About the unitary dual of $GL(r, D)$

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Introduction.

F : p-adic field (finite extension of $\mathbb{Q}_p$).

D : central division algebra over F, $\dim_F D = d^2$.

$G_n = GL(n, F), \quad G'_r = GL(r, D).$

INTERESTING PROBLEMS :

- Classification of the unitary dual of G'

- Properties of the Jacquet-Langlands correspondence with respect to unitarity.

ref : -Bernstein-Zelevinski, Tadic, Deligne-Kazhdan-Vigneras, Badulescu.

Notation for $G_n = GL(n, F)$.

$\mathcal{M}(G_n)$: category of smooth rep. of G_n.

R_n : Grothendieck group of $\mathcal{M}(G_n)$.

$\mathbf{Irr_n}$: equiv. classes of irr. rep. in $\mathcal{M}(G_n)$.

$\mathbf{Irr_n^u}$: unitary rep. in $\mathbf{Irr_n}$:

$$\mathcal{C}_n \subset \mathcal{D}_n \subset \mathbf{Irr_n}$$

respectively cuspidal and essentially square integrable representations.

$$\mathcal{C}_n^u \subset \mathcal{D}_n^u \subset \mathbf{Irr_n^u}$$

respectively cuspidal unitary and square integrable representations.

$$R = \bigoplus_{n \in \mathbb{N}} R_n$$

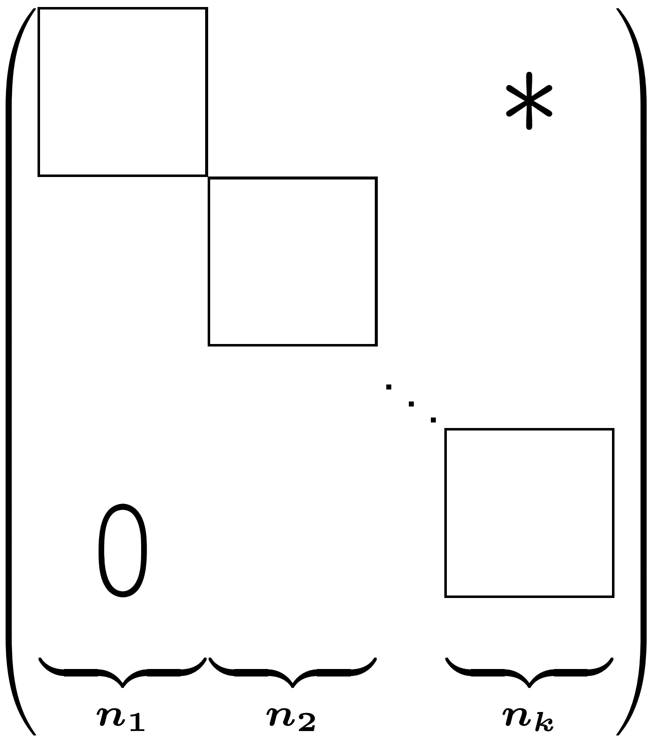
$$\mathcal{C} = \coprod_{n \in \mathbb{N}} \mathcal{C}_n, \quad \mathcal{D} = \coprod_{n \in \mathbb{N}} \mathcal{D}_n.$$

$$\mathcal{C}^u = \coprod_{n \in \mathbb{N}} \mathcal{C}_n^u, \quad \mathcal{D}^u = \coprod_{n \in \mathbb{N}} \mathcal{D}_n^u.$$

Standard parabolic and Levi subgroups of G_n :

$\beta = (n_1, n_2, \ldots n_k)$ partition of n.

$$P_\beta = L_\beta U_\beta =$$



$$L_\beta \simeq G_{n_1} \times G_{n_2} \times \ldots \times G_{n_k}$$

$$\lambda = (n_1, n_2), \quad n_1 + n_2 = n,$$

$$P = P_\lambda, \quad L = L_\lambda \simeq G_{n_1} \times G_{n_2}.$$

$$i_P^G : \mathcal{M}(G_{n_1}) \times \mathcal{M}(G_{n_2}) \to \mathcal{M}(G_n)$$

$$(\pi_1, \pi_2) \mapsto i_P^G(\pi_1 \otimes \pi_2)$$

extend to a bilinear associative and commutative product $R_{n_1} \times R_{n_2} \to R_n$ and to :

$$R \times R \to R \quad (\pi_1, \pi_2) \mapsto \pi_1 \times \pi_2.$$

Determinant character :

$$\nu : G \to \mathbb{R}_+^\times, \quad g \mapsto |\det(g)|_F$$

Notation : $\nu^\alpha \pi$ for $\nu^\alpha \otimes \pi$, $\alpha \in \mathbb{R}$.

$$\nu(\pi_1 \times \pi_2) = \nu\pi_1 \times \nu\pi_2$$

Classification of square integrable rep. of G.

$$m_1 \leq m_2 \in \mathbb{R}, \quad m_2 - m_1 \in \mathbb{Z}$$

segment :

$$\|m_1, m_2\| := \{m_1, m_1 + 1, \ldots, m_2 - 1, m_2\}$$

$\rho \in \mathcal{C}^u$,

$$\|m_1, m_2\|^\rho := \{\nu^{m_1}\rho, \nu^{m_1+1}\rho, \ldots, \nu^{m_2-1}\rho, \nu^{m_2}\rho\}$$

Zelevinski segments.

$$(*) \ \nu^{m_1}\rho \times \nu^{m_1+1}\rho \times \cdots, \nu^{m_2-1}\rho \times \nu^{m_2}\rho$$

has a unique irreducible quotient

$$\delta(\|m_1, m_2\|^\rho) \in \mathcal{D}$$

and all rep. in $\mathcal{D}$ are obtained uniquely in this way.

$(*)$ also have a unique irr. subrep.

$$Z(\|m_1, m_2\|^\rho)$$

Langlands classification for G.

$\delta \in \mathcal{D}$, there exists unique $e(\delta) \in \mathbb{R}$, $\delta^u \in \mathcal{D}^u$ s.t.

$$\delta = \nu^{e(\delta)} \delta^u.$$

If $\delta = \delta(\|m_1, m_2\|^\rho)$, $e(\delta) = \frac{m_2 - m_1}{2}$.

$\delta_1, \ldots \delta_k \in \mathcal{D}$, s.t. $e(\delta_1) \geq \ldots \geq e(\delta_k)$

$\underline{d} = (\delta_1, \ldots \delta_k)$.

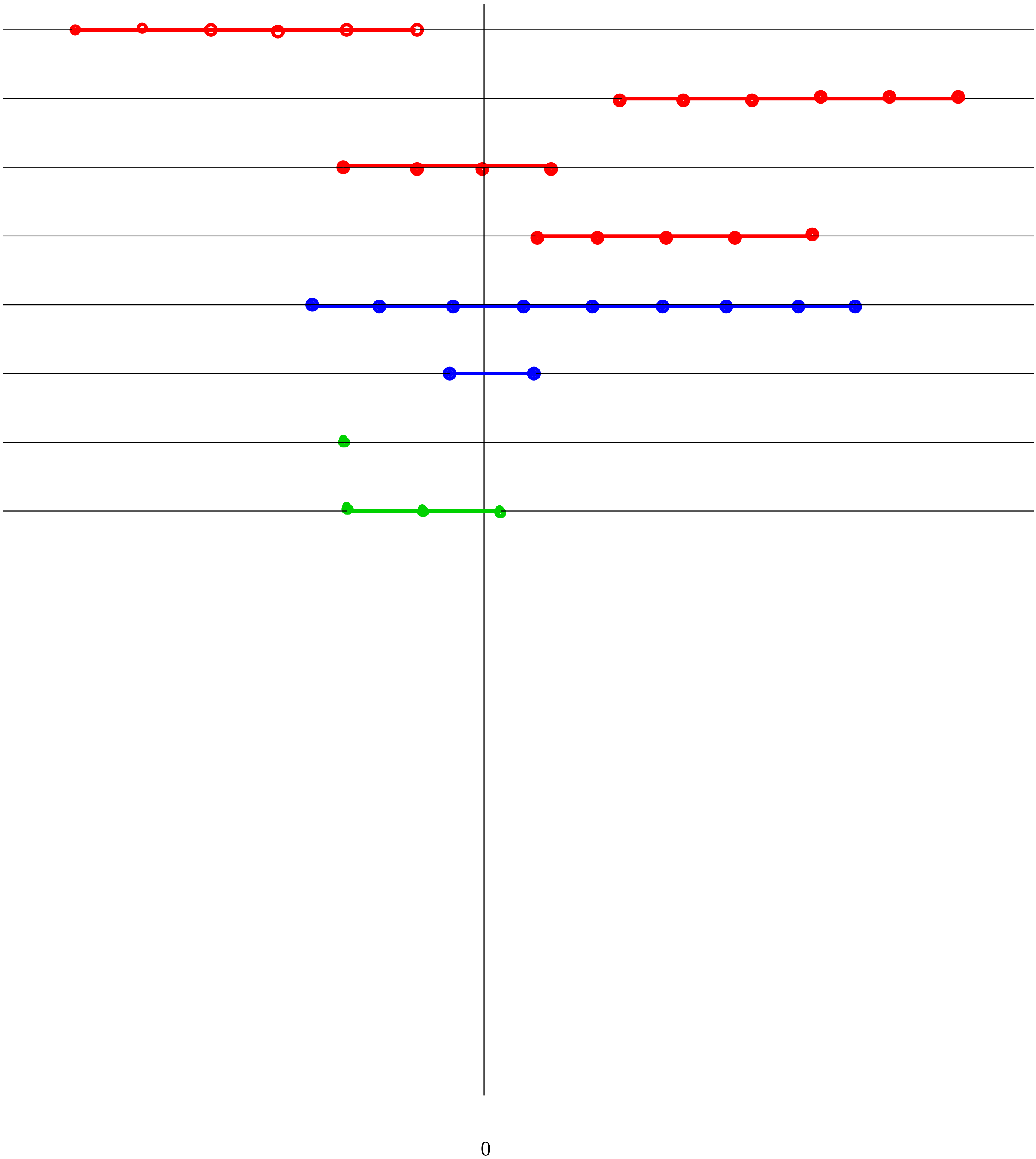
$$\lambda(\underline{d}) = \delta_1 \times \cdots \times \delta_k, \qquad \text{(standard rep.)}$$

has a unique irreducible quotient

$$L(\underline{d}), \qquad \text{(Langlands quotient.)}$$

- All irreducible rep. can be obtained in this way.
- R is a polynomial ring over $\mathcal{D}$.
- $(\lambda(\underline{d}))_{\underline{d}}$ and $(L(\underline{d}))_{\underline{d}}$ are bases of R.

Classification by multisegments.



Notation for $G'_r = GL(r, D)$.

$\mathcal{M}(G'_r)$: category of smooth rep. of G'_r.

R'_r : Grothendieck group of $\mathcal{M}(G'_r)$.

$\mathbf{Irr}'_{\mathbf{r}}$: equiv. classes of irr. rep. in $\mathcal{M}(G'_r)$.

${\mathbf{Irr}'_{\mathbf{r}}}^{\mathbf{u}}$: unitary rep. in $\mathbf{Irr}'_{\mathbf{r}}$:

$$\mathcal{C}'_r \subset \mathcal{D}'_r \subset \mathbf{Irr}'_{\mathbf{r}}$$

respectively cuspidal and essentially square integrable representations.

$$\mathcal{C}'^u_r \subset \mathcal{D}'_r \subset \mathbf{Irr}'_{\mathbf{r}}$$

respectively cuspidal unitary and square integrable representations.

$$R' = \bigoplus_{r \in \mathbb{N}} R'_r$$

$$\mathcal{C}' = \coprod_{r \in \mathbb{N}} \mathcal{C}'_r, \quad \mathcal{D}' = \coprod_{r \in \mathbb{N}} \mathcal{D}_r.$$

$$\mathcal{C}'^u = \coprod_{r \in \mathbb{N}} \mathcal{C}'^u_r, \quad \mathcal{D}'^u = \coprod_{r \in \mathbb{N}} \mathcal{D}'^u_r.$$

Similar notation for standard Levi and parabolic subgroups...

$$R' \times R' \to R' \quad (\pi_1, \pi_2) \mapsto \pi_1 \times \pi_2.$$

Associative and commutative bilinear product.

"Determinant" character of G' :

$$\nu' : G' \to \mathbb{R}_+^\times, \quad g \mapsto |R.N(\det(g))|_F$$

$$\nu'(\pi'_1 \times \pi'_2) = \nu'\pi'_1 \times \nu'\pi_2$$

Langlands Classification for G'

$\delta' \in \mathcal{D}'$ decomposes as $\nu'^{e(\delta')}\delta'^u$, $e(\delta') \in \mathbb{R}$, $\delta'^u \in \mathcal{D}'^u$.

$\delta'_1, \ldots \delta'_k \in \mathcal{D}'$, s.t. $e(\delta'_1) \geq \ldots \geq e(\delta'_k)$

$\underline{d'} = (\delta'_1, \ldots \delta'_k)$

$$\lambda(\underline{d'}) = \delta'_1 \times \cdots \times \delta'_k$$

has a unique irreducible quotient

$$L(\underline{d'})$$

- All irreducible rep. can be obtained in this way.
- R' is a polynomial ring over $\mathcal{D}'$.
- $(\lambda(\underline{d'}))_{\underline{d'}}$ and $(L(\underline{d'}))_{\underline{d}}$ are bases of R'.

Jacquet-Langlands correspondence

$n = rd$

Theorem(DKV) There is a bijection
$JL : \mathcal{D}'_r \simeq \mathcal{D}_n$, with a lot of nice properties.

Basic exemple of Langlands functoriality principle.

Gives an injection $\mathcal{D}' \hookrightarrow \mathcal{D}$

Extends canonically to a ring morphism
$JL : R' \hookrightarrow R$.

In practice : if $JL(\delta'_i) = \delta_i$

$$\lambda(\delta'_1, \ldots, \delta'_k) \mapsto \lambda(\delta_1, \ldots, \delta_k).$$

Inverse : $LJ : R \longrightarrow R'$

$LJ(\lambda(\delta_1, \ldots, \delta_k)) = 0$ if $\delta_i \in \mathcal{D}_k$, $k \nmid d$ for some i.

Remark : JL and LJ are easy to compute on standard rep. To compute them on irr.rep. requires Kazhdan-Lusztig algorithm for G and G'.

Classification of square integrable rep. of G'.

Fix $\rho' \in \mathcal{C}' \subset \mathcal{D}'$ and $JL(\rho') = \delta = \delta(\|a, b\|^{\rho})$, $\rho \in \mathcal{C}^u$.

Put $b - a + 1 = s(\rho')$.

Tadic segments : $m_1, m_2 \in \mathbb{R}$, $m_2 - m_1 \in \mathbb{N}$, $\rho' \in \mathcal{C}^u$, $\nu'_{\rho'} = (\nu')^{s(\rho')}$,

$$\|m_1, m_2\|^{\rho'} := \{{\nu'_{\rho'}}^{m_1}\rho', {\nu'_{\rho'}}^{m_1+1}\rho', \ldots, {\nu'_{\rho'}}^{m_2}\rho'\}$$

$(*)$ ${\nu'_{\rho'}}^{m_1}\rho' \times {\nu'_{\rho'}}^{m_1+1}\rho' \times \cdots, {\nu'_{\rho'}}^{m_2-1}\rho' \times {\nu'_{\rho'}}^{m_2}\rho'$

has a unique irreducible quotient

$$\delta'(\|m_1, m_2\|^{\rho'}) \in \mathcal{D}'$$

and all rep. in $\mathcal{D}'$ are obtained uniquely in this way.

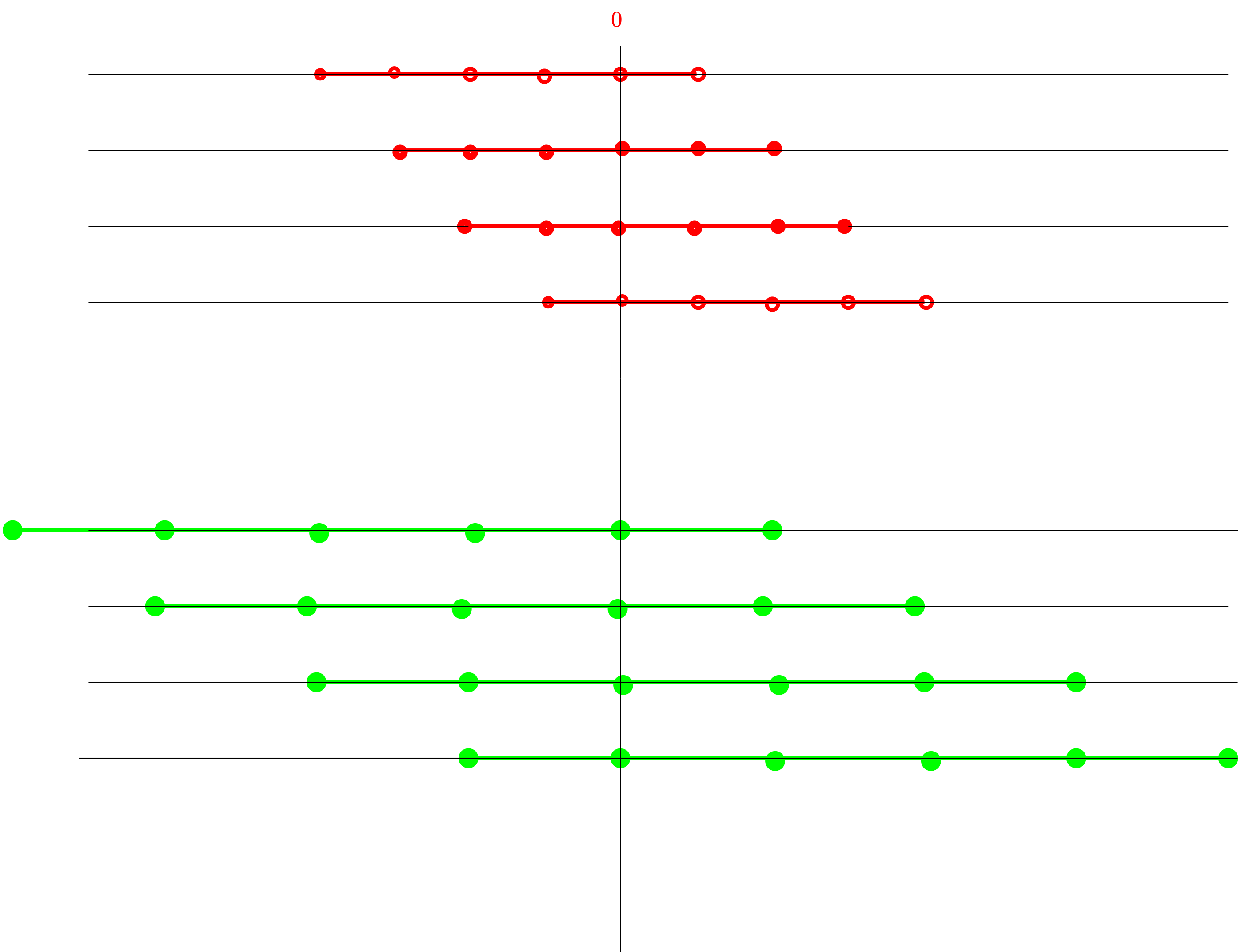
Some distinguished rep.

Let $\delta \in \mathcal{D}^u$, say $\delta = \delta(\|m_1, m_2\|^\rho)$,
$m_1 = -m_2$.

$$k \in \mathbb{N}^\times, u(\delta, k) = L(\nu^{\frac{k-1}{2}}\delta, \nu^{\frac{k-3}{2}}\delta, \ldots \nu^{-\frac{k-1}{2}}\delta)$$

For G' : put $'$ everywhere, and ν'_ρ instead of ν

EX : for G, $u(\delta, 4)$, $\quad \delta = \delta(\| -\frac{5}{2}, \frac{5}{2}\|^\rho)$:
for G', $u(\delta', 4)$, $\quad \delta' = \delta'(\| -\frac{5}{2}, \frac{5}{2}\|^{\rho'})$,
$s(\rho') = 2$



Classification of $\mathbf{Irr^u}$

Consider the following statements for G and G' (adding the $'$ in place)

$U(0)$ If $\tau, \sigma \in \mathbf{Irr^u}$, then $\tau \times \sigma \in \mathbf{Irr^u}$.

$U(1)$ $\delta \in \mathcal{D}^u$, $n \in \mathbb{N}$, then $u(\delta, n) \in \mathbf{Irr^u}$.

$U(2)$ $\delta \in \mathcal{D}^u$, $n \in \mathbb{N}$, $\alpha \in]0, \frac{1}{2}[$ then

$$u(\delta, n, \alpha) := \nu^{\alpha} u(\delta, n) \times \nu^{-\alpha} u(\delta, n) \in \mathbf{Irr^u}.$$

(Complementary series)

$U(3)$ $\delta \in \mathcal{D}^u$, $n \in \mathbb{N}$, then $u(\delta, n)$ prime in R.

$U(4)$ $\underline{d_1}, \underline{d_2}$ multisets in $\mathcal{D}$, then $L(\underline{d_1} \cup \underline{d_2})$ subquotient of $L(\underline{d_1}) \times L(\underline{d_2})$

Theorem (Tadic). Suppose $U(0), \dots U(4)$ hold for G or G'. Set

$$B = \{u(\delta, n), u(\delta, n, \alpha) \mid n \in \mathbb{N}, \alpha \in]0, \frac{1}{2}[, \rho \in \mathcal{C}^u\}.$$

Then

(i) If $\tau_1, \dots \tau_k \in B$, $\tau_1 \times \cdots \tau_k \in \mathbf{Irr^u}$.

(ii) If $\sigma \in \mathbf{Irr^u}$, there exists $\tau_1, \dots \tau_k \in B$, s.t. $\sigma = \tau_1 \times \cdots \tau_k$

(iii) If $\sigma = \tau_1 \times \cdots \tau_k = \tau'_1 \times \cdots \tau'_m$, $\tau_i, \tau'_j \in B$, then $k = m$ and $\tau_i = \tau'_i$ after renumeration.

$U(2), U(3), U(4)$ are established by Tadic for G and G', assuming $U(0)$ and $U(1)$.

For G, $U(0)$ is due to Bernstein, $U(1)$ to Tadic using a global argument due to Speh.

$\rightsquigarrow$ Classification of the unitary dual of G by Tadic.

On $U(1)$

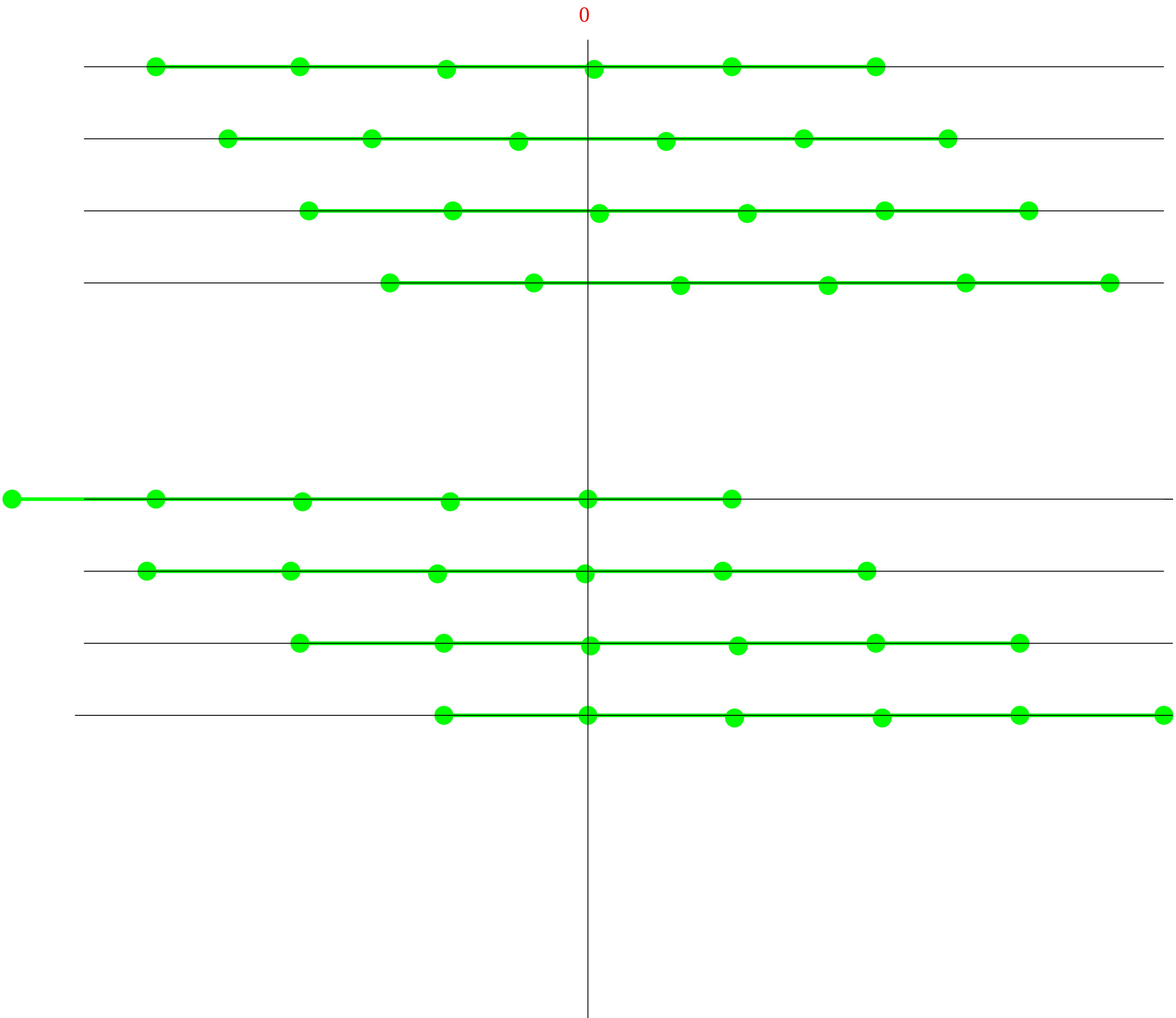
For G : $u(\delta, n)$ appears as local component of a global automorphic rep. $\rightsquigarrow$ unitarity.

For G' and $u(\delta', n)$: this is false if $s(\rho') \neq 1$

Define $\bar{u}(\delta', n)$ by

$$k \in \mathbb{N}^{\times}, u(\delta', k) = L(\nu'^{\frac{k-1}{2}}\delta, \nu'^{\frac{k-3}{2}}\delta, \ldots \nu'^{-\frac{k-1}{2}}\delta)$$

EX : $\bar{u}(\delta', 4)$ and $u(\delta', 4)$, $\delta' = \delta'(|| - \frac{5}{2}, \frac{5}{2}||^{\rho'})$, $s(\rho') = 2$

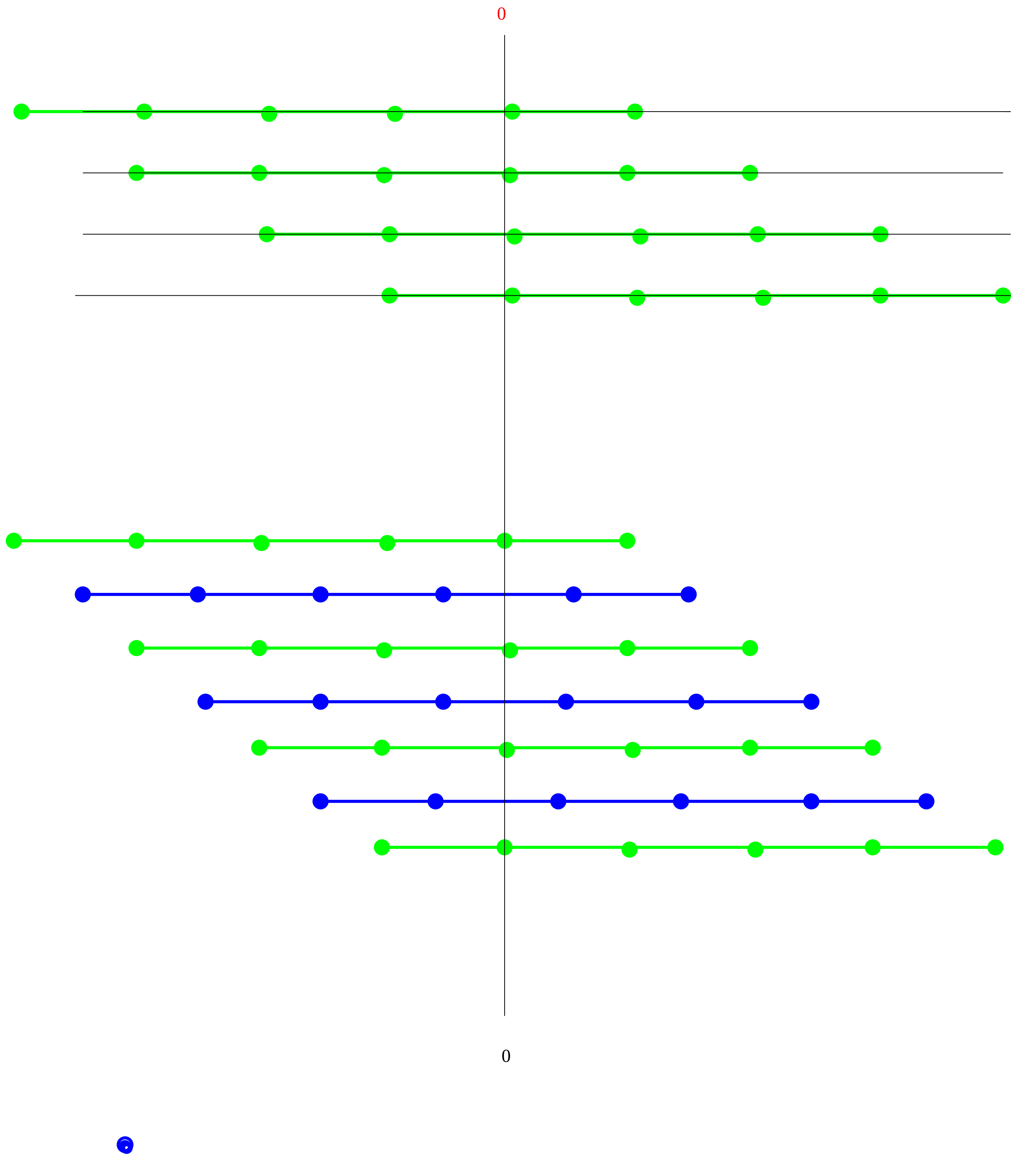


Theorem(Badulescu) The rep. $\bar{u}(\delta', n)$ are unitary.

Proof by global methods.

Corollary(Badulescu-Renard) The rep. $u(\delta', n)$ are unitary.

Proof by combinatorics on segments.



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Related results

Theorem (Badulescu) $\delta \in \mathcal{D}^u$. Suppose $\delta = JL(\delta')$, $\delta' \in \mathcal{D}'^u$. then :

$$LJ(u(\delta, k)) = \bar{u}(\delta', k)$$

Cor • $u(\delta, k) \in \mathbf{Irr^u}$, then $LJ(u(\delta, k))$ is unitary irreducible or $\mathbf{0}$.

• If $\pi \in \mathbf{Irr^u}$, then $LJ(\pi) \in \mathbf{Irr'^u}$ or $\mathbf{0}$.

Also obtained by Tadic assuming $U(0)$: Tadic writes a closed formula for the character of $u(\delta', k)$ (and thus of all irr. unitary reps) in terms of standard rep. , much simplier than KL algorithm.

Surprising from the point of view of KL algorithm, but similar things happen in other exemples of Langlands functoriality (Kazhdan-Patterson lifting).