of categories.

The functor $p\text{Sol}$ is the solution functor shifted by the dimension of $X$ (analyzed in [11, §3.3]), and the functor $\text{RH}^S$ was introduced in [13, §3.4]. A particular case of this correspondence was proved in loc. cit., namely that, if $M$ underlies a regular mixed twistor $\mathcal{D}$-module, then $M$ can be recovered from $p\text{Sol}(M)$ up to isomorphism by the formula $M \cong \text{RH}^S(p\text{Sol}(M))$.

The methods used in the present paper rely on the previous works [11], [13] as well as [15], [10], [2].

One of the reasons that make $\mathcal{D}_{X \times S/S}$-modules interesting is that they frame families of holomorphic (or distributions) solutions for instance of the form $x^{\alpha(s)}$, where $x$ and $s$ are (one or several) complex (or real analytic) variables and $\alpha$ is holomorphic (or real analytic) in some open subset of $S$.

Similarly, according to [2, Lem.2.10], we can regard a strict (i.e., $\mathcal{O}_S$-flat) relative regular holonomic $\mathcal{D}_{X \times S/S}$-module as an $\mathcal{O}_S$-flat family of regular holonomic $\mathcal{D}_X$-modules with a same characteristic variety, and a relative perverse sheaf whose dual complex is also perverse as an $\mathcal{O}_S$-flat family of perverse sheaves on $X$ with the same microsupport.

Hence, if $\dim S = 1$, another interesting consequence of Theorem 1 (more precisely, its particular case Proposition 4.2) is an equivalence between $\mathcal{O}_S$-flat holomorphic families of regular holonomic $\mathcal{D}_X$-modules having the same characteristic variety $\Lambda$ and ($\mathcal{O}_S$-flat) holomorphic families of perverse sheaves on $X$ with the same microsupport $\Lambda$.

The main tools in the proof of [13] are the good functorial properties satisfied by holonomic $\mathcal{D}$-modules underlying a mixed twistor $\mathcal{D}$-module, which include stability under localization along an hypersurface, pushforward and direct image by projective morphisms.

In our previous attempt to prove Theorem 1, the difficulty was that, contrary to the case $\dim S = 0$, we had not the tool of $b$-functions for $\mathcal{D}_{X \times S/S}$-modules and the inverse image functor does not preserve the holonomicity in general ([12, Ex.2.4]). However, a consequence of this work is that, if we add the regularity assumption, then the category of complexes with regular holonomic cohomology is stable by inverse image under closed immersions (hence localization). Let us indicate the main points in the proof of Theorem 1. We assume that $\dim S = 1$.

The first tool is [13, Th.3] which asserts that there exists a natural transformation
\[
\text{Id}_{\mathcal{D}_{\text{C-c}}^b(p_X^{-1}\mathcal{O}_S)} \xrightarrow{\alpha} p\text{Sol} \circ \text{RH}^S
\]
which provides a functorial isomorphism
\[
F \xrightarrow{\sim} p\text{Sol}(\text{RH}^S(F))
\]
for any $F \in \mathcal{D}^b_{\mathcal{C},c}(p_X^{-1}\mathcal{O}_S)$.

We are then reduced to proving that there exists a natural transformation

$$\text{Id}_{\mathcal{D}^b_{\text{hol}}(\mathcal{D}_{X \times S/S})} \xrightarrow{\beta} \mathcal{R}

\text{Hom}_{\mathcal{D}_{X \times S/S}}(M, \mathcal{R}

\text{Hom}_{\mathcal{D}_{X \times S/S}}^S(F)) \xrightarrow{\alpha} \mathcal{R}

\text{Hom}_{\mathcal{D}_{X \times S/S}}(M, \mathcal{R}

\text{Hom}_{\mathcal{D}_{X \times S/S}}^S(\mathcal{P}\text{Sol}(M)))$$

such that, for any $M \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_{X \times S/S})$, denoting by

$$\beta_M : M \to \mathcal{R}

\text{Hom}_{\mathcal{D}_{X \times S/S}}^S(\mathcal{P}\text{Sol}(M))$$

the unique morphism such that $\mathcal{P}\text{Sol}(\beta_M) \circ \alpha_{\mathcal{P}\text{Sol}(M)} = \text{Id}_{\mathcal{P}\text{Sol}(M)}$, we have

$$M \xrightarrow{\beta_M} \mathcal{R}

\text{Hom}_{\mathcal{D}_{X \times S/S}}^S(\mathcal{P}\text{Sol}(M)).$$

An argument of [13, §4.3]—more precisely, the proof of (a) \Rightarrow (b) in the proof of Lemma 4.8 of loc.cit.—which is given for $\mathcal{D}_{X \times S/S}$-modules underlying mixed twistor $\mathcal{D}$-modules, can be adapted in a straightforward way to the present more general setting, so the proof of the existence of such a $\beta$ is reduced to that of the following theorem.

**Theorem 2.** For any $M \in \mathcal{D}^b_{\text{hol}}(\mathcal{D}_{X \times S/S})$ and for any $F \in \mathcal{D}^b_{\mathcal{C},c}(p_X^{-1}\mathcal{O}_S)$ the complex $\mathcal{R}

\text{Hom}_{\mathcal{D}_{X \times S/S}}(M, \mathcal{R}

\text{Hom}_{\mathcal{D}_{X \times S/S}}^S(F))$ belongs to $\mathcal{D}^b_{\mathcal{C},c}(p_X^{-1}\mathcal{O}_S)$.

Although we directly prove Theorem 1 in the torsion-free (strict) case and in the torsion case separately, the general case needs Theorem 2 in order to apply induction on the dimension of the support, following Kashiwara’s proof [4, §8.3]. The proof of Theorem 2 uses the torsion-free case of Theorem 1 for the induction step and proceeds by considering the case of $\mathcal{D}_{X \times S/S}$-modules of D-type (normal crossing case).

As a consequence, we obtain the good behaviour of the functor $\mathcal{R}

\text{Hom}_{\mathcal{D}_{X \times S/S}}^S$ with respect to Poincaré-Verdier duality on the one hand, and duality for $\mathcal{D}_{X \times S/S}$-modules on the other hand.

**Corollary 3.** For any $F \in \mathcal{D}^b_{\mathcal{C},c}(p_X^{-1}\mathcal{O}_S)$, there exists an isomorphism in $\mathcal{D}^b_{\text{hol}}(\mathcal{D}_{X \times S/S})$

$$\mathcal{D}(\mathcal{R}

\text{Hom}_{\mathcal{D}_{X \times S/S}}^S(F)) \simeq \mathcal{R}

\text{Hom}_{\mathcal{D}_{X \times S/S}}^S(\mathcal{D}F)$$

which is functorial in $F$.

1. **Preliminaries**

1.a. **Preliminaries on the $\mathcal{D}_{X \times S/S}$-module side of the correspondence.** Let us recall the main definitions and tools concerning the $\mathcal{D}_{X \times S/S}$-module side of the relative Riemann-Hilbert correspondence.

We denote by $\mathcal{D}_{X \times S/S}$ the subsheaf of $\mathcal{D}_{X \times S}$ of relative differential operators with respect to the projection $p_X : X \times S \to S$. A coherent $\mathcal{D}_{X \times S/S}$-module $M$ is holonomic if $\text{Char}(M) \subseteq \Lambda \times S$ for some closed conic Lagrangian complex analytic subset $\Lambda$ of $T^*X$. We denote by $\mathcal{D}^b_{\text{hol}}(\mathcal{D}_{X \times S/S})$ the full subcategory of $\mathcal{D}^b_{\text{coh}}(\mathcal{D}_{X \times S/S})$ whose complexes have holonomic cohomologies. In particular, according to [2, Lem. 2.10], given a holonomic $\mathcal{D}_{X \times S/S}$-module $M$, there exists, locally on $X$, a finite set $I$ and, for each $i \in I$,
• a closed \( \mathbb{C}^* \)-conic irreducible Lagrangian subvariety \( \Lambda_i = T_{Y_i}^* X \) of \( T^* X \), where \( Y_i \) is a closed analytic subset of \( X \),

• a closed analytic subset \( T_i \) of \( S \),

such that \( \text{Char}(M) = \bigcup_{i \in I} (\Lambda_i \times T_i) \). Hence \( \text{Supp}(M) = \bigcup_{i \in I} (Y_i \times T_i) \). If \( q : X \times S \to X \) denotes the projection, then \( Z_{M} := q(\text{Supp} M) = \bigcup_{i \in I} Y_i \) is a closed analytic subset of \( X \).

Given \( s_o \in S \), let \( i_{s_o} \) denote the inclusion \( X \times \{s_o\} \hookrightarrow X \times S \). Following [11], we denote by \( L_i^{s_o} \) the derived functor

\[
p^{-1}(\mathcal{O}_S/m_{s_o}) \otimes_{p^{-1}\mathcal{O}_S} (\cdot),
\]

where \( m_{s_o} \) is the maximal ideal of functions vanishing at \( s_o \). Thanks to the variant of Nakayama’s lemma [13, Prop. 1.9, Cor. 1.10], the family of functors \( L_i^{s_o} \) for \( s_o \in S \) is a conservative family, i.e., if \( \phi : M \to N \) is a morphism in \( \mathcal{D}_{\text{hol}}(\mathcal{D}_{X/S}) \) such that \( L_i^{s_o} \phi \) is an isomorphism in \( \mathcal{D}_{\text{hol}}(\mathcal{D}_X) \) for each \( s_o \in S \) then \( \phi \) is an isomorphism (or, equivalently, using the mapping cone: if \( M \in \mathcal{D}_{\text{hol}}(\mathcal{D}_{X/S}) \) is such that \( L_i^{s_o} M = 0 \) for each \( s_o \in S \) then \( M = 0 \)).

Recall (cf. [11]) that a coherent \( \mathcal{D}_{X \times S/S} \)-module is said to be strict if it is \( p^{-1}\mathcal{O}_S \)-flat. If \( \dim S = 1 \), this is equivalent to have no \( p^{-1}\mathcal{O}_S \)-torsion. In that case, we shall denote by \( t(M) \) its (coherent) submodule of germs of sections which are torsion elements for the \( p^{-1}\mathcal{O}_S \)-action, and \( f(M) := M/t(M) \) called its strict (or torsion-free) quotient. Therefore, if \( \dim S = 1 \), the \( \mathcal{D}_{X \times S/S} \)-module \( M \) is strict if and only if \( M \simeq f(M) \).

Given \( M \in \mathcal{D}_{\text{hol}}^b(\mathcal{D}_{X \times S/S}) \), the functor

\[
D(M) := R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(M, \mathcal{D}_{X \times S/S} \otimes_{\mathcal{O}_{X \times S}} \Omega_{X \times S/S}^{-1})[dX]
\]

provides a duality in \( \mathcal{D}_{\text{hol}}^b(\mathcal{D}_{X \times S/S}) \) but, contrary to the absolute case (i.e., \( \dim S = 0 \)), this functor is not \( t \)-exact. The lack of exactness of the dual functor is due to the fact that, if \( \dim S = 1 \) for example, the dual of a torsion holonomic \( \mathcal{D}_{X \times S/S} \)-module is not concentrated in degree zero: if \( M \simeq t(M) \) we have \( D(M) \simeq \mathcal{H}^1(D(M))[-1] \); on the other hand, if \( N \) is strict holonomic, then \( D(N) \simeq \mathcal{H}^0(D(N)) \) and \( D(N) \) is strict (see [13, Prop. 2]), that is, \( N \) is also dual holonomic: recall that a complex \( N \) in \( \mathcal{D}_{\text{hol}}^b(\mathcal{D}_{X \times S/S}) \) is called \textit{dual holonomic} if it is in the heart of the \( t \)-structure \( \Pi \) (see [2, §2]) which, by definition, is the \( t \)-structure dual to the canonical \( t \)-structure.

We will later need the following lemma.

**Lemma 1.1.** Let \( M \) be a coherent \( \mathcal{D}_{X \times S/S} \)-module and let \( Z \) be a closed analytic subset of \( X \). Then \( \Gamma_{Z \times S} M \) is \( \mathcal{D}_{X \times S/S} \)-coherent.

**Proof.** The question is local. Let \( F_k \mathcal{M} \) be a good filtration of \( \mathcal{M} \). According e.g. to [11, Prop. 1.9], \( \Gamma_{Z \times S} F_k \mathcal{M} \) is \( \mathcal{O}_{X \times S} \)-coherent for each \( k \), and thus \( \mathcal{D}_{X \times S/S} \cdot \Gamma_{Z \times S} F_k \mathcal{M} \) is a \( \mathcal{D}_{X \times S/S} \)-coherent submodule of \( \mathcal{M} \). Since \( \Gamma_{Z \times S} \mathcal{M} = \bigcup_k \mathcal{D}_{X \times S/S} \cdot \Gamma_{Z \times S} F_k \mathcal{M} \), we conclude that \( \Gamma_{Z \times S} \mathcal{M} \) is \( \mathcal{D}_{X \times S/S} \)-coherent.

q.e.d.
Remark 1.2. By the previous lemma, if \( Y \) is a hypersurface in \( X \) locally defined by an equation \( f = 0 \), we have \( \Gamma_{[Y \times S]} M \simeq \Gamma_{X \times S} M \) for any coherent \( \mathcal{D}_{X \times S/S'} \)-module \( M \). This equality still holds true in the case of a \( \mathcal{D}_{X \times S/S'} \)-module \( N = \lim_{\to\in N} N_n \), where each \( N_n \) is coherent and the inductive limit is associated to inclusions \( N_k \subset N_{k+1} \). To see it, let us consider a germ of local section \( m \) of \( N_n \) supported by \( Y \times S \); then there exists \( k \in \mathbb{N} \) such that \( m \) is represented by \( m = m_k \), with \( m_k \) being a germ of local section of \( N_k \) supported by \( Y \times S \). Since \( N_k \) is coherent, we can choose a power \( f^N \) such that \( f^N m_k = 0 \) and hence \( f^N m = 0 \).

1.b. Preliminaries on the topological side of the correspondence.

Let us recall the main definitions and tools concerning the constructibility side of the relative Riemann-Hilbert correspondence.

Following [11] we say that a sheaf \( p_X^{-1}(\mathcal{O}_S) \)-module \( F \) is \( S \)-locally constant coherent if, for each point \( (x_o, s_o) \in X \times S \) there exists a neighborhood \( U = V_{x_o} \times T_{s_o} \) and a coherent sheaf \( G^{(x_o, s_o)} \) of \( O_{T_{s_o}} \)-modules such that \( F_{|U} \simeq p_{x_o}^{-1}(G^{(x_o, s_o)}) \). By definition, \( \mathcal{D}^b_{\text{coh}}(p_X^{-1}(\mathcal{O}_S)) \) is the full subcategory of \( \mathcal{D}^b(\mathcal{O}_S) \) whose complexes have \( S \)-locally constant coherent cohomologies (notice that, for such an \( F \), \( F_{|(x_o) \times S} \in \mathcal{D}^b_{\text{coh}}(\mathcal{O}_S) \)). We denote by \( \mathcal{D}^b_{\text{C^*-conic}}(p_X^{-1}(\mathcal{O}_S)) \) the the full subcategory of \( \mathcal{D}^b(\mathcal{O}_S) \) whose objects \( F \) admit a \( \mu \)-stratification \( (X_\alpha) \) of \( X \) such that \( i^{-1}_\alpha(F) \in \mathcal{D}^b_{\text{C^*-conic}}(p_X^{-1}(\mathcal{O}_S)) \) for any \( \alpha \); the full subcategory \( \mathcal{D}^b_{\text{C^*-conic}}(p_X^{-1}(\mathcal{O}_S)) \) consists of objects whose micro-support is \( \mathbb{C}^* \)-conic.

An analogue of the variant of Nakayama’s lemma holds for \( \mathcal{D}^b_{\text{C^*-conic}}(p_X^{-1}(\mathcal{O}_S)) \) [11, Prop. 2.2]. Hence the family of functors \( L_i^s \) for \( s \in S \) is a conservative family, i.e., if \( \psi : F \to G \) is a morphism in \( \mathcal{D}^b_{\text{C^*-conic}}(p_X^{-1}(\mathcal{O}_S)) \) such that \( L_i^s \psi \) is an isomorphism in \( \mathcal{D}^b_{\text{C^*-conic}}(\mathcal{O}_X) \) for each \( s \in S \), then \( \psi \) is an isomorphism (or equivalently using the mapping cone: if \( F \in \mathcal{D}^b_{\text{C^*-conic}}(p_X^{-1}(\mathcal{O}_S)) \) such that \( L_i^s F = 0 \) for each \( s \in S \) then \( F = 0 \)).

The category \( \mathcal{D}^b_{\text{C^*-conic}}(p_X^{-1}(\mathcal{O}_S)) \) is endowed with a perverse \( t \)-structure defined in [11, §2.7] as the relative analogue to the middle perverse \( t \)-structure in the absolute case:

- \( \mathcal{D}^b_{\text{C^*-conic}}(p_X^{-1}(\mathcal{O}_S)) \) is the full subcategory of objects \( F \) of \( \mathcal{D}^b_{\text{C^*-conic}}(p_X^{-1}(\mathcal{O}_S)) \) such that there exists an adapted \( \mu \)-stratification \( (X_\alpha) \) of \( X \) for which \( i^{-1}_x F \in \mathcal{D}^c_{\text{coh}}(\mathcal{O}_S) \) for any \( x \in X_\alpha \) and any \( \alpha \).
- \( \mathcal{D}^b_{\text{C^*-conic}}(p_X^{-1}(\mathcal{O}_S)) \) is the full subcategory of objects \( F \) of \( \mathcal{D}^b_{\text{C^*-conic}}(p_X^{-1}(\mathcal{O}_S)) \) such that there exists an adapted \( \mu \)-stratification \( (X_\alpha) \) of \( X \) for which \( i^{-1}_x F \in \mathcal{D}^c_{\text{coh}}(\mathcal{O}_S) \) for any \( x \in X_\alpha \) and any \( \alpha \).

The heart of this \( t \)-structure is the abelian category of perverse sheaves denoted by \( \text{per}(p_X^{-1}(\mathcal{O}_S)) \).

Let us assume that \( \dim S = 1 \) up to the end of this section. In analogy with the \( \mathcal{D}_{X \times S/S'} \)-module counterpart, following [2, Prop. 3.12], we say that a perverse sheaf is torsion if it belongs to the subcategory \( \text{per}(p_X^{-1}(\mathcal{O}_S)) \) of \( \text{per}(p_X^{-1}(\mathcal{O}_S)) \) whose objects \( F \) satisfy \( \text{codim} p_X(\text{Supp} F) \geq 1 \) (cf. [2, Cor. 3.1]
for this condition), while a perverse sheaf is called strict (or torsion-free) if it belongs to the full subcategory $\text{perv}(p_X^{-1}O_S)_t$ of $\text{perv}(p_X^{-1}O_S)$ whose objects $F$ satisfy $\mathcal{L} i_s^* F \in \text{perv}(C_X)$ for all $s \in S$. The category $\text{perv}(p_X^{-1}O_S)_t$ is a full thick abelian subcategory of the category $\text{perv}(p_X^{-1}O_S)$.

We denote by $D^b_{\mathcal{C},c}(p_X^{-1}O_S)_t$ the thick subcategory of $D^b_{\mathcal{C},c}(p_X^{-1}O_S)$ whose objects have support in $X \times T$, where $T$ is a subset of $S$ with $\dim T = 0$ or, equivalently, whose perverse cohomologies belong to $\text{perv}(p_X^{-1}O_S)_t$.

Given $F \in D^b_{\mathcal{C},c}(p_X^{-1}O_S)$, the functor

$$D(F) = R\mathcal{H}\text{om}_{p_X^{-1}O_S}(F, p_X^{-1}O_S)[2d_X]$$

provides a duality in $D^b_{\mathcal{C},c}(p_X^{-1}O_S)$ which is not $t$-exact with respect to the perverse $t$-structure. In particular, if $F$ is a torsion perverse sheaf, then $D(F) \simeq p\mathcal{H}\text{om}((D(F))[-1]$ (it is a perverse sheaf concentrated in degree 1) while if $L$ is a strict perverse sheaf $D(L) \simeq p\mathcal{H}\text{om}((D(L))$ is perverse too. Let us recall that an object $L$ of $D^b_{\mathcal{C},c}(p_X^{-1}O_S)$ is called dual perverse if it is in the heart of the $t$-structure $\pi$ which by definition is the $t$-structure dual to the perverse $t$-structure introduced in [11, §2.7]. By [13, Lem. 1.4], a complex $L \in D^b_{\mathcal{C},c}(p_X^{-1}O_S)$ is perverse and dual perverse if and only if it is strict. By [2, Cor. 4.3], the solution functor

$$p\text{Sol} : D^b_{\text{hol}}(\mathcal{D}_{X \times S/S}) \rightarrow D^b_{\mathcal{C},c}(p_X^{-1}O_S)$$

$$M \mapsto R\mathcal{H}\text{om}_{D_{X \times S/S}}(M, O_{X \times S})[d_X]$$

is $t$-exact with respect to the $t$-structure $\Pi$ in $D^b_{\text{hol}}(\mathcal{D}_{X \times S/S})$ and the perverse one $p$ in $D^b_{\mathcal{C},c}(p_X^{-1}O_S)$.

2. Regular holonomic $\mathcal{D}_{X \times S/S}$-modules

Let us recall that, given a triangulated category $\mathcal{C}$, by Rickard’s criterion ([14]), a full triangulated category $\mathcal{C}'$ of $\mathcal{C}$ is a thick subcategory if and only if it is closed under direct factors in $\mathcal{C}$ (which means that any direct summand of an object in $\mathcal{C}'$ is in $\mathcal{C}'$). In our case the category $\mathcal{C}$ is $D^b_{\text{hol}}(\mathcal{D}_{X \times S/S})$ and we aim at studying the thick subcategory of regular holonomic complexes.

When the triangulated category $\mathcal{C}$ is endowed with a bounded $t$-structure $D = (D^<0, D^>0)$ one can require that the subcategory $\mathcal{C}'$ is compatible with the truncation functors of the $t$-structure $\mathcal{D}$, i.e., for any $M \in \mathcal{C}'$ we have $\tau^<0 M, \tau^>1 M \in \mathcal{C}'$. Due to the fact that any object in $\mathcal{C}$ has only a finite number of non zero cohomologies, the compatibility of $\mathcal{C}'$ with the truncation functors of $\mathcal{D}$ is equivalent to requiring that $3\mathcal{C}'(M) \in \mathcal{C}'$ for any $i \in \mathbb{Z}$. This condition is essential in order to proceed by induction on the cohomological length of the complex.

According to [13], we say that an object $M \in D^b_{\text{hol}}(\mathcal{D}_{X \times S/S})$ is regular if it satisfies the property (Reg 1) below:

(Reg 1) For each $s_o \in S$ and any $j \in \mathbb{Z}$, $Li_{s_o}^* \mathcal{H}^j(M) \in D^b_{\text{hol}}(\mathcal{D}_X)$.

An alternative and natural property of regularity would be the following:

(Reg 2) For each $s_o \in S$, $Li_{s_o}^* M \in D^b_{\text{hol}}(\mathcal{D}_X)$. 
Lemma 2.2. Proposition 2.1. For any \( \tau \in \text{tors} \) while Property (Reg 1) is by definition compatible with the truncation functors while Property (Reg 2) is compatible with base change on \( S \), meaning that, for any morphism \( g : S' \to S \) of complex manifolds and any object \( M \in D^b_{\text{hol}}(\mathcal{D}_{X \times S/S}) \) which satisfies (Reg 2), the pullback \( L(\text{Id}_X \times g)^*(M) \in D^b_{\text{hol}}(\mathcal{D}_{X \times S'/S'}) \) satisfies (Reg 2) too.

**Proposition 2.1.** For any \( S \),

(i) Condition (Reg 1) is equivalent to (Reg 2),

(ii) the category of regular holonomic \( \mathcal{D}_{X \times S/S} \)-modules is closed under sub-quotients in the category \( \text{Mod}(\mathcal{D}_{X \times S/S}) \).

Let us start with the easy direction in (i).

**Lemma 2.2.** For any \( S \), condition (Reg 1) implies (Reg 2).

**Proof.** To see this, we argue by induction on the amplitude of the complex \( M \). Without loss of generality, we may assume that \( M \in D^0_{\text{hol}}(\mathcal{D}_{X \times S/S}) \) and we consider the following distinguished triangle

\[
\begin{align*}
\tau^0 M &\longrightarrow M &\longrightarrow \tau^{\geq 1} M &\longrightarrow \\
\end{align*}
\]

(where \( \tau^{\geq 1} \) is the truncation functor with respect to the natural \( t \)-structure on \( D^b_{\text{hol}}(\mathcal{D}_{X \times S/S}) \)). Let us assume that \( M \) satisfies (Reg 1), hence by definition both \( \tau^0 M \) and \( \tau^{\geq 1} M \) satisfy (Reg 1). As remarked, \( \tau^0 M \) satisfies (Reg 2) too and by induction on the amplitude of \( M \), \( \tau^{\geq 1} M \) satisfies (Reg 2), which permits to conclude that \( M \) also satisfies (Reg 2). q.e.d.

**Proof of Proposition 2.1.** For each \( d \geq 1 \) we define inductively the following properties for a complex \( M \in D^b_{\text{hol}}(\mathcal{D}_{X \times S/S}) \):

(Reg 1)_d \( \dim S = d \) and for all \( s_o \in S \) and any smooth codimension-one germ \( i_T : (T, s_o) \hookrightarrow (S, s_o), L i_T^* \mathcal{T} \mathcal{O} M \in D^b_{\text{hol}}(\mathcal{D}_X) \) satisfies (Reg 2)_d for all \( j \);

(Reg 2)_d \( \dim S = d \) and for all \( s_o \in S \) and any smooth codimension-one germ \( i_T : (T, s_o) \hookrightarrow (S, s_o), L i_T^* M \in D^b_{\text{hol}}(\mathcal{D}_X) \) satisfies (Reg 2)_d.

By definition, (Reg 1)_0 = (Reg 2)_0 reduces to require that \( M \in D^b_{\text{hol}}(\mathcal{D}_X) \). One checks that, if \( \dim S = d \), (Reg 2)_d is equivalent to (Reg 2) for a complex \( M \in D^b_{\text{hol}}(\mathcal{D}_{X \times S/S}) \). For the first assertion, according to Lemma 2.2, it is now enough to prove that (Reg 2)_d implies (Reg 1)_d for each \( d \geq 1 \). Indeed, by applying \( L i_{s_o}^* \) in (Reg 1)_d we see that (Reg 1)_d implies (Reg 1).

We proceed by induction on \( d := \dim S \), denoting by (i)_d and (ii)_d the statements of the Proposition restricted to \( \dim S = d \). We will prove the following implications for \( d \geq 1 \):

\[
(i)_{d-1} \land (ii)_{d-1} \implies (i)_d; \quad (i)_d \implies (ii)_d;
\]

For the initial step \( d = 0 \) both (i)_0 and (ii)_0 are satisfied since we are reduced to the classical case. Let us prove that \( (i)_{d-1} \land (ii)_{d-1} \implies (i)_d \). Due to the induction hypothesis (ii)_{d-1} we can simply write (Reg)_{d-1}.

Let \( M \in D^b_{\text{hol}}(\mathcal{D}_{X \times S/S}) \) be such that \( L i_T^* M \) satisfies (Reg)_{d-1} for each smooth codimension-one germ \( (T, s_o) \). We have to prove that \( L i_T^* \mathcal{T} \mathcal{O} (M) \)
satisfies \((\text{Reg})_{d-1}\) for any \(j\). We shall argue by induction on the cohomological length of \(M\). As above we may assume that \(M \in D^b_{\text{hol}}(\mathcal{D}_{X \times S/S})\) and we consider the distinguished triangle \((A)\). We deduce \(\mathcal{H}^{-1}Li_T^*\mathcal{H}^0(M) \simeq \mathcal{H}^{-1}Li_T^*(M)\) and an exact sequence
\[
0 \rightarrow \mathcal{H}^0 Li_T^* \mathcal{H}^0 M \rightarrow \mathcal{H}^0 Li_T^* M \rightarrow \mathcal{H}^0 Li_T^* \mathcal{H}^1 M \rightarrow 0.
\]
(Note that \(\mathcal{H}^k Li_T^* \mathcal{H}^0(M) = 0\) for \(k \neq 0, -1\).) By induction on \(d\), \(Li_T^* M\) satisfies \((\text{Reg})_{d-1}\), hence so does \(\mathcal{H}^0 Li_T^* M\), and since \((ii)_{d-1}\) is assumed to hold, so does any holonomic submodule, hence so does \(\mathcal{H}^0 Li_T^* \mathcal{H}^0 M\), as well as \(\mathcal{H}^1 Li_T^* \mathcal{H}^1 M\) by the previous isomorphism. From Lemma 2.2 we conclude that \(Li_T^* \mathcal{H}^0(M)\) satisfies \((\text{Reg})_{d-1}\). On the other hand, induction on the cohomological length implies that \(Li_T^* \mathcal{H}^j(M)\) satisfies \((\text{Reg})_{d-1}\) for any \(j \geq 1\). This concludes the proof of the first assertion and we now simply write \((\text{Reg})_d\).

Let us now prove that \((i)_d \Rightarrow (ii)_d\). Let \(M \in \text{Mod}(\mathcal{D}_{X \times S/S})\) satisfying \((\text{Reg})_d\). Given any short exact sequence
\[
(2.1) \quad 0 \rightarrow M_1 \rightarrow M \rightarrow M_2 \rightarrow 0
\]
of holonomic \(\mathcal{D}_{X \times S/S}\)-modules, we wish to prove that \(M_1\) and \(M_2\) satisfy \((\text{Reg})_d\).

Let \((T, s_0) \subset (S, s_0)\) be any smooth codimension-one germ. From (2.1) we obtain the long exact sequence:
\[
(2.2) \quad 0 \rightarrow \mathcal{H}^{-1} Li_T^* M_1 \rightarrow \mathcal{H}^{-1} Li_T^* M \rightarrow \mathcal{H}^{-1} Li_T^* M_2 \\
\quad \quad \quad \rightarrow \mathcal{H}^0 Li_T^* M_1 \rightarrow \mathcal{H}^0 Li_T^* M \rightarrow \mathcal{H}^0 Li_T^* M_2 \rightarrow 0.
\]

We wish to prove:

\((*)_T\) if \(Li_T^* M\) satisfies \((\text{Reg})_{d-1}\) then so do \(Li_T^* M_1\) and \(Li_T^* M_2\).

Due to the induction hypothesis, if \(Li_T^* M\) satisfies \((\text{Reg})_{d-1}\) then so do \(\mathcal{H}^{-1} Li_T^* M_1\) and \(\mathcal{H}^0 Li_T^* M_2\), and \((*)_T\) is equivalent to
\[\mathcal{H}^{-1} Li_T^* M_2, \text{ or } \mathcal{H}^0 Li_T^* M_1, \text{ satisfies } (\text{Reg})_{d-1}.
\]

\(Step\ 1\). Let \(s\) be a local coordinate on \(s\) vanishing on \(T\). We note that, by a standard argument, we can reduce the study of a \(T\)-torsion coherent (resp. holonomic) \(\mathcal{D}_{X \times S/S}\)-module \(M\) to the case of one satisfying the condition \(sM = 0\), which is a coherent (resp. holonomic) \(\mathcal{D}_{X \times T/T}\)-module. Therefore, by the induction hypothesis, \((ii)\) holds — and in particular \((*)_T\) also — if \(M\) is equal to its \(T\)-torsion.

\(Step\ 2\). Let us assume that, in (2.1), \(M_1\) is equal to the \(T\)-torsion part \(t_T(M)\) of \(M\) and \(M_2\) is the \(T\)-torsion free quotient \(f_T(M) := M/t_T(M)\). Since \(\mathcal{H}^{-1} Li_T^* f_T(M_2) = 0\) by \(T\)-torsion freeness, \((**)_T\) holds, hence \((*)_T\) also.

\(Conclusion\ of\ Steps\ 1\ and\ 2\). For any exact sequence (2.1), \(Li_T^* t_T(M_1)\) satisfies \((\text{Reg})_{d-1}\).
Step 3. We prove that, if $M = f_T(M)$, then $(*)_T$ holds. Let us denote by $M'$ the pullback of $t_T(M_2)$ in $M$. The diagram below is commutative cartesian and its columns and rows are short exact sequences:

$$
\begin{array}{c}
0 \to M_1 \to M' \to t_T(M_2) \to 0 \\
\bigg\| \bigg\| \bigg\| \\
0 \to M_1 \to M \to M_2 \to 0 \\
\downarrow \downarrow \downarrow \\
\downarrow \downarrow \downarrow \\
M_1 \to M_2 \to f_T(M_2) \to f_T(M_2)
\end{array}
$$

(2.3)

Since $f_T(M_2)$ is $T$-torsion free, $(**)_T$ holds for the middle column, so $Li^*_T$ of $f_T(M_2)$ and $M'$ satisfy (Reg)$_{d-1}$.

We now prove that the first row satisfies $(**)_T$. Let $s$ be a local coordinate on $s$ vanishing on $T$. We argue by induction on $n$ such that $s^n t_T(M_2) = 0$.

If $n = 1$, $(**)_{T'}$, and thus $(*)_T$, holds for the first row, since we have $\mathcal{K}^{-1}Li^*_T t_T(M_2) \simeq \mathcal{K}^0 Li^*_T t_T(M_2)$.

If $n > 1$, we argue similarly with the following commutative diagram, analogous to (2.3),

$$
\begin{array}{c}
0 \to M_1 \to M'' \to \mathcal{K}_{t_T(M_2)} \to 0 \\
\bigg\| \bigg\| \bigg\| \\
0 \to M_1 \to M' \to t_T(M_2) \to 0 \\
\downarrow \downarrow \downarrow \\
\downarrow \downarrow \downarrow \\
M_1 \to M_2 \to s t_T(M_2) \to s t_T(M_2)
\end{array}
$$

(2.4)

By the induction hypothesis, $Li^*_T$ of $s t_T(M_2)$ and $\mathcal{K}_{t_T(M_2)}$ satisfy (Reg)$_{d-1}$.

From the first property we deduce that $(**)_{T'}$ applies to the middle vertical sequence, so $Li^*_TM''$ satisfies (Reg)$_{d-1}$. Then the second property implies that $(**)_T$ applies to the first row, thus $Li^*_TM_1$ satisfies (Reg)$_{d-1}$ too. This concludes the proof of Step 3.

Step 4. For a holonomic $D_{X \times S/S}$-module $M$ satisfying (Reg)$_d$ and a holonomic submodule $M_1 \subseteq M$, $Li^*_T$ of the $T$-torsion part $t_T(M_1) = M_1 \cap t(M)$ satisfies (Reg)$_{d-1}$ by Steps 1 and 2, and $Li^*_T t_T(M_1)$ also satisfies (Reg)$_{d-1}$, according to Step 3, since it injects into $f_T(M)$. Therefore, so does $Li^*_T M_1$, and this concludes the proof.

q.e.d.

3. The functor $RH^S$

In this section we briefly recall the relative Riemann-Hilbert functor $RH^S(\cdot)$ introduced in [13] and state some complementary results needed in the sequel.

3a. Reminder on relative subanalytic sites and relative subanalytic sheaves. For details on this subject we refer to [10]. We also refer to [8] as a foundational paper and to [9] for a detailed exposition on the general theory of sheaves on sites.

Let $X$ and $S$ be real or complex analytic manifolds where we consider the family of open subanalytic subsets. On $X \times S$, $T$ is the family consisting of
finite unions of open relatively compact subsets and the family $\mathcal{T}'$ consists of finite unions of open relatively compact sets of the form $U \times V$. The associated sites $(X \times S)_\gamma$ and $(X \times S)_{\gamma'}$ are nothing more than, respectively, $(X \times S)_{\text{sa}}$ and the product of sites $X_{\text{sa}} \times S_{\text{sa}}$.

We shall denote by $\rho$, without reference to $X \times S$ unless otherwise specified, the natural functor of sites $\rho : X \times S \to (X \times S)_{\text{sa}}$ associated to the inclusion $\text{Op}((X \times S)_{\text{sa}}) \subset \text{Op}(X \times S)$. Accordingly, we shall consider the associated functors $\rho_*,\rho^{-1},\rho!$.

We shall also denote by $\rho' : X \times S \to (X \times S)_{\gamma'}$ the natural functor of sites. Following [9] we have functors $\rho'_*,\rho'_!$ from $\text{Mod}(C_{X\times S})$ to $\text{Mod}(C_{X_{\text{sa}}\times S_{\text{sa}}})$. Subanalytic sheaves are defined on the subanalytic site of a real analytic manifold, and relative subanalytic sheaves are defined on the relative subanalytic site recalled above. We refer to [10] for the detailed construction of the relative subanalytic site.

Following [9] we have functors $\rho'_*,\rho'_!$ from $\text{Mod}(C_{X\times S})$ to $\text{Mod}(C_{X_{\text{sa}}\times S_{\text{sa}}})$. Subanalytic sheaves are defined on the subanalytic site of a real analytic manifold, and relative subanalytic sheaves are defined on the relative subanalytic site recalled above. We refer to [10] for the detailed construction of the relative subanalytic sheaves $\mathcal{D}_{X \times S}^t$ (where $X$ and $S$ are real analytic) and $\mathcal{O}^t_{S, X \times S}$ (in the complex framework)

They are both $\rho'_!\mathcal{D}_{X \times S/S}$-modules (either in the real or the complex case) as well as a $\rho'_*\rho'^{-1}\mathcal{O}_S$-module and both structures commute. Moreover, when $X$ is complex, considering the complex conjugate structure $\overline{X}$ on $X$ (resp. $\overline{S}$ on $S$) and the underlying real analytic structure $X_{\mathbb{R}}$ (resp. $S_{\mathbb{R}}$), we have

$$\mathcal{O}^t_{S, X \times S} = R\mathcal{H}\text{om}_{\mathcal{D}_{X \times S}}(\rho'_!\mathcal{O}_{X \times S}, \mathcal{D}_{X \times S}^t)$$

where we omit the reference to the real structures.

3.b. Reminder on $RH^S$ and complementary properties. In the real framework ($X$ and $S$ being real analytic manifolds with $\dim S = 1$), for $F \in D_{-c}^b(p^{-1}\mathcal{O}_S)$ we set

$$\text{TH}^S(F) := \rho'^{-1} R\mathcal{H}\text{om}_{\rho'^{-1}\mathcal{O}_S}(\rho'_! F, \mathcal{D}_{X \times S}^t).$$

If $X$ is a complex manifold of complex dimension $d_X$ and $S$ is a complex manifold of dimension one, $RH^S : D_{-c}^b(p^{-1}\mathcal{O}_S) \to D_{\text{hol}}^b(\mathcal{D}_{X \times S/S})$ is given by the assignment

$$F \mapsto RH^S(F) := \rho'^{-1} R\mathcal{H}\text{om}_{\rho'^{-1}\mathcal{O}_S}(\rho'_! F, \mathcal{D}_{X \times S}^t)[d_X]$$

$$\simeq R\mathcal{H}\text{om}_{\mathcal{D}_{X \times S}}(\mathcal{O}_{X \times S}, \text{TH}^S(F)),$$

the last isomorphism being called here “realification procedure” for short (cf. [13, (3.16)])

Proposition 3.1. Let $Y$ be a complex hypersurface of $X$. Then, for any $F \in D_{-c}^b(p^{-1}\mathcal{O}_S)$ there is a natural isomorphism

$$RH^S(F)(*(Y \times S)) \simto RH^S(F \otimes C_{(X \times Y)\times S}).$$

In particular, if $F \in D_{-c}^b(p^{-1}\mathcal{O}_S)$,

1. $RH^S(F)(*(Y \times S))$ belongs to $D_{\text{hol}}^b(\mathcal{D}_{X \times S/S})$.

2. There is a natural isomorphism $RH^S(F \otimes C_{Y \times S}) \simeq R\Gamma_{Y \times S}(RH^S(F))$ and so $R\Gamma_{Y \times S}(RH^S(F))$ also belongs to $D_{\text{hol}}^b(\mathcal{D}_{X \times S/S})$. 
(3) If the natural morphism $\text{RH}^S(F) \to \text{RH}^S(F)(*(Y \times S))$ is an isomorphism, then so is the natural morphism $F \otimes \mathcal{C}_{(X,Y)\times S} \to F$.

Proof. Let $f = 0$ be a local defining equation of $Y$. We start by assuming that $F = p_X^{-1}\mathcal{O}_S \otimes \mathcal{C}_{\Omega \times S}$ for a relatively compact open subanalytic subset $\Omega$ of $X$. Noting that $f$ is invertible on $T\mathcal{H}om(\mathcal{C}_{(\Omega \times Y)\times S}, \mathcal{D}b_{X \times S})$, according to [4, Prop. 3.23], the natural restriction morphism $f : \mathcal{D}b_{X \times S} \to \mathcal{D}b_{Y \times S}$ is an isomorphism. According to [13, Prop. 3.5], the natural restriction morphism $\mathcal{D}b_{X \times S} \to \mathcal{D}b_{Y \times S}$ is an isomorphism for a general $F$. The existence of the morphism $(*)$ and the fact that it is an isomorphism then follows by [13, (3.16)] and functoriality.

The remaining statements (1) and (2) follow straightforwardly (see also [13, Ex. 3.20]), while (3) is obtained by applying $\mathcal{R}\mathcal{S}ol$ to the isomorphism $(*)$, and by using [13, Th. 3]. q.e.d.

We now adapt the proof of the relative version of [7, Prop. 5.9 (5.20)].

Proposition 3.2. For any $F \in \mathcal{D}_{X,c}(p_X^{-1}\mathcal{O}_S)$ and for any morphism $f : Y \to X$ of complex manifolds, there exists a natural morphism in $\mathcal{D}^b(\mathcal{D}_{Y \times S/S})$:

$$\mathcal{D}f^* \text{RH}^S(F) \to \text{RH}^S(f^{-1}F).$$

Proof. We start by decomposing $f$ as the graph embedding $Y \to Y \times X$ followed by the projection $Y \times X \to X$, reducing to the case of (i) a closed immersion and (ii) a smooth morphism.

Let us treat (i). We shall prove that

$$\text{RH}^S(f^{-1}F) \simeq \mathcal{D}f^* \text{RH}^S(F)$$

by a natural isomorphism in $\mathcal{D}^b(\mathcal{D}_{Y \times S/S})$ functorially in $F$. We start by noticing that

$$\mathcal{D}f^* \text{RH}^S(F \otimes \mathcal{C}_{(X,Y)\times S}) = 0.$$

To check this local statement we may assume, by induction on $\text{codim}Y$, that $Y$ is smooth of codimension one, and (3.1) follows from Proposition 3.1.

Hence we conclude that

$$\mathcal{D}f^* \text{RH}^S(F) \simeq \mathcal{D}f^* \text{RH}^S(F \otimes \mathcal{C}_{(X,Y)\times S}) \simeq \mathcal{D}f^* \text{RH}^S(f^{-1}F)$$

$$\simeq \mathcal{D}f^* \mathcal{D}f_* \text{RH}^S(f^{-1}F) \quad \text{by [13, Th. 3.28]}$$

$$\simeq \text{RH}^S(f^{-1}F) \quad \text{since } \mathcal{D}f^* \mathcal{D}f_* \simeq \text{Id} \text{ as functors on } \mathcal{D}^b(\mathcal{D}_{Y \times S/S}).$$

Let us now treat (ii). Recall that in that case we have a natural transformation of functors on $\mathcal{D}^b(\mathcal{D}_{Y \times S/S})$

$$\mathcal{D}_{(Y \to X)\times S/S} \otimes f^{-1}\mathcal{D}_{X \times S/S} : \mathcal{R}\mathcal{H}om_{\mathcal{D}_{Y \times S/S}}(\mathcal{D}_{(Y \to X)\times S/S}, \cdot) \to \text{Id}$$

The statement will follow provided we prove that there exists a natural isomorphism in $\mathcal{D}^b(f^{-1}\mathcal{D}_{X \times S/S})$, functorial in $F$:

$$\mathcal{R}\mathcal{H}om_{\mathcal{D}_{Y \times S/S}}(\mathcal{D}_{(Y \to X)\times S/S}, \text{RH}^S(f^{-1}F)) \simeq f^{-1} \text{RH}^S(F)$$
The proof of this is performed by mimicking the proof of [7, Th. 5.8 (5.14)], thanks to the relative version of Proposition 4.3 of loc. cit. which was already implicit in [13, Prop 3.27]. Therefore, the lemma below concludes the proof.

q.e.d.

**Lemma 3.3.** For any morphism $f : Y \to X$ of real analytic manifolds and for any $F \in D^b_{\mathbb{R}, c}(p_X^{-1}O_S)$ there exists a canonical morphism in $D^b(D_{X \times S}/S)$

$$\rho_f^! TH^S(f^{-1}F) \to TH^S(F).$$

Proof. One replaces $F$ by a resolution as in [13, Prop. 3.5] and then apply [7, Prop. 4.3].

q.e.d.

**Corollary 3.4.** Let $F \in D^b_{\mathbb{C}, c}(p_X^{-1}O_S)$, let $Y$ be a closed hypersurface of $X$ and let us assume that $RH^S(F)$ is localized along $Y \times S$, i.e., $RH^S(F) \simeq RH^S(F)(*(Y \times S))$. Let $Z$ be a closed analytic subset of $X$ such that $\text{Supp } F \subset Z \times S$, $Z \cap Y$ has dimension strictly smaller than $Z$ and $Z^* := Z \setminus Y$ is smooth. Let $\pi : X' \to X$ be a morphism such that $X'$ is a complex manifold, $Y' := \pi^{-1}(Y)$ is a divisor in $X'$, and $\pi$ induces an isomorphism $\pi_* : X' \setminus Y' \xrightarrow{\sim} Z^*$ (thus $d_{X'} = d_Z$). Then the natural morphism $\rho_\pi^* RH^S(F) \to RH^S(\pi^{-1}F)$ of Proposition 3.2 is an isomorphism.

Proof. By Proposition 3.1(3), the natural morphism $F \otimes \mathbb{C}_{(X \setminus Y) \times S} \to F$ is an isomorphism, hence so is the natural morphism $\pi^{-1}F \otimes \mathbb{C}_{(X \setminus Y') \times S} \to \pi^{-1}F$. Therefore so is the natural morphism $RH^S(\pi^{-1}F) \to RH^S(\pi^{-1}F)(*(Y' \times S))$, according to ($\ast$) in Proposition 3.1. On the other hand, by functoriality, $\rho_\pi^* RH^S(F) \to \rho_\pi^* RH^S(F)(*(Y' \times S))$ is also an isomorphism.

The cone $\mathcal{M}'$ of the natural morphism $\rho_\pi^* RH^S(F) \to RH^S(\pi^{-1}F)$ is supported on $Y' \times S$ and satisfies $\mathcal{M}' \simeq \mathcal{M}'(+(Y' \times S))$, hence it is zero (cf. Remark 1.2).

q.e.d.

**Corollary 3.5.** For any $F \in D^b_{\mathbb{C}, c}(p_X^{-1}O_S)$ and for any closed submanifold of $X$, $R\Gamma_{Y \times S}(RH^S(F))$ is a complex with regular holonomic $D_{X \times S/S^*}$-cohomologies. Equivalently $\rho_{\pi_*}^- RH^S(F)$ is a complex with regular holonomic $D_{Y \times S/S^*}$-cohomologies.

Proof. The statement being local, we may assume that $Y$ is an intersection of smooth hypersurfaces of $X$ and then conclude by Proposition 3.1(2) that $R\Gamma_{Y \times S}(RH^S(F)) \simeq RH^S(F \otimes \mathbb{C}_{Y \times S})$ which implies the first statement.

According to the relative version of [3, Prop. 4.3], the second statement is equivalent to the first one.

q.e.d.

Recall that, for $F \in D^b_{\mathbb{C}, c}(p_X^{-1}O_S)$, $RH^S(F)$ is a strict regular holonomic module if and only if $F$ is perverse and dual perverse ([13, Prop. 2]), that is, $F$ is strict, since dim $S = 1$ (cf. Section 1.b).

**Corollary 3.6.** Let $F, G \in D^b_{\mathbb{C}, c}(p_X^{-1}O_S)$ and let us assume that $G$ is perverse and $F$ is perverse and perverse dual (and hence strict). Then

$$\mathfrak{H}om_{perv(p_X^{-1}O_S)}(F, G) := \mathfrak{H}om^{\mathfrak{D}}_{p_X^{-1}O_S}(F, G)$$
is strict.

Proof. Let us consider a morphism \( \Phi : F \to G \) of perverse sheaves such that, for some \( s_0 \in S \) and some local coordinate \( s \) vanishing at \( s_0 \), \( s^s \Phi = 0 \). Then \( \Phi \) is the zero morphism away from \( s_0 \), so \( \text{RH}^S(\Phi) : \text{RH}^S(G) \to \text{RH}^S(F) \) is also zero away from \( s_0 \). This means that the image of \( \text{RH}^S(\Phi) \) is a torsion submodule of the strict module \( \text{RH}^S(F) \), so it is zero, hence \( \Phi \) is zero since \( ^p\text{Sol} \text{RH}^S \simeq \text{Id} \).

q.e.d.

3.c. \( \mathcal{D}_{X \times S/S} \)-modules of D-type. Let us recall the following results in [13]. Let \( D \) be a normal crossing divisor in \( X \) and let \( j : X^* := X \setminus D \hookrightarrow X \) denote the inclusion (we will also denote by \( j \) the morphism \( j \times \text{Id}_S \)). Let \( F \) be a coherent \( S \)-locally constant sheaf on \( X^* \times S \) and let \( (V, \nabla) = (\mathcal{O}_{X^* \times S} \otimes_{\mathcal{O}_S} F, d_{X/S}) \) be the associated coherent \( \mathcal{O}_{X^* \times S} \)-module with flat relative connection. There exists a coherent \( \mathcal{O}_S \)-module \( G \) such that, locally on \( X^* \times S \), \( F \simeq p^{-1}G \) and if \( U \) is any contractible open set of \( X^* \), then \( F|_{U \times S} \simeq p_U^{-1}G \).

Let \( \varpi : \tilde{X} \to X \) denote the real blowing up of \( X \) along the components of \( D \). Denote by \( \tilde{j} : X^* \hookrightarrow \tilde{X} \) the inclusion, so that \( j = \varpi \circ \tilde{j} \). Let \( x^o \in D \), \( \tilde{x}^o \in \tilde{x}^{-1}(x^o) \) and let \( s^o \in S \). Choose local coordinates \( (x_1, \ldots, x_n) \) at \( x^o \) such that \( D = \{x_1 \cdots x_{\ell} = 0\} \) and consider the associated polar coordinates \( (p, \theta, x^\prime) := (p_1, \theta_1, \ldots, p_\ell, \theta_\ell, x_{\ell+1}, \ldots, x_n) \) so that \( \tilde{x}^o \) has coordinates \( p^o = 0, \theta^o, x^o = 0 \).

A local section \( \tilde{v} \) of \( (\tilde{j}_* V)(\tilde{x}^o, s^o) \) is said to have moderate growth if for some system of generator of \( G_{s^o} \), and some neighbourhood

\[
U_\varepsilon := \{\|\rho\| < \varepsilon, \|x^\prime\| < \varepsilon, \|\theta - \theta^o\| < \varepsilon\}
\]

(\( \varepsilon \) small enough) on which it is defined, its coefficients on the chosen generators of \( G_{s^o} \) (these are sections of \( \mathcal{O}(U^*_\varepsilon \times U(s^o)) \) for a small enough neighbourhood \( U(s^o) \) of \( s^o \) in \( S \), and \( U^*_\varepsilon := U_\varepsilon \setminus \{p_1 \cdots p_\ell = 0\} \) are bounded by \( C p^{-N} \), for some \( C, N > 0 \). A local section \( v \) of \( (j_* V)(x^o, s^o) \) is said to have moderate growth if for each \( \tilde{x}^o \) in \( \tilde{x}^{-1}(x^o) \), the corresponding germ in \( (\tilde{j}_* V)(\tilde{x}^o, s^o) \) has moderate growth.

The following result is proved in [13] (Theorem 2.6 and Corollary 2.8):

**Theorem 3.7.** The subsheaf \( \tilde{V} \) of \( j_* V \) consisting of local sections having moderate growth is stable by \( \nabla \) and \( \mathcal{O}_{X \times S}(*D) \)-coherent and it is a regular holonomic \( \mathcal{D}_{X \times S/S} \)-module with characteristic variety contained in \( \Lambda \times S \), where \( \Lambda \) is the union of the conormal spaces of the natural stratification of \( (X, D) \).

**Definition 3.8.** ([13, Def.2.10]) A coherent \( \mathcal{D}_{X \times S/S} \)-module \( \mathcal{L} \) is said to be of D-type with singularities along a normal crossing divisor \( D \subset X \) if it satisfies the following conditions:

1. \( \text{Char}(\mathcal{L}) \subset (\pi^{-1}(D) \times S) \cup (T^*_X X \times S) \),
2. \( \mathcal{L} \) is regular holonomic and strict,
3. \( \mathcal{L} \simeq \mathcal{L}(*(D \times S)) \).
Let us fix a normal crossing divisor $D$. The category of holonomic $\mathcal{D}_{X \times S/S}$-modules $\mathcal{L}$ of D-type with singularities along $D$ is equivalent to the category of locally free $p_X^{-1}\mathcal{O}_S$ with $X^* := X \setminus D$ ([13, Prop. 2.11 & Lem. 4.2]) under the correspondence

$$(3.2) \quad \mathcal{L} \mapsto \mathcal{H}^0 \text{DR}(\mathcal{L})|_{X^* \times S}, \quad F \mapsto \mathcal{R}H^S(\mathcal{H}(F|d_X))) = \mathcal{L}.$$ 

3.d. **Behaviour of $\mathcal{R}H^S$ under finite ramification over $S$.** We shall now prove a result which will be useful in the sequel: let $N$ be a natural number, let $\delta : S' := \mathbb{C} \to S = \mathbb{C}$ be the ramification $s' \mapsto s'^N$ of degree $N$. Then $\mathcal{O}_{X \times S'}$ is $\delta^{-1}\mathcal{O}_{X \times S}$-flat and $\delta_* \mathcal{O}_{X \times S'}$ is $\mathcal{O}_{X \times S}$. In particular, $\mathcal{O}_{X \times S}$ is a direct summand of $\delta_* \mathcal{O}_{X \times S'}$.

For a $\mathcal{D}_{X \times S/S}$-module $\mathcal{M}$, resp. an object $F \in \mathcal{D}^b_{\text{coh}}(p_X^{-1}\mathcal{O}_S)$, the pullback is defined by

$$(\rho \delta)_*^* \mathcal{M} := \mathcal{O}_{X \times S'} \otimes_{\delta^{-1}\mathcal{O}_{X \times S}} \delta^{-1} \mathcal{M} \quad \text{resp.} \quad \rho \delta_*^* F := p_X^{-1}\mathcal{O}_{S'} \otimes_{p_X^{-1}\mathcal{O}_S} \delta^{-1} F.$$ 

It satisfies

$$(\rho \delta)_*^* \delta_*^* \mathcal{M} \simeq \delta_* \mathcal{O}_{X \times S'} \otimes_{\mathcal{O}_{X \times S}} \mathcal{M} \quad \text{resp.} \quad (\rho \delta)_*^* \delta_*^* F \simeq \delta_* \mathcal{O}_{S'} \otimes_{\mathcal{O}_S} \delta_*^* F,$$

so that $\mathcal{M}$ resp. $F$ is a direct summand in $(\rho \delta)_*^* \delta_*^* \mathcal{M}$ resp. in $(\rho \delta)_*^* \delta_*^* F$.

The first pullback induces a well-defined exact functor from $\mathcal{D}^b_{\text{coh}}(\mathcal{D}_{X \times S/S})$ to $\mathcal{D}^b_{\text{coh}}(\mathcal{D}_{X \times S'/S'})$, as already used in [13] in a particular situation (proof of Corollary 2.8, where $\delta$ is denoted by $\rho$), and the second one a well-defined functor $\mathcal{D}^b_{\text{perf}}(p_X^{-1}\mathcal{O}_S) \to \mathcal{D}^b_{\text{perf}}(p_X^{-1}\mathcal{O}_{S'})$.

**Proposition 3.9.** With the notation above, for any $F \in \mathcal{D}^b_{\text{perf}}(p_X^{-1}\mathcal{O}_S)$, there is an isomorphism

$$(\rho \delta)^* \mathcal{R}H^S(F) \simeq \mathcal{R}H^{S'}(\delta_*^* F).$$

**Proof.** Given an almost free resolution $\tilde{F}$ of $F$ (which by definition is that constructed in [13, Prop. 3.5]) then $\delta_*^* \tilde{F}$ is an almost free resolution of $\delta_*^* F$. So we are reduced to prove the statement for $F = C_{\Omega \times S} \otimes p_X^{-1}\mathcal{O}_S$ where $\Omega$ is subanalytic relatively compact open subset in $X$, in which case

$$\mathcal{R}H^S(F) \simeq T\mathcal{H}\text{om}(\mathcal{C}_{\Omega \times S}, \mathcal{O}_{X \times S}) \text{ and } \mathcal{R}H^{S'}(\delta_*^* F) \simeq T\mathcal{H}\text{om}(\mathcal{C}_{\Omega \times S'}, \mathcal{O}_{X \times S'}).$$

In view of [7, (5.20) of Prop. 5.9], there exists a natural morphism

$$(\rho \delta)^* \quad T\mathcal{H}\text{om}(\mathcal{C}_{\Omega \times S}, \mathcal{O}_{X \times S}) \to T\mathcal{H}\text{om}(\mathcal{C}_{\Omega \times S'}, \mathcal{O}_{X \times S}).$$

It is therefore enough to prove that this morphism is an isomorphism.

By the realification procedure (see Section 3.b) and [7, Th. 10.5], we are reduced to proving that the natural morphism $(\rho \delta)^* T\mathcal{H}\text{om}(\mathcal{C}_{\Omega \times S}, \mathcal{O}_{X \times S}) \to T\mathcal{H}\text{om}(\mathcal{C}_{\Omega \times S'}, \mathcal{O}_{X \times S'})$ is an isomorphism, which holds true since the ramification $\delta$ in the variable $s$ does not interfere with the growth with respect to the boundary of $\Omega$.

q.e.d.

We shall now come back to the situation described at the beginning of this section, and we keep the same notations.
Corollary 3.10. Let $M, N$ be coherent $\mathcal{D}_{X \times S/S}$-modules. If the complex $R\mathcal{H}\text{om}_{\mathcal{D}_{X \times S'/S}}(p_{X}^{\ast}M, p_{S}^{\ast}N)$ belongs to $\mathbf{D}^{b}_{\mathcal{C}, c}(p_{X}^{\ast}\mathcal{O}_{S'})$, then the complex $R\mathcal{H}\text{om}_{\mathcal{D}_{X \times S/S}}(M, N)$ belongs to $\mathbf{D}^{b}_{\mathcal{C}, c}(p_{X}^{\ast}\mathcal{O}_{S})$.

Proof. Thanks to the functorial properties of the tensor product and $R\mathcal{H}\text{om}$, there is a natural morphism

\begin{equation}
\delta_{\ast}\delta^{\ast} R\mathcal{H}\text{om}_{\mathcal{D}_{X \times S'/S}}(M, N) \longrightarrow \delta_{\ast} R\mathcal{H}\text{om}_{\mathcal{D}_{X \times S/S'}}(p_{X}^{\ast}M, p_{S}^{\ast}N).
\end{equation}

Let us prove that it is an isomorphism. Since the statement is local we reduce to the case of $\delta^{\ast}D_{X \times S/S} = D_{X \times S/S'}$. In that case, $\delta_{\ast}\delta^{\ast} R\mathcal{H}\text{om}_{\mathcal{D}_{X \times S/S}}(M, N) \simeq \delta_{\ast}(p_{X}^{\ast}(\mathcal{O}_{S}) \otimes \delta^{-1}(p_{S}^{\ast}\mathcal{O}_{S})^{-1})$ and $\delta_{\ast} R\mathcal{H}\text{om}_{\mathcal{D}_{X \times S/S'}}(p_{X}^{\ast}M, p_{S}^{\ast}N) \simeq \delta_{\ast} \delta^{\ast} N$ and the statement is obvious in that case.

Under the assumption of the corollary, the right-hand side of (3.3) is an object of $\mathbf{D}^{b}_{\mathcal{C}, c}(p_{X}^{\ast}\mathcal{O}_{S})$, hence so is the left-hand side. Since each cohomology sheaf of $R\mathcal{H}\text{om}_{\mathcal{D}_{X \times S/S}}(M, N)$ is a direct summand of the corresponding cohomology of $\delta_{\ast}\delta^{\ast} R\mathcal{H}\text{om}_{\mathcal{D}_{X \times S/S}}(M, N)$, it is also $S$-constructible. q.e.d.

4. Relative Riemann-Hilbert correspondence

In this section, we assume that $\dim S = 1$.

4.a. A finiteness theorem.

Theorem 4.1. Let $M, N$ be strict holonomic $\mathcal{D}_{X \times S/S}$-modules. Then $\mathcal{H}\text{om}_{\mathcal{D}_{X \times S/S}}(M, N)$ is $S$-constructible.

Proof. We adapt the proof of [5, Th. 4.45]. Let $(X_{a})_{a \in A}$ be a $\mu$-stratification of $X$ compatible with $M$ and $N$, let $U$ be the union of strata of maximal dimension on which $M$ and $N$ are not identically zero and let $Z$ denote its closure in $X$. Then $Z$ is a closed analytic subset of $X$. We argue by induction on the dimension of $Z$. Set $Z' = Z \setminus U$. This is a closed analytic subset of $Z$ of dimension $< \dim Z$.

Let $i : U \hookrightarrow X' := X \setminus Z'$ denote the inclusion. By Kashiwara’s equivalence theorem in the relative setting (cf. [13, Th. 1.5]), we have

$$\mathcal{H}\text{om}_{\mathcal{D}_{X' \times S/S}}(M_{|X' \times S}, N_{|X' \times S}) \simeq i_{\ast}\mathcal{H}\text{om}_{\mathcal{D}_{U \times S/S}}(i^{\ast}M_{|X' \times S}, i^{\ast}N_{|X' \times S}).$$

Moreover, for any open subset $\tilde{U}$ in $X$ such that $\tilde{U} \cap Z = U$, the restriction $M_{|\tilde{U} \times S}$ is holonomic and its characteristic variety is contained in the union of the sets $T_{X_{a}} \times S$ for any stratum $X_{a}$ in $U$, hence $i^{\ast}M_{|X' \times S}, i^{\ast}N_{|X' \times S}$ are $\mathcal{O}_{U \times S}$-coherent, and thus of the form $\mathcal{O}_{U \times S} \otimes_{\mathcal{O}_{\tilde{U}}} \mathcal{F}$ and $\mathcal{O}_{U \times S} \otimes_{\mathcal{O}_{\tilde{U}}} \mathcal{G}$ for some $p_{U}^{-1}\mathcal{O}_{S}$-coherent $S$-local systems $\mathcal{F}, \mathcal{G}$ on $U \times S$, according to [1, Th. I.2.23(iii)]. Moreover, Kashiwara’s equivalence implies that

$$\mathcal{H}\text{om}_{\mathcal{D}_{U \times S/S}}(i^{\ast}M_{|X' \times S}, i^{\ast}N_{|X' \times S}) \simeq \mathcal{H}\text{om}_{p_{U}^{-1}\mathcal{O}_{S}}(\mathcal{F}, \mathcal{G}),$$

which is thus also a $p_{U}^{-1}\mathcal{O}_{S}$-coherent local system. Therefore,

$$\mathcal{H}\text{om}_{\mathcal{D}_{X' \times S/S}}(M_{|X' \times S}, N_{|X' \times S}) \simeq i_{\ast}\mathcal{H}\text{om}_{p_{U}^{-1}\mathcal{O}_{S}}(\mathcal{F}, \mathcal{G})$$
is $S$-$C$-constructible.

Let $j : X' \hookrightarrow X$ denote the open inclusion. Then, according to [11, Cor. 2.8 & Prop. 2.20], the sheaf $j_*\mathcal{H}om_{\mathcal{D}_{X' \times S/S}}(M|_{X' \times S}, N|_{X' \times S})$ is $S$-$C$-constructible. By considering the exact sequence

$$0 \rightarrow \mathcal{H}om_{\mathcal{D}_{X \times S/S}}(M, \Gamma_{Z' \times S}N) \rightarrow \mathcal{H}om_{\mathcal{D}_{X \times S/S}}(M, N) \rightarrow j_*j^1\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(M, N)$$

we are reduced to proving the result in the case where $N$ is replaced with $N' := \Gamma_{Z' \times S}N$, which is holonomic by Lemma 1.1. Note that, since $\dim S = 1$, strictness is equivalent to the absence of $O_S$-torsion, so that $N'$ is strict if $N$ is so.

Since $M$ is strict holonomic, $DM$ is also strict holonomic (cf. [13, Cor. 1.12]), hence so is $\Gamma_{Z' \times S}DM$, as well as $M' := D\Gamma_{Z' \times S}DM$. On the other hand, since $N'$ is strict holonomic, $DN'$ satisfies the same properties, and the natural morphism

$$\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(DN', \Gamma_{Z' \times S}DM) \rightarrow \mathcal{H}om_{\mathcal{D}_{X \times S/S}}(DN', DM)$$

is an isomorphism. Since $N'$ and $\Gamma_{Z' \times S}DM$ are $\mathcal{D}_{X \times S/S}$-coherent, they satisfy the biduality isomorphism and by [11, (3)] we conclude

$$\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(M', N') \rightarrow \mathcal{H}om_{\mathcal{D}_{X \times S/S}}(M, N').$$

Now the induction hypothesis applies to the pair $(M', N')$ with corresponding set $Z'$.

q.e.d.

4.b. Proof of Theorem 1 in the strict case.

**Proposition 4.2.** Let $M, N$ be regular holonomic $\mathcal{D}_{X \times S/S}$-modules, $N$ being strict. Then the natural morphism

$$\beta_{M,N} : \mathcal{H}om_{\mathcal{D}_{X \times S/S}}(M, N) \rightarrow \mathcal{H}om_{perv(p^{-1}_X O_S)}(p^*\text{Sol} N, p^*\text{Sol} M)$$

is an isomorphism. In particular, if $M$ is strict, taking $N := RH^{S}(p^*\text{Sol} M)$, there exists an isomorphism $\beta_M : M \rightarrow RH^{S}(p^*\text{Sol} M)$ which is compatible with Kashiwara’s morphism in the absolute case.

**Proof.** Since $N$ is holonomic and strict, $p^*\text{Sol} N$ is perverse and dual perverse, after [13, Prop. 2] and by Corollary 3.6 the target of $\beta_{M,N}$, which is $S$-$C$-constructible, is strict. By an argument similar to that of Corollary 3.6, the source of $\beta_{M,N}$ is also strict, and it is $S$-$C$-constructible according to Theorem 4.1. Hence, in particular, for each $s \in S$ we have

$$Li^*_s(\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(M, N)) \simeq i^*_s(\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(M, N))$$

and

$$Li^*_s(\mathcal{H}om_{perv(p^{-1}_X O_S)}(p^*\text{Sol} N, p^*\text{Sol} M)) \simeq i^*_s(\mathcal{H}om_{perv(p^{-1}_X O_S)}(p^*\text{Sol} N, p^*\text{Sol} M)).$$

Since $i^*_s(\beta_{M,N})$ is an isomorphism according to the regularity assumption and the absolute case, the first statement follows. For the second statement, we use the isomorphism $\alpha_F$ recalled in the introduction for $F = p^*\text{Sol}(M)$,
so that, for $N = \text{RH}^S(p\text{Sol}M)$, we find $p\text{Sol}N \simeq p\text{Sol}M$. Note that $F$ is perverse and dual perverse, according to [13, Prop.2], that is, strict, and therefore $N$ is also strict (cf. [13, Cor.4]). We have thus obtained an isomorphism

$$\text{Hom}_{\mathcal{D}_{X \times S/S}}(M, \text{RH}^S(p\text{Sol}M)) \to \text{Hom}_{\text{perv}(p_X^{-1}O_S)}(p\text{Sol}M, p\text{Sol}M),$$

and we define $\beta_M$ as being the unique morphism corresponding to $\text{Id} \in \text{Hom}_{\text{perv}(p_X^{-1}O_S)}(p\text{Sol}M, p\text{Sol}M)$. It is an isomorphism by checking the reduction to each $s_o \in S$.

A special case of Corollary 3 can now be proved (the general case is proved at the end of Section 4.e).

**Corollary 4.3.** For each $F \in \mathcal{D}^b_{\text{hol}}(p_X^{-1}O_S)$ which is perverse and dual perverse there is a functorial (when restricting to this quasi-abelian category) isomorphism which is compatible with the canonical morphism in the absolute case:

$$D(\text{RH}^S(F)) \simeq \text{RH}^S(DF)$$

where in the left hand side $D$ stands for the duality functor in $\mathcal{D}^b_{\text{hol}}(\mathcal{D}_{X \times S/S})$ while in the right hand side stands for the duality functor on $\mathcal{D}^b_{\text{c},\text{c}}(p_X^{-1}O_S)$.

**Proof.** We consider the isomorphism $\beta_{MN}$ in the case of $M := D \text{RH}^S(F)$ and $N := \text{RH}^S(DF)$. Since $p\text{Sol}(M) \simeq p\text{Sol}(N) \simeq DF$, the desired morphism will be the unique one corresponding to the identity of $DF$. The fact that it is an isomorphism is again a consequence of the reduction to the absolute case for each value of $s_o$ in $S$.

q.e.d.

4.c. **Proof of Theorems 1 and 2 in the torsion case.**

**Proposition 4.4.** Let $M \in \text{Mod}_{\text{rhol}}(\mathcal{D}_{X \times S/S})$ be such that $\text{Supp}(M) \subseteq X \times T$ with $\dim T = 0$. Then $\tilde{M} := \mathcal{D}_{X \times S} \otimes_{\mathcal{D}_{X \times S/S}} M$ is a regular holonomic $\mathcal{D}_{X \times S}$-module.

**Proof.** The statement being local, we may assume that $T = \{s_o\}$. Hence $\text{Char}(M) = \Lambda \times \{s_o\}$, where $\Lambda$ is a Lagrangian $\mathcal{C}^*$-conic closed analytic subset in $T^*X$, and, taking a local coordinates $s$ on $S$ vanishing at $s_o$, there exists $n \in \mathbb{N}$ such that $s^nM = 0$. Since we are dealing with triangulated categories, by an easy argument by induction on $n$ we may assume that $n = 1$. In that case, we have $M \simeq M_0 \boxtimes O_S/O_{SS}$, where, by the assumption of relative regularity, $M_0$ is a regular holonomic $\mathcal{D}_{X}$-module satisfying $\text{Char}(M_0) = \Lambda$. By construction $\tilde{M} \simeq M_0 \boxtimes \mathcal{D}_{S} \boxtimes \mathcal{D}_{SS}$ and $\text{Char}(\tilde{M}) = \Lambda \times T^*_s S =: \Lambda$.

Therefore $\tilde{M}$ is a regular holonomic $\mathcal{D}_{\times S}$-module since the category of regular holonomic $\mathcal{D}_{\times S}$-modules is closed under external tensor product.

q.e.d.

We denote by $\mathcal{D}^b_{\text{rhol}}(\mathcal{D}_{X \times S/S})_t$ the thick subcategory of $\mathcal{D}^b_{\text{rhol}}(\mathcal{D}_{X \times S/S})$ whose objects have support in $X \times T$ with $\dim T = 0$.
Proposition 4.5. The solution functor $^p\text{Sol}$ restricted to $\mathcal{D}^b_{\text{rhhol}}(\mathcal{D}_{X \times S/S})_t$ is an equivalence of categories

$$^p\text{Sol} : \mathcal{D}^b_{\text{rhhol}}(\mathcal{D}_{X \times S/S})_t \rightarrow \mathcal{D}^b_{\text{c-c}}(\mathcal{P}_{X}^{-1}\mathcal{O}_S)_t$$

with quasi-inverse the restriction of the functor $\text{RH}^S$ to $\mathcal{D}^b_{\text{c-c}}(\mathcal{P}_{X}^{-1}\mathcal{O}_S)_t$.

Proof. It is sufficient to prove that the restriction of $\text{RH}^S$ to $\mathcal{D}^b_{\text{c-c}}(\mathcal{P}_{X}^{-1}\mathcal{O}_S)_t$ is fully faithful. Indeed $^p\text{Sol}$ is essentially surjective since for any $\mathcal{F} \in \mathcal{D}^b_{\text{c-c}}(\mathcal{P}_{X}^{-1}\mathcal{O}_S)$ we have $\mathcal{F} \simeq ^p\text{Sol}\text{RH}^S(\mathcal{F})$ (cf. [13, Th. 3]) and in the case of a torsion object $\mathcal{F}$ in $\mathcal{D}^b_{\text{c-c}}(\mathcal{P}_{X}^{-1}\mathcal{O}_S)_t$ we have $\text{RH}^S(\mathcal{F}) \in \mathcal{D}^b_{\text{rhhol}}(\mathcal{D}_{X \times S/S})_t$.

For the full faithfulness it is enough to prove that the morphism:

$$\text{(4.1)} \quad \text{R}^i\text{Hom}_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \text{RH}^S(\mathcal{G}))$$

$$\longrightarrow \text{R}^i\text{Hom}_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \text{R}^i\text{Hom}_{\mathcal{P}^{-1}\mathcal{O}_S}(\mathcal{G}, \mathcal{O}_{X \times S})[d \mathcal{X}])$$

is an isomorphism for any $\mathcal{M} \in \mathcal{D}^b_{\text{rhhol}}(\mathcal{D}_{X \times S/S})_t$ and for any $\mathcal{G} \in \mathcal{D}^b_{\text{c-c}}(\mathcal{P}_{X}^{-1}\mathcal{O}_S)$.

The cohomologies of $\mathcal{M}$ are regular holonomic $\mathcal{D}_{X \times S/S}$-modules and, according to Proposition 4.4, $\mathcal{D}_{X \times S} \otimes_{\mathcal{D}_{X \times S/S}} \mathcal{M}$ is a complex whose cohomologies are regular holonomic.

Thanks to [13, Prop. 3.5], we may assume that $\mathcal{G} = \mathcal{C}_{\Omega \times S} \otimes \mathcal{P}_{X}^{-1}\mathcal{O}_S$ for some open subanalytic subset $\Omega$ of $X$, hence $\text{RH}^S(\mathcal{G}) = T\text{Hom}(\mathcal{C}_{\Omega \times S}, \mathcal{O}_{X \times S})[d \mathcal{X}]$ which is a complex with $\mathcal{D}_{X \times S}$-modules as cohomologies and we get a chain of isomorphisms

$$\text{R}^i\text{Hom}_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \text{RH}^S(\mathcal{G})) \simeq \text{R}^i\text{Hom}_{\mathcal{D}_{X \times S/S}}(\mathcal{D}_{X \times S} \otimes_{\mathcal{D}_{X \times S/S}} \mathcal{M}, \text{RH}^S(\mathcal{G}))$$

$$\simeq \text{R}^i\text{Hom}_{\mathcal{D}_{X \times S/S}}(\mathcal{D}_{X \times S} \otimes_{\mathcal{D}_{X \times S/S}} \mathcal{M}, \text{R}^i\text{Hom}(\mathcal{C}_{\Omega \times S}, \mathcal{O}_{X \times S})[d \mathcal{X}])$$

$$\simeq \text{R}^i\text{Hom}_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \text{R}^i\text{Hom}_{\mathcal{P}^{-1}\mathcal{O}_S}(\mathcal{G}, \mathcal{O}_{X \times S})[d \mathcal{X}])$$

$$\simeq \text{R}^i\text{Hom}_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \text{R}^i\text{Hom}_{\mathcal{P}^{-1}\mathcal{O}_S}(\mathcal{G}, \mathcal{O}_{X \times S})[d \mathcal{X}])$$

where isomorphism (*) follows by [4, Cor. 8.6]. q.e.d.

Remark 4.6. By [6, (2.6.7)], we have a natural isomorphism in $\mathcal{D}^b(\mathcal{P}_{X}^{-1}\mathcal{O}_S)$:

$$\text{R}^i\text{Hom}_{\mathcal{D}_{X \times S/S}}(\mathcal{M}, \text{R}^i\text{Hom}_{\mathcal{P}^{-1}\mathcal{O}_S}(\mathcal{G}, \mathcal{O}_{X \times S})[d \mathcal{X}]) \simeq \text{R}^i\text{Hom}_{\mathcal{P}^{-1}\mathcal{O}_S}(\mathcal{G}, \text{Sol} \mathcal{M})$$

and the latter complex belongs to $\mathcal{D}^b_{\text{c-c}}(\mathcal{P}_{X}^{-1}\mathcal{O}_S)$. Therefore, Theorem 2 holds when $\mathcal{M}$ is a torsion module.

4d. Proof of Theorems 1 and 2 in the case of a module of D-type.

Let us fix a reduced divisor $D$ in $X$. We will prove that Theorem 2 holds for any $\mathcal{D}_{X \times S/S}$-module $\mathcal{L}$ of D-type and for any $\mathcal{F} \in \mathcal{D}^b_{\text{c-c}}(\mathcal{P}_{X}^{-1}\mathcal{O}_S)$.

Proof. The statement has a local nature so we can argue as in the proof of [13, Cor. 2.8].

Let us consider a finite ramification in the $s$ variable so that, in local coordinates $(x, s)$ on $X \times S$ vanishing at $p = (x_0, s_0) \in D \times S$ such that $D = \{x \in X, x_1 \cdots x_l = 0\}$, $\mathcal{L}' := \delta^*\mathcal{L}$ is isomorphic to $\mathcal{D}_{X \times S'/S'} \sum_{i=1}^d \mathcal{D}_{X \times S'/S'}(x_i \partial_{x_i} - \alpha_i(s')) + \sum_{i=d+1}^n \mathcal{D}_{X \times S'/S'} \partial_{x_i}$ for some
holomorphic functions $\alpha_i(s')$, $i = 1, \ldots, \ell$, on a fixed open neighbourhood of 0 in $S$.

We will prove a stronger result in this framework (assumption of $\mathbb{R}$-constructibility). Let us now work with $S'$ denoted by $S$ for a moment.

**Lemma 4.7.** For $L'$ and for any $F \in D^{b}_{\mathbb{R}, c}(p^{-1}\mathcal{O}_S)$, the canonical morphism:

$$\beta_{L', F} : R\mathcal{H}om_{\mathcal{D}_{X \times S}/S}(L', R\mathcal{H}S(F)) \rightarrow R\mathcal{H}om_{\mathcal{D}_{X \times S}/S}(L', R\mathcal{H}om_{p^{-1}\mathcal{O}_S}(F, \mathcal{O}_{X \times S})[d_X])$$

is an isomorphism.

**Proof.** We may again reduce to the case $F = \mathbb{C}_{\Omega \times S} \times p^{-1}\mathcal{O}_S$ for some open subanalytic relatively compact subset $\Omega$ of $X$, so $F = \mathbb{C}_{\Omega \times S} \times p^{-1}\mathcal{O}_S$. We have

$$R\mathcal{H}om_{\mathcal{D}_{X \times S}/S}(L', R\mathcal{H}S(F)) \cong R\mathcal{H}om_{\mathcal{D}_{X \times S}/S}(\mathcal{L}', R\mathcal{H}om_{\mathcal{O}_{X \times S}/S}(\mathcal{O}_{X \times S}, THS(F)[d_X])) \cong R\mathcal{H}om_{\mathcal{D}_{X \times S}/S}(\mathcal{L}', \mathcal{O}_{X \times S}, THS(F)[d_X]) \cong R\mathcal{H}om_{\mathcal{D}_{X \times S}/S}(\mathcal{L}', \mathcal{O}_{X \times S}, T\mathcal{H}om(\mathbb{C}_{\Omega \times S}, \mathcal{D}\mathfrak{b}_{X \times S}[d_X])).$$

Remark that

$$(\mathcal{D}_{X \times S} \otimes \mathcal{D}_{X \times S}/S \mathcal{M}) \otimes \mathcal{O}_{X \times S}$$

is a Fuchsian system along each hypersurface $D_j \times S := \{(x, s), x_j = 0\}$, $j = 1, \ldots, \ell$, in the sense of [16]. It is clear (cf. [10, Ex. 5.1]) that the solutions of the homogeneous system defining $\mathcal{L}'$ belong to $T\mathcal{H}om(\mathbb{C}_{\Omega \times S}, \mathcal{D}\mathfrak{b}_{X \times S})$. The result is then an application of [16, Th. 1] which entails the solvability in $\mathcal{D}\mathfrak{b}_{(0,0)}$ of the same system. q.e.d.

According to Lemma 4.7, the morphism $(\beta_{\mathcal{D}, \mathcal{L}, \mathcal{S}^F})$ is an isomorphism. Therefore $R\mathcal{H}om_{\mathcal{D}_{X \times S}/S}(\mathcal{D}, \mathcal{L}, \mathcal{S}^F)$ is $\mathcal{L}^F$-constructible provided that $\mathcal{S}^F$ is so. The statement now follows from Corollary 3.10. q.e.d.

4.e. **End of the proof of Theorem 2.** Let $\text{Char}(\mathcal{M}) = \bigcup_{i \in I} \Lambda_i \times T_i$, $\text{Supp}(\mathcal{M}) = \bigcup_{i \in I} Y_i \times T_i$ and $Z_{\mathcal{M}} := \bigcup_{i \in I} Y_i$. We proceed by induction on the dimension of $Z_{\mathcal{M}}$. If $\dim Z_{\mathcal{M}} < 0$ we have $\text{Supp}(\mathcal{M}) = \emptyset$ and so the statement is trivially verified since $\mathcal{M} = 0$. Let us suppose the statement true for any $\mathcal{N} \in D_{\mathfrak{hol}}(\mathcal{D}_{X \times S}/S)$ such that with $\dim Z_{\mathcal{N}} < k$ (with $k \geq 0$) and let us prove the statement for $\mathcal{M} \in D_{\mathfrak{hol}}(\mathcal{D}_{X \times S}/S)$ with $\dim Z_{\mathcal{M}} = k$.

**Step 1. Reduction to a single module.** Without loss of generality we can suppose that $\mathcal{M} \in D_{\mathfrak{hol}}(\mathcal{D}_{X \times S}/S)$ and we can proceed by induction on the length of $\mathcal{M}$. Using the distinguished triangle $\mathcal{I}^{\mathfrak{b}}(\mathcal{M}) \rightarrow \mathcal{M} \rightarrow \tau^{\geq 1} \mathcal{M}$ we are reduced to prove the statement in the case of $\mathcal{M}$ a regular holonomic $\mathcal{D}_{X \times S}/S$-module with $\dim Z_{\mathcal{M}} = k$. 

Step 2. Reduction to the strict case. Following the notations of Section 1 let \( t(M) \) (respectively \( f(M) \)) be the torsion part (respectively the strict quotient) of \( M \). According to Remark 4.6 we have \( R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(t(M), RH^S(F)) \in \mathcal{D}_{C, c}^{b}(p_X^{-1}O_S) \) for any \( F \in \mathcal{D}_{C, c}^{b}(p_X^{-1}O_S) \). Hence \( R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(M, RH^S(F)) \) belongs to \( \mathcal{D}_{C, c}^{b}(p_X^{-1}O_S) \) if and only if \( R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(f(M), RH^S(F)) \) does. We notice that \( Z_f(M) \subseteq Z_M \); hence we are reduced to prove the statement in the case where \( M = f(M) \) is a strict regular holonomic \( \mathcal{D}_{X \times S/S} \)-module.

Step 3. Reduction to the strict localized case. Let \( M \) be a strict regular holonomic \( \mathcal{D}_{X \times S/S} \)-module such that \( \dim Z_M = k \). Locally there exists a hypersurface \( Y \) in \( X \) such that \( Y \cap Z_M \) is smooth with dimension strictly smaller than \( k \).

Let us consider the distinguished triangle in \( \mathcal{D}^{b}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \)

\[
R\Gamma[Y \times S](M) \rightarrow M \rightarrow M(\ast (Y \times S)) \xrightarrow{+1}.
\]

and let us notice that since \( Z_{M(\ast (Y \times S))} \subseteq Z_M \) the complex \( R\Gamma[Y \times S](M) \) satisfies the induction hypothesis since, by Propositions 4.2 and 3.1, \( R\Gamma[Y \times S](M) \in \mathcal{D}^{b}_{\text{hol}}(\mathcal{D}_{X \times S/S}) \), and \( \dim Z_{R\Gamma[Y \times S](M)} < k \). Hence we are reduced to prove Theorem 2 with \( M = M(\ast (Y \times S)) \).

Step 4. Reduction to the case of systems of \( D \)-type. Let \( M \simeq M(\ast (Y \times S)) \) be a strict regular \( \mathcal{D}_{X \times S/S} \)-module. Let us consider the distinguished triangle

\[
R\Gamma[Y \times S](RH^S(F)) \rightarrow RH^S(F) \rightarrow RH^S(F)(\ast (Y \times S)) \xrightarrow{+1}.
\]

According to [11, (3)], we have

\[
R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(M, R\Gamma[Y \times S](RH^S(F))) \\
\simeq R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(M, R\Gamma[(Y \cap Z_M) \times S](RH^S(F))) \\
\simeq R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(D\mathcal{R}\Gamma[(Y \cap Z_M) \times S](RH^S(F)), DM),
\]

where the first isomorphism holds true since \( \text{Supp}(M) \subseteq Z_M \times S \). Thus \( R\mathcal{H}om_{\mathcal{D}_{X \times S/S}}(D\mathcal{R}\Gamma[(Y \cap Z_M) \times S](RH^S(F)), DM(\mathcal{D})) \) belongs to \( \mathcal{D}_{C, c}^{b}(p_X^{-1}O_S) \) by induction hypothesis since \( D\mathcal{R}\Gamma[(Y \cap Z_M) \times S](RH^S(F)) \in \mathcal{D}_{\text{hol}}^{b}(\mathcal{D}_{X \times S/S}) \) and it is supported on \( (Y \cap Z_M) \times S \). Hence it remains to prove the statement for \( G = F \otimes C_{(X \setminus Y) \times S} \) so that, after Proposition 3.1,

\[
RH^S(G) := RH^S(F)(\ast (Y \times S)).
\]

Following the argument of [13, §4.4], let us consider the following commutative diagram

\[
\begin{array}{ccc}
X' & \xrightarrow{i} & X' \\
\downarrow{j'} & & \downarrow{j} \\
Z'_M & \xrightarrow{i} & U \xrightarrow{j} X
\end{array}
\]

\[
\pi Z' \xleftarrow{\pi j} Y' \xrightarrow{j'} X' \xrightarrow{\pi} X
\]
where \( \pi \) is a projective morphism, \( Y' \) is a normal crossing hypersurface, \( Z^*_{\text{M}} := Z_\text{M} \setminus Y, U := X \setminus Y \) and \( \tau'_Z \) is biholomorphic. Note that the assumption entails that \( \pi_* \pi^{-1}G = G \). By adjunction and according to Corollary 3.4, we have

\[
R\pi_* R\mathcal{H}om_{D_X \times S/S}(D\pi^*\mathcal{M}, D\pi^* \mathcal{R}H^S(G)) \simeq R\pi_* R\mathcal{H}om_{D_X \times S/S}(D\pi^*\mathcal{M}, \mathcal{R}H^S(\pi^{-1}G)).
\]

It is thus enough to prove that the right-hand side belongs to \( D_b^{B,C}(p_X^{-1}\mathcal{O}_S) \).

Conclusion. We can now apply the result of Section 4.d for the case where \( \mathcal{M} \) is of D-type, and conclude the proof of Theorem 2, and consequently that of Theorem 1. q.e.d.

Proof of Corollary 3. Since, for any \( F \in D_b^{B,C}(p_X^{-1}\mathcal{O}_S) \), we have functorial isomorphisms

\[
\mathcal{P}\text{Sol} D(\mathcal{R}H^S(F)) \simeq D \mathcal{P}\text{Sol}(\mathcal{R}H^S(F)) \simeq DF \simeq \mathcal{P}\text{Sol} \mathcal{R}H^S(DF),
\]

we obtain Corollary 3. q.e.d.

References
[2] Luisa Fiorot and Teresa Monteiro Fernandes, \( t \)-structures for relative \( \mathcal{D} \)-modules and \( t \)-exactness of the de Rham functor, J. Algebra 509 (2018), 419–444.

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