UNITARY DUAL OF GL(n) AT ARCHIMEDEAN PLACES AND GLOBAL JACQUET-LANGLANDS CORRESPONDENCE

I. A BADULESCU AND D. RENARD

Abstract. In [7], results about the global Jacquet-Langlands correspondence, (weak and strong) multiplicity-one theorems and the classification of automorphic representations for inner forms of the general linear group over a number field are established, under the condition that the local inner forms are split at archimedean places. In this paper, we extend the main local results of [7] to archimedean places so that this assumption can be removed. Along the way, we collect several results about the unitary dual of general linear groups over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ of independent interest.

Contents

Date: November 16, 2009.
1991 *Mathematics Subject Classification.* 11F70, 22E50, 22E55. Keywords : Jacquet-Langlands correspondence, Automorphic Representations, Unitary Dual, Langlands functoriality.

1. Introduction

In [7], results about the global Jacquet-Langlands correspondence, (weak and strong) multiplicity-one theorems and the classification of automorphic representations for inner forms of the general linear group over a number field are established, under the condition that the local inner forms are split at archimedean places. The main goal of this paper is to remove this hypothesis. The paper consists of two parts. In the first part, we extend the main local results of [7] to archimedean places. In the second part, we explain how to use these local results to establish the global results in their full generality. Along the way, we collect several results about the unitary dual of general linear groups over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ of independent interest. Let us now explain in more details the content of this paper.

1.1. **Some notation.** Let A be one of the division algebras $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. If $A = \mathbb{R}$ or $A = \mathbb{C}$ and $n \in \mathbb{N}^\times$, we denote by det the determinant map on $\mathbf{GL}(n, A)$ (taking values in A). If $A = \mathbb{H}$, let RN be the reduced norm map on $\mathbf{GL}(n, \mathbb{H})$ (taking values in $\mathbb{R}_+^\times$).

If $n \in \mathbb{N}$ and $\sum_{i=1}^s n_i = n$ is a partition of n, the group $\mathbf{GL}(n_1, A) \times \mathbf{GL}(n_2, A) \times ... \times \mathbf{GL}(n_s, A)$ is identified with the subgroup of $\mathbf{GL}(n, A)$ of bloc diagonal matrices of size $n_1, \dots, n_s$. Let us denote $G_{(n_1,\dots,n_s)}$ this subgroup and $P_{(n_1,\dots,n_s)}$ the parabolic subgroup of $GL(n, A)$ containing $G_{(n_1,\dots,n_s)}$ and the Borel subgroup of invertible upper triangular matrices. For $1 \leq i \leq s$, let π_i be an admissible representation of $\mathbf{GL}(n_i, A)$ of finite length. We write then $\pi_1 \times \pi_2 \times ... \times \pi_s$ for the representation parabolically induced from the representation $\pi_1 \otimes \pi_2 \otimes ... \otimes \pi_s$ of $G_{(n_1,\dots,n_s)}$ with respect to $P_{(n_1,\dots,n_s)}$. We will also use this notation for the image of representations in the Grothendieck group of virtual characters, which makes the above product commutative. We also often do not distinguish between a representation and its isomorphy class and write "equal" for "isomorphic".

1.2. **Classification of unitary representations.** We recall first Tadić classification of the unitary dual of the groups $\mathbf{GL}(n, \mathbb{R})$ and $\mathbf{GL}(n, \mathbb{C})$, following [47]. The classification is similar to the one for non archimedean local fields ([42],[44]) and is explained in details in Section 7. Part of the arguments do not appear in the literature in the case of $\mathbf{GL}(n, \mathbb{H})$, so we give the complete proofs in Section 10, 11 and 12, using Vogan's classification [52].

Let $X_\mathbb{C}$ be the set of unitary characters of $\mathbb{C}^\times$. If $\chi \in X_\mathbb{C}$, $n \in \mathbb{N}^\times$ let χ_n be the character $\chi \circ \det$ of $\mathbf{GL}(n, \mathbb{C})$. Let ν_n be the character of $\mathbf{GL}(n, \mathbb{C})$ given by the square of the module of the determinant. If σ is a representation of $\mathbf{GL}(n, \mathbb{C})$ and $\alpha \in \mathbb{R}$, write $\pi(\sigma, \alpha)$ for the representation $\nu_n^\alpha \sigma \times \nu_n^{-\alpha} \sigma$ of $\mathbf{GL}(2n, \mathbb{C})$. Set

$$\mathcal{U}_\mathbb{C} = \{\chi_n, \pi(\chi_n, \alpha) \mid \chi \in X_\mathbb{C}, \, n \in \mathbb{N}^\times, \, \alpha \in]0, \frac{1}{2}[\}.$$

Let $X_\mathbb{R}$ be the set of unitary characters of $\mathbb{R}^\times$. Let sgn denote the sign character. If $\chi \in X_\mathbb{R}$, $n \in \mathbb{N}^\times$ let χ_n be the character $\chi \circ \det$ of $\mathbf{GL}(n, \mathbb{R})$ and χ'_n the character $\chi \circ RN$ of $\mathbf{GL}(n, \mathbb{H})$. For fixed n, the map $\chi \mapsto \chi_n$ is an isomorphism from the group of unitary characters of $\mathbb{R}^\times$ to the group of unitary characters of $\mathbf{GL}(n, \mathbb{R})$, while $\chi \mapsto \chi'_n$ is a surjective map from the group of unitary characters of $\mathbb{R}^\times$ to the group of unitary characters of $\mathbf{GL}(n, \mathbb{H})$, with kernel $\{1, \mathrm{sgn}\}$.

Let ν_n (resp. ν'_n) be the character of $\mathbf{GL}(n, \mathbb{R})$ (resp. $\mathbf{GL}(n, \mathbb{H})$) given by the absolute value (resp. the reduced norm) of the determinant. If σ is a representation of

$\mathbf{GL}(n,\mathbb{R})$ (resp. $\mathbf{GL}(n,\mathbb{H})$) and $\alpha\in\mathbb{R}$, write $\pi(\sigma,\alpha)$ for the representation $\nu_n^{\alpha}\sigma\times\nu_n^{-\alpha}\sigma$ of $\mathbf{GL}(2n,\mathbb{R})$ (resp. the representation ${\nu'}_n^{\alpha}\sigma\times{\nu'}_n^{-\alpha}\sigma$ of $\mathbf{GL}(2n,\mathbb{H})$).

Let D_2^u be the set of isomorphy classes of square integrable (modulo center) representations of $\mathbf{GL}(2,\mathbb{R})$. For $\delta\in D_2^u$ and $k\in\mathbb{N}^\times$, write $u(\delta,k)$ for the Langlands quotient of the representation $\nu_2^{\frac{k-1}{2}}\delta\times\nu_2^{\frac{k-3}{2}}\delta\times\nu_2^{\frac{k-5}{2}}\delta\times...\times\nu_2^{-\frac{k-1}{2}}\delta$. Then $u(\delta,k)$ is a representation of $\mathbf{GL}(2k,\mathbb{R})$. Set

$$\begin{aligned}\mathcal{U}_{\mathbb{R}}=&\{\chi_n,\pi(\chi_n,\alpha)\mid\ \chi\in X_{\mathbb{R}},\,n\in\mathbb{N}^\times,\,\alpha\in]0,\frac{1}{2}[\}\\ &\cup\{u(\delta,k),\pi(u(\delta,k),\alpha)\mid\ \delta\in D_2^u,\,k\in\mathbb{N}^\times,\ \alpha\in]0,\frac{1}{2}[\}.\end{aligned}$$

Let now D be the set of isomorphism classes of irreducible unitary representations of $\mathbb{H}^\times$ *which are not one-dimensional*. For $\delta\in D$ and $k\in\mathbb{N}^\times$, write $u(\delta,k)$ for the Langlands quotient of the representation ${\nu'}_1^{\frac{k-1}{2}}\delta\times{\nu'}_1^{\frac{k-3}{2}}\delta\times{\nu'}_1^{\frac{k-5}{2}}\delta\times...\times{\nu'}_1^{-\frac{k-1}{2}}\delta$. Then $u(\delta,k)$ is a representation of $\mathbf{GL}(k,\mathbb{H})$.

Set

$$\begin{aligned}\mathcal{U}_{\mathbb{H}}=&\{\chi'_n,\pi(\chi'_n,\alpha)\mid\ \chi\in X_{\mathbb{R}},\,n\in\mathbb{N}^\times,\,\alpha\in]0,1[\}\\ \cup\ \ &\{u(\delta,k),\pi(u(\delta,k),\alpha)\mid\ \delta\in D,\,k\in\mathbb{N}^\times\,\alpha\in]0,\frac{1}{2}[\}.\end{aligned}$$

Theorem 1.1. *For $A=\mathbb{C},\mathbb{R},\mathbb{H}$, any representation in $\mathcal{U}_A$ is irreducible and unitary, any product of representations in $\mathcal{U}_A$ is irreducible and unitary, and any irreducible unitary representation π of $\mathbf{GL}(n,A)$ can be written as a product of elements in $\mathcal{U}_A$. Moreover, π determines the factors of the product (up to permutation).*

Notice the two different ranges for the the possible values of α in the case $A=\mathbb{H}$.

1.3. **Jacquet-Langlands correspondence for unitary representations.** Any element in $\mathbf{GL}(n,\mathbb{H})$ has a characteristic polynomial of degree $2n$ with coefficients in $\mathbb{R}$. We say that two elements $g\in\mathbf{GL}(2n,\mathbb{R})$ and $g'\in\mathbf{GL}(n,\mathbb{H})$ correspond (to each other) if they have the same characteristic polynomial and this polynomial has distinct roots in $\mathbb{C}$ (this last condition means that g and g' are regular semisimple). We then write $g\leftrightarrow g'$.

Let $\mathbf{C}$ denote the Jacquet-Langlands correspondence between irreducible square integrable representations of $\mathbf{GL}(2,\mathbb{R})$ and irreducible unitary representations of $\mathbb{H}^\times$ ([23]). This correspondence can be extended to a correspondence $|\mathbf{LJ}|$ between *all* irreducible unitary representations of $\mathbf{GL}(2n,\mathbb{R})$ and $\mathbf{GL}(n,\mathbb{H})$ (it comes from a ring morphism $\mathbf{LJ}$ between the respective Grothendieck groups, defined in Section 4, which explain the notation). In what follows, it is understood that each time we write the relation $|\mathbf{LJ}|(\pi)=\pi'$ for π and π' representations of $\mathbf{GL}(2n,\mathbb{R})$ and $\mathbf{GL}(n,\mathbb{H})$ respectively, then π and π' satisfy the character relation $\Theta_\pi(g)=\varepsilon(\pi)\Theta'_\pi(g')$ for all $g\leftrightarrow g'$ where $\varepsilon(\pi)$ is an explicit sign (π clearly determines π' and ε). The correspondence $|\mathbf{LJ}|$ for unitary representations is given first on elements in $\mathcal{U}_{\mathbb{R}}$:

(a) $|\mathbf{LJ}|(\chi_{2n})=\chi'_n$ and $|\mathbf{LJ}|(\pi(\chi_{2n},\alpha))=\pi(\chi'_n,\alpha)$ for all $\chi\in X_{\mathbb{R}}$ and $\alpha\in]0,\frac{1}{2}[$;

(b) If $\delta\in D_2^u$ is such that $\mathbf{C}(\delta)$ is in D (i.e. is not one-dimensional) then $|\mathbf{LJ}|(u(\delta,k))=|\mathbf{LJ}|(\mathbf{C}(\delta),k)$ and $\mathbf{LJ}(\pi(u(\delta,k),\alpha))=\pi(u(\mathbf{C}(\delta),k),\alpha)$ for all $\alpha\in]0,\frac{1}{2}[$;

(c) If $\delta \in D_2^u$ is such that $\mathbf{C}(\delta)$ is a one-dimensional representation χ'_1, then

— $|\mathbf{LJ}|(u(\delta,k)) = \pi(\chi'_{\frac{k}{2}}, \frac{1}{2})$, $|\mathbf{LJ}|(\pi(u(\delta,k),\alpha)) = \pi(\pi(\chi'_{\frac{k}{2}}, \frac{1}{2}),\alpha)$ if k is even and $\alpha \in]0, \frac{1}{2}[$.

— $|\mathbf{LJ}|(u(\delta,k)) = \chi'_{\frac{k+1}{2}} \times \chi'_{\frac{k-1}{2}}$, $|\mathbf{LJ}|(\pi(u(\delta,k),\alpha)) = \pi(\chi'_{\frac{k+1}{2}},\alpha) \times \pi(\chi'_{\frac{k-1}{2}},\alpha)$ if $k \neq 1$ odd and $\alpha \in]0, \frac{1}{2}[$.

— $|\mathbf{LJ}|(\delta) = \chi'_1$, $|\mathbf{LJ}|(\pi(\delta,\alpha)) = \pi(\chi'_1,\alpha)$, $\alpha \in]0, \frac{1}{2}[$,.

Let π be an irreducible unitary representation of $\mathbf{GL}(2n,\mathbb{R})$. If writing π as a product of elements in $\mathcal{U}_{\mathbb{R}}$ involves a factor not listed in (a), (b) or (c) it is easy to show that π has a character which vanishes on elements which correspond to elements of $\mathbf{GL}(n,\mathbb{H})$, and we set $|\mathbf{LJ}|(\pi) = 0$. If all the factors σ_i of π are in (a) (b) (c) above, $|\mathbf{LJ}|(\pi)$ is the product of the $|\mathbf{LJ}|(\sigma_i)$ (an irreducible unitary representation of $\mathbf{GL}(n,\mathbb{H})$). Elements of $\mathcal{U}_{\mathbb{R}}$ not listed at (a) (b) (c) are of type χ or $\pi(\chi,\alpha)$, with χ a character of some $\mathbf{GL}(k,\mathbb{R})$ and k odd.

Notice that some unitary irreducible representations of $\mathbf{GL}(n,\mathbb{H})$ are not in the image of this map (if $n \geq 2$). For instance, when $\chi \in X_{\mathbb{R}}$ and $\frac{1}{2} < \alpha < 1$, then both $\pi(\chi_2,\alpha)$ and $\pi(\chi'_1,\alpha)$ are irreducible and correspond to each other by the character relation, but $\pi(\chi'_1,\alpha)$ is unitary while $\pi(\chi_2,\alpha)$ is not. Using the classification of unitary representations for $\mathbf{GL}(4,\mathbb{R})$ and basic information from the infinitesimal character, it is clear that no (possibly other) unitary representation of $\mathbf{GL}(4,\mathbb{R})$ has matching character with $\pi(\chi'_1,\alpha)$.

As a consequence of the above results, we get:

Theorem 1.2. *Let u be a unitary irreducible representation of $\mathbf{GL}(2n,\mathbb{R})$. Then either the character Θ_u of u vanishes on the set of elements of $\mathbf{GL}(2n,\mathbb{R})$ which correspond to some element of $\mathbf{GL}(n,\mathbb{H})$, or there exists a unique irreducible unitary (smooth) representation u' of $\mathbf{GL}(n,\mathbb{H})$ such that*

$$\Theta_u(g) = \varepsilon(u)\Theta_{u'}(g')$$

for all $g \leftrightarrow g'$, where $\varepsilon(u) \in \{-1,1\}$.

The above results are proved in Section 13 and are based on the fact that $\mathbf{GL}(2n,\mathbb{R})$ and $\mathbf{GL}(n,\mathbb{H})$ share Levi subgroups (of θ-stable parabolic subgroups, the ones used in cohomological induction ([27])) which are products of $\mathbf{GL}(n_i,\mathbb{C})$. The underlying principle (a nice instance of Langlands' functoriality) is that the Jacquet-Langlands morphism $\mathbf{LJ}$ commutes with cohomological induction. The same principle, with Kazhdan-Patterson lifting instead of Jacquet-Langlands correspondence, was already used in [1].

1.4. **Character identities and ends of complementary series.** In Section 14, we give the composition series of ends of complementary series in most cases. This is not directly related to the main purpose of the paper, which is the global theory of the second part, but it solves some old conjectures of Tadić which are important in understanding the topology of the unitary dual of the groups $\mathbf{GL}(n,A)$, $A = \mathbb{R},\mathbb{C},\mathbb{H}$. The starting point is Zuckerman formula for the trivial representation of $\mathbf{GL}(n,A)$. Together with cohomological induction, it gives character formulas for unitary representations of the groups $\mathbf{GL}(n,A)$. In case $A = \mathbb{C}$, Zuckerman formula is given by a determinant (see formula (14.2)), and the Lewis Carroll identity of [14] allows us to deduce formulas (14.3), (14.5), (14.6), (14.7), (14.10) for the ends of complementary series.

1.5. **Global results.** Let $\mathcal{F}$ be a global field of characteristic zero and $\mathcal{D}$ a central division algebra over $\mathcal{F}$ of dimension d^2. Let $n \in \mathbb{N}^*$. Set $A' = M_n(\mathcal{D})$. For each place v of $\mathcal{F}$ let $\mathcal{F}_v$ be the completion of $\mathcal{F}$ at v and set $A'_v = A' \otimes \mathcal{F}_v$. For every place v of $\mathcal{F}$, A'_v is isomorphic to the matrix algebra $M_{r_v}(\mathcal{D}_v)$ for some positive number r_v and some central division algebra $\mathcal{D}_v$ of dimension d_v^2 over $\mathcal{F}_v$ such that $r_v d_v = nd$. We will fix once and for all an isomorphism and identify these two algebras. Let V be the (finite) set of places where $M_n(\mathcal{D})$ is not split (i.e. $d_v \neq 1$).

Let $G'(\mathcal{F})$ be the group $A'^{\times} = \mathbf{GL}(n, \mathcal{D})$. For every place $v \in V$, set $G'_v = A'^{\times}_v = \mathbf{GL}(r_v, \mathcal{D}_v)$ and $G_v = \mathbf{GL}(n, \mathcal{F}_v)$. For a given place v (clear from the context) write $g \leftrightarrow g'$ if $g \in G_v$ and $g' \in G'_v$ are regular semisimple and have equal characteristic polynomial.

If $v \notin V$, the groups G_v and G'_v are isomorphic and we fix once and for all an isomorphism which allows us to identify them.

Theorem 1.2 has been proved in the p-adic case too ([46], [7]). So, if $v \in V$, with the same notation and conventions in the p-adic case as in the archimedean case :

Theorem 1.3. *Let u be a unitary irreducible smooth representation of G_v. Then we have one and only one of the two following possibilities:*

(i) *the character Θ_u of u vanishes on the set of elements of G_v which correspond to elements of G'_v,*

(ii) *there exists a unique unitary smooth irreducible representation u' of G'_v such that*

$$\Theta_u(g) = \varepsilon(u)\Theta_{u'}(g')$$

for any $g \leftrightarrow g'$, where $\varepsilon(u) \in \{-1, 1\}$.

In the second case (ii) we say u is **compatible**. We denote the map $u \mapsto u'$ defined on the set of compatible (unitary) representations by $|\mathbf{LJ}_v|$.

Let $\mathbb{A}$ be the ring of adeles of $\mathcal{F}$. The group $G'(\mathcal{F})$ (resp. $G(\mathcal{F})$) is a discrete subgroup of $G'(\mathbb{A})$ (resp. $G(\mathbb{A})$). The centers of G' and G consist of scalar nonzero matrices and can be identified, so both will be denoted by Z, which we consider as an algebraic group defined over $\mathcal{F}$.

We endow these local and global groups with measures as in [3] and for every unitary continuous character ω of $Z(\mathbb{A})$ trivial on $Z(\mathcal{F})$, we let $L^2(G'(\mathcal{F})Z(\mathbb{A})\backslash G'(\mathbb{A}); \omega)$ be the space of functions f defined on $G'(\mathbb{A})$ with values in $\mathbb{C}$ such that

i) f is left invariant under $G'(\mathcal{F})$,

ii) $f(zg) = \omega(z)f(g)$ for all $z \in Z(\mathbb{A})$ and all $g \in G'(\mathbb{A})$,

iii) $|f|^2$ is integrable over $G'(\mathcal{F})Z(\mathbb{A})\backslash G'(\mathbb{A})$.

Let us denote by R'_ω the representation of $G'(\mathbb{A})$ on $L^2(G'(F)Z(\mathbb{A})\backslash G'(\mathbb{A}); \omega)$ by right translations. A *discrete series of* $G'(\mathbb{A})$ is the equivalence class of an irreducible subrepresentation of R'_ω for some smooth unitary character ω of $Z(\mathbb{A})$ trivial on $Z(\mathcal{F})$. Then ω is the central character of π. Let $R'_{\omega,disc}$ be the subrepresentation of R'_ω generated by irreducible subrepresentations. It is known that a discrete series representation of $G'(\mathbb{A})$ appears with finite multiplicity in $R'_{\omega,disc}$ ([18]).

Similar definitions and statements can be made with G instead of G', with obvious notation. Every discrete series π of $G'(\mathbb{A})$ (resp. $G(\mathbb{A})$) is "isomorphic" to a restricted Hilbert tensor product of irreducible unitary smooth representations π_v of the groups G'_v (resp. G_v) - see [17] for a precise statement and proof. The local components π_v are determined by π.

Denote DS (resp. DS') the set of discrete series of $G(\mathbb{A})$ (resp. $G'(\mathbb{A})$). Let us say that a discrete series π of $G(\mathbb{A})$ is **$\mathcal{D}$-compatible** if π_v is compatible for all places $v \in V$.

Theorem 1.4. (a) *There exists a unique map $\mathbf{G} : DS' \to DS$ such that for every $\pi' \in DS'$, if $\pi = \mathbf{G}(\pi')$, one has*
— π is $\mathcal{D}$-compatible,
— if $v \notin V$, then $\pi_v = \pi'_v$ and
— if $v \in V$, then $|\mathbf{LJ}_v|(\pi_v) = \pi'_v$.
The map $\mathbf{G}$ is injective. The image of $\mathbf{G}$ is the set of all $\mathcal{D}$-compatible discrete series of $G(\mathbb{A})$.

(b) *If $\pi' \in DS'$, then the multiplicity of π' in the discrete spectrum is one (Multiplicity One Theorem).*

(c) *If $\pi', \pi'' \in DS'$, if $\pi'_v \simeq \pi''_v$ for almost all v, then $\pi' = \pi''$ (Strong Multiplicity One Theorem).*

With $\mathcal{D}$ fixed, we need now to consider all possible $n \in \mathbb{N}^\times$ at the same time and we add a subscript in the notation : $A_n = M_n(\mathcal{F})$, $A'_n = M_n(\mathcal{D})$, G_n, G'_n, DS_n, DS'_n... We recall the Moeglin-Waldspurger classification of the residual spectrum for the groups $G_n(\mathbb{A})$, $n \in \mathbb{N}^*$. Let ν be the character of $G_n(\mathbb{A})$ or $G'_n(\mathbb{A})$ given by the restricted product of characters $\nu_v = |\det|_v$ where $|\ \ |_v$ is the v-adic norm and det is the reduced norm at the place v. Let $m \in \mathbb{N}^*$ and $\rho \in DS_m$ be a cuspidal representation. If $k \in \mathbb{N}^*$, then the induced representation to $G_{mk}(\mathbb{A})$ from $\otimes_{i=0}^{k-1}(\nu^{\frac{k-1}{2}-i}\rho)$ has a unique constituent (in the sense of [30]) π which is a discrete series (i.e. $\pi \in DS_{mk}$). We set then $\pi = MW(\rho, k)$. Discrete series π of groups $G_n(\mathbb{A})$, $n \in \mathbb{N}^*$, are all of this type, k and ρ are determined by π. The discrete series π is cuspidal if $k = 1$ and residual if $k > 1$. These results are proved in [32].

The proof of the following propositions and corollary is the same as in [7], once the local and global transfer are established without condition on archimedean places. Firstly, concerning cuspidal representations of $G'(\mathbb{A})$, we get:

Proposition 1.5. *Let $m \in \mathbb{N}^*$ and let $\rho \in DS_m$ be a cuspidal representation. Then*

(a) *There exists $s_{\rho,\mathcal{D}} \in \mathbb{N}^*$ such that, for $k \in \mathbb{N}^*$, $MW(\rho, k)$ is $\mathcal{D}$-compatible if and only if $s_{\rho,\mathcal{D}}|k$. We have $s_{\rho,\mathcal{D}}|d$.*

(b) *$\mathbf{G}^{-1}(MW(\rho, s_{\rho,\mathcal{D}})) = \rho' \in DS'_{\frac{ms_{\rho,\mathcal{D}}}{d}}$ is cuspidal. The map $\mathbf{G}^{-1}$ sends cuspidal $\mathcal{D}$-compatible representations to cuspidal representations.*

(c) *Every cuspidal representation in $DS'_{\frac{ms_{\rho,\mathcal{D}}}{d}}$ is obtained as in* (b).

Let us call *essentially cuspidal* representation the twist of a cuspidal representation by a real power of ν. If $n_1, n_2, ..., n_k$ are positive integers such that $\sum_{i=1}^k n_i = n$, then the subgroup L of $G'_n(\mathbb{A})$ of diagonal matrices by blocks of sizes n_1, n_2,...,n_k will be called *standard Levi subgroup* of $G'_n(\mathbb{A})$. We identify then L to $\times_{i=1}^k G'_{n_i}(\mathbb{A})$. All the definitions extend in an obvious way to L. The two statements in the following Proposition generalize respectively [32] and Theorem 4.4 in [24].

Proposition 1.6. (a) *Let $\rho' \in DS'_m$ be a cuspidal representation and let $k \in \mathbb{N}^*$. The induced representation from $\otimes_{i=0}^{k-1}(\nu_{\rho'}^{\frac{k-1}{2}-i}\rho')$ has a unique irreducible quotient π' (also characterized among irreducible subquotients by being in the discrete series), denoted by $\pi' = MW'(\rho', k)$. Every discrete* *series* π' *of a group* $G'_n(\mathbb{A})$, $n \in \mathbb{N}^*$, *is of this type, and* k *and* ρ' *are determined by* π'. *The representation* π' *is cuspidal if* $k = 1$, *and residual if* $k > 1$. *If* $\pi' = MW'(\rho', k)$, *then* $\mathbf{G}(\rho') = MW(\rho, s_{\rho,\mathcal{D}})$ *if and only if* $\mathbf{G}(\pi') = MW(\rho, ks_{\rho,\mathcal{D}})$.

(b) *Let* (L_i, ρ'_i), $i = 1, 2$, *be such that* L_i *is a standard Levi subgroup of* $G'_n(\mathbb{A})$ *and* ρ'_i *is an essentially cuspidal representation of* L_i *for* $i = 1, 2$. *Fix any finite set of places* V' *containing the infinite places and all the finite places* v *where* $\rho'_{1,v}$ *or* $\rho'_{2,v}$ *is ramified (i.e. has no non-zero vector fixed under* K_v*). If, for all places* $v \notin V'$, *the unramified subquotients of the representation of* $G'_n(\mathbb{A})$ *induced from the* $\rho'_{i,v}$ *are equal, then* (L_1, ρ'_1) *and* (L_2, ρ'_2) *are conjugate.*

We know by [30] that if π' is an automorphic representation of G'_n, then there exists (L, ρ') where L is a standard Levi subgroup of G'_n and ρ' is an essentially cuspidal representation of L such that π' is a constituent of the representation of G'_n induced from ρ'. A corollary of the point (b) of the proposition is

Corollary 1.7. (L, ρ') *is unique up to conjugation.*

1.6. **Final comment and acknowledgment.** Let us say a word about the length of the paper which can be explained by our desire to give complete proofs or/and references of all the statements. For instance, the proof of $U(3)$ for $\mathbf{GL}(n, \mathbb{H})$ in Section 10 is quite long in itself, and requires the material about Bruhat G-order introduced in the previous section, not needed otherwise. We could have saved four or five pages by referring to [47] which gives the proof of $U(3)$ for $\mathbf{GL}(n, \mathbb{R})$ and $\mathbf{GL}(n, \mathbb{C})$, but [47] is at the time still unpublished, and our arguments using Bruhat G-order could be used to simplify the proofs in [47]. Our exposition is also intended for the reader who is interested in comparing the archimedean and non-archimedean theory, by making them as similar as possible. Our discussion of Vogan's classification in Section 12 is also longer than strictly needed, but we feel that it is important that the relation between Vogan and Tadić classifications is explained somewhere in some details.

We would like to thank D. Vogan for answering many questions concerning his work.

2. Notation

2.1. **Multisets.** Let X be a set. We denote by $M(X)$ the set of functions from X to $\mathbb{N}$ with finite support, and we consider an element $m \in M(X)$ as a 'set with multiplicities'. Such an element $m \in M(X)$ will be typically denoted by

$$m = (x_1, x_2, \dots, x_r)$$

It is a (non ordered) list of elements x_i in X.

The multiset $M(X)$ is endowed with the structure of a monoid induced from the one on $\mathbb{N}$: if $m = (x_1, , \dots, x_r)$, $n = (y_1, , \dots, y_s)$ are in $M(X)$, we get

$$m + n = (x_1, \dots, x_r, y_1, \dots, y_s).$$

2.2. **Local fields and division algebras.** In the sequel, we will use the following notation : F is a local field, $|\,.\,|_F$ is the normalized absolute value on F and A is a central division algebra over F with $\dim_F(A) = d^2$.

If F is archimedean, then either $F = \mathbb{R}$ and $A = \mathbb{R}$ or $A = \mathbb{H}$, the algebra of quaternions, or $F = A = \mathbb{C}$.

2.3. **GL.** For $n \in \mathbb{N}^\times$, we set $G_n = \mathbf{GL}(n, A)$ and $G_0 = \{1\}$. We denote the reduced norm on G_n by

$$RN : G_n \to F^\times.$$

We set :

$$\nu_n : G_n \to |RN(g)|_F.$$

When the value of n is not relevant to the discussion, we will simply put $G = G_n$ and $\nu = \nu_n$.

Remark 2.1. If $A = F$, the reduced norm is just the determinant.

The character ν of G is unramified and in fact the group of unramified characters of G is

$$\mathcal{X}(G) = \{\nu^s,\ s \in \mathbb{C}\}.$$

If G is one of the groups G_n, or more generally, the group of rational points of any reductive algebraic connected group defined over F, we denote by $\mathcal{M}(G)$ the category of smooth representations of G (in the non archimedean case), or the category of Harish-Chandra modules (in the archimedean case), with respect to a fixed maximal compact subgroup K of G. For $\mathbf{GL}(n, \mathbb{R})$, $\mathbf{GL}(n, \mathbb{C})$ and $\mathbf{GL}(n, \mathbb{H})$, these maximal compact subgroups are respectively chosen to be $\mathbf{O}(n)$, $\mathbf{U}(n)$ and $\mathbf{Sp}(n)$, embedded in the standard way. Then $\mathcal{R}(G)$ denotes the Grothendieck group of the category of finite length representations in $\mathcal{M}(G)$. This is the free $\mathbb{Z}$-module with basis $\mathbf{Irr}(G)$, the set of equivalence classes of irreducible representations in $\mathcal{M}(G)$. If $\pi \in \mathcal{M}(G)$, of finite length, we will again denote by π its image in $\mathcal{R}(G)$. When confusion may occur, we will state precisely if we consider π as a representation or as an element in $\mathcal{R}(G)$.

Set

$$\mathbf{Irr}_n = \mathbf{Irr}(G_n), \qquad \mathbf{Irr} = \coprod_{n\in\mathbb{N}} \mathbf{Irr}_n, \quad \mathcal{R} = \bigoplus_{n\in\mathbb{N}} \mathcal{R}(G_n), .$$

If $\tau \in \mathcal{M}(G_n)$ or $\mathcal{R}(G_n)$, we set $\deg \tau = n$.

2.4. **Standard parabolic and Levi subgroups.** Let $n \in \mathbb{N}$ and let $\sum_{i=1}^s n_i = n$ be a partition of n. The group

$$\prod_{i=1}^s G_{n_i}$$

is identified with the subgroup of G_n of bloc diagonal matrices of respective size $n_1, \dots, n_s$. Let us denote $G_{(n_1,\dots,n_s)}$ this subgroup and $P_{(n_1,\dots,n_s)}$ (resp. $\bar{P}_{(n_1,\dots,n_s)}$) the parabolic subgroup of G_n generated by $G_{(n_1,\dots,n_s)}$ and the Borel subgroup of invertible upper triangular matrices (resp. lower triangular). The subgroup $G_{(n_1,\dots,n_s)}$ is a Levi factor of the standard parabolic subgroup $P_{(n_1,\dots,n_s)}$.

In this setting, we denote by $i_{(n_1,\dots,n_s)}$ (resp. $\underline{i}_{(n_1,\dots,n_s)}$) the functor of normalized parabolic induction from $\mathcal{M}(G_{(n_1,\dots,n_s)})$ to $\mathcal{M}(G_n)$ with respect to the parabolic subgroup $P_{(n_1,\dots,n_s)}$ (resp. $\bar{P}_{(n_1,\dots,n_s)}$).

Definition 2.2. Let $\pi_1 \in \mathcal{M}(G_{n_1})$ and $\pi_2 \in \mathcal{M}(G_{n_2})$, both of finite length. We can then form the induced representation :

$$\pi_1 \times \pi_2 := i_{(n_1,n_2)}(\pi_1 \otimes \pi_2).$$

We still denote by $\pi_1 \times \pi_2$ the image of $i_{n_1,n_2}(\pi_1 \otimes \pi_2)$ in the Grothendieck group $\mathcal{R}_{n_1+n_2}$. This extends linearly to a product

$$\times : \mathcal{R} \times \mathcal{R} \to \mathcal{R}.$$

Remark 2.3. Again we warn the reader that it is important to know when we consider $\pi_1 \times \pi_2$ as a representation or an element in $\mathcal{R}$. For instance $\pi_1 \times \pi_2 = \pi_2 \times \pi_1$ in $\mathcal{R}$ (see below), but $i_{(n_1,n_2)}(\pi_1 \otimes \pi_2)$ is not isomorphic to $i_{(n_1,n_2)}(\pi_2 \otimes \pi_1)$ in general.

Proposition 2.4. *The ring $(\mathcal{R}, \times)$ is graded commutative. Its identity is the unique element in $\mathbf{Irr}_0$.*

3. Langlands classification

We recall how to combine Langlands classification of **Irr** in terms of irreducible essentially tempered representations, and the fact that for the groups G_n, tempered representations are induced fully from irreducible square integrable modulo center representations to give a classification of **Irr** in terms of irreducible essentially square integrable modulo center representations.

Let us denote respectively by

$$D_n^u \subset \mathbf{Irr}_n, \qquad D_n \subset \mathbf{Irr}_n,$$

the set of equivalence classes of irreducible, square integrable modulo center (respectively essentially square integrable modulo center) representations of G_n, and set

$$D^u = \coprod_{n \in \mathbb{N}^\times} D_n^u, \qquad D = \coprod_{n \in \mathbb{N}^\times} D_n.$$

Similarly,

$$T_n^u \subset \mathbf{Irr}_n, \qquad T_n \subset \mathbf{Irr}_n,$$

denote respectively the sets of equivalence classes of irreducible tempered representations of G_n and equivalence classes of irreducible essentially tempered representations of G_n. Set

$$T^u = \coprod_{n \in \mathbb{N}} T_n^u, \qquad T = \coprod_{n \in \mathbb{N}} T_n.$$

For all $\tau \in T$, there exists a unique $e(\tau) \in \mathbb{R}$ and a unique $\tau^u \in T^u$ such that

$$\tau = \nu^{e(\tau)} \tau^u.$$

Theorem 3.1. *Let $d = (\delta_1, \ldots, \delta_l) \in M(D^u)$. Then*

$$\delta_1 \times \delta_2 \times \ldots \times \delta_l$$

is irreducible, therefore in T^u. This defines a one-to-one correspondence between $M(D^u)$ and T^u.

This is due to Jacquet and Zelevinsky in the case $A = F$ non archimedean ([21] or [53]). For a non archimedean division algebra, this is established in [16]. In the archimedean case, reducibility of induced from square integrable representations are well-understood in terms of R-groups (Knapp-Zuckermann [28]), and for the groups G_n, the R-groups are trivial.

Definition 3.2. Let $t = (\tau_1, \ldots, \tau_l) \in M(T)$. We say that t is written in a standard order if

$$e(\tau_1) \geq \ldots \geq e(\tau_l).$$

Theorem 3.3. *Let $d = (d_1, \ldots, d_l) \in M(D)$ written in a standard order, i.e.*

$$e(d_1) \geq e(d_2) > \ldots \geq e(d_l).$$

Then :
(i) the representation :

$$\lambda(d) = d_1 \times \ldots \times d_l$$

has a unique irreducible quotient $\mathbf{Lg}(d)$, appearing with multiplicity one in a Jordan-Hölder sequence of $\lambda(d)$. It is also the unique subrepresentation of

$$d_l \times d_{l-1} \times \ldots \times d_2 \times d_1.$$

(ii) Up to a multiplicative scalar, there is an unique intertwining operator

$$J : d_1 \times \ldots \times d_l \longrightarrow d_l \times \ldots \times d_1.$$

We have then $\mathbf{Lg}(d) \simeq \lambda(d)/\ker J \simeq \operatorname{Im} J$.
(iii) The map

$$d \mapsto \mathbf{Lg}(d)$$

is a bijection between $M(D)$ and $\mathbf{Irr}$.

For a proof in the non archimedean case, the reader may consult [36].

Representations of the form $\lambda(d) = d_1 \times \ldots \times d_l$ with $d = (d_1, \ldots, d_l) \in M(D)$ written in a standard order are called *standard representations*.

Remark 3.4. If d is a multiset of representations in $\mathbf{Irr}$, we denote by $\deg d$ the sum of the degrees of representations in d. Let $M(D)_n$ be the subset of $M(D)$ of multisets of degree n. Then the theorem gives a one-to-one correspondence between $M(D)_n$ and $\mathbf{Irr}_n$.

Proposition 3.5. *The ring R is isomorphic to $\mathbb{Z}[D]$, the ring of polynomials in X_d, $d \in D$ with coefficients in $\mathbb{Z}$, i.e. $\{[\lambda(d)]\}_{d \in D}$ is a $\mathbb{Z}$-basis of $\mathcal{R}$.*

See [53], Prop. 8.5 for a proof.

We give some easy consequences of the proposition :

Corollary 3.6. *(i) The ring R is a factorial domain.*
— *(ii) If $\delta \in D$, $[\delta]$ is prime $\mathcal{R}$.*
— *(iii) If $\pi \in \mathcal{R}$ is homogeneous and $\pi = \sigma_1 \times \sigma_2$ in $\mathcal{R}$, the σ_1 and σ_2 are homogeneous.*
— *(iv) The group of invertible elements in $\mathcal{R}$ is $\{\pm \mathbf{Irr}_0\}$.*

4. Jacquet-Langlands correspondence

In this section, we fix a central division algebra A of dimension d^2 over the local field F. We recall the Jacquet-Langlands correspondence between $\mathbf{GL}(n, A)$ and $\mathbf{GL}(nd, F)$. Since we need simultaneously both F and A in the notation, we set G_n^A, G_n^F respectively for $\mathbf{GL}(n, A)$, $\mathbf{GL}(n, F)$, and similarly with other notation *e.g.* $\mathcal{R}(G_n^A)$ or $\mathcal{R}(G_n^F)$, D_n^A or D_n^F, etc.

There is a standard way of defining the determinant and the characteristic polynomial for elements of G_n^A, in spite of A being non commutative (see for example [34] Section 16), and the reduced norm RN introduced above is just given by the constant term of the characteristic polynomial. If $g \in G_n^A$, then the characteristic polynomial of g has coefficients in F, it is monic and has degree nd. If $g \in G_n^A$ for some n, we say g is *regular semisimple* if the characteristic polynomial of g has distinct roots in an algebraic closure of F.

If $\pi \in \mathcal{R}(G_n)$, then we let Θ_π denote the function character of π, as a locally constant map, stable under conjugation, defined on the set of regular semisimple elements of G_n.

We say that $g' \in G_n^A$ *corresponds* to $g \in G_{nd}^F$ if g and g' are regular semisimple and have the same characteristic polynomial, and we write then $g' \leftrightarrow g$. Notice that if $g' \leftrightarrow g$ and if g_1' and g_1 are respectively conjugate to g' and g, then $g_1' \leftrightarrow g_1$. Said otherwise, it means that $\leftrightarrow$ is really a correspondence between conjugacy classes.

Theorem 4.1. *There is a unique bijection* $\mathbf{C} : D_{nd}^F \to D_n^A$ *such that for all* $\pi \in D_{nd}^F$ *we have*

$$\Theta_\pi(g) = (-1)^{nd-n}\Theta_{\mathbf{C}(\pi)}(g')$$

for all $g \in G_{nd}^F$ *and* $g' \in G_n^A$ *such that* $g' \leftrightarrow g$.

For the proof, see [16] if the characteristic of the base field F is zero and [4] for the non zero characteristic case. In the archimedean case, see sections 9.2 and 9.3, and Remark 9.6 for more details about this correspondence ([23],[16]).

We identify the centers of G_{nd}^F and G_n^A via the canonical isomorphisms with $F^\times$. Then the correspondence $\mathbf{C}$ preserves central characters so in particular σ is unitary if and only if $\mathbf{C}(\sigma)$ is.

The correspondence $\mathbf{C}$ may be extended in a natural way to a correspondence $\mathbf{LJ}$ between Grothendieck groups :

- If $\sigma \in D_{nd}^F$, viewed as an element in $\mathcal{R}(G_{nd}^F)$, we set

$$\mathbf{LJ}(\sigma) = (-1)^{nd-n}\mathbf{C}(\sigma),$$

viewed as an element in $\mathcal{R}(G_n^A)$.

- If $\sigma \in D_r^F$, where r is not divisible by d, we set $\mathbf{LJ}(\sigma) = 0$.

- Since $\mathcal{R}^F$ is a polynomial algebra in the variables $d \in D^F$, one can extend $\mathbf{LJ}$ in a unique way to an algebra morphism between $\mathcal{R}^F$ and $\mathcal{R}^A$. It is clear that $\mathbf{LJ}$ is surjective.

The fact that $\mathbf{LJ}$ is a ring morphism means that "it commutes with parabolic induction". Let us describe how to compute (theoretically) $\mathbf{LJ}(\pi)$, $\pi \in \mathcal{R}^F$. Since $\{\lambda(a)\}_{a \in M(D^F)}$ is a basis of $\mathcal{R}^F$, we first write π in this basis as

$$\pi = \sum_{a \in M(D^F)} M(a, \pi)\lambda(a),$$

with $M(a,\pi)\in\mathbb{Z}$ (see Section 6). Since $\mathbf{LJ}$ is linear,

$$\mathbf{LJ}(\pi)=\sum_{a\in M(D^F)} M(a,\pi)\,\mathbf{LJ}(\lambda(a)),$$

so it remains to describe $\mathbf{LJ}(\lambda(a))$. If $a=(d_1,\dots d_k)$, then

$$\lambda(a)=d_1\times\ldots\times d_k$$

(since we consider $\lambda(a)$ as an element in $\mathcal{R}^F$, the order of the d_j is not important). Since $\mathbf{LJ}$ is an algebra morphism

$$\mathbf{LJ}(\lambda(a))=\mathbf{LJ}(d_1)\times\ldots\times\mathbf{LJ}(d_k).$$

If d does not divide one of the $\deg d_i$, this is 0, and if d divides all the $\deg d_i$, setting $\sum_i \deg d_i=n$, we get

$$\mathbf{LJ}(\lambda(a))=\prod_{i=1}^{k}(-1)^{d\deg d_i-\deg d_i}\mathbf{C}(d_i)=(-1)^{nd-n}\mathbf{C}(d_1)\times\ldots\times\mathbf{C}(d_k).$$

5. Support and infinitesimal character

The goal of this section is again to introduce some notation and to recall well known results, but we want to adopt a uniform terminology for archimedean and non archimedean case. In the non archimedean case, some authors, by analogy with the archimedean case, call ‘infinitesimal character’ the cuspidal support of a representation (a multiset of irreducible supercuspidal representations). We take the opposite view of considering infinitesimal characters in the archimedean case as multisets of complex numbers.

5.1. Non archimedean case.

We start with the case F non archimedean. We denote by C (resp. C^u) the subset of $\mathbf{Irr}$ of supercuspidal representations (resp. unitary supercuspidal, *i.e.* such that $e(\rho)=0$).

For all $\pi\in\mathbf{Irr}$, there exist $\rho_1,\dots,\rho_n\in C$ such that π is a subquotient of $\rho_1\times\rho_2\times\ldots\times\rho_n$. The multiset $(\rho_1,\dots,\rho_n)\in M(C)$ is uniquely determined by π, and we denote it by $\mathbf{Supp}\,(\pi)$. It is called the cuspidal support of π. When π is a finite length representation whose irreducible subquotients have same cuspidal support, we denote it by $\mathbf{Supp}\,(\pi)$. If $\tau=\pi_1\times\pi_2$, with $\pi_1,\pi_2\in\mathbf{Irr}$ we have

$$\mathbf{Supp}\,(\tau)=\mathbf{Supp}\,(\pi_1)+\mathbf{Supp}\,(\pi_2) \tag{5.1}$$

For all $\omega\in M(C)$, denote by $\mathbf{Irr}_\omega$ the set of $\pi\in\mathbf{Irr}$ whose cuspidal support is ω. We obtain a decomposition :

$$\mathbf{Irr}=\coprod_{\omega\in M(C)}\mathbf{Irr}_\omega. \tag{5.2}$$

Set

$$\mathcal{R}_\omega=\bigoplus_{\pi\in\mathbf{Irr}_\omega}\mathbb{Z}\,\pi.$$

Then

$$\mathcal{R}=\bigoplus_{\omega\in M(C)}\mathcal{R}_\omega \tag{5.3}$$

is a graduation of $\mathcal{R}$ by $M(C)$.

We recall the following well known result.

Proposition 5.1. *Let $\omega\in M(C)$. Then $\mathbf{Irr}_\omega$ is finite.*

5.2. **Archimedean case.** Denote by $\mathfrak{g}_n$ the complexification of the Lie algebra of G_n, $\mathfrak{U}_n = \mathfrak{U}(\mathfrak{g}_n)$ its enveloping algebra, and $\mathfrak{Z}_n$ the center of the latter. Let $\mathfrak{h}_n$ be a Cartan subalgebra of $\mathfrak{g}_n$, and $W_n = W(\mathfrak{g}_n, \mathfrak{h}_n)$ its Weyl group. Harish-Chandra has defined an algebra isomorphism from $\mathfrak{Z}_n$ to the Weyl group invariants in the symmetric algebra over $\mathfrak{h}_n$:

$$\mathrm{HC}_n \,:\, \mathfrak{Z}_n \longrightarrow S(\mathfrak{h}_n)^{W_n}.$$

Using this isomorphism, every character of $\mathfrak{Z}_n$ (*i.e.* a morphism of algebra with unit $\mathfrak{Z}_n \to \mathbb{C}$) is identified with a character of $S(\mathfrak{h}_n)^{W_n}$. Such characters are given by orbits of W_n in $\mathfrak{h}_n^*$, by evaluation at a point of the orbit.

A representation (recall that in the archimedean case, this means a Harish-Chandra module) admits an infinitesimal character if the center of the enveloping algebra acts on it by scalars. Irreducible representations admit infinitesimal character. For all $\lambda \in \mathfrak{h}_n^*$, let us denote by $\mathbf{Irr}_\lambda$ the set of $\pi \in \mathbf{Irr}$ whose infinitesimal character is given by λ.

We are now going to identify infinitesimal characters with multisets of complex numbers.

— $A = \mathbb{R}$. In this case, $\mathfrak{g}_n = M_n(\mathbb{C})$ and we can choose $\mathfrak{h}_n$ to be the space of diagonal matrices, identified with $\mathbb{C}^n$. Its dual space is also identified with $\mathbb{C}^n$ by the canonical duality

$$\mathbb{C}^n \times \mathbb{C}^n \to \mathbb{C}, \quad ((x_1, \dots, x_n), (y_1, \dots y_n)) \mapsto \sum_{i=1}^n x_i y_i.$$

The Weyl group W_n is then identified with the symmetric group $\mathfrak{S}_n$, acting on $\mathbb{C}^n$ by permuting coordinates. Thus, an infinitesimal character for G_n is given by a multiset of n complex numbers.

— $A = \mathbb{C}$. In this case, $\mathfrak{g}_n = M_n(\mathbb{C}) \oplus M_n(\mathbb{C})$, and we can choose $\mathfrak{h}_n$ to be the space of couples of diagonal matrices, identified with $\mathbb{C}^n \times \mathbb{C}^n$. Its dual space is also identified with $\mathbb{C}^n \times \mathbb{C}^n$ as above. The Weyl group is then identified with $\mathfrak{S}_n \times \mathfrak{S}_n$, acting on $\mathfrak{h}_n^* \simeq \mathbb{C}^n \times \mathbb{C}^n$ by permuting coordinates. Thus, an infinitesimal character for G_n is given by a couple of multisets of n complex numbers.

— $A = \mathbb{H}$. The group G_n is a real form of $\mathbf{GL}(2n, \mathbb{C})$, so $\mathfrak{g}_n = M_{2n}(\mathbb{C})$. The discussion is then the same as for $F = \mathbb{R}$, with $2n$ replacing n.

By analogy with the non archimedean case, we denote by $M(C)$ the set of multisets (or couple of multisets if $A = \mathbb{C}$) described above.

Definition 5.2. Let $\omega \in M(C)$ be a multiset (or a couple of multisets of the same cardinality, if $A = \mathbb{C}$) of complex numbers. If $\pi \in \mathbf{Irr}_n$, we set

$$\mathbf{Supp}\,(\pi) = \omega$$

where $\omega \in M(C)$ is the multiset (or couple of multisets of the same cardinality if $F = \mathbb{C}$) defined by the infinitesimal character of π. We say that ω is the support of π. When π is a finite length representation whose subquotients have all same support, we denote it by $\mathbf{Supp}\,(\pi)$. If $\pi \in \mathbf{Irr}$, $\pi = \mathbf{Lg}(a)$ for $a \in M(D)$, we set

$$\mathbf{Supp}\,(a) := \mathbf{Supp}\,(\pi).$$

We denote by $M(D)_\omega$ the set of $a \in M(D)$ with support ω.

Proposition 5.3. *The results of 5.1 are valid in the archimedean case.*

By this, we mean (5.1), (5.2), (5.3) and Prop. 5.1 above.

6. Bruhat G-order

We continue with the notation of the previous section. In the sequel, we will use a partial order $\leq$ on $M(D)$, called Bruhat G-order, obtained from partial orders on each $M(D)_\omega$, $\omega \in M(C)$ whose main properties are described in the following :

Proposition 6.1. *Let $a \in M(D)$. Then the decomposition of $\lambda(a)$ in the basis $\{\mathbf{Lg}(b)\}_{b\in M(D)}$ of $\mathcal{R}$ is of the form*

$$\lambda(a) = \sum_{b\leq a} m(b,a)\ \mathbf{Lg}(b),$$

where the $m(a,b)$ are non negative integers. The decomposition of $\mathbf{Lg}(a)$ in the basis $\{\lambda(b)\}_{b\in M(D)}$ of $\mathcal{R}$ is of the form

$$\mathbf{Lg}(a) = \sum_{b\leq a} M(b,a)\ \lambda(b),$$

where the $M(b,a)$ are integers. In particular, all factors $\mathbf{Lg}(b)$ (resp. $\lambda(b)$) appearing in the decomposition of $\lambda(a)$ (resp. $\mathbf{Lg}(a)$) have same support. Furthermore, $m(a,a) = M(a,a) = 1$.

In the non archimedean case, Bruhat G-order is described by Zelevinsky [53] ($A = F$) and Tadić [44] in terms of linked segments. On arbitrary real reductive groups, Bruhat G-order is defined by Vogan on a different sets of parameters, in terms of integral roots (see [51], def. 12.12). In all cases, Bruhat G-order is constructed by defining first *elementary operations*, starting from an element $a \in M(D)$ and obtaining another element $a' \in M(D)$. This is written

$$a' \prec a.$$

Bruhat G-order is then generated by $\prec$[1]. Another important property of Bruhat G-order is the following. One can define on all $M(D)_\omega$ a length function :

$$l:\ M(D)_\omega \to \mathbb{N}$$

such that if $b \leq a$, then $l(b) \leq l(a)$, if $b \leq a$ and $l(b) = l(a)$ then $b = a$ and finally if $b \leq a$, and $l(b) = l(a) - 1$ then $b \prec a$. In particular, if $b \prec a$, there is no $c \in M(D)_\omega$ such that $b \leq c < a$ but $b = c$.

We have then

Proposition 6.2. *Let $a, b \in M(D)_\omega$ such that $b \prec a$. Then $m(b,a) \neq 0$ and $M(b,a) \neq 0$.*

Proof. The first assertion follows from the recursion formulas for Kazhdan-Lusztig-Vogan polynomials in the archimedean case [48]. We even have $m(b,a) = 1$ in this case. It is established by Zelevinsky [53] or Tadić [44] in the non archimedean case, and the second assertion follows using Prop. 6.1. $\square$

[1] In the case $A = \mathbb{R}$, the situation is a little more complicated.

7. Unitary dual

7.1. **Representations $u(\delta,n)$ and $\pi(\delta,n;\alpha)$.** Let $\delta \in D$. Then $\delta \times \delta$ is irreducible. Indeed, if $\delta \in D^u$, this is 3.1, and the general case follows by tensoring with an unramified character. Consider $\delta \times \nu^\alpha \delta$, with $\alpha > 0$. There exists a smallest $\alpha_0 > 0$ such that $\delta \times \nu^{\alpha_0}\delta$ is reducible.

Definition 7.1. Let $\delta \in D$. Set $\nu_\delta = \nu^{\alpha_0}$, where $\alpha_0 > 0$ is the smallest real number $\alpha > 0$ such that $\delta \times \nu^\alpha \delta$ is reducible.

For all $\delta \in D$, and for all $n \in \mathbb{N}^\times$ we set

$$a(\delta,n) = (\nu_\delta^{\frac{n-1}{2}}\delta, \nu_\delta^{\frac{n-1}{2}-1}\delta, \ldots, \nu_\delta^{-\frac{n-1}{2}}\delta) \in M(D), \tag{7.1}$$

$$u(\delta,n) = \mathbf{Lg}(a(\delta,n)). \tag{7.2}$$

For all $\delta \in D$, for all $n \in \mathbb{N}^\times$, and for all $\alpha \in \mathbb{R}$, set

$$\pi(\delta,n;\alpha) = \nu_\delta^\alpha u(\delta,n) \times \nu_\delta^{-\alpha} u(\delta,n). \tag{7.3}$$

7.2. **Tadić hypotheses $U(0),\ldots,U(4)$ and classification of the unitary dual.** We recall Tadić's classification of the unitary dual of the groups G_n. For a fixed division algebra A, consider the following hypotheses :
$U(0)$: if $\sigma,\tau \in \mathbf{Irr}^u$, then $\sigma \times \tau \in \mathbf{Irr}^u$.
$U(1)$: if $\delta \in D^u$ and $n \in \mathbb{N}^\times$, then $u(\delta,n) \in \mathbf{Irr}^u$.
$U(2)$: if $\delta \in D^u$, $n \in \mathbb{N}^\times$ and $\alpha \in]0,1/2[$, then $\pi(\delta,n;\alpha) \in \mathbf{Irr}^u$.
$U(3)$: if $\delta \in D$, $u(\delta,n)$ is prime in $\mathcal{R}$.
$U(4)$: if $a,b \in M(D)$, then $L(a) \times L(b)$ contains $L(a+b)$ as a subquotient.

Suppose Tadić's hypotheses are satisfied for A. We have then the following :

Theorem 7.2. *The set $\mathbf{Irr}^u$ is endowed with the structure of a free commutative monoid, with product $(\sigma,\tau) \mapsto \sigma \times \tau$ and with basis*

$$\mathcal{B} = \{u(\delta,n),\ \pi(\delta,n;\alpha) \mid \delta \in D^u, n \in \mathbb{N}^\times, \alpha \in]0,1/2[\ \}.$$

More explicitly, if $\pi_1,\ldots,\pi_k \in \mathcal{B}$, then $\pi_1 \times \ldots \times \pi_k \in \mathbf{Irr}^u$ and if $\pi \in \mathbf{Irr}^u$, there exists $\pi_1,\ldots,\pi_k \in \mathcal{B}$, unique up to permutation, such that $\pi = \pi_1 \times \ldots \times \pi_k$.

This is proved in [45], prop 2.1. The proof is formal.

Let us first notice that $U(4)$ is a quite simple consequence of Langlands classification, established by Tadić for all A in [46] (the proof works also for archimedean A, see [47]). It is also easy to see that $U(2)$ can be deduced from $U(0)$ and $U(1)$ by the following simple principle : if $(\pi_t)_{t\in I}$, is a family of hermitian representations in $\mathcal{M}(G)$, where I is an open interval containing 0, continuous in a sense that we won't make precise here, and if π_0 is unitary and irreducible, then π_t is unitary on the largest interval $J \subset I$ containing 0 where π_t is irreducible (the signature of the hermitian form can change only when crossing reducibility points). Representations $\pi(\delta,n;\alpha)$, $\alpha \in \mathbb{R}$ are hermitian,

$$\pi(\delta,n;0) = u(\delta,n) \times u(\delta,n)$$

is unitary and irreducible ($U(0)$ and $U(1)$), and $\pi(\delta,n;\alpha)$ is irreducible for $\alpha \in]-\frac{1}{2},\frac{1}{2}[$. See [47] and the references given there for details.

For the remaining $U(0)$, $U(1)$ and $U(3)$, the situation is more complicated.

— $U(3)$ is proved by Tadić in the non archimedean case in [42], and for $A = \mathbb{R}, \mathbb{C}$ in [47]. We give below the proof for $A = \mathbb{H}$, following Tadić's ideas.

— $U(1)$ is proved by Tadić in the non archimedean case in [42] for the field case $A = F$. The generalization to all division algebra over F is given by the authors in [9], using unitarity of some distinguished representations closely related to the $u(\delta, n)$ established by the first named author in [6] by global methods. For $F = \mathbb{C}$, $u(\delta, n)$ is a unitary character, so the statement is obvious. For $F = \mathbb{R}$, $U(1)$ was first proved by Speh in [41] using global method. It can also be proved using Vogan's results on cohomological induction (see details below). Finally, for $A = \mathbb{H}$, $U(1)$ can be established using again the general results on cohomological induction, and the argument in [9]. A more detailed discussion of the archimedean case is in section 11.

— $U(0)$ is by far the most delicate point. For $A = F$ non archimedean, it is established by Bernstein in [12], using reduction to the mirabolic subgroup. For $A = \mathbb{R}$ or $\mathbb{C}$, the same approach can be used, but some serious technical difficulties remained unsolved until the paper of Baruch [11]. For A a general non archimedean division algebra, $U(0)$ is established by V. Sécherre [39] using his deep results on Bushnell-Kutzko's type theory for the groups $\mathbf{GL}(n, A)$, which give Hecke algebras isomorphisms and allow one to reduce the problem to the field case (the proof also uses in a crucial way Barbash-Moy results on unitarity for Hecke algebras representations [10]). In the case $A = \mathbb{H}$, there is to our knowledge no written references, but it is well-known to some experts that this can be deduced from Vogan's classification of the unitary dual of G_n in the archimedean case ([52]). Vogan's classification is conceptually very different from Tadić's classification. It has its own merits, but the final result is quite difficult to state and to understand, since it uses sophisticated concepts and techniques of the theory of real reductive groups. So, for people interested mainly in applications, to automorphic forms for instance, Tadić's classification is much more convenient. In the literature, before Baruch's paper was published, one can often find the statement of Tadić's classification, with reference to Vogan's paper [52] for the proof. It might not be totally obvious for non experts to derive Tadić's classification from Vogan's. We take this opportunity to explain in this paper (see §12 below) some aspects of Vogan's classification's, how it is related to Tadić's classification and how to deduce $U(0)$ from it. Of course, an independent proof of $U(0)$ would be highly desirable in this case. It would be even better to have an uniform proof of $U(0)$ for all cases, but for this, new ideas are clearly needed.

— all these results are true if the characteristic of F is positive (as explained in [8]).

8. Classification of generic irreducible unitary representations

From the classification of the unitary dual of $\mathbf{GL}(n, \mathbb{R})$ given above and the classification of irreducible generic representations of a real reductive groups ([49], [29]), we deduce the classification of generic irreducible unitary representations of $\mathbf{GL}(n, \mathbb{R})$. Let us first recall that Vogan gives a classification of 'large' irreducible representations of a quasi-split real reductive group (*i.e.* having maximal Gelfand-Kirillov dimension), that Kostant shows that such a group admits generic representations if and only if the group is quasi-split, and that "generic" is equivalent to "large". Therefore, Vogan's result can be stated as follows :

Theorem 8.1. *Any generic irreducible representation of any quasisplit real reductive group is irreducibly induced from a generic limit of discrete series,* *and conversely, a representation which is irreducibly induced from a generic limit of discrete series is generic.*

Let us notice that in the above theorem, one can replace "limit of discrete series" by "essentially tempered", because according to [28], any tempered representation is fully induced from a limit of discrete series. In the case of $\mathbf{GL}(n,\mathbb{R})$ all discrete series are generic, so by Theorem 3.1, all essentially tempered representations are generic.

Let us denote by $\mathbf{Irr}^u_{gen}$ the subset of $\mathbf{Irr}^u$ consisting of generic representations. We have then the following specialization of theorem 7.2.

Theorem 8.2. *The set $\mathbf{Irr}^u_{gen}$ is endowed with the structure of a free commutative monoid, with product $(\sigma,\tau)\mapsto\sigma\times\tau$ and with basis*

$$\mathcal{B}_{gen}=\{u(\delta,1),\ \pi(\delta,1;\alpha)\,|\,\delta\in D^u,\alpha\in]0,1/2[\ \}.$$

More explicitly, if $\pi_1,\ldots,\pi_k\in\mathcal{B}_{gen}$, then $\pi_1\times\ldots\times\pi_k\in\mathbf{Irr}^u_{gen}$ and if $\pi\in\mathbf{Irr}^u_{gen}$, there exists $\pi_1,\ldots,\pi_k\in\mathcal{B}_{gen}$, unique up to permutation, such that $\pi=\pi_1\times\ldots\times\pi_k$.

9. Classification of discrete series : archimedean case

In this section, we describe explicitly square integrable modulo center irreducible representations of G_n in the archimedean case. In the case $A=\mathbb{H}$, we give also details about supports, Bruhat G-order... Since the Bruhat G-order is defined by Vogan on a set of parameters for irreducible representations consisting of (conjugacy classes of) characters of Cartan subgroups, we also describe the bijections between the various sets of parameters.

9.1. $A=\mathbb{C}$. There are square integrable modulo center irreducible representations of $\mathbf{GL}(n,\mathbb{C})$ only when $n=1$. Thus

$$D=D_1=\mathbf{Irr}_1.$$

An element $\delta\in D$ is then a character

$$\delta:\mathbf{GL}(1,\mathbb{C})\simeq\mathbb{C}^\times\to\mathbb{C}^\times$$

Let $\delta\in D$. Then there exists a unique $n\in\mathbb{Z}$ and a unique $\beta\in\mathbb{C}$ such that

$$\delta(z)=|z|^{2\beta}\left(\frac{z}{|z|}\right)^n=|z|_{\mathbb{C}}^{\beta}\left(\frac{z}{|z|}\right)^n.$$

Let $x,y\in\mathbb{C}$ satisfying

$$\begin{cases}x+y&=2\beta\\ x-y&=n.\end{cases}$$

We set, with the above notation (and abusively writing a complex power of a complex number),

$$\delta(z)=\gamma(x,y)=z^x\bar{z}^y.$$

The following is well-known.

Proposition 9.1. *Let $\delta=\gamma(x,y)\in D$ as above. Then $\delta\times\nu^\alpha\delta$ is reducible for $\alpha=1$ and irreducible for $0\le\alpha<1$. Thus $\nu_\delta=\nu$ (cf. 7.1). In the case of reducibility $\alpha=1$, we have in $\mathcal{R}$:*

$$\gamma(x,y)\times\gamma(x+1,y+1)=\mathbf{Lg}((\gamma(x,y),\gamma(x+1,y+1)))+\gamma(x,y+1)\times\gamma(x+1,y).$$

9.2. $A = \mathbb{R}$**.** There are square integrable modulo center irreducible representations of $\mathbf{GL}(n, \mathbb{R})$ only when $n = 1, 2$:

$$D = D_1 \coprod D_2 = \mathbf{Irr}_1 \coprod D_2.$$

Let us start with the parametrization of D_1. An element $\delta \in D_1$ is a character

$$\delta : \mathbf{GL}(1, \mathbb{R}) \simeq \mathbb{R}^\times \to \mathbb{C}^\times$$

Let $\delta \in D_1$. Then there exists a unique $\epsilon \in \{0, 1\}$ and a unique $\alpha \in \mathbb{C}$ such that

$$\delta(x) = |x|^\alpha \mathrm{sgn}(x)^\epsilon, \quad (x \in \mathbb{R}^\times).$$

We set

$$\delta = \delta(\alpha, \epsilon).$$

Let us now give a parametrization of D_2. Let $\delta_1, \delta_2 \in D_1$. Then $\delta_1 \times \delta_2$ is reducible if and only if there exists $p \in \mathbb{Z} \setminus \{0\}$ such that

$$\delta_1 \delta_2^{-1}(x) = x^p \mathrm{sgn}(x), \quad (x \in \mathbb{R}^\times)$$

If $\delta_i = \delta(\alpha_i, \epsilon_i)$, we rewrite these conditions as

$$\alpha_1 - \alpha_2 = p, \quad \epsilon_1 - \epsilon_2 = p + 1 \mod 2 \tag{9.1}$$

If $\delta_1 \times \delta_2$ is reducible, we have in R,

$$\delta_1 \times \delta_2 = \mathbf{Lg}((\delta_1, \delta_2)) + \eta(\delta_1, \delta_2) \tag{9.2}$$

where $\eta(\delta_1, \delta_2) \in D_2$ and $\mathbf{Lg}((\delta_1, \delta_2))$ is an irreducible finite dimensional representation (of dimension $|p|$ with the notation above).

Definition 9.2. If $\alpha_1, \alpha_2 \in \mathbb{C}$ satisfy $\alpha_1 - \alpha_2 \in \mathbb{Z} \setminus \{0\}$, we set

$$\eta(\alpha_1, \alpha_2) = \eta(\delta_1, \delta_2) \tag{9.3}$$

where $\delta_1(x) = |x|^{\alpha_1}$ and $\delta_2(x) = |x|^{\alpha_2} \mathrm{sgn}(x)^{\alpha_1 - \alpha_2 + 1}$. This define a surjective map from

$$\{(\alpha_1, \alpha_2) \in \mathbb{C}^2 \,|\, \alpha_1 - \alpha_2 \in \mathbb{Z} \setminus \{0\}\}$$

to D_2 and

$$\eta(\alpha_1, \alpha_2) = \eta(\alpha_1', \alpha_2') \Leftrightarrow \{\alpha_1, \alpha_2\} = \{\alpha_1', \alpha_2'\}.$$

This gives a parametrization of D_2 by pairs of complex numbers α_1, α_2 satisfying $\alpha_1 - \alpha_2 \in \mathbb{Z} \setminus \{0\}$.

Remark 9.3. The representation $\eta(x, y) \in D_2$, $x, y \in \mathbb{C}$, $x - y \in \mathbb{Z} \setminus \{0\}$ is obtained from the character $\gamma(x, y)$ of $\mathbb{C}^\times$ by some appropriate functor of cohomological induction. But, even when $x = y$, the functor of cohomological induction maps $\gamma(x, x)$ to an irreducible essentially tempered representation of $\mathbf{GL}(2, \mathbb{R})$, namely the *limit of discrete series* $\delta(x, 0) \times \delta(x, 1)$, which is an irreducible principal series.

For that reason, we set for $x \in \mathbb{C}$:

$$\eta(x, x) := \delta(x, 0) \times \delta(x, 1) \in \mathbf{Irr}_2 \tag{9.4}$$

Proposition 9.4. *Let $\delta \in D$. Then $\delta \times \nu^\alpha \delta$ is reducible for $\alpha = 1$ and irreducible for $0 \leq \alpha < 1$. Thus $\nu_\delta = \nu$ (cf. 7.1).*

This is also well-known. Let us be more precise, by giving the composition series for $\delta \times \nu\delta$. We start with the case $\delta = \delta(\alpha, \epsilon) \in D_1$. Then we get from (9.2) that we have in $\mathcal{R}$,

$$\delta(\alpha, \epsilon) \times \delta(\alpha + 1, \epsilon) = \mathbf{Lg}(\delta(\alpha, \epsilon), \delta(\alpha + 1, \epsilon)) + \eta(\alpha, \alpha + 1). \tag{9.5}$$

In the case where $\delta = \eta(x, y) \in D_2$, $x - y = r \in \mathbb{N}^\times$, we get if $r \neq 1$,

$$\begin{gathered} \eta(x, y) \times \eta(x + 1, y + 1) = \\ \mathbf{Lg}(\eta(x, y), \eta(x + 1, y + 1)) + \eta((x, y + 1) \times \eta(x + 1, y)). \end{gathered} \tag{9.6}$$

If $r = 1$, the situation degenerates, but the following formulas remain valid by coherent continuation (see Section 13.1) :

$$\begin{gathered} \eta(x, y) \times \eta(x + 1, y + 1) = \\ \mathbf{Lg}(\eta(x, y), \eta(x + 1, y + 1)) + \eta(x, y + 1) \times \eta(x + 1, y). \end{gathered}$$

Recall that our convention is that

$$\eta(y + 1, y + 1) = \delta(y + 1, 0) \times \delta(y + 1, 1)$$

is a limit of discrete series, thus :

$$\begin{gathered} \eta(y + 1, y) \times \eta(y + 2, y + 1) = \\ \mathbf{Lg}(\eta(y + 1, y), \eta(y + 2, y + 1)) + \delta(y + 1, 0) \times \delta(y + 1, 1) \times \eta(y + 2, y). \end{gathered} \tag{9.7}$$

9.3. $A = \mathbb{H}$**.** Let us identify quaternions and 2×2 matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}, \quad \alpha, \beta \in \mathbb{C}.$$

The reduced norm is given by

$$RN \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = |\alpha|^2 + |\beta|^2.$$

The group of invertible elements $\mathbb{H}^\times$ contains $\mathbf{SU}(2)$, the kernel of the reduced norm. Thus we have an exact sequence

$$1 \to \mathbf{SU}(2) \hookrightarrow \mathbb{H}^\times \xrightarrow{RN} \mathbb{R}_+^\times \to 1,$$

and we can identify $\mathbb{H}^\times$ with the direct product $\mathbf{SU}(2) \times \mathbb{R}_+^\times$.

The group $\mathbf{GL}(n, \mathbb{H})$ is a real form of $\mathbf{GL}(2n, \mathbb{C})$, its elements are $2n \times 2n$-matrices composed of 2×2 quaternionic matrices described above. Complex conjugacy on $\mathbf{GL}(2n, \mathbb{C})$ for this real form is given on the 2×2 blocs by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \bar{\delta} & -\bar{\gamma} \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

A maximal compact subgroup of $\mathbf{GL}(n, \mathbb{H})$ is then

$$\mathbf{Sp}(n) \simeq \mathbf{U}(2n) \cap \mathbf{GL}(n, \mathbb{H}).$$

Its rank is n, the rank of $\mathbf{GL}(n, \mathbb{H})$ is $2n$ and the split rank of the center is 1. Thus there are square integrable modulo center representations only when $n = 1$.

For $n = 1$, $D_1 = \mathbf{Irr}_1$, all irreducible representations of $\mathbb{H}^\times$ are essentially square integrable modulo center, since $\mathbb{H}^\times$ is compact modulo center. Harish-Chandra's parametrization in this case is as follows : irreducible representations of $\mathbb{H}^\times$ are parametrized by some characters of a fundamental Cartan subgroup, here we choose

$$\mathbb{C}^\times \hookrightarrow \mathbb{H}^\times, \quad \alpha \mapsto \begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix},$$

which is connected. Characters of $\mathbb{C}^\times$ were described in section $F = \mathbb{C}$. They are of the form $\gamma(x,y)$, $x - y \in \mathbb{Z}$. An irreducible representation of $\mathbb{H}^\times$ is then parametrized by a couple of complex numbers (x,y), such that $x - y \in \mathbb{Z}$. The couples (x,y) and (x',y') parametrize the same representation if and only if the characters $\gamma(x,y)$ and $\gamma(x',y')$ are conjugate under the Weyl group, *i.e.* if the multisets (x,y) and (x',y') are equal. Furthermore $\gamma(x,y)$ corresponds to an irreducible representation if and only if $x \neq y$. Let us denote $\eta'(x,y)$ the representation parametrized by the multiset (x,y), $x - y \in \mathbb{Z} \setminus \{0\}$. It is obtained from the character $\gamma(x,y)$ of the Cartan subgroup $\mathbb{C}^\times$ by cohomological induction.

Remark 9.5. As opposed to the case $A = \mathbb{R}$, when we induced cohomologically the character $\gamma(x,x)$ of the Cartan subgroup $\mathbb{C}^\times$ to $\mathbb{H}^\times$, we get 0 : there is no limits of discrete series. Thus we set $\eta'(x,x) = 0$.

Remark 9.6. Jacquet-Langlands correspondence (see Section 4) between representations of $\mathbf{GL}(1,\mathbb{H}) = \mathbb{H}^\times$ and essentially square integrable modulo center irreducible representations $\mathbf{GL}(2,\mathbb{R})$ is given by

$$\mathbf{C}(\eta(x,y)) = \eta'(x,y), \quad x, y \in \mathbb{C}, x - y \in \mathbb{Z} \setminus \{0\}.$$

The representations $\eta(x,y)$ and $\eta'(x,y)$ are obtained by cohomological induction from the same character $\gamma(x,y)$ of the Cartan subgroup $\mathbb{C}^\times$ of $\mathbf{GL}(2,\mathbb{R})$ and $\mathbb{H}^\times$. In the case $x = y$ the construction still respect the Jacquet-Langlands character relation since both sides are equal to zero.

More generally let us give now the parametrization of irreducible representations of $\mathbf{GL}(n,\mathbb{H})$ by conjugacy classes of characters of Cartan subgroups. The group $\mathbf{GL}(n,\mathbb{H})$ has only one conjugacy class of Cartan subgroups, a representative being T_n, which consist of 2×2 bloc diagonal matrices of the form $\begin{pmatrix} \alpha & 0 \\ 0 & \bar{\alpha} \end{pmatrix}$. Thus $T_n \simeq (\mathbb{C}^\times)^n$ is connected, $\mathfrak{t}_n = \mathrm{Lie}(T) \simeq \mathbb{C}^n$ and $(\mathfrak{t}_n)_\mathbb{C} \simeq (\mathbb{C} \oplus \mathbb{C})^n$.

Let Λ be a character of T_n. Its differential

$$\lambda = d\Lambda \, : \, \mathfrak{t}_n \to \mathbf{Lie}(\mathbb{C}^\times) \simeq \mathbb{C},$$

is a $\mathbb{R}$-linear map, with complexification the $\mathbb{C}$-linear map

$$\lambda = d\Lambda \, : \, \mathfrak{t}_\mathbb{C} \simeq (\mathbb{C} \oplus \mathbb{C})^n \to \mathbf{Lie}(\mathbb{C}^\times) \simeq \mathbb{C}.$$

Such a linear form is given by a n-tuple of couples (λ_i, μ_i) such that $\lambda_i - \mu_i \in \mathbb{Z}$.

Since T_n is connected, a character Λ of T_n is determined by its differential. We write

$$\Lambda = \Lambda(\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n) = \Lambda((\lambda_i, \mu_i)_{1 \le i \le n})$$

if its differential is given by the n-tuple of couples (λ_i, μ_i) such that $\lambda_i - \mu_i \in \mathbb{Z}$.

Let $\mathcal{P}$ be the set of characters $\Lambda = \Lambda((\lambda_i, \mu_i)_{1 \le i \le n})$ of the Cartan subgroup T_n, such that $\lambda_i - \mu_i \in \mathbb{Z} \setminus \{0\}$.

Irreducible representations of $\mathbf{GL}(n,\mathbb{H})$ are parametrized by $\mathcal{P}$, two characters Λ_1 and Λ_2 giving the same irreducible representations if and only if they are conjugate under $W(\mathbf{GL}(2n,\mathbb{C}),T_n)$. This group is isomorphic to $\{\pm 1\}^n\times\mathfrak{S}_n$. Its action on $\mathfrak{t}_{\mathbb{C}}\simeq(\mathbb{C}\oplus\mathbb{C})^n$ is as follows : each factor $\{\pm 1\}$ acts inside the corresponding factor $\mathbb{C}\oplus\mathbb{C}$ by permutation, and $\mathfrak{S}_n$ acts by permuting the n factors $\mathbb{C}\oplus\mathbb{C}$. Thus we see that irreducible representations of $\mathbf{GL}(n,\mathbb{H})$ are parametrized by multisets of cardinality n of pairs of complex numbers (λ_i,μ_i) such that $\lambda_i-\mu_i\in\mathbb{Z}\backslash\{0\}$. Since such a pair (λ_i,μ_i) parametrizes the representation $\eta'(\lambda_i,\mu_i)$, we recover the Langlands parametrization of **Irr** by $M(D)$. Let us denote by $\sim$ the equivalence relation on $\mathcal{P}$ given by the Weyl group action $W(\mathbf{GL}(2n,\mathbb{C}),T)$. We have described one-to-one correspondences

$$\mathcal{P}/\sim\;\simeq\mathbf{Irr}_n\simeq M(D)_n$$

Recall that a support for $\mathbf{GL}(n,\mathbb{H})$ is a multiset of $2n$ complex numbers, *i.e.* an element of the quotient of $\mathfrak{t}_{\mathbb{C}}^*\simeq(\mathbb{C}\oplus\mathbb{C})^n\simeq\mathbb{C}^{2n}$, by the action of the Weyl group $W_{\mathbb{C}}\simeq\mathfrak{S}_{2n}$.

Definition 9.7. The support of a character $\Lambda=\Lambda((\lambda_i,\mu_i)_{1\leq i\leq n})\in\mathcal{P}$ is the multiset

$$(\lambda_1,\mu_1,\dots,\lambda_n,\mu_n).$$

It does not depend on the equivalence class of Λ for $\sim$. If $\Lambda\in\mathcal{P}$ parametrizes the irreducible representation π, we denote $\mathbf{Supp}\,(\Lambda)=\mathbf{Supp}\,(\pi)$.

This describes explicitly the map

$$\mathcal{P}\to M(C),\quad\Lambda\mapsto\mathbf{Supp}\,(\Lambda)$$

and its fibers : two parameters

$$\Lambda_1((\lambda_i^1,\mu_i^1))\quad\text{ and }\quad\Lambda_2((\lambda_i^2,\mu_i^2)),$$

have same support if and only if the multisets

$$(\lambda_1^1,\dots,\lambda_n^1,\mu_1^1,\dots,\mu_n^1,)\quad\text{ and }\quad(\lambda_1^2,\dots,\lambda_n^2,\mu_1^2,\dots,\mu_n^2)$$

are equal. We denote by $\mathcal{P}(\omega)$ the fiber at ω.

Let us give now the description of the Bruhat G-order, in terms of integral roots. We have the decomposition of Lie algebra :

$$\mathrm{Lie}(\mathbf{GL}(2n,\mathbb{C}))=(\mathfrak{g}_{2n})_{\mathbb{C}}=(\mathfrak{t}_n)_{\mathbb{C}}\oplus(\bigoplus_{\alpha\in R}\mathfrak{g}_{\mathbb{C}}^{\alpha})$$

where $R=\{\pm(e_i-e_j),\ 1\leq i<j\leq 2n\}$ is the usual root system of type A_{2n-1}. Let us denote by σ the non-trivial element of the Galois group of $\mathbb{C}/\mathbb{R}$.

The roots $\pm(e_{2j-1}-e_{2j})$, $j=1,\dots,n$ are imaginary compact, thus

$$\sigma\cdot e_{2j-1}=e_{2j},\quad j=1,\dots,n.$$

Other roots are complex: for all j,l, $1\leq j\neq l\leq l$,

$$\sigma\cdot(e_{2j-1}-e_{2l-1})=e_{2j}-e_{2l},\quad\sigma\cdot(e_{2j-1}-e_{2l})=e_{2j}-e_{2l-1}.$$

Let us fix a support ω and let Λ be a character of T_n such that $\mathbf{Supp}\,(\Lambda)=\omega$, say $\Lambda=\Lambda((\lambda_i,\mu_i)_{i=1,\dots,n})$, $\lambda_i-\mu_i\in\mathbb{Z}\setminus\{0\}$. Notice that $W_{\mathbb{C}}\simeq\mathfrak{S}_{2n}$ doesn't act on $\mathcal{P}(\omega)$, since the condition

$$\lambda_i-\mu_i\in\mathbb{Z}$$

might not hold anymore after some permutation of the λ_i.

Denote by W_Λ the subgroup of $W_{\mathbb{C}}$ consisting of elements w such that

$$w \cdot (\lambda_i, \mu_i)_i - (\lambda_i, \mu_i)_i \in (\mathbb{Z} \times \mathbb{Z})^n.$$

Then W_Λ is the Weyl group of the root system R_Λ of *integral roots* for Λ. A root $\alpha = e_k - e_l$ in R is integral for Λ if, when writing

$$\lambda_1, \mu_1, \lambda_2, \mu_2, \ldots, \lambda_n, \mu_n = \nu_1, \ldots, \nu_{2n}$$

then $\nu_k - \nu_l \in \mathbb{Z}$.

Suppose that the support ω is regular, *i.e.* all the ν_i, $1 \leq i \leq 2n$ are distinct. We choose as a positive root system $R_\Lambda^+ \subset R_\Lambda$ the roots $e_k - e_l$ such that $\nu_k - \nu_l > 0$. This defines simple roots.

Let us state first a necessary and sufficient condition for reducibility of standard modules (for regular support).

Proposition 9.8. *Let $a = (\eta'(\lambda_i, \mu_i)_{i=1,\ldots,n}) \in M(D)_\omega$, parametrized by the character $\Lambda = \Lambda((\lambda_i, \mu_i)_{i=1,\ldots,n})$ of T_n. Suppose that the support*

$$\omega = (\lambda_1, \mu_1, \ldots, \lambda_n, \mu_n)$$

is regular. Then $\lambda(a)$ is reducible if and only if there exists a simple root $e_k - e_l$ in R_Λ^+, which is complex, such that, if $e_k - e_l = e_{2i-1} - e_{2j-1}$, or $e_k - e_l = e_{2i} - e_{2j}$, $i \neq j$, then

$$\lambda_i - \lambda_j > 0 \quad \textit{and} \quad \mu_i - \mu_j > 0,$$

and if $e_k - e_l = e_{2i-1} - e_{2j}$, or $e_k - e_l = e_{2i} - e_{2j-1}$, $i \neq j$, then

$$\lambda_i - \mu_j > 0 \quad \textit{and} \quad \mu_i - \lambda_j > 0.$$

When ω is not regular, we still have a necessary condition for reducibility: if $\lambda(a)$is reducible, then there exists a root $e_k - e_l$ in R_Λ^+, not necessarily simple, but still satisfying the condition above.

See [51].

Definition 9.9. We still assume $\omega \in M(C)$ to be regular, and suppose that $\Lambda \in \mathcal{P}(\omega)$ satisfies the reducibility criterion above for the simple integral complex root $e_k - e_l$. Write

$$\Lambda = \Lambda((\lambda_1, \mu_1), \ldots, (\lambda_n, \mu_n)) = \Lambda((\nu_1, \nu_2), \ldots, (\nu_{2n-1}, \nu_{2n}))$$

Let $\Lambda' \in \mathcal{P}(\omega)$, obtained from Λ by exchanging ν_k and ν_l, and let $a' \in M(D)_\omega$ correspond to Λ'. We say that a' is obtained from a by an *elementary operation*, and we write $a' \prec a$. The Bruhat G-order on $M(D)_\omega$ is the partial order generated by $\prec$.

Let us now deduce from the reducibility criterion above the invariant ν_δ attached (*cf.* definition 7.1) to an essentially square integrable modulo center irreducible representation $\delta = \eta'(x, y)$, $x, y \in \mathbb{C}$, $x - y \in \mathbb{Z} \setminus \{0\}$. We may suppose that $x - y = r > 0$, since $\eta'(x, y) = \eta'(y, x)$.

Proposition 9.10. *With the previous notation, $\nu_\delta = \nu$ if $r > 1$ and $\nu_\delta = \nu^2$ if $r = 1$. Since r is the dimension of δ, we see that $\nu_\delta = \nu$ except when δ is a one-dimensional representation of* $\mathbf{GL}(1, \mathbb{H})$.

Proof. We want to study the reducibility of

$$\pi = \eta'(y+r,y) \times \eta'(y+r+\alpha, y+\alpha)$$

for $\alpha > 0$. The support of this representation is regular if and only if $y+r, y, y+r+\alpha, y+\alpha$ are distinct, but since

$$y+r+\alpha > y+\alpha > y, \quad y+r+\alpha > y+r > y,$$

the support is regular except when $r=\alpha$. The representation π is the standard representation attached to the character

$$\Lambda = \Lambda((y+r+\alpha, y+\alpha),(y+r,y)).$$

If $\alpha \notin \mathbb{Z}$, the support is regular, all integral roots are imaginary compact for Λ, and then π is irreducible.

If $\alpha = 1$ and $r \neq 1$, the support is regular, all the roots are integral for

$$\Lambda((y+r+1,y+1),(y+r,y)),$$

and $e_1 - e_3$ is a complex root, simple in

$$R^+_\Lambda = \{e_1-e_3, e_1-e_2, e_1-e_4, e_3-e_2, e_3-e_4, e_2-e_4\},$$

satisfying the reducibility criterion, since

$$(\sigma\cdot(e_1-e_3))(y+r+1,y+1,y+r,y) = (e_2-e_4)(y+r+1,y+1,y+r,y) = 1 > 0.$$

The only smaller element than Λ in the Bruhat G-order is

$$\Lambda' = \Lambda((y+r,y+1),(y+r+1,y)),$$

and we get

$$\begin{gathered}\eta'(y+r,y) \times \eta'(y+r+1,y+1) = \\ \mathbf{Lg}(\eta'(y+r,y),\eta'(y+r+1,y+1)) + \eta'(y+r,y+1) \times \eta'(y+r+1,y).\end{gathered} \tag{9.8}$$

If $\alpha = 1$ and $r=1$, the support is singular. Applying Zuckerman translation functors (see [27] for instance), we get

$$\begin{gathered}\eta'(y+2,y) \times \eta'(y+3,y+1) = \\ \mathbf{Lg}(\eta'(y+2,y),\eta'(y+3,y+1)) + \eta'(y+1,y+1) \times \eta'(y+2,y).\end{gathered}$$

But, according to our convention, $\eta'(y+1,y+1) = 0$ (this is really what we get applying translation functor to the wall), thus

$$\eta'(y+1,y) \times \eta'(y+2,y+1) = \mathbf{Lg}(\eta'(y+1,y),\eta'(y+2,y+1)).$$

is irreducible.

The next possibility of reducibility for $r=1$ is then $\alpha = 2$, but then the support is regular and we see as above that π is reducible, more precisely

$$\begin{gathered}\eta'(y+3,y+2) \times \eta'(y+1,y) = \\ \mathbf{Lg}(\eta'(y+3,y+2),\eta'(y+1,y)) + \eta'(y+2,y+1) \times \eta'(y+3,y).\end{gathered} \tag{9.9}$$

□

10. $U(3)$ for $A = \mathbb{H}$

We follow Tadić [47] who gives a proof of $U(3)$ for $A = \mathbb{C}, \mathbb{R}$ to deal with the case $A = \mathbb{H}$.

Theorem 10.1. *Let $\delta = \eta'(y+r, y) \in D$, $y \in \mathbb{C}$ and $r \in \mathbb{N}^\times$ and let $n \in \mathbb{N}^\times$. Then $u(\delta, n)$ is a prime in the ring $\mathcal{R}$.*

<u>*Proof*</u>. We know that δ is prime in $\mathcal{R}$, thus we start with $n \geq 2$. Let us first deal with the case $r = 1$. Then $\nu_\delta = 2$ and

$$a_0 = a(\delta, n) = (\nu_\delta^{\frac{n-1}{2}}\delta, \nu_\delta^{\frac{n-1}{2}-1}\delta, \ldots, \nu_\delta^{-\frac{n-1}{2}}\delta) =$$

$$(\eta'(y+n, y+n-1), \eta'(y+n-2, y+n-3), \ldots \eta'(y-n+2, y-n+1)).$$

Set $a_0 = a(\delta, n) = (X_1, \ldots, X_n)$ with

$$X_i = \gamma\left(y+n+2-2i, y+n+1-2i\right), \quad i = 1, \ldots, n.$$

Remark 10.2. The support of $u(\delta, n)$ is the multiset

$$(y+n+2-2i, y+n+1-2i)_{i=1,\ldots n}.$$

This support is regular.

Suppose that $u(\delta, n)$ is not prime in $\mathcal{R}$. Then there exists polynomials P and Q in the variables $d \in D$, non invertible, such that $u(\delta, n) = PQ$. Since $u(\delta, n)$ is homogeneous in $\mathcal{R}$ for the natural graduation, the same holds for P and Q.

Let us write

$$P = \sum_{c \in M(D)} m(c, P)\lambda(c), \quad Q = \sum_{d \in M(D)} m(d, Q)\lambda(d). \tag{10.1}$$

Set $S_P = \{a \in M(D) | m(a, P) \neq 0\}$, $S_Q = \{a \in M(D) | m(a, Q) \neq 0\}$. We get

$$\mathbf{Lg}(a_0) = X_1 \times X_2 \ldots \times X_n + \sum_{a \in M(D),\, a < a_0} M(a, a_0)\, \lambda(a).$$

Thus there exists $c_0 \in S_P$ and $d_0 \in S_Q$ such that

$$c_0 + d_0 = a_0 = (X_1, \ldots, X_n).$$

Since $\deg P > 0$, $\deg Q > 0$, c_0 and d_0 are not empty and the polynomials P and Q are not constant Denote by S_1 the set of X_i such that $X_i \in c_0$ and by S_2 the set of X_i such that $X_i \in d_0$. We get a partition of the X_i's into two non empty disjoint sets. Thus we can find $1 \leq i \leq n-1$ such that

$$\{X_i, X_{i+1}\} \not\subset S_j, \quad j = 1, 2,$$

and without any loss of generality, we may suppose that $X_i \in S_1$, $X_{i+1} \in S_2$. Furthermore, we have

$$|S_1| = \deg P, \quad |S_2| = \deg Q, \quad \deg P + \deg Q = n$$

We get from (9.9) that $X_i \times X_{i+1}$ is reducible, more precisely

$$X_i \times X_{i+1} = \mathbf{Lg}(X_i, X_{i+1}) + Y_i \times Y_{i+1}$$

where $Y_i = \eta'(y+n+2-2i, y+n-1-2i)$, $Y_{i+1} = \eta'(y+n+1-2i, y+n-2i)$.

We have $a_1 := (Y_i, Y_{i+1}) \prec (X_i, X_{i+1})$. Set

$$a_{i,i+1} = a_1 + (X_1, \dots, X_{i-1}, X_{i+2}, \dots, X_n).$$

Then $a_{i,i+1} \prec a_0$ and thus $M(a_1, a_0) \neq 0$ by Prop. 6.2. Therefore, there exists non empty $c_1 \in S_P$, $d_1 \in S_Q$ and such that

$$c_1 + d_1 = a_{i,i+1}.$$

We suppose now that Y_i divides $\lambda(c_1)$ in $\mathcal{R}$. The case where Y_i divides $\lambda(d_1)$ is similar.

Suppose that also $\lambda(Y_{i+1})$ divide $\lambda(c_1)$. We get a partition of the X_j, $j \neq i, i+1$ into two non empty sets S_1' and S_2', such that

$$c_1 = \{X_j, j \in S_1'\} + Y_i + Y_{i+1}, \quad d_1 = \{X_j, j \in S_2'\}.$$

The polynomials P and Q being homogeneous, we get

$$\deg(P) = |S_1'| + 2, \quad \deg(Q) = |S_2'|.$$

We see that $X_{i+1} \notin T := S_1 \cup S_2'$, thus $\{X_1, \dots, X_n\} \not\subset T$. For $r \in \mathcal{R}$, let us denote by $\deg_T(r)$ the degree of r in the variables $X_j \in T$. We get $\deg_T P \geq |S_1| = \deg P$, $\deg_T Q \geq |S_2'| = \deg Q$, thus $\deg_T(\mathbf{Lg}(a_0)) \geq n$. But the fact that the total degree of $\mathbf{Lg}(a_0)$ is n implies $\deg_T(\mathbf{Lg}(a_0)) = n$. The expression of $\mathbf{Lg}(a_0)$ in the basis $\lambda(b)$, $b \leq a_0$ shows that we can find $b_0 \in M(D)$, such that $M(b_0, a_0) \neq 0$, $\deg(b_0) = n$ and $\deg_T \lambda(b_0) = n$. Furthermore $\lambda(b_0)$ can be written

$$\lambda(b_0) = X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n}, \quad \alpha_j \in \mathbb{N}, \alpha_1 + \dots + \alpha_n = n.$$

Since $T \neq \{X_1, \dots, X_n\}$, there exists j such that $\alpha_j > 1$. But then X_j appears with multiplicity at least two in b_0. Since $\mathbf{Supp}\,(b_0) = \mathbf{Supp}\,(a_0)$ is regular, we get a contradiction.

Suppose now that $\lambda(Y_{i+1})$ doesn't divide $\lambda(c_1)$. We get a partition of the X_j, $j \neq i, i+1$ into two non empty sets S_1' and S_2', such that

$$c_1 = \{X_j, j \in S_1'\} + Y_i, \quad d_1 = \{X_j, j \in S_2'\} + Y_{i+1}.$$

We set now $T = S_1' \cup S_2$, and we see that X_{i+1} doesn't belong to T , thus $\{X_1, \dots, X_n\} \not\subset T$. For $r \in \mathcal{R}$, denote by $\deg_T(r)$ the degree of r in the variables $X_j \in T$ and Y_i. As above, we get that $\deg_T(\mathbf{Lg}(a_0)) = n$ and that there exists $b_0 \in M(D)$, such that $M(b_0, a_0) \neq 0$, $\deg(b_0) = n$ and $\deg_T(\lambda(b_0)) = n$. We can write

$$\lambda(b_0) = X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n} Y_i^{\alpha}, \quad \alpha_j \in \mathbb{N}, \alpha_1 + \dots + \alpha_n + \alpha = n.$$

Since $\{X_1, \dots, X_n\} \not\subset T$, there exists j such that $\alpha_j = 0$. If $\alpha = 0$, we get a contradiction as above. Thus $\alpha \geq 1$, but since multiplicities in $\mathbf{Supp}\,(a_0)$ are at most 1, we get $\alpha = 1$, $\alpha_j = 1$ if $j \neq i+1, i$, $\alpha_i = \alpha_{i+1} = 0$ and we still get a contradiction. This finishes the case $r = 1$.

Let us deal with briefly the case $r > 1$. Then $\nu_\delta = \nu$ and

$$a(\delta, n) = (\nu^{\frac{n-1}{2}}\delta, \nu^{\frac{n-1}{2}-1}\delta, \dots, \nu^{-\frac{n-1}{2}}\delta) =$$

$$(\eta'(x + \frac{n-1}{2}, y + \frac{n-1}{2}), \eta'(x + \frac{n-1}{2} - 1, y + \frac{n-1}{2} - 1), \dots \eta'(x - \frac{n-1}{2}, y - \frac{n-1}{2})).$$

Set $a_0 = a(\delta, n) = (X_1, \dots, X_n)$ with

$$X_i = \gamma\left(y + r + \frac{n-1}{2} + 1 - i, y + \frac{n-1}{2} + 1 - i\right), \quad i = 1, \dots, n.$$

We proceed as above, using now formula (9.8) for the reducibility of $\lambda(X_i, X_{i+1})$:

$$\lambda(X_i, X_{i+1}) = \mathbf{Lg}(X_i, X_{i+1}) + \lambda(Y_i, Y_{i+1})$$

where $Y_i = \eta'(y+r+\frac{n-1}{2}+1-i, y+\frac{n-1}{2}-i)$ and $Y_{i+1} = \eta'(y+r+\frac{n-1}{2}-i, y+\frac{n-1}{2}+1-i)$. In all cases, we get contradictions by inspecting multiplicities in the support. We leave the details to the reader. □

11. $U(1)$: archimedean case

We recall briefly the arguments for $A = \mathbb{C}$ and $\mathbb{R}$, even if it is well known and done elsewhere, because we will need the notation anyway. We give the complete argument when $A = \mathbb{H}$.

11.1. $A = \mathbb{C}$**.** This case is easy because for $\gamma = \gamma(x, y)$, $x, y \in \mathbb{C}$, $x - y \in \mathbb{Z}$, a character of $\mathbb{C}^\times$, we have

$$u(\gamma, n) = \gamma \circ \det .$$

Representations $u(\gamma, n)$ are thus 1-dimensional representations of $\mathbf{GL}(n, \mathbb{C})$. Furthermore, if γ is unitary (*i.e.* $\Re e(x + y) = 0$) then $u(\gamma, n)$ is unitary.

11.2. $A = \mathbb{R}$**.** There are two cases to consider. The first is $\delta \in D_1$, $\delta = \delta(\alpha, \epsilon)$, $\alpha \in \mathbb{C}$, $\epsilon \in \{0, 1\}$. This case is similar to the case $A = \mathbb{C}$ above, since

$$u(\delta, n) = \delta \circ \det .$$

Representations $u(\delta, n)$ are 1-dimensional representations of $\mathbf{GL}(n, \mathbb{R})$. Furthermore, if δ is unitary (*ie.* $\Re e(\alpha) = 0$) then $u(\delta, n)$ is unitary.

The second case is $\delta = \eta(x, y) \in D_2$, $x, y \in \mathbb{C}$, $x - y = r \in \mathbb{N}^\times$. We have already mentioned without giving any details that $\eta(x, y)$ is obtained by cohomological induction from the character $\gamma(x, y)$ of the Cartan subgroup $\mathbb{C}^\times$ of $\mathbf{GL}(2, \mathbb{R})$. Let us be now more precise. Cohomological induction functors considered here are normalized as in [27], (11.150b): if $(\mathfrak{g}_\mathbb{C}, K)$ is a reductive pair associated to a real reductive group G, if $\mathfrak{q}_\mathbb{C} = \mathfrak{l}_\mathbb{C} \oplus \mathfrak{u}_\mathbb{C}$ is a θ-stable parabolic subalgebra of $\mathfrak{g}_\mathbb{C}$, with Levi factor $\mathfrak{l}_\mathbb{C}$, and if L is the normalizer in G of $\mathfrak{q}_\mathbb{C}$, we define the cohomological induction functor :

$$\begin{aligned} \mathcal{R}_{\mathfrak{q}_\mathbb{C}} : \mathcal{M}(\mathfrak{l}_\mathbb{C}, K \cap L) &\longrightarrow \mathcal{M}(\mathfrak{g}_\mathbb{C}, K) \\ X &\mapsto \Gamma^S \circ \mathrm{pro}(X \otimes \tilde{\tau}) \end{aligned}$$

where $S = \dim(\mathfrak{u}_\mathbb{C} \cap \mathfrak{k}_\mathbb{C})$, Γ^S is the S-th Zuckerman derived functor from $\mathcal{M}(\mathfrak{g}_\mathbb{C}, K \cap L)$ to $\mathcal{M}(\mathfrak{g}_\mathbb{C}, K)$, pro is the parabolic induction functor from $\mathcal{M}(\mathfrak{l}_\mathbb{C}, K \cap L)$ to $\mathcal{M}(\mathfrak{g}_\mathbb{C}, K \cap L)$, and $\tilde{\tau}$ is a character of L, square root of the character $\bigwedge^{top}(\mathfrak{u}_\mathbb{C}/\mathfrak{u}_\mathbb{C} \cap \mathfrak{k}_\mathbb{C})$ (such a square root is usually defined only on a double cover of L, but for the cases we are interested in here, *i.e.* products of $G = \mathbf{GL}(n, \mathbb{R})$, $\mathbf{GL}(n, \mathbb{C})$ or $\mathbf{GL}(n, \mathbb{H})$, we can find such a square root on L). This normalization preserves infinitesimal character.

With this notation, for $G = \mathbf{GL}(2, \mathbb{R})$, $L \simeq \mathbb{C}^\times$ and $\mathfrak{u}_\mathbb{C} = \mathfrak{g}_\mathbb{C}^{e_1 - e_2}$, we get

$$\mathcal{R}_{\mathfrak{q}_\mathbb{C}}(\gamma(x, y)) = \eta(x, y), \quad x, y \in \mathbb{C}, x - y \in \mathbb{N}.$$

Recall the convention $\eta(x, x) = \delta(x, 0) \times \delta(x, 1)$ for limits of discrete series, so this formula is also valid when $x = y$.

Set $a_0 = a(\eta(x, y), n) \in M(D)$. The standard representation $\lambda(a_0)$ is obtained by parabolic induction from the representation

$$\eta = \eta(x + \frac{n-1}{2}, y + \frac{n-1}{2}) \otimes \eta(x + \frac{n-3}{2}, y + \frac{n-3}{2}) \otimes \ldots \otimes \eta(x - \frac{n-1}{2}, y - \frac{n-1}{2})$$

of $\mathbf{GL}(2,\mathbb{R})\times\ldots\times\mathbf{GL}(2,\mathbb{R})$, the representation η being from what has just been said obtained by cohomological induction from the character

$$\gamma=\gamma(x+\frac{n-1}{2},y+\frac{n-1}{2})\otimes\gamma(x+\frac{n-3}{2},y+\frac{n-3}{2})\otimes\ldots\otimes\gamma(x-\frac{n-1}{2},y-\frac{n-1}{2})$$

of $\mathbb{C}^\times\times\ldots\times\mathbb{C}^\times$. Furthermore $u(\eta(x,y),n)$ is the unique irreducible quotient of $\lambda(a_0)$.

Independence of polarization results in [27], chapter 11 show that the standard representation $\lambda(a_0)$ could be also obtained from the character γ of $(\mathbb{C}^\times)^n$ in the following way : first use parabolic induction from $(\mathbb{C}^\times)^n$ to $\mathbf{GL}(n,\mathbb{C})$ (with respect to the usual upper triangular Borel subgroup) to get the standard representation

$$(11.1)\quad \gamma(x+\frac{n-1}{2},y+\frac{n-1}{2})\times\gamma(x+\frac{n-3}{2},y+\frac{n-3}{2})\times\ldots\times\gamma(x-\frac{n-1}{2},y-\frac{n-1}{2})$$

whose unique irreducible quotient is $u(\gamma(x,y),n)$, and then the cohomological induction functor $\mathcal{R}_{\mathfrak{q}_\mathbb{C}}$ from $\mathbf{GL}(n,\mathbb{C})$ to $\mathbf{GL}(2n,\mathbb{R})$ (the reader can guess which θ-stable parabolic subalgebra $\mathfrak{q}_\mathbb{C}$ we use). This shows also that $u(\delta,n)$ is the unique irreducible quotient of $\mathcal{R}_{\mathfrak{q}_\mathbb{C}}(u(\gamma(x,y),n))$. Now, irreducibility and unitarizability theorems of [27] also imply, the character $u(\gamma(x,y),n)$ of $\mathbf{GL}(n,\mathbb{C})$ being in the weakly good range, that $\mathcal{R}_{\mathfrak{q}_\mathbb{C}}(u(\gamma(x,y),n))$ is irreducible and unitary if $u(\gamma(x,y),n)$ is unitary. Thus we get

$$\mathcal{R}_{\mathfrak{q}_\mathbb{C}}(u(\gamma(x,y),n))=u(\eta(x,y),n)$$

and this representation is unitary if and only if $\Re e(x+y)=0$.

In the degenerate case $x=y$ (see (9.4)), we get

$$\mathcal{R}_{\mathfrak{q}_\mathbb{C}}(u(\gamma(x,y),n))=u(\delta(x,0),n)\times u(\delta(x,1),n).$$

11.3. $A=\mathbb{H}$. Let $\delta=\eta'(x,y)$, $x,y\in\mathbb{C}$, $x-y\in\mathbb{N}^\times$, be an irreducible representation of $\mathbb{H}^\times$. Consider the representation $u(\eta'(x,y),n)$, and recall the invariant ν_δ of definition 7.1. We have seen that $\nu_\delta=\nu$ when $x-y>1$, $\nu_\delta=\nu^2$ when $x-y=1$. In the first case, the discussion for the unitarizability of $u(\eta'(x,y),n)$ is exactly the same as in the case $A=\mathbb{R}$: the standard representation $\lambda(a_0)$ whose unique irreducible quotient is $u(\eta'(x,y),n)$ is obtained by cohomological induction from $\mathbf{GL}(n,\mathbb{C})$ to $\mathbf{GL}(n,\mathbb{H})$ of the representation γ defined in (11.1). Furthermore $u(\eta'(x,y),n)$ is the unique irreducible quotient of $R_{\mathfrak{q}'_\mathbb{C}}(u(\gamma(x,y),n))$ and is unitary if and only if $\Re e(x+y)=0$.

When $\nu_\delta=\nu^2$, *i.e.* $x-y=1$, we get the same results, not for $u(\eta'(x,y),n)$, but for $\bar{u}(\eta'(x,y),n)$, the Langlands quotient of the standard representation

$$\eta'(x+\frac{n-1}{2},y+\frac{n-1}{2})\times\eta'(x+\frac{n-3}{2},y+\frac{n-3}{2})\times\ldots\times\eta'(x-\frac{n-1}{2},y-\frac{n-1}{2})$$

$$=\nu^{\frac{n-1}{2}}\eta'(x,y)\times\nu^{\frac{n-3}{2}}\eta'(x,y)\times\ldots\times\nu^{-\frac{n-1}{2}}\eta'(x,y).$$

Recall that $u(\eta'(x,y),n)$ is the Langlands quotient of

$$\begin{aligned}&\nu_\delta^{\frac{n-1}{2}}\eta'(x,y)\times\nu_\delta^{\frac{n-3}{2}}\eta'(x,y)\times\ldots\times\nu_\delta^{-\frac{n-1}{2}}\eta'(x,y)\\&=\nu^{n-1}\eta'(x,y)\times\nu^{n-3}\eta'(x,y)\times\ldots\times\nu^{-(n-1)}\eta'(x,y).\end{aligned}$$

With the two conditions $x-y=1$ and $\Re e(x+y)=0$, we see that, up to a twist by a unitary character, we only have to study the case $u(\eta',n)$ with $\eta'=\eta'(\frac{1}{2},-\frac{1}{2})$. Unitarity of $u(\eta',n)$ can be deduced from the unitarity of the $\bar{u}(\eta',k)$ as in [9], using the facts that

$$(11.2)\qquad \bar{u}(\eta',2n+1)=u(\eta',n+1)\times u(\eta',n)$$

$$\bar{u}(\eta', 2n) = \nu^{\frac{1}{2}} u(\eta', n) \times \nu^{-\frac{1}{2}} u(\eta', n). \tag{11.3}$$

One could also observe (as it was done by the referee) that $u(\eta', n)$ is the trivial representation, and so is certainly unitarizable.

12. Vogan's classification and $U(0)$ in the archimedean case

As we have already said, $U(0)$ is established in the case $A = \mathbb{R}$ or $\mathbb{C}$ by the work of M. Baruch filling the serious technical gap that remained in Kirillov's treatment of Bernstein approach ([25]). It is also possible to establish $U(0)$ from Vogan's classification, and this will work also for $A = \mathbb{H}$. Of course, this might seem a rather convoluted and unnatural approach, if the final goal is to prove the classification of the unitary dual in Tadić's form, since a direct comparison between the classifications is possible. But let us notice that :

— One of the main difficulty of Vogan's paper is to prove some special cases of $U(0)$ (the other difficult point is the exhaustion of the list of unitary almost spherical representations). The rest of his paper uses only standard and general techniques of the representation of real reductive groups, mainly cohomological induction.

— The argument which allows the comparison between the two classifications ("independence of polarizations") is also the one leading to $U(0)$ from Vogan's classification.

— There is still some hope to find an uniform proof of $U(0)$ for all A.

In this section, we give a brief overview of Vogan's paper [52], and how it implies $U(0)$. Here, $A = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Let us fix a unitary character

$$\delta : \mathbf{GL}(1, A) \simeq A^\times \to \mathbb{C}^\times.$$

It extends canonically to a family of unitary characters

$$\delta_n : \mathbf{GL}(n, A) \to \mathbb{C}^\times,$$

by composing with the determinant $\mathbf{GL}(n, A) \to \mathbf{GL}(1, A)$ (non commutative determinant of Dieudonné if $F = \mathbb{H}$).

The *basic blocs* of Vogan's classification are the representations :

$$\nu^{i\beta}\delta_n, \quad \beta \in \mathbb{R}$$

(with Tadić's notation, $\nu^{i\beta}\delta_n = u(\nu^{i\beta}\delta, n)$: it is a unitary character of $\mathbf{GL}(n, A)$), and the representations

$$\pi(\nu^{i\beta}\delta, n; \alpha) = \nu^{-\alpha}\nu^{i\beta}\delta_n \times \nu^{\alpha}\nu^{i\beta}\delta_n, \quad 0 < \alpha < \frac{1}{2}$$

of $\mathbf{GL}(2n, F)$. These are Stein's complementary series.

Vogan considers first parabolically induced representations of the form

$$\tau = \tau_1 \times \tau_2 \times \ldots \times \tau_r \tag{12.1}$$

where each τ_j is either a unitary character

$$\tau_j = \nu^{\beta_j}\delta_{n_j}, \quad \beta_j \in i\mathbb{R},$$

or a Stein's complementary series

$$\tau_j = \pi(\nu^{\beta_j}\delta, n_j; \alpha), \quad \beta_j \in i\mathbb{R}, \ 0 < \alpha < \frac{1}{2}.$$

The reason for these conditions is the following : recall our choices of maximal compact subgroups $K(n,A)$ of $\mathbf{GL}(n,A)$ respectively for $A=\mathbb{R},\mathbb{C}$ and $\mathbb{H}$:

$$\mathbf{O}(n), \mathbf{U}(n) \text{ and } \mathbf{Sp}(n)$$

and denote by μ_n the restriction of δ_n to $K(n,A)$. We say that μ_n is a *special* 1-dimensional representation of $K(n,A)$. If $A=\mathbb{R}$, since μ_n factorizes through the determinant, there are two special representations of $\mathbf{O}(n)$: the trivial representation, and the sign of the determinant. If $A=\mathbb{C}$, special representations of $\mathbf{U}(n)$ are obtained by composing the determinant (with values in $\mathbf{U}(1)$), and a character of $\mathbf{U}(1)$ (given by an integer). Finally, if $A=\mathbb{H}$ the only special representation of $\mathbf{Sp}(n)$ is the trivial one. A representation of $\mathbf{GL}(n,A)$ is said to be *almost spherical* (of type μ_n) if it contains the special K-type μ_n. This generalizes spherical representations. The characters $\delta_n\nu^\beta$ are exactly the ones whose restriction to $K(n,A)$ is μ_n. The τ_i's above are thus either almost spherical unitary characters of type μ (the family $\mu=(\mu_n)_n$ is fixed), or almost spherical Stein's complementary series of type μ.

Then Vogan shows the following ([52], Theorem 3.8):

Theorem 12.1. *The representations $\tau=\tau_1\times\tau_2\times\ldots\times\tau_r$ are*
(i) *unitary*
(ii) *irreducible*

Furthermore, every irreducible, almost spherical of type μ, unitary representation is obtained in this way, and two irreducible, almost spherical of type μ, unitary representations

$$\tau=\tau_1\times\tau_2\times\ldots\times\tau_r$$

and

$$\tau'=\tau'_1\times\tau'_2\times\ldots\times\tau'_s$$

are equivalent if and only if the multisets $\{\tau'_i\}$ and $\{\tau_j\}$ are equal.

Let us notice that this theorem contains a special case of $U(0)$: this is the point (ii). It can be proved using Proposition 2.13 in [7] and results of S. Sahi ([37], Thm 3A).

Furthermore, the classification of irreducible, almost spherical, unitary representations it gives coincide with Tadić's classification. (One has to notice that an irreducible, almost spherical, unitary representation is such with respect to an unique special K-type : special K-types are minimal, and minimal K-types for $\mathbf{GL}(n,A)$ are unique, and appear with multiplicity 1).

Vogan classification of the unitary dual of $\mathbf{GL}(n,A)$ reduces matters to this particular case of almost spherical representations using cohomological induction functors preserving irreducibility and unitarity. More precisely, let us recall some material about Vogan's classification of the admissible dual of a real reductive group G by minimal K-types ([50]). To each irreducible representation of G is attached a finite number of minimal K-types. As we said above, for $G=\mathbf{GL}(n,A)$, the minimal K-type is unique, and appears with multiplicity 1. This gives a partition (which can be explicitly given in terms of Langlands classification) of the admissible dual of $\mathbf{GL}(n,A)$.

Vogan's classification of the unitary dual deals with each term of this partition separately. To each irreducible representation μ of the compact group $K(n,A)$ is attached a subgroup L of $\mathbf{GL}(n,A)$ with maximal compact subgroup $K_L:=K(n,A)\cap L$, and an irreducible representation μ_L of K_L. The subgroup L is a product of groups of the form

$\mathbf{GL}(n_i, A_i)$,

$$K(n,A) \cap L \simeq \prod_i K(n_i, A_i)$$

and μ_L is a tensor product of special representations of the $K(n_i, A_i)$.

As opposed to Tadić's classification which uses only parabolic induction functors, Vogan's classification of $\mathbf{GL}(n,\mathbb{R})$ for instance, will use classification of the almost spherical unitary dual of groups $\mathbf{GL}(k,\mathbb{C})$. More precisely :

- For $F = \mathbb{R}$, the subgroups L are products of $\mathbf{GL}(k,\mathbb{R})$ and $\mathbf{GL}(m,\mathbb{C})$.
- For $F = \mathbb{C}$, the subgroups L are products of $\mathbf{GL}(k,\mathbb{C})$.
- For $F = \mathbb{H}$, the subgroups L are products of $\mathbf{GL}(k,\mathbb{H})$ and $\mathbf{GL}(m,\mathbb{C})$.

A combination of parabolic and cohomological induction functors then defines a functor

$$\mathcal{I}_L^G$$

from $\mathcal{M}(L)$ to $\mathcal{M}(\mathbf{GL}(n,A))$ with the following properties :

- $\mathcal{I}_L^G$ sends an irreducible (resp. unitary) representation of L with minimal K_L-type μ_L to an irreducible (resp. unitary) representation of $\mathbf{GL}(n,F)$ with minimal K-type μ.
- $\mathcal{I}_L^G$ realizes a bijection between equivalence classes of irreducible unitary representations of L with minimal K_L-type μ_L and equivalence classes of irreducible unitary representations of $\mathbf{GL}(n,F)$ with minimal K-type μ.

From this point of view, to establish $U(0)$, the first thing to do is to check that products of representations of the form (12.1) for different families of special K-types μ are irreducible. For $F = \mathbb{H}$, there is nothing to check since there is only one family of special K-types $\mu = (\mu_n)_n$. For $F = \mathbb{R}$, there are two families of special K-types, the trivial and sign characters of the determinant of $\mathbf{O}(n)$. The relevant result is then lemma 16.1 of [52]. For $F = \mathbb{R}$, we have now obtained all irreducible unitary representations which are products of $u(\delta,k)$ and $\pi(\delta,k;\alpha)$ with δ any unitary character of $\mathbf{GL}(1,\mathbb{R}) = \mathbb{R}^\times$.

The case $A = \mathbb{C}$ is simpler and dealt with as follows. Let us notice first that since square integrable modulo center representations of $\mathbf{GL}(n,\mathbb{C})$ exist only for $n = 1$, the above assertion shows that we get all representations of Tadić's classification, and this establishes $U(0)$. In that case, the subgroups L from which we use cohomological induction are of the form

$$L = \mathbf{GL}(n_1,\mathbb{C}) \times \ldots \mathbf{GL}(n_r,\mathbb{C}).$$

The cohomological induction setting is that $\mathfrak{l}_\mathbb{C} = \mathrm{Lie}(L)_\mathbb{C}$ is a Levi factor of a θ-stable parabolic subalgebra $\mathfrak{q}_\mathbb{C}$ of $\mathfrak{g}_\mathbb{C} = \mathrm{Lie}(\mathbf{GL}(n,\mathbb{C}))_\mathbb{C}$. But L is also a Levi factor of a parabolic subgroup of $\mathbf{GL}(n,\mathbb{C})$. Thus there are two ways of inducing from L to $\mathbf{GL}(n,\mathbb{C})$: parabolic and cohomological induction. An 'independence of polarization' result ([52], Theorem 17.6, see [27], Chapter 11 for a proof), asserts that the two coincide. This finishes the case $A = \mathbb{C}$.

Let us now finish to discuss the cases $A = \mathbb{R}$ and $A = \mathbb{H}$. Representations from Tadić's classification which are still missing are the ones built from $u(\delta,k)$'s and $\pi(\delta,k;\alpha)$'s with δ a square integrable modulo center representation of $\mathbf{GL}(2,\mathbb{R})$ or $\mathbb{H}^\times$. As we have seen in 11.2 a square integrable modulo center representation of $\mathbf{GL}(2,\mathbb{R})$ or $\mathbb{H}^\times$ is obtained by cohomological induction from the subgroup $L \simeq \mathbb{C}^\times$ of $\mathbf{GL}(2,\mathbb{R})$ or $\mathbf{GL}(1,\mathbb{H}) = \mathbb{H}^\times$. This explains somehow why cohomological induction will produce the missing representations. Let us explain this :

— case $F = \mathbb{R}$: we start with representations of the form

$$u(\chi_a, k_a),\ \pi(\chi_b, k_b; \alpha_b),\ u(\chi_c, k_c),\ \pi(\chi_d, k_d; \alpha_d),\ u(\chi_e, k_e),\ \pi(\chi_f, k_f; \alpha_f))$$

where $u(\chi_a, k_a)$ are unitary characters of $\mathbf{GL}(k_a, \mathbb{C})$, $\pi(\chi_b, k_b; \alpha_b)$ are Stein complementary series of $\mathbf{GL}(2k_b, \mathbb{C})$, $u(\chi_c, k_c)$ are unitary characters of $\mathbf{GL}(k_c, \mathbb{R})$ of trivial type μ, $\pi(\chi_d, k_d; \alpha_d)$ are Stein complementary series of $\mathbf{GL}(2k_d, \mathbb{R})$ of trivial type μ, $u(\chi_e, k_e)$ are unitary characters of $\mathbf{GL}(k_c, \mathbb{R})$ of type $\mu = \mathrm{sgn}$, $\pi(\chi_f, k_f; \alpha_f)$ are Stein complementary series of $\mathbf{GL}(2k_f, \mathbb{R})$ of type $\mu = \mathrm{sgn}$.

The tensor product

$$(\bigotimes_a u(\chi_a, k_a)) \otimes (\bigotimes_b \pi(\chi_b, k_b; \alpha_b)) \otimes (\bigotimes_c u(\chi_c, k_c))$$
$$\otimes(\bigotimes_d \pi(\chi_d, k_d; \alpha_d)) \otimes (\bigotimes_e u(\chi_e, k_e)) \otimes (\bigotimes_f \pi(\chi_f, k_f; \alpha_f))$$

is a representation of the Levi subgroup

$$\prod_a \mathbf{GL}(k_a, \mathbb{C}) \prod_b \mathbf{GL}(2k_b, \mathbb{C}) \prod_c \mathbf{GL}(k_c, \mathbb{R}) \prod_d \mathbf{GL}(2k_d, \mathbb{R}) \prod_e \mathbf{GL}(k_e, \mathbb{R}) \prod_f \mathbf{GL}(2k_f, \mathbb{R})$$

of $\mathbf{GL}(n, \mathbb{R})$, where $n = \sum_a 2k_a + \sum_b 4k_b + \sum_c k_c + \sum_d 2k_d + \sum_e k_e + \sum_k 2k_f$.

As we saw, we first form almost spherical representations of a given type by parabolic induction. Thus we induce

$$(\bigotimes_c u(\chi_c, k_c)) \otimes (\bigotimes_d \pi(\chi_d, k_d; \alpha_d))$$

from

$$\prod_c \mathbf{GL}(k_c, \mathbb{R}) \prod_d \mathbf{GL}(2k_d, \mathbb{R})$$

to $\mathbf{GL}(q_0, \mathbb{R})$, where $q_0 = \sum_c k_c + \sum_d 2k_d$, obtaining an irreducible unitary spherical representation π_0, and similarly

$$(\bigotimes_e u(\chi_e, k_e)) \otimes (\bigotimes_f \pi(\chi_f, k_f; \alpha_f))$$

from

$$\prod_e \mathbf{GL}(k_e, \mathbb{R}) \prod_d \mathbf{GL}(2k_f, \mathbb{R})$$

to $\mathbf{GL}(q_1, \mathbb{R})$, where $q_1 = \sum_e k_e + \sum_f 2k_f$, obtaining an irreducible unitary almost spherical type $\mu = \mathrm{sgn}$ representation.

Then we mix spherical and almost spherical of type $\mu = \mathrm{sgn}$ representations inducing parabolically $\pi_0 \times \pi_1$ from $\mathbf{GL}(q_0, \mathbb{R}) \times \mathbf{GL}(q_1, \mathbb{R})$ to $\mathbf{GL}(q_0 + q_1, \mathbb{R})$: we get an irreducible unitary representation π of $\mathbf{GL}(q_0 + q_1, \mathbb{R})$.

The group $\prod_a \mathbf{GL}(k_a, \mathbb{C}) \prod_b \mathbf{GL}(2k_b, \mathbb{C}) \times \mathbf{GL}(q_0 + q_1, \mathbb{R})$ is denoted by L_θ in [52]. Applying cohomological induction functor $\mathcal{I}_{L_\theta}^G$ to the representation

$$(\bigotimes_a u(\chi_a, k_a)) \otimes (\bigotimes_b \pi(\chi_b, k_b; \alpha_b)) \otimes \pi$$

of L_θ, we get an irreducible unitary representation ρ of $\mathbf{GL}(n, \mathbb{R})$.

Independence of polarization theorems ([52], Theorem 17.6, Theorem 17.7 and 17.9, see [27], Chapter 11), allows us to invert the order of the two types of induction. We could in fact start with cohomological induction, inducing each

$$u(\chi_a, k_a)$$

from $\mathbf{GL}(k_a,\mathbb{C})$ to $\mathbf{GL}(2k_a,\mathbb{R})$. In non degenerate case, following the terminology of [52], definition 17.3, we get representations $u(\delta_a, 2k_a)$, where δ_a is a square integrable modulo center irreducible representation of $\mathbf{GL}(2,\mathbb{R})$. In the degenerate case, δ_a is a limit of discrete series (9.4). These are almost spherical representations that we had before (see [52], prop. 17.10).

In the same way, we induce all

$$\pi(\chi_b, k_b; \alpha_b)$$

from $\mathbf{GL}(2k_b,\mathbb{C})$ to $\mathbf{GL}(4k_b,\mathbb{R})$. In the non degenerate case, we get representations $\pi(\delta_b, 2k_b;\alpha_b)$, where δ_b is as above. In the degenerate case, we still get almost spherical representations.

The parabolically induced representation from

$$\prod_a \mathbf{GL}(2k_a,\mathbb{R}) \prod_b \mathbf{GL}(4k_b,\mathbb{R}) \times \mathbf{GL}(q_0+q_1,\mathbb{R})$$

to $\mathbf{GL}(n,\mathbb{R})$ of

$$(\bigotimes_a u(\delta_a,k_a)) \otimes (\bigotimes_b \pi(\delta_b,k_b;\alpha_b)) \otimes \pi$$

is ρ (and thus irreducible), see [52], Theorem 17.6 .

This finishes the comparison of the two classifications. The case $A=\mathbb{H}$ is entirely similar.

We deduce $U(0)$ using again independence of polarization. We want to show that $\rho = \rho_1 \times \rho_2$ is irreducible if ρ_1 and ρ_2 are irreducible and unitary. We write ρ_1 and ρ_2 as above using first cohomological induction and then, parabolic induction. Using parabolic induction by stage, we see that $\rho_1 \times \rho_2$ is also written in this form. Using again independence of polarization we write ρ as a parabolically then cohomologically induced representation, and we see that as such, this is a representation appearing in Vogan's classification which is therefore irreducible.

13. Jacquet-Langlands correspondence in the archimedean case

Ideas in this section are taken from [1] which deals with a similar problem (Kazhdan-Patterson lifting).

13.1. **Jacquet-Langlands correspondence and coherent families.** Since we need to consider simultaneously the case $A=\mathbb{R}$ and $A=\mathbb{H}$, we add relevant superscripts to the notation when needed, as in Section 4. We have noticed that Jacquet-Langlands correspondence between essentially square integrable modulo center irreducible representations of $\mathbf{GL}(2,\mathbb{R})$ and irreducible representations of $\mathbb{H}^\times$ is given at the level of Grothendieck groups by

$$\mathbf{LJ}(\eta(x,y)) = -\eta'(x,y)$$

Representations in $D_1^{\mathbb{R}}$ are sent to 0. We extend this linearly to an algebra morphism :

$$\mathcal{R}^{\mathbb{R}} \to \mathcal{R}^{\mathbb{H}}.$$

Lemma 13.1. *Jacquet-Langlands correspondence preserves supports.*

Proof. $a \in M(D)$, $a = (\eta(x_1,y_1),\ldots,\eta(x_r,y_r))$. We have then

$$\mathbf{LJ}(\lambda(a)) = (-1)^r \lambda(a')$$

where $a = (\eta'(x_1, y_1), \dots, \eta'(x_r, y_r))$. The support of a is $(x_1, y_1, \dots, x_r, y_r)$, and this is also the support of a'.

We recall now the definition of a coherent family of Harish-Chandra modules.

Definition 13.2. Let G be a real reductive group, H a Cartan subgroup, $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{h}_{\mathbb{C}}$ the respective complexification of their Lie algebras and Λ the lattice of weights of H in finite dimensional representations of G. A coherent family of (virtual) Harish-Chandra modules based at $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$ is a family

$$\{\pi(\lambda + \mu) \,|\, \mu \in \Lambda\}$$

($\lambda + \mu$ is just a formal symbol, since the two terms are not in the same group) in the Grothendieck group $\mathcal{R}(G)$ such that

— The infinitesimal character of $\pi(\lambda + \mu)$ is given by $\lambda + d\mu$.

— For any finite dimensional representation F of G, we have, with $\Delta(F)$ denoting the set of weights of H in F, the following identity in $\mathcal{R}(G)$:

$$\pi(\lambda + \mu) \otimes F = \sum_{\gamma \in \Delta(F)} \pi(\lambda + \mu + \gamma).$$

Jacquet-Langlands correspondence preserves coherent families :

Lemma 13.3. *Let us identify two Cartan subgroups H and H' respectively of $\mathbf{GL}(2n, \mathbb{R})$ and $\mathbf{GL}(n, \mathbb{H})$ isomorphic to $(\mathbb{C}^\times)^n$. Let $\pi(\lambda+\mu)$ be a coherent family of Harish-Chandra modules for $\mathbf{GL}(2n, \mathbb{R})$ based at $\lambda \in \mathfrak{h}_{\mathbb{C}}^*$. Then $\mathbf{LJ}(\pi(\lambda + \mu))$ is a coherent family for $\mathbf{GL}(n, \mathbb{H})$.*

Proof. The first property of coherent families is satisfied by $\mathbf{LJ}(\pi(\lambda + \mu))$ because of the previous lemma. For the second property, let us remark first that $\mathbf{GL}(2n, \mathbb{R})$ and $\mathbf{GL}(n, \mathbb{H})$ being two real forms of $\mathbf{GL}(2n, \mathbb{C})$, a finite dimensional representation F of one of these two groups is in fact the restriction of a finite dimensional representation of $\mathbf{GL}(2n, \mathbb{C})$. We get for all regular element g' of $\mathbf{GL}(n, \mathbb{H})$ corresponding to an element g in $\mathbf{GL}(2n, \mathbb{R})$,

$$\begin{aligned}\sum_{\gamma \in \Delta(F)} \Theta_{\mathbf{LJ}(\pi(\lambda+\mu+\gamma))}(g') &= \sum_{\gamma \in \Delta(F)} \Theta_{\pi(\lambda+\mu+\gamma)}(g) = \Theta_{\pi(\lambda+\mu)\otimes F}(g) \\ &= \Theta_{\pi(\lambda+\mu)}(g)\, \Theta_F(g) = \Theta_{\mathbf{LJ}(\pi(\lambda+\mu))}(g')\Theta_F(g') = \Theta_{\mathbf{LJ}(\pi(\lambda+\mu))\otimes F}(g'),\end{aligned}$$

so $\sum_{\gamma \in \Delta(F)} \mathbf{LJ}(\pi(\lambda + \mu + \gamma)) = \mathbf{LJ}(\pi(\lambda + \mu)) \otimes F$. □

13.2. **Jacquet-Langlands correspondence and cohomological induction.** The cohomological induction functor $\mathcal{R}_{\mathfrak{q}_{\mathbb{C}}}$ introduced in 11.2 preserves irreducibility and unitarity when the infinitesimal character of the induced module satisfies certain positivity properties with respect to $\mathfrak{q}_{\mathbb{C}}$ ("weakly good range"). Furthermore, with the same conditions, other derived functors $\Gamma^i(\mathrm{pro}(\bullet \otimes \tilde{\tau}))$, $i \neq S$, vanish. This is not true in general, and this is the reason why we need to consider Euler-Poincaré characteristic :

$$\widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}} := \sum_i (-1)^i \Gamma^i(\mathrm{pro}(\bullet \otimes \tilde{\tau})).$$

This is not a functor between $\mathcal{M}(L)$ and $\mathcal{M}(G)$ anymore, but simply a morphism between the Grothendieck groups $\mathcal{R}(L)$ and $\mathcal{R}(G)$.

Lemma 13.4. *The morphism $\widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}} : \mathcal{R}(L) \to \mathcal{R}(G)$ preserves coherent families.*

Proof. The functors $\Gamma^i(\mathrm{pro}(\bullet \otimes \tilde{\tau}))$ are normalized in order to preserve infinitesimal character, and thus the first property of coherent family is preserved.

Let $\pi(\lambda+\mu)$ be a coherent family of Harish-Chandra for $(\mathfrak{l}, L\cap K)$. We want to show that for any finite dimensional representation F of G,

$$\widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}(\pi(\lambda+\mu)) \otimes F = \sum_{\gamma\in\Delta(F)} \widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}(\pi(\lambda+\mu+\gamma)) \tag{13.1}$$

But

$$\begin{aligned}\sum_{\gamma\in\Delta(F)} \widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}(\pi(\lambda+\mu+\gamma)) &= \widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}\left(\sum_{\gamma\in\Delta(F)} \pi(\lambda+\mu+\gamma)\right)\\ =&\widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}(\pi(\lambda+\mu)\otimes F)\end{aligned}$$

It is then enough to show that for any $(\mathfrak{l}, L\cap K)$-module X,

$$\widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}(X)\otimes F = \widehat{\mathcal{R}}_{\mathfrak{q}_{\mathbb{C}}}(X\otimes F) \tag{13.2}$$

Let U be a any $(\mathfrak{g}, K)$-module. Let us compute, using adjunction properties of the functors involved:

$$\begin{aligned}&\mathrm{Hom}_{\mathfrak{g},K}(U, \Gamma(\mathrm{pro}((X\otimes F)\otimes\tilde{\tau}))) \simeq \mathrm{Hom}_{\mathfrak{l},L\cap K}(U, X\otimes F\otimes\tilde{\tau})\\ &\simeq \mathrm{Hom}_{\mathfrak{l},L\cap K}(U, X\otimes (F^*)^*\otimes\tilde{\tau}) \simeq \mathrm{Hom}_{\mathfrak{l},L\cap K}(U, \mathrm{Hom}_{\mathbb{C}}(F^*, X\otimes\tilde{\tau})\\ &\simeq \mathrm{Hom}_{\mathfrak{l},L\cap K}(U\otimes F^*, X\otimes\tilde{\tau})) \simeq \mathrm{Hom}_{\mathfrak{g},K}(U\otimes F^*, \Gamma(\mathrm{pro}(X\otimes\tilde{\tau})))\\ &\simeq \mathrm{Hom}_{\mathfrak{g},K}(U, \Gamma(\mathrm{pro}(X\otimes\tilde{\tau}))\otimes F)\end{aligned}$$

We deduce from this that $\Gamma(\mathrm{pro}(X\otimes\tilde{\tau}\otimes F)) \simeq \Gamma(\mathrm{pro}(X\otimes\tilde{\tau}))\otimes F$.

The same is true for Γ^i replacing Γ in the computation above. This can be seen using general arguments and the exactness of the functor $\bullet\otimes F$. Thus, for all $i\geq 0$, $\Gamma^i(\mathrm{pro}(X\otimes\otimes\tilde{\tau}\otimes F)) \simeq \Gamma^i(\mathrm{pro}(X\otimes\tilde{\tau}))\otimes F$, which implies (13.2).

□

Let us now denote $\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}$ and $\widehat{\mathcal{R}}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}$ the Euler-Poincaré morphisms of cohomological induction between $\mathbf{GL}(1,\mathbb{C})$ and respectively $\mathbf{GL}(2,\mathbb{R})$ and $\mathbf{GL}(1,\mathbb{H})$, where $\mathfrak{q}_{\mathbb{C}}$ and $\mathfrak{q}'_{\mathbb{C}}$ are as 11.2 and 11.3.

Lemma 13.5. *With the above notation, and $x, y\in\mathbb{C}$, $x-y\in\mathbb{Z}$,*

$$\mathbf{LJ}(\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}(\gamma(x,y))) = -\widehat{\mathcal{R}}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}(\gamma(x,y))$$

Proof. When $x-y\geq 0$, we have

$$\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}(\gamma(x,y)) = -\mathcal{R}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}(\gamma(x,y)) = -\eta(x,y)$$

and

$$\widehat{\mathcal{R}}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}(\gamma(x,y)) = -\mathcal{R}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}(\gamma(x,y)) = -\eta'(x,y).$$

The formula is thus true in this case. The case $x-y<0$ follows because $\mathbf{LJ}(\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_{\mathbb{C}}}(\gamma(x-n, y+n)))$ and $\widehat{\mathcal{R}}^{\mathbb{H}}_{\mathfrak{q}'_{\mathbb{C}}}(\gamma(x-n, y+n))$ are two coherent families which coincide for $n\geq 0$, and are therefore equal.

Theorem 13.6. *Let $\mathcal{R}_{\mathfrak{q}_\mathbb{C}}^{\mathbb{R}}$ and $\mathcal{R}_{\mathfrak{q}'_\mathbb{C}}^{\mathbb{H}}$ be the cohomological induction functors from $\mathbf{GL}(n,\mathbb{C})$ to respectively $\mathbf{GL}(2n,\mathbb{R})$ and $\mathbf{GL}(n,\mathbb{H})$. We have then*

$$\mathbf{LJ}\circ\widehat{\mathcal{R}}_{\mathfrak{q}_\mathbb{C}}^{\mathbb{R}} = (-1)^n\widehat{\mathcal{R}}_{\mathfrak{q}'_\mathbb{C}}^{\mathbb{H}}.$$

Proof. It is enough to show that the formula holds on the basis $\lambda(a)$, $a\in M(D)$ of $\mathcal{R}^{\mathbb{C}}$). Let $a\in M(D)$, $a=(\gamma(x_1,y_1),\dots,\gamma(x_r,y_r))$. We compute

$$\begin{aligned}\mathbf{LJ}\circ\widehat{\mathcal{R}}_{\mathfrak{q}_\mathbb{C}}^{\mathbb{R}}(\lambda(a)) &= \mathbf{LJ}\circ\widehat{\mathcal{R}}_{\mathfrak{q}_\mathbb{C}}^{\mathbb{R}}(\gamma(x_1,y_1)\times\ldots\times\gamma(x_r,y_r))\\ &= \mathbf{LJ}(i_{\mathbf{GL}(2,\mathbb{R})^r}^{\mathbf{GL}(2r,\mathbb{R})}\circ\widehat{\mathcal{R}}_{\mathfrak{q}_\mathbb{C}}^{\mathbb{R}}(\gamma(x_1,y_1)\otimes\ldots\otimes\gamma(x_r,y_r))\\ &= i_{\mathbf{GL}(1,\mathbb{H})^r}^{\mathbf{GL}(r,\mathbb{H})}\circ\mathbf{LJ}(\widehat{\mathcal{R}}_{\mathfrak{q}_\mathbb{C}}^{\mathbb{R}}(\gamma(x_1,y_1)\otimes\ldots\otimes\gamma(x_r,y_r))\\ &= (-1)^r i_{\mathbf{GL}(1,\mathbb{H})^r}^{\mathbf{GL}(r,\mathbb{H})}\circ\widehat{\mathcal{R}}_{\mathfrak{q}'_\mathbb{C}}^{\mathbb{H}}(\gamma(x_1,y_1)\otimes\ldots\otimes\gamma(x_r,y_r))\\ &= (-1)^r\widehat{\mathcal{R}}_{\mathfrak{q}'_\mathbb{C}}^{\mathbb{H}}(\gamma(x_1,y_1)\times\ldots\times\gamma(x_r,y_r))\\ &= (-1)^r\widehat{\mathcal{R}}_{\mathfrak{q}'_\mathbb{C}}^{\mathbb{H}}(\lambda(a)).\end{aligned}$$

We have used independence of polarization theorem of [27], Chapter 11, to replace a part of cohomological induction by parabolic induction, and the fact that $\mathbf{LJ}$ commutes with parabolic induction. □

Corollary 13.7. *Recall the representations $\bar{u}(\eta',n)$ introduced in 11.3. We have*

$$\mathbf{LJ}(u(\eta(x,y),n) = (-1)^n\,\bar{u}(\eta'(x,y),n),$$

$x,y\in\mathbb{C}$, $x-y\in\mathbb{N}$.

Recall that when $x-y\neq 1$, then $\bar{u}(\eta'(x,y),n)=u(\eta'(x,y),n)$ (see 11.3).
Proof. This follows from the theorem and the formulas $\mathcal{R}_{\mathfrak{q}_\mathbb{C}}^{\mathbb{R}}(u(\gamma(x,y))=u(\eta(x,y),n)$, $\mathcal{R}_{\mathfrak{q}'_\mathbb{C}}^{\mathbb{H}}(u(\gamma(x,y))=\bar{u}(\eta'(x,y),n)$ obtained in 11.2 and 11.3. □

To be able to compute the transfer to $\mathbf{GL}(n,\mathbb{H})$ of any irreducible unitary representation of $\mathbf{GL}(2n,\mathbb{R})$, we need to compute the transfer of the $u(\delta,k)$ when $\delta\in D_1^{\mathbb{R}}$. But, in this case, if $\delta=\delta(\alpha,\epsilon)$,

$$u(\delta(\alpha,\epsilon),2k)=\delta(\alpha,\epsilon)\circ\det,$$

and we know from [16] that the transfer of this character is the character

$$\delta(\alpha,\epsilon)\circ RN$$

(RN is the reduced norm) which is

$$u(\eta'(\alpha+\frac{1}{2},\alpha-\frac{1}{2}),k).$$

From this, we get

Theorem 13.8. *Let u be an irreducible unitary representation of $\mathbf{GL}(2n,\mathbb{R})$. Then $\mathbf{LJ}(u)$ is either 0, or up to a sign, an irreducible unitary representation of $\mathbf{GL}(n,\mathbb{H})$. For representations $u(\delta,k)$, we get:*
— if $\delta=\delta(\alpha,\epsilon)\in D_1^{\mathbb{R}}$,

$$\mathbf{LJ}(u(\delta(\alpha,\epsilon),2k))=u(\eta'(\alpha+\frac{1}{2},\alpha-\frac{1}{2}),k),$$

— *if* $\delta = \eta(x, y) \in D_2^{\mathbb{R}}$,

$$\mathbf{LJ}(u(\eta(x, y)), k) = (-1)^k \bar{u}(\eta'(x, y), k).$$

To make it simple, a character is sent by $\mathbf{LJ}$ on the corresponding character, while if $\delta \in D_2^F$ and $\delta' = \mathbf{C}(\delta) = -\mathbf{LJ}(\delta)$, then $\mathbf{LJ}(u(\delta, k)) = (-1)^k \bar{u}(\delta', k)$.

In the first case note that we deal with a slightly different situation from non archimedean fields, since the reduced norm of $\mathbb{H}$ is *not* surjective, but has image in $\mathbb{R}_+^*$. In particular, if s is the character sign of the determinant on $GL_{2k}(\mathbb{R})$, then $\mathbf{LJ}(s)$ is the trivial character of $GL_k(\mathbb{H})$. In the non archimedean case, it is easy to check that $\mathbf{LJ}$ is injective on the set of representations $u(\delta, k)$.

The above theorem gives a correspondence between irreducible unitary representations of $\mathbf{GL}(2n, \mathbb{R})$ and of $\mathbf{GL}(n, \mathbb{H})$, by forgetting the signs. As in the introduction, we denote this correspondence by $|\mathbf{LJ}|$. Using (11.2) and (11.3), we easily reformulate the result as in the introduction.

14. Character formulas and ends of complementary series

From Tadić's classification of the unitary dual, and the character formula for induced representations, the character of any irreducible unitary representation of $\mathbf{GL}(n, A)$ can be computed from the characters of the $u(\delta, n)$, $\delta \in D$, $n \in \mathbb{N}$. It is remarkable that the characters of the $u(\delta, n)$ can be computed, or more precisely, expressed in terms of characters of square integrable modulo center representations. We give also composition series of ends of complementary series. This information is important for the topology of the unitary dual (see [43]).

14.1. $A = \mathbb{C}$**.** Let $\gamma = \gamma(x, y)$ be a character of $\mathbb{C}^\times$, $x, y \in \mathbb{C}$, $x - y = r \in \mathbb{Z}$. The representation $u(\gamma(x, y), n)$ is the character

$$\det \circ \gamma$$

of $\mathbf{GL}(n, \mathbb{C})$. There is a formula, due to Zuckerman, for the trivial character of any real reductive group, obtained from a finite length resolution of the trivial representation by standard modules in the category $\mathcal{M}(G)$.

For $\mathbf{GL}(n, \mathbb{C})$, this formula is, denoting $\mathbf{1}_{\mathbf{GL}(n,\mathbb{C})}$ the trivial representation

$$\mathbf{1}_{\mathbf{GL}(n,\mathbb{C})} = u(\gamma(0,0), n) = \sum_{w \in \mathfrak{S}_n} (-1)^{l(w)} \prod_{i=1}^n \gamma(\frac{n-1}{2} - i + 1, \frac{n-1}{2} - w(i) + 1)$$

From this, we get by tensoring with $\gamma(x, y)$,

(14.1)

$$u(\gamma(x, y), n) = \sum_{w \in \mathfrak{S}_n} (-1)^{l(w)} \prod_{i=1}^n \gamma(x + \frac{n-1}{2} - i + 1, y + \frac{n-1}{2} - w(i) + 1)$$

Set $\gamma_{i,j} = \gamma(x + \frac{n-1}{2} - i + 1, y + \frac{n-1}{2} - j + 1) \in \mathcal{R}$. The formula above becomes :

$$u(\gamma(x, y), n) = \det((\gamma_{i,j})_{1 \leq i,j \leq n}) \tag{14.2}$$

From the Lewis Carroll identity ([14]), we deduce easily from this a formula for composition series of ends of complementary series. This was obtained previously by Tadić [45], using partial results of Sahi [38], but the proof was complicated. For an easy formula, set

$$\gamma(x, y) = \delta(\beta, r)$$

with, $r = x - y$, $2\beta = x + y$.

Proposition 14.1. *With the above notation, and $n \geq 2$*

$$\begin{aligned}(14.3) \qquad \nu^{-\frac{1}{2}}u(\delta(\beta,r),n) \times \nu^{\frac{1}{2}}u(\delta(\beta,r),n) \\ = u(\delta(\beta,r),n+1) \times u(\delta(\beta,r),n-1) \\ + u(\delta(\beta,r+1),n) \times u(\delta(\beta,r-1),n)\end{aligned}$$

14.2. $A = \mathbb{R}$. Let $\eta(x,y)$ be an essentially square integrable modulo center representation of $\mathbf{GL}(2,\mathbb{R})$, $x, y \in \mathbb{C}$, $x - y = r \in \mathbb{N}^\times$. Since

$$u(\eta(x,y),n) = -\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_\mathbb{C}}(u(\gamma(x,y))),$$

we get from (14.1) that

$$u(\eta(x,y),n) = -\sum_{w\in\mathfrak{S}_n}(-1)^{l(w)}\prod_{i=1}^n \widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_\mathbb{C}}(\gamma(x+\frac{n-1}{2}-i+1, y+\frac{n-1}{2}-w(i)+1)).$$

We have noticed in the proof of Lemma 13.5 that $-\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_\mathbb{C}}(\gamma(x-n,y+n))$ is a coherent family of representation of $\mathbf{GL}(2,\mathbb{R})$ such that $-\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_\mathbb{C}}(\gamma(x-n,y+n)) = \eta(x-n,y+n)$ when $x - n > y + n$. Set $\tilde{\eta}(x-n,y+n) = -\widehat{\mathcal{R}}^{\mathbb{R}}_{\mathfrak{q}_\mathbb{C}}(\gamma(x-n,y+n))$. Then we get

$$u(\eta(x,y),n) = (-1)^{n+1}\sum_{w\in\mathfrak{S}_n}(-1)^{l(w)}\prod_{i=1}^n \tilde{\eta}(x+\frac{n-1}{2}-i+1, y+\frac{n-1}{2}-w(i)+1).$$

Set $\tilde{\eta}_{i,j} = \tilde{\eta}(x + \frac{n-1}{2} - i + 1, y + \frac{n-1}{2} - j + 1)$. The formula above becomes :

$$(14.4) \qquad u(\eta(x,y),n) = (-1)^{n+1}\det((\tilde{\eta}_{i,j})_{1\leq i,j\leq n})$$

Again from the Lewis Carroll identity ([14]), we deduce easily from this a formula for composition series of ends of complementary series

Proposition 14.2. *With the above notation, $n \geq 2$, $x - y > 1$,*

$$\begin{aligned}(14.5) \qquad \nu^{-\frac{1}{2}}u(\eta(x,y),n) \times \nu^{\frac{1}{2}}u(\eta(x,y),n) \\ = u(\eta(x,y),n+1) \times u(\eta(x,y),n-1) \\ + u(\eta(x+\frac{1}{2},y-\frac{1}{2}),n) \times u(\eta(x-\frac{1}{2},y+\frac{1}{2}),n).\end{aligned}$$

If $x = y + 1$, recall the convention that

$$\eta(x-\frac{1}{2},x-\frac{1}{2}) = \delta(x-\frac{1}{2},0) \times \delta(x-\frac{1}{2},1).$$

We get

(14.6)

$$\begin{aligned}\nu^{-\frac{1}{2}}u(\eta(x,x-1),n) \times \nu^{\frac{1}{2}}u(\eta(x,x-1),n) \\ = u(\delta(x,x-1),n+1) \times u(\eta(x,x-1),n-1) \\ + u(\eta(x+\frac{1}{2},x-\frac{3}{2}),n) \times [u(\delta(x-\frac{1}{2},0),n) \times u(\delta(x-\frac{1}{2},1),n)]\end{aligned}$$

Remark 14.3. We cannot deduce by our method the composition series of the ends of complementary series for $u(\delta, n)$ when $\delta \in D_1$. There is still a formula for the character of $u(\delta, n)$, since $u(\delta, n) = \delta \circ \det$ is a one-dimensional representation (Zuckerman), but no interpretation for the right-hand-side of this formula as a determinant, so we cannot apply the Lewis Carroll identity.

14.3. $A = \mathbb{H}$. The discussion is similar to the real case for the $u(\eta'(x, y), n)$ when $x - y \geq 2$.

Proposition 14.4. *With the above notation, $n \geq 2$, $x - y \geq 2$,*

$$\begin{aligned}(14.7)\qquad & \nu^{-\frac{1}{2}} u(\eta'(x, y), n) \times \nu^{\frac{1}{2}} u(\eta'(x, y), n) \\ &= u(\eta'(x, y), n + 1) \times u(\eta'(x, y), n - 1) \\ &\quad + u(\eta'(x + \frac{1}{2}, y - \frac{1}{2}), n) \times u(\eta'(x - \frac{1}{2}, y + \frac{1}{2}), n).\end{aligned}$$

If $y = x - 1$, we get the same kind of character formulas, but for the $\bar{u}(\eta'(x, y), n)$:

$$(14.8)\qquad \bar{u}(\eta'(x, x - 1), n) = (-1)^{n+1} \det((\tilde{\eta}'_{i,j})_{1 \leq i,j \leq n}),$$

where $\tilde{\eta}'_{i,j} = \tilde{\eta}'(x + \frac{n-1}{2} - i + 1, y + \frac{n-1}{2} - j + 1)$, and $\tilde{\eta}'$ denotes the coherent family coinciding with η when $x - y$ is positive, as in the real case.

Again from the Lewis Carroll identity, we deduce the following (with $2n$ in place of n):

$$\begin{aligned}(14.9)\qquad & \nu^{-\frac{1}{2}} \bar{u}(\eta'(x, x - 1), 2n) \times \nu^{\frac{1}{2}} \bar{u}(\eta'(x, x - 1), 2n) \\ &= \bar{u}(\eta'(x, x - 1), 2n + 1) \times \bar{u}(\eta(x, x - 1), 2n - 1) \\ &\quad + \bar{u}(\eta'(x + \frac{1}{2}, x - \frac{1}{2}), 2n) \times \bar{u}(\eta(x - \frac{1}{2}, x - \frac{3}{2}), 2n).\end{aligned}$$

The representations $\bar{u}(\eta'(.,.), .)$ in this expression can be expressed as products of $u(\eta'(.,.), .)$ explicitly in the following way:

$$\begin{aligned}\bar{u}(\eta'(x, x - 1), 2n) &= u(\eta'(x + \frac{1}{2}, x - \frac{1}{2}), n) \times u(\eta'(x - \frac{1}{2}, x - \frac{3}{2}), n), \\ \bar{u}(\eta'(x, x - 1), 2n + 1) &= u(\eta'(x, x - 1), n + 1) \times u(\eta'(x, x - 1), n).\end{aligned}$$

Substituting this in (14.9), and using the fact that the ring $\mathcal{R}$ is a domain, we find that :

Proposition 14.5.

$$\begin{aligned}(14.10)\qquad & \nu^{-1} u(\eta'(x, x - 1), n) \times \nu u(\eta'(x, x - 1), n) \\ &= u(\eta'(x, x - 1), n + 1) \times u(\eta'(x, x - 1), n - 1) \\ &\quad + u(\eta'(x + \frac{1}{2}, x - \frac{1}{2}), n) \times u(\eta'(x - \frac{1}{2}, x - \frac{3}{2}), n).\end{aligned}$$

15. Compatibility and further comments

Let F be a local field (archimedean or non archimedean of any characteristic) and A a central division algebra of dimension d^2 over F (if F is archimedean, then $d \in \{1,2\}$). If $g \in G^F_{nd}$ is a regular semisimple element, we say that g *transfers* if there exists an element g' of G^A_n which corresponds to g (see Section 4). Then g transfers if and only if its characteristic polynomial breaks into a product of irreducible polynomials of degrees divisible by d. We say that $\pi \in \mathcal{R}(G^F_{nd})$ is *d-compatible* if $\mathbf{LJ}(\pi) \neq 0$. Otherwise stated, π is d-compatible if and only if its character does not identically vanish on the set of elements of G^F_{nd} which transfer. This justify the dependence of the definition only on d (and not on D). We then have the following results:

Proposition 15.1. *Let $\pi_i \in \mathbf{Irr}^F_{n_i}$, $1 \leq i \leq k$, with $\sum_i n_i = n$. Then $\pi_1 \times \pi_2 \times ... \times \pi_k$ is d-compatible if and only if for all $1 \leq i \leq k$, d divides n_i and π_i is d-compatible.*

<u>*Proof*</u>. If an element $g \in G^F_n$ is conjugated with an element of a Levi subgroup of G^F_n, say $(g_1, g_2, ..., g_k) \in G_{(n_1,n_2,...,n_k)}$ with $g_i \in G^F_{n_i}$, then the characteristic polynomial of g is the product of the characteristic polynomials of g_i. It follows that, if g is semisimple regular, it transfers if and only if $d|n_i$ for all i and g_i transfers.

It is a general fact that for a fully induced representation of a group G from a Levi subgroup M, the character is zero on regular semisimple elements which are not conjugated in G to some element in M. Moreover, one has a precise formula of the character of the fully induced representation in terms of the character of the inducing representation (see [20] and [15], Proposition 3 for non-archimedean F, [26] Section 13, for archimedean F). The proposition follows. □

We define now an order $<<$ finer than the Bruhat order on $<$ on $\mathbf{Irr}^A_n$. If $\pi = \mathbf{Lg}(\delta_1, \delta_2, ..., \delta_k)$ and $\pi' = \mathbf{Lg}(\delta'_1, \delta'_2, ..., \delta'_{k'})$ are in $\mathbf{Irr}^A_n$, we set $\pi << \pi'$ if

$$\mathbf{Lg}(\mathbf{C}^{-1}(\delta_1), \mathbf{C}^{-1}(\delta_2), ..., \mathbf{C}^{-1}(\delta_k)) < \mathbf{Lg}(\mathbf{C}^{-1}(\delta'_1), \mathbf{C}^{-1}(\delta'_2), ..., \mathbf{C}^{-1}(\delta'_{k'}))$$

in $\mathbf{Irr}^F_{nd}$.

Proposition 15.2. *Let $\delta_i \in D^F_{n_i}$, $1 \leq i \leq k$. Assume for all $1 \leq i \leq k$ we have $d|n_i$, and set $\delta'_i = \mathbf{C}(\delta_i) \in D^A_{n_i}$. Then $\mathbf{Lg}(\delta_1, \delta_2, ..., \delta_k)$ is compatible and one has:*

$$\mathbf{LJ}(\mathbf{Lg}(\delta_1, \delta_2, ..., \delta_k)) = (-1)^{nd-n}\mathbf{Lg}(\delta'_1, \delta'_2, ..., \delta'_k) + \sum_{j \in J} m_j \pi'_j$$

where J is empty or finite, $m_j \in \mathbb{Z}^*$, $\pi'_j \in \mathbf{Irr}^A_{\sum n_i}$ and $\pi'_j << \mathbf{Lg}(\delta'_1, \delta'_2, ..., \delta'_k)$ for all $j \in J$.

<u>*Proof*</u>. One applies Theorem 6.1 and an induction on the number of representations smaller than $\mathbf{Lg}(\delta_1, \delta_2, ..., \delta_k)$. See [6], Proposition 3.10. □

Proposition 15.3. *If $\delta \in D^F_n$, set $\deg(\delta) = n$ and let $l(\delta)$ be the length of $\mathbf{Supp}(\delta)$ (notice that $l(\delta)|\deg(\delta)$). Then*

a) $u(\delta, k)$ is d-compatible if and only if either $d|\deg(\delta)$ or $d|k\frac{\deg(\delta)}{l(\delta)}$.

b) there exists $k_\delta \in \mathbb{N}^*$ such that $u(\delta, k)$ is d-compatible if and only if $k_\delta|k$. Moreover, $k_\delta|d$.

<u>*Proof*</u>. a) is in Section 3.5 of [7] for the non-archimedean case. It follows from Theorem 13.8 in the archimedean case.

b) follows easily from a). For the archimedean (non trivial *i.e.* $A = \mathbb{H}$) case, $d = 2$ and the transfer theorem 13.8 shows that

— if $\deg(\delta) = 2$, then $u(\delta, k)$ is 2-compatible for all k (hence $k_\delta = 1$) and

— if $\deg(\delta) = 1$ then $u(\delta, k)$ is 2-compatible if (and *only* if, because of the dimension of G_k^F) k is even (hence $k_\delta = 2$). □

Let γ be an irreducible generic unitary representation of G_n^F. As γ is generic, it is fully induced from an essentially square integrable representation ([53] for non archimedean fields, section 8 for archimedean fields). Then as γ is unitary, thanks to the classification of the unitary spectrum ([42], [52] and Section 8 the present paper), γ is an irreducible product $\sigma_1 \times \sigma_2 \times ... \times \sigma_p \times \pi_1 \times \pi_2 \times ... \times \pi_l$, where, for $1 \leq i \leq p$, $\sigma_i \in D^{u,F}$, and, for $1 \leq j \leq l$, $\pi_j = \pi(\delta_j, 1; \alpha_j)$ for some $\delta_j \in D^{u,F}$ and some $\alpha_i \in]0, \frac{1}{2}[$.

Using the Langlands classification, it is easy to see that the representation

$$\nu^{\frac{k-1}{2}}\gamma \times \nu^{\frac{k-1}{2}-1}\gamma \times ... \times \nu^{-\frac{k-1}{2}}\gamma$$

has a unique quotient $u(\gamma, k)$, and one has

$$u(\gamma, k) = u(\sigma_1, k)\times u(\sigma_2, k)\times ...\times u(\sigma_p, k)\times \pi(\delta_1, k; \alpha_1)\times \pi(\delta_2, k; \alpha_2)\times ...\times \pi(\delta_l, k; \alpha_l)$$

(see for instance [6] Section 4.1). The local components of cuspidal automorphic representations of $\mathbf{GL}_n$ over adeles of global fields are unitary generic representations ([40]). According to the classification of the residual spectrum ([32]), it follows that local component of residual automorphic representations of the linear group are of type $u(\gamma, k)$.

Proposition 15.4. *Let γ be a unitary generic representation of G_n^F for some $n \in \mathbb{N}^\times$. There exists k_γ such that $u(\gamma, k)$ is d-compatible if and only if $k_\gamma | k$. Moreover, $k_\gamma | d$.*

Proof. The (easy) proof given in [7] Section 3.5 for non-archimedean fields works also for archimedean fields. If

$$u(\gamma, k) = u(\sigma_1, k)\times u(\sigma_2, k)\times ...\times u(\sigma_p, k)\times \pi(\delta_1, k; \alpha_1)\times \pi(\delta_2, k; \alpha_2)\times ...\times \pi(\delta_l, k; \alpha_l),$$

then $u(\gamma, k)$ is d-compatible if and only if all the $u(\sigma_i, k)$ and $u(\delta_j, k)$ are compatible (Proposition 15.1). Then Prop. 15.3 implies Prop. 15.4. If $F = \mathbb{R}$, $k_\gamma = 1$ if and only if all the σ_i and δ_j are in D_2. If not, $k_\gamma = 2$. □

16. Notation for the global case

Let F be a global field *of characteristic zero* and D a central division algebra over F of dimension d^2. Let $n \in \mathbb{N}^*$. Set $A = M_n(D)$. For each place v of F let F_v be the completion of F at v and set $A_v = A \otimes F_v$. For every place v of F, A_v is isomorphic to $M_{r_v}(D_v)$ for some positive integer r_v and some central division algebra D_v of dimension d_v^2 over F_v such that $r_v d_v = nd$. We fix once for all an isomorphism $A_v \simeq M_{r_v}(D_v)$ and identify these two algebras. We say that $M_n(D)$ *is split* at a place v if $d_v = 1$. The set V of places where $M_n(D)$ is not split is finite. For each v, d_v divides d, and moreover d is the smallest common multiple of the d_v over all the places v.

Let $G'(F)$ be the group $A^\times = \mathbf{GL}_n(D)$. For every finite place v of F, set $G'_v = A_v^\times = \mathbf{GL}_{r_v}(D_v)$. For every finite place v of F, we set $K_v = \mathbf{GL}_{r_v}(O_v)$, where O_v is the ring of integers of D_v. Let $\mathbb{A}$ be the ring of adèles of F. We define the group $G'(\mathbb{A})$ of adèles of $G'(F)$ as the restricted product of the G'_v over all v, with respect to the family of open compact subgroups K_v, v finite.

Let G'_∞ be the direct product of G'_v over the set of infinite places of F and and G'_f the restricted product of G'_v over finite places, with respect to the open compact subgroups K_v. The group $G'(\mathbb{A})$ decomposes into the direct product

$$G'(\mathbb{A}) = G'_\infty \times G'_f.$$

Fix maximal compact subgroups K_v at archimedean places v like before, $K_v = \mathbf{O}(n), \mathbf{U}(n), \mathbf{Sp}(n)$ according to G'_v being $\mathbf{GL}_n(\mathbb{R})$, $\mathbf{GL}_n(\mathbb{C})$ or $\mathbf{GL}_n(\mathbb{H})$. Let K_∞ (resp. K_f) be the compact subgroup of G_∞ (resp. of G'_f) which is the direct product of K_v over the infinite places (resp. finite places) v. Let K be $K_\infty \times K_f$ as a (compact) subgroup of $G'(\mathbb{A})$. Let $\mathfrak{g}_\infty$ be the Lie algebra of G_∞.

An *admissible* $G'(\mathbb{A})$*-module* is a linear space V which is both a $(\mathfrak{g}_\infty, K_\infty)$-module and a smooth G'_f-module such that the actions of $(\mathfrak{g}_\infty, K_\infty)$ and G'_f commute and for all irreducible equivalence class of continuous representations π of K the π isotypic component of V is of finite dimension. It is *irreducible* if it has no proper $G'(\mathbb{A})$-submodule, and *unitary* if admits a Hermitian product which is invariant under both actions of $(\mathfrak{g}_\infty, K_\infty)$ and G'_f.

If V is an irreducible admissible $G'(\mathbb{A})$-module, then V is isomorphic with a tensor product $V_\infty \otimes V_f$, where V_∞ is an irreducible $(\mathfrak{g}_\infty, K_\infty)$-module and V_f is an irreducible smooth representation of V_f.

If (π, H) is a unitary irreducible admissible G_f-module, then π breaks into a restricted tensor product $\otimes_{v \text{ finite}} \pi_v$ where π_v is a unitary irreducible representation of G'_v ([23], [31], [18] or [17]). For almost all v, π_v has a fixed vector under the maximal compact subgroup K_v. Such a representation is called *spherical*. The π_v are determined by π. Such a π_v is called the *local component* of π at the place v. The set of local components π_v determines π.

Let $Z(F)$ be the center of $G'(F)$ and, for every place v, let Z_v be the center of G'_v. Then we identify the center $Z(\mathbb{A})$ of $G'(\mathbb{A})$ with the restricted product of the Z_v, with respect to the open compact subgroups $Z_v \cap K_v$ at finite places. For any finite v, we fix a Haar measure dg_v on G'_v such that the volume of K_v is one, and a Haar measure dz_v on Z_v such that the volume of $Z_v \cap K_v$ is one. The set of measures $\{dg_v\}_{v \text{ finite}}$ induce a well defined Haar measure on the locally compact group G'_f and $\{dz_v\}_{v \text{ finite}}$ induce a well defined measure on its center (see for instance [35] where measures on restricted products are explained).

For the archimedean groups we chose Duflo-Vergne's normalization, defined as follows: let G be a reductive group (complex or real), and pick a G-invariant symmetric, non-degenerate bilinear form κ on the Lie algebra $\mathfrak{g}$. Then $\mathfrak{g}$ will be endowed with the Lebesgue measure dX such that the volume of a parallelotope supported by a basis $\{X_1, \ldots, X_n\}$ of $\mathfrak{g}$ is equal to $|\det(\kappa(X_i, X_j))|^{\frac{1}{2}}$ and G will be endowed with the Haar measure tangent to dX. If G' is a closed subgroup of G, such that κ is non-degenerate on its Lie algebra $\mathfrak{g}'$, we endow G' with the Haar measure determined by κ as above. This gives measures on G'_∞ and its center.

We fix now the measure dg on $G'(\mathbb{A}) = G'_\infty \times G'_f$ (resp. dz on $Z(\mathbb{A})$) which is the product of measures chosen before for the infinite and the finite part. We fix a measure on $Z(\mathbb{A})\backslash G'(\mathbb{A})$ which is the quotient measure $dz\backslash dg$.

We view $G'(F)$ as a subgroup of $G'(\mathbb{A})$ via the diagonal embedding. As $G'(F) \cap Z(\mathbb{A})\backslash G'(F)$ is a discrete subgroup of $Z(\mathbb{A})\backslash G'(\mathbb{A})$, $dz\backslash dg$ defines a measure on the quotient space $Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A})$. The measure of the space $Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A})$ is finite.

Fix a unitary continuous character ω of $Z(\mathbb{A})$, trivial on $Z(F)$.

Let $L^2(Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A});\omega)$ be the space of classes of functions f defined on $G'(\mathbb{A})$ with values in $\mathbb{C}$ such that
i) f is left invariant under $G'(F)$,
ii) f satisfies $f(zg) = \omega(z)f(g)$ for all $z \in Z(\mathbb{A})$ and almost all $g \in G'(\mathbb{A})$,
iii) $|f|^2$ is integrable over $Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A})$.

Let R'_ω be the representation of $G'(\mathbb{A})$ in $L^2(Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A});\omega)$ by right translations. As explained in [13], each irreducible subspace of $L^2(Z(\mathbb{A})G'(F)\backslash G'(\mathbb{A});\omega)$ gives rise to a unique unitary irreducible admissible $G'(\mathbb{A})$-module. We call such a $G'(\mathbb{A})$-module a *discrete series of* $G'(\mathbb{A})$.

Every discrete series of $G'(\mathbb{A})$ with the central character ω appears in R'_ω with a finite multiplicity ([18]).

Let $R'_{\omega,disc}$ be the subrepresentation of R'_ω generated by the discrete series. If π is a discrete series we call the *multiplicity of* π *in the discrete spectrum* the multiplicity with which π appears in $R'_{\omega,disc}$.

Notation. Fix n and D as before. The same constructions work obviously starting with $A = \mathbf{GL}_{nd}(F)$ instead of $A = \mathbf{GL}_n(D)$. We denote $G(\mathbb{A})$ the group of invertible elements of A and modify all notations accordingly.

17. Second insight of some local results

We would like to point out that some of the archimedean results described in this paper may be proved by global methods and local tricks as in the non-archimedean case ([7] and [6]), avoiding any reference to cohomological induction. These are $U(1)$ for $\mathbf{GL}(n,\mathbb{H})$, the fact that products of representations in $\mathcal{U}_{\mathbb{H}}$ are irreducible and the Jacquet-Langlands transfer of unitary representations (using $U(0)$ for $\mathbf{GL}(n,\mathbb{R})$ - [11] - but not on $\mathbf{GL}(n,\mathbb{H})$) We sketch here these proofs.

17.1. $U(1)$ **and the transfer of** $u(\delta,k)$**.** Let $\mathbf{LJ} : \mathcal{R}_{2n}^{\mathbb{R}} \to \mathcal{R}_n^{\mathbb{H}}$ be the morphism between Grothendieck groups extending the classical Jacquet-Langlands correspondence for square integrable representations (Section 4). We give here a second proof of the

Proposition 17.1. (a) If $\chi \in D_1$, then $\mathbf{LJ}(u(\chi,2n)) = \chi'_n$.
(b) If $\delta \in D_2$ and $\delta' = \mathbf{C}(\delta)$, then $\mathbf{LJ}(u(\delta,n)) = (-1)^n\bar{u}(\delta',n)$.
(c) The statement $U(1)$, i.e. $u(\delta',n)$ are unitary, is true for $\mathbf{GL}(n,\mathbb{H})$.

The first assertion (a) is obvious since $u(\chi,2n) = \chi_{2n}$ and the equality of characters may be checked directly. To prove (c), recall we have

$$\mathbf{LJ}(u(\delta,n)) = (-1)^n(\bar{u}(\delta',n) + \sum_{i=1}^{k} a_i u_i), \tag{17.1}$$

where the u_i are irreducible non-equivalent representations of $\mathbf{GL}(n,\mathbb{H})$, non equivalent to $\bar{u}(\delta',n)$, and a_i are non-zero integers (Proposition 15.2).

We now claim that all the irreducible representations on the right hand side of the equality are unitary and the a_i are all positive. One may proceed like in [6]: choose a global field F and a division algebra D over F such that, if $G'(\mathbb{A})$ is the adele group of $D^\times$, we have $G'_v = GL_n(\mathbb{H})$ for some place v. As $\delta \in D_2$, there exists a cuspidal representation ρ of $G(\mathbb{A}) = GL_{2n}(\mathbb{A})$ such that $\rho_v = \delta$. According to the classification of the residual spectrum for $G(\mathbb{A})$ ([32]) there exists a residual representation π of $G(\mathbb{A})$ such that $\pi_v = u(\delta, n)$. Comparing then the trace formula from [3] (or the simple trace formula from [2]) of $G(\mathbb{A})$ and $G'(\mathbb{A})$, one gets using standard simplifications and multiplicity one on the $G(\mathbb{A})$ side a local formula $\mathbf{LJ}(u(\delta, n)) = \pm \sum_{j=1}^{k} b_j w_i$, where the b_j are multiplicities of representations - hence positive, and w_j are local component of global discrete series - hence unitary. By linear independence of characters on $\mathbf{GL}(n, \mathbb{H})$, this formula is the same as the formula (17.1) which implies in particular $\bar{u}(\delta', n)$ is unitary (see [6], Cor. 4.8(a)). This implies the assertion $U(1)$, since when δ' is not a character one has $\bar{u}(\delta', n) = u(\delta', n)$, while when δ' is a (unitary) character we know $u(\delta', k)$ is the unitary character $\delta' \circ RN$. So (c) is proved.

We now prove (b). We want to prove that on the right hand side of the equality (17.1) there is just one term, $\bar{u}(\delta', n)$. If π is an irreducible unitary representation of $\mathbf{GL}(n, \mathbb{R})$ we say π is *semirigid* if it is a product of representations $u(\delta, k)$. We already showed in the previous paragraph that all these representations $u(\delta, k)$ correspond by $\mathbf{LJ}$ to zero or a sum of unitary representations. As $\mathbf{LJ}$ commutes with products and a product of irreducible unitary representations is a sum of irreducible unitary representations, it follows that any sum of semirigid irreducible unitary representation of some $\mathbf{GL}(2n, \mathbb{R})$ correspond to zero or a sum of unitary representations of $\mathbf{GL}(n, \mathbb{H})$. The relation (17.1) shows now that for all $\alpha \in \mathbb{R}$, $\mathbf{LJ}(\pi(\delta, n; \alpha)) = \nu'^{\alpha}(\sum_{i=0}^{k} a_i u_i) \times \nu'^{-\alpha}(\sum_{i=0}^{k} a_i u_i)$ where $a_0 = 1$, $u_0 = \bar{u}(\delta', n)$. When $\alpha = \frac{1}{2}$ on the left hand side of the equality we obtain a sum of semirigid unitary representations (see Proposition 14.5 for precise formula), so on the right hand side we should have a sum of unitary representations. But this is impossible as soon as the sum $\sum_{i=1}^{k} a_i u_i$ contains a representation u_1, since then the mixed product $\nu'^{-\frac{1}{2}} u_0 \times \nu'^{\frac{1}{2}} u_1$ contains a non hermitian subquotient (the "bigger" one for the Bruhat order for example). This shows there is only one u_i, $i = 0$, and so $\mathbf{LJ}(u(\delta, n)) = (-1)^n \bar{u}(\delta', n)$. □

17.2. **Irreducibility and transfer of all unitary representations.** We know now that the representations in $\mathcal{U}_{\mathbb{H}}$ are all unitary. To show that their products remain irreducible, we may use the irreducibility trick in [7], Proposition 2.13 which reduces the problem to show that $u(\delta', k) \times u(\delta', k)$ is irreducible for all discrete series δ' of $\mathbf{GL}(1, \mathbb{H})$ and all $k \in \mathbb{N}^{\times}$. Let δ be a square integrable representation of $\mathbf{GL}(2, \mathbb{R})$ such that $\mathbf{LJ}(\delta) = \delta'$. It follows that we have the equality $\mathbf{LJ}(u(\delta, k) \times u(\delta, k)) = \bar{u}(\delta', k) \times \bar{u}(\delta', k)$. On the left hand side we have the irreducible representation $M = u(\delta, k) \times u(\delta, k)$. On the right hand side we have a sum of unitary representations, the product $M' = \bar{u}(\delta', k) \times \bar{u}(\delta', k)$ (we already know $\bar{u}(\delta', k)$ is unitary), which we want to show has actually a single term. Apply the same α trick like before : we know that $\pi(M, \alpha)$ corresponds to $\pi(M', \alpha)$. For $\alpha = \frac{1}{2}$ the first representation breaks into a sum of semirigid unitary representations, while the second is a sum containing non unitary representations unless M' contains a single term. Notice that the Langlands quotient theorem and $U(4)$ guarantee M' has a subquotient which appears with multiplicity one, so either M' is a sum containing two different terms, or is irreducible. So the square of $\bar{u}(\delta', k)$ is irreducible for all k. If δ' is not a character, then $u(\delta', k) = \bar{u}(\delta', k)$ so the square of $u(\delta', k)$ is irreducible. If δ' is a character then we saw $\bar{u}(\delta', 2k+1) = u(\delta', k) \times u(\delta', k+1)$ and the result follows again.

This implies now: *if u is an irreducible unitary representation of* $\mathbf{GL}(2n, \mathbb{R})$*, then* $\mathbf{LJ}(u)$ *is either zero, or plus or minus irreducible unitary representation of* $\mathbf{GL}(n, \mathbb{H})$.

The proofs here are based on the trace formula and do not involve cohomological induction. However, the really difficult result is $U(0)$ on $\mathbf{GL}(n, \mathbb{H})$, and it does.

18. Global results

18.1. **Global Jacquet-Langlands, multiplicity one and strong multiplicity one for inner forms.** For all $v \in V$, denote $\mathbf{LJ}_v$ (resp. $|\mathbf{LJ}|_v$) the correspondence $\mathbf{LJ}$ (resp. $|\mathbf{LJ}|$), as defined in Sections 4 and 13, applied to G_v and G'_v.

If π is a discrete series of $G(\mathbb{A})$, we say π is *D-compatible* if, for all $v \in V$, π_v is d_v-compatible. Then $\mathbf{LJ}(\pi_v) \neq 0$ and $|\mathbf{LJ}|_v(\pi_v)$ is an irreducible representation of G'_n.

Here are the Jacquet-Langlands correspondence and the multiplicity one theorems for $G'(\mathbb{A})$ (already known for $G(\mathbb{A})$: [40], [33]).

Theorem 18.1. (a) *There exists a unique map $\mathbf{G}$ from the set of discrete series of $G'(\mathbb{A})$ into the set of discrete series of $G(\mathbb{A})$ such that $\mathbf{G}(\pi') = \pi$ implies $|\mathbf{LJ}|_v(\pi_v) = \pi'_v$ for all places $v \in V$, and $\pi_v = \pi'_v$ for all places $v \notin V$. The map $\mathbf{G}$ is injective and onto the set of D-compatible discrete series of $G(\mathbb{A})$.*

(b) *The multiplicity of every discrete series of $G'(\mathbb{A})$ in the discrete spectrum is 1. If two discrete series of $G'(\mathbb{A})$ have isomorphic local component at almost every place, then they are equal.*

The proof is the same as the proof of Theorem 5.1 in [7] with the following minor changes: Lemma 5.2 [7] is obviously still true when the inner form is not split at infinite places using the Proposition 15.1 here. For the finiteness property quoted in [7], p. 417 as [BB], one has to replace this reference with [5], where the case of ramified at infinite places inner form is addressed. We do not need here the claim (d) in [7], Theorem 5.1 which is now a particular case of Tadić classification of unitary representation for inner forms. At the bottom of pages 417 and 419 in [7], the independence of characters on a product of connected p-adic groups is used. Here the product involves also real, sometimes non connected groups like $\mathbf{GL}(n, \mathbb{R})$. The linear independence of characters on each of these $\mathbf{GL}_n$ is enough to ensure the linear independence of characters on the product, as at infinite places representations are Harish-Chandra modules so for all these groups, real or p-adic, irreducible representations correspond to irreducible modules on a well chosen algebra with idempotents.

As in [7], the hard core of the proof is the powerful equality 17.8 from [3] (comparison of trace formulae of $G(\mathbb{A})$ and $G'(\mathbb{A})$).

Let us show now the classification of cuspidal representations of $G'(\mathbb{A})$ in terms of cuspidal representations of $G(\mathbb{A})$. Let ν (resp. ν') be the global character of $G(\mathbb{A})$ (resp. $G'(\mathbb{A})$) given by the product of local characters like before (i.e. absolute value of the reduced norm). Recall that, according to Moeglin-Waldspurger classification, every discrete series π of $G(\mathbb{A})$ is the unique irreducible quotient of an induced representation $\nu^{\frac{k-1}{2}}\rho \times \nu^{\frac{k-3}{2}}\rho \times ... \times \nu^{-\frac{k-1}{2}}\rho$ where ρ is cuspidal. Then k and ρ are determined by π, so π is cuspidal if and only if $k = 1$. We set $\pi = MW(\rho, k)$.

Proposition 18.2. (a) Let $n \in \mathbb{N}^\times$ and let ρ be a cuspidal representation of $G_n(\mathbb{A})$. Then there exists k_ρ such that, if $k \in \mathbb{N}^\times$, then $MW(\rho, k)$ is D-compatible if and only if $k_\rho | k$. Moreover, $k_\rho | d$.

(b) Let π' be a discrete series of $G'(\mathbb{A})$ and $\pi = \mathbf{G}(\pi')$. Then π' is cuspidal if and only if π is of the form $MW(\rho, k_\rho)$.

(c) Let ρ' be a cuspidal representation of some $G'_n(\mathbb{A})$. Write $\mathbf{G}(\rho') = MW(\rho, k_\rho)$ and then set $\nu_{\rho'} = \nu^{k_\rho}$. For every $k \in \mathbb{N}^\times$, the induced representation

$$\nu_{\rho'}^{\frac{k-1}{2}} \rho' \times \nu_{\rho'}^{\frac{k-3}{2}} \rho' \times ... \times \nu_{\rho'}^{-\frac{k-1}{2}} \rho'$$

has a unique irreducible quotient which we will denote $MW'(\rho', k)$. It is a discrete series and all discrete series are obtained from some cuspidal ρ' like that. If $\mathbf{G}(\rho') = MW(\rho, k_\rho)$ we have $\mathbf{G}(MW'(\rho', k)) = MW(\rho, kk_\rho)$.

Proof. (a) This follows from the Proposition 15.4 and the fact that for all $v \in V$ $d_v | d$.

(b) This is the proposition 5.5 in [7], with "cuspidal" in place of "basic cuspidal" thanks to Grbac's appendix. Both the proof of the claim and the proof in the appendix work the same way here.

(c) When $G'_n(\mathbb{A})$ is split at infinite places, this is the claim (a) of Proposition 5.7 in [7]. We follow the same idea which reduces the problem to local computation. As [7] makes use of Zelevinsky involution, we have to give here a proof in the archimedean case (where the involution doesn't exist). First, to show that the induced representation

$$\nu_{\rho'}^{\frac{k-1}{2}} \rho' \times \nu_{\rho'}^{\frac{k-3}{2}} \rho' \times ... \times \nu_{\rho'}^{-\frac{k-1}{2}} \rho'$$

has a constituent which is a discrete series, we will directly show that $\mathbf{G}^{-1}(MW(\rho, kk_\rho))$, which is a discrete series indeed, is a constituent of

$$\nu_{\rho'}^{\frac{k-1}{2}} \rho' \times \nu_{\rho'}^{\frac{k-3}{2}} \rho' \times ... \times \nu_{\rho'}^{-\frac{k-1}{2}} \rho'.$$

We will show it place by place, local component by local component. Fix a place v and let γ be the local component of ρ at the place v. It is an irreducible unitary generic representation, and we know that $u(\gamma, k_\rho)$ transfers. Set $\pi = \mathbf{LJ}(u(\gamma, k_\rho))$. What we want to prove is that $\mathbf{LJ}(u(\gamma, kk_\rho))$ is a subquotient of $\nu^{k_\rho \frac{k-1}{2}} \pi \times \nu^{k_\rho \frac{k-3}{2}} \pi \times ... \times \nu^{k_\rho (-\frac{k-1}{2})} \pi$. The unitary generic representation γ may be written as $\gamma = (\times_i \sigma_i) \times (\times_j \pi(\tau_j, 1, \alpha_j))$, with σ_i and τ_j square integrable representations and $\alpha_j \in]0, \frac{1}{2}[$. So it is enough to prove the result when γ is a square integrable representation. Let us suppose γ is square integrable. To prove that $\pi = \mathbf{LJ}(u(\gamma, k_\rho))$ implies $\mathbf{LJ}(u(\gamma, kk_\rho))$ is a quotient of $\nu^{k_\rho \frac{k-1}{2}} \pi \times \nu^{k_\rho \frac{k-3}{2}} \pi \times ... \times \nu^{k_\rho (-\frac{k-1}{2})} \pi$ we would like to show that the essentially square integrable support of the representation $\mathbf{LJ}(u(\gamma, kk_\rho))$ is the union of the square integrable support of the representations $\{\nu^{k_\rho (\frac{k-1}{2} - i)} \pi\}_{i \in \{0,1,...,k-1\}}$. Then, as the essentially square integrable support of $\times_{i=0}^{k-1} [\nu^{k_\rho (\frac{k-1}{2} - i)} \pi]$ is in standard order, $\mathbf{LJ}(u(\gamma, kk_\rho))$ will be the unique quotient of the product.

If γ lives on a group of a size such that it transfers to some $\mathbf{C}(\gamma)$, then $\pi = \bar{u}(\mathbf{C}(\gamma), k_\rho)$, $\mathbf{LJ}(u(\gamma, kk_\rho)) = \bar{u}(\mathbf{C}(\gamma), kk_\rho)$ ([7] Proposition 3.7 (a) and second case of transfer in Theorem 13.8 of this paper), and the result is straightforward. If not, then $u(\gamma, k_\rho)$ verifies the "twisted" case of transfer [7], Proposition 3.7 (b) for non archimedean field, first case of Theorem 13.8 in this paper for archimedean field. In the non archimedean case, one may compute more explicit formulas for the transfer ([7] formula (3.9)) and see that it works. In the archimedean case γ is a character of $\mathbf{GL}_1(\mathbb{R})$ and so $\pi = \gamma \circ RN_{\frac{k_\rho}{2}}$ and $\mathbf{LJ}(u(\gamma, kk_\rho)) = \gamma \circ RN_{\frac{kk_\rho}{2}}$. □

Let us recall the uniqueness of the cuspidal support for automorphic representations. According to a result of Langlands [30] particularized to our case, we know that any automorphic representation of $G'(\mathbb{A})$ is a constituent of an induced representation of the form $\nu'^{a_1}\rho_1 \times \nu'^{a_2}\rho_2 \times ... \times \nu'^{a_k}\rho_k$ where a_i are real numbers and ρ_i are cuspidal representations. In [24] the authors prove that, for $G(\mathbb{A})$, the couples (ρ_i, a_i) are unique which in particular solves the question of the existence of CAP representations. In [7], it is shown that the result is true (more or less by transfer) for the more general case $G'(\mathbb{A})$, if the inner form is split at infinite places. Using the previous results, the same proof now works with no condition on the infinite places.

19. L-functions ϵ-factors and transfer

The fundamental work of Jacquet, Langlands and Godement of L-functions and ϵ-factors of linear groups over division algebras easily implies the following theorem. What we call ϵ'-factors following [19] are sometimes called γ-factors in literature. The value of all functions depend on the choice of some additive non trivial character ψ of $\mathbb{R}$ which is not relevant for the results.

Theorem 19.1. (a) Let u be a 2-compatible irreducible unitary representation of $\mathbf{GL}_{2n}(\mathbb{R})$ and u' the irreducible unitary representation of $\mathbf{GL}_n(\mathbb{H})$ such that $\mathbf{LJ}(u) = \pm u'$. Then the ϵ' factors of u and u' are equal.

(b) Let $\delta \in D_2$ and set $\delta' = \mathbf{C}(\delta)$. Then for all $k \in \mathbb{N}^\times$ the L-functions of $u(\delta, k)$ and $\bar{u}(\delta', k)$ are equal and the ϵ-factors of $u(\delta, k)$ and $\bar{u}(\delta', k)$ are equal.

(c) If χ is a character of $\mathbf{GL}(2n, \mathbb{R})$ and $\chi' = \mathbf{LJ}(\chi)$, then the ϵ'-factors of χ and χ' are equal.

Proof. If we prove (b) and (c), then (a) follows by the corollary 8.9 from [19] and classifications of unitary representations in Tadić setting explained in the present paper.

(b) is proved in [23] for $k = 1$. As a particular case of [22] (5.4) page 80, the L-function (resp. ϵ-factor) of a Langlands quotient $u(\delta, k)$ is the product to the L-functions (resp. ϵ-factors) of representations $\nu^{i-\frac{k-1}{2}}\delta$, $0 \leq i \leq k-1$. The same proof given there for $\mathbf{GL}_{2n}(\mathbb{R})$ works for $\mathbf{GL}_n(\mathbb{H})$ as well, so the case $k = 1$ imply the general case.

(c) In case χ is the trivial character, this is the corollary 8.10 page 121 in [19]. The general case follows easily by torsion with χ (or by reproducing the same proof). □

References

[1] J. Adams and J.-S. Huang. Kazhdan-Patterson lifting for GL$(n, \mathbf{R})$. *Duke Math. J.*, 89(3):423–444, 1997.

[2] J. Arthur. The invariant trace formula. II. Global theory. *J. Amer. Math. Soc.*, 1(3):501–554, 1988.

[3] J. Arthur and L. Clozel. *Simple algebras, base change, and the advanced theory of the trace formula*, volume 120 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 1989.

[4] A. I. Badulescu. Correspondance de Jacquet-Langlands pour les corps locaux de caractéristique non nulle. *Ann. Sci. École Norm. Sup. (4)*, 35(5):695–747, 2002.

[5] A. I. Badulescu. Un théorème de finitude dans le spectre automorphe pour les formes intérieures de GL_n sur un corps global. *Bull. London Math. Soc.*, 37(5):651–657, 2005.

[6] A. I. Badulescu. Jacquet-Langlands et unitarisabilité. *J. Inst. Math. Jussieu*, 6(3):349–379, 2007.

[7] A. I. Badulescu. Global Jacquet-Langlands correspondence, multiplicity one and classification of automorphic representations. *Invent. Math.*, 172(2):383–438, 2008. With an appendix by Neven Grbac.

[8] A. I. Badulescu, G. Henniart, B. Lemaire, and S. V. Sur le dual unitaire de GL(r,D). *American Journal of Math., to be published*, 2009.

[9] A. I. Badulescu and D. A. Renard. Sur une conjecture de Tadić. *Glas. Mat. Ser. III*, 39(59)(1):49–54, 2004.

[10] D. Barbasch and A. Moy. A unitarity criterion for p-adic groups. *Invent. Math.*, 98(1):19–37, 1989.

[11] E. M. Baruch. A proof of Kirillov's conjecture. *Ann. of Math. (2)*, 158(1):207–252, 2003.

[12] J. N. Bernstein. P-invariant distributions on $\mathrm{GL}(N)$ and the classification of unitary representations of $\mathrm{GL}(N)$ (non-Archimedean case). In *Lie group representations, II (College Park, Md., 1982/1983)*, volume 1041 of *Lecture Notes in Math.*, pages 50–102. Springer, Berlin, 1984.

[13] A. Borel and H. Jacquet. Automorphic forms and automorphic representations. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 189–207. Amer. Math. Soc., Providence, R.I., 1979. With a supplement "On the notion of an automorphic representation" by R. P. Langlands.

[14] G. Chenevier and D. Renard. Characters of Speh representations and Lewis Carroll identity. *Represent. Theory*, 12:447–452, 2008.

[15] L. Clozel. Théorème d'Atiyah-Bott pour les variétés p-adiques et caractères des groupes réductifs. *Mém. Soc. Math. France (N.S.)*, (15):39–64, 1984. Harmonic analysis on Lie groups and symmetric spaces (Kleebach, 1983).

[16] P. Deligne, D. Kazhdan, and M.-F. Vignéras. Représentations des algèbres centrales simples p-adiques. In *Representations of reductive groups over a local field*, Travaux en Cours, pages 33–117. Hermann, Paris, 1984.

[17] D. Flath. Decomposition of representations into tensor products. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 179–183. Amer. Math. Soc., Providence, R.I., 1979.

[18] I. M. Gel′fand, M. I. Graev, and I. I. Pyatetskii-Shapiro. *Representation theory and automorphic functions*, volume 6 of *Generalized Functions*. Academic Press Inc., Boston, MA, 1990. Translated from the Russian by K. A. Hirsch, Reprint of the 1969 edition.

[19] R. Godement and H. Jacquet. *Zeta functions of simple algebras*. Lecture Notes in Mathematics, Vol. 260. Springer-Verlag, Berlin, 1972.

[20] Harish-Chandra. *Harmonic analysis on reductive p-adic groups*. Lecture Notes in Mathematics, Vol. 162. Springer-Verlag, Berlin, 1970. Notes by G. van Dijk.

[21] H. Jacquet. Generic representations. In *Non-commutative harmonic analysis (Actes Colloq., Marseille-Luminy, 1976)*, pages 91–101. Lecture Notes in Math., Vol. 587. Springer, Berlin, 1977.

[22] H. Jacquet. Principal L-functions of the linear group. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2*, Proc. Sympos. Pure Math., XXXIII, pages 63–86. Amer. Math. Soc., Providence, R.I., 1979.

[23] H. Jacquet and R. P. Langlands. *Automorphic forms on* GL(2). Lecture Notes in Mathematics, Vol. 114. Springer-Verlag, Berlin, 1970.

[24] H. Jacquet and J. A. Shalika. On Euler products and the classification of automorphic forms. II. *Amer. J. Math.*, 103(4):777–815, 1981.

[25] A. A. Kirillov. Infinite-dimensional representations of the complete matrix group. *Dokl. Akad. Nauk SSSR*, 144:37–39, 1962.

[26] A. W. Knapp. *Representation theory of semisimple groups*. Princeton Landmarks in Mathematics. Princeton University Press, Princeton, NJ, 2001. An overview based on examples, Reprint of the 1986 original.

[27] A. W. Knapp and D. A. Vogan, Jr. *Cohomological induction and unitary representations*, volume 45 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1995.

[28] A. W. Knapp and G. J. Zuckerman. Classification of irreducible tempered representations of semisimple groups. *Ann. of Math. (2)*, 116(2):389–455, 1982.

[29] B. Kostant. On Whittaker vectors and representation theory. *Invent. Math.*, 48(2):101–184, 1978.

[30] R. P. Langlands. On the notion of automorphic representation. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 203–208. Amer. Math. Soc., Providence, R.I., 1979.

[31] R. P. Langlands. *Base change for* GL(2), volume 96 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, N.J., 1980.

[32] C. Mœglin and J.-L. Waldspurger. Le spectre résiduel de GL(n). *Ann. Sci. École Norm. Sup. (4)*, 22(4):605–674, 1989.

[33] I. I. Piatetski-Shapiro. Multiplicity one theorems. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 209–212. Amer. Math. Soc., Providence, R.I., 1979.

[34] R. S. Pierce. *Associative algebras*, volume 88 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1982. Studies in the History of Modern Science, 9.

[35] D. Ramakrishnan and R. J. Valenza. *Fourier analysis on number fields*, volume 186 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1999.

[36] D. Renard. Representations of p-adic reductive groups.

[37] S. Sahi. Jordan algebras and degenerate principal series. *J. Reine Angew. Math.*, 462:1–18, 1995.

[38] S. Sahi. Jordan algebras and degenerate principal series. *J. Reine Angew. Math.*, 462:1–18, 1995.

[39] V. Sécherre. Proof of the tadic conjecture U0 on the unitary dual of GL(m,D). *J. Reine Angew. Math.*, 626:187–204, 2009.

[40] J. A. Shalika. The multiplicity one theorem for GL_n. *Ann. of Math. (2)*, 100:171–193, 1974.

[41] B. Speh. Unitary representations of Gl(n, $\mathbf{R}$) with nontrivial ($\mathfrak{g}$, K)-cohomology. *Invent. Math.*, 71(3):443–465, 1983.

[42] M. Tadić. Classification of unitary representations in irreducible representations of general linear group (non-Archimedean case). *Ann. Sci. École Norm. Sup. (4)*, 19(3):335–382, 1986.

[43] M. Tadić. Topology of unitary dual of non-Archimedean GL(n). *Duke Math. J.*, 55(2):385–422, 1987.

[44] M. Tadić. Induced representations of GL(n, A) for p-adic division algebras *A. J. Reine Angew. Math.*, 405:48–77, 1990.

[45] M. Tadić. On characters of irreducible unitary representations of general linear groups. *Abh. Math. Sem. Univ. Hamburg*, 65:341–363, 1995.

[46] M. Tadić. Representation theory of GL(n) over a p-adic division algebra and unitarity in the Jacquet-Langlands correspondence. *Pacific J. Math.*, 223(1):167–200, 2006.

[47] M. Tadić. $\mathrm{GL}(n,\mathbb{C})\hat{}$ and $\mathrm{GL}(n,\mathbb{R})\hat{}$. Contemporary Mathematics. Amer. Math. Soc., Providence, R.I., 2009.

[48] D. A. Vogan. Irreducible characters of semisimple Lie groups. III. Proof of Kazhdan-Lusztig conjecture in the integral case. *Invent. Math.*, 71(2):381–417, 1983.

[49] D. A. Vogan, Jr. Gel′fand-Kirillov dimension for Harish-Chandra modules. *Invent. Math.*, 48(1):75–98, 1978.

[50] D. A. Vogan, Jr. The algebraic structure of the representation of semisimple Lie groups. I. *Ann. of Math. (2)*, 109(1):1–60, 1979.

[51] D. A. Vogan, Jr. Irreducible characters of semisimple Lie groups. IV. Character-multiplicity duality. *Duke Math. J.*, 49(4):943–1073, 1982.

[52] D. A. Vogan, Jr. The unitary dual of $\mathrm{GL}(n)$ over an Archimedean field. *Invent. Math.*, 83(3):449–505, 1986.

[53] A. V. Zelevinsky. Induced representations of reductive $\mathfrak{p}$-adic groups. II. On irreducible representations of $\mathrm{GL}(n)$. *Ann. Sci. École Norm. Sup. (4)*, 13(2):165–210, 1980.

Centre de Mathmatiques Laurent Schwartz, Ecole Polytechnique