KAZHDAN-LUSZTIG ALGORITHMS FOR NONLINEAR GROUPS AND APPLICATIONS TO KAZHDAN-PATTERSON LIFTING

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0. Introduction

The purpose of this paper is to establish an algorithm to compute characters of irreducible Harish-Chandra modules for a large class of nonlinear (that is, nonalgebraic) real reductive Lie groups. We then apply this theory to study a particular group (the universal cover $\widetilde{\mathrm{GL}}(n,\mathbb{R})$ of $\mathrm{GL}(n,\mathbb{R})$), and discover a symmetry of the character computations encoded in a character multiplicity duality for $\widetilde{\mathrm{GL}}(n,R)$ and a nonlinear double cover $\widetilde{\mathrm{U}}(p,q)$ of $\mathrm{U}(p,q)$. Using this duality theory, we reinterpret a kind of representation-theoretic Shimura correspondence for $\widetilde{\mathrm{GL}}(n,\mathbb{R})$ geometrically, and find that it is dual to an analogous lifting for $\widetilde{\mathrm{U}}(p,q)$. It seems likely that this example is illustrative of a general framework for studying similar correspondences.

One of the main issues (as we explain below) in computing irreducible characters of reductive Lie groups centers on finding a natural class groups to study. Certainly the class of groups obtained as the real points of connected reductive algebraic groups defined over $\mathbb{R}$ is extremely natural. (Henceforth we will simply call these groups algebraic.) Yet from several perspectives, the algebraic condition is unsatisfactorily restrictive. For instance, one of the only known ways to construct automorphic representations of algebraic groups is by means of the theta correspondence. This immediately brings the nonalgebraic metaplectic double cover of the symplectic group into the fold. A different kind of reason for studying nonalgebraic groups has its origins in the representation-theoretic formulation of the classical Shimura correspondence of irreducible unitary spherical representations of $\mathrm{SL}(2,\mathbb{R})$ and its metaplectic double cover. Subsequent work of a number of people suggests an intricate interaction between the unitary duals of algebraic groups and certain nonalgebraic covering groups.

For these reasons (and in fact many others) one is led to study groups outside the class of algebraic groups, even if one is ultimately interested only in (say) the automorphic spectrum in the algebraic case. Abstractly, the structure theory of algebraic groups and their nonalgebraic coverings is more or less uniform. Yet in practice, the structure theory of nonalgebraic groups leads to complications that rapidly become unmanageable. For instance, Cartan subgroups of connected linear groups are always abelian. Yet in connected nonlinear covers, they may become nonabelian. Moreover, when one considers coverings of disconnected groups, the disconnectedness in the covering becomes more intractable. (The main issue amounts to understanding finite extensions of the component group and how those extensions act as automorphisms of the identity component.) One needs to restrict the class of groups under consideration to avoid pathologies, but at the same time one needs to make certain that the

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restrictions do not rule out interesting groups. But what constitutes "interesting"? We seek external guidance for an answer.

The oracle we consult is the representation-theoretic part of the local Langlands conjecture. This theory provides profound organizing principles for automorphic representations of algebraic groups (and often suggests hidden structure on the set of all irreducible unitary representations). Despite the efforts of a number of people, it is difficult to see how to extend Langlands' original formulation of his parameters to the nonalgebraic case. If one instead works within the equivalent framework suggested first in the work of Kazhdan-Lusztig and Zelevinsky (and sharpened substantially in [V5] and [ABV]), there are natural constructions to imitate. More precisely, the main result of [ABV] interprets the local Langlands formalism as consequence of a character multiplicity duality theory for algebraic groups. Roughly speaking such a theory implies that the computation of characters of irreducible representations of a group is equivalent to the corresponding computation for a "dual" group. (We offer more details around Equation (0.1) below.) This becomes our main guiding principle: we seek a class of groups for which there exists a character multiplicity duality theory and which is closed under the passage from a group to its "dual" group. A remarkable result of [ABV] implies that the class of algebraic groups satisfies this desideratum, but as discussed above we seek to enlarge this class to include the kinds of nonlinear covering groups that might arise in automorphic applications (for instance, the nonlinear double covering of the split real form of a reductive algebraic group).

It is worth remarking that there are some rather natural groups for which one can prove that no character multiplicity duality theory holds. (The nonlinear triple cover of the real symplectic group is one example.) In addition the dual of a connected group is often disconnected, so for the class of groups we consider we are forced to confront disconnectedness and the kinds of structure-theoretic complications alluded to above.

The class of groups we eventually settle on is as follows. Let $G_{\mathbb{R}}$ be a linear group in the class that Vogan considers in [Vgr]; that is, $G_{\mathbb{R}}$ is a linear group in Harish-Chandra's class with abelian Cartan subgroups. We let our class of groups consist of all two-fold covers of $G_{\mathbb{R}}$ such that when the cover is nontrivial, it is nonlinear in the sense of Definition 6.3. (We further impose one likely superfluous technical condition; see the discussion around Hypothesis 6.15.) It turns out that this class of groups meets our desideratum: in future joint work with Adams, we show it is essentially closed under the character duality theory mentioned above. For instance, for groups in this class with simply laced Lie algebras, the extra technical condition is redundant (Proposition 6.16), and the results of Part II of this paper can be extended to conclude that the class of simply laced groups is closed under duality.

We now turn to a more detailed description of the computation of irreducible character and corresponding character duality theories. The computation of characters is extremely technical and thus it is appropriate to highlight carefully the main subtleties involved in our setting. For orientation, we must first frame the problem precisely, as well as recall the deep results of Vogan in the algebraic case.

Suppose $G_{\mathbb{R}}$ is a real reductive Lie group in our class. Next let $\widehat{G}_{\mathbb{R}}^{\chi}$ denote the set of irreducible Harish-Chandra modules for $G_{\mathbb{R}}$ with fixed infinitesimal character χ. The work of a number of people (perhaps most notably Miličić) building on Langlands' classification for algebraic groups showed that each irreducible $\pi \in \widehat{G}_{\mathbb{R}}^{\chi}$ arises as the quotient of a standard module. More precisely (in the formulation of Speh and Vogan), there is a finite parameter set $\mathcal{P}_\chi$, and two bases of the Grothendieck group of Harish-Chandra modules for $\widehat{G}^\chi_\mathbb{R}$: one consists of the standard modules $\{X(\gamma) \mid \gamma \in \mathcal{P}_\chi\}$, the other of the irreducibles $\{\overline{X}(\gamma) \mid \gamma \in \mathcal{P}_\chi\}$. They are related by an integral change of basis matrix

$$\overline{X}(\gamma) = \sum_{\delta \in \mathcal{P}_\chi} M(\gamma, \delta) X(\delta),$$

which is, in fact, upper triangular in an appropriate ordering. Roughly speaking the characters of standard modules are computable in principle, so the determination of the characters of each $\overline{X}(\gamma)$ amounts to computing the integers $M(\gamma, \delta)$.

When $G_\mathbb{R}$ is a complex, an algorithm for determining the numbers $M(\gamma, \delta)$ was proposed by Kazhdan-Lusztig [KL] and established by Brylinski-Kashiwara [BK]. In fact, the problem is equivalent to one for highest weight modules, and the relevant combinatorics is that of the Bruhat order on the Weyl group W. When $G_\mathbb{R}$ is linear and has abelian Cartan subgroups (for instance, if $G_\mathbb{R}$ is algebraic), the algorithm was proposed and established by Vogan [V1]–[V3]. He developed the combinatorics of the so-called Bruhat $\mathcal{G}$-order on $\mathcal{P}_\chi$. The only additional case where an algorithm has been established is that of the metaplectic double cover of the symplectic group [RT1]–[RT2] (where, incidentally, all Cartans are abelian).

The complications associated with nonabelian Cartans and nonlinear groups are serious. (Clearly Vogan had these kinds of groups in mind during the course of the series [V1]–[V4]; compare the remarks before Definition 0.1.3 in [Vgr].) Most concretely, nonabelian Cartans can potentially complicate the combinatorics of the set $\mathcal{P}_\chi$. To take but one example, a key combinatorial construction on $\mathcal{P}_\chi$ is the so-called Cayley transform. In particular cases, this essentially amounts to computing how an irreducible representation of an index two-subgroup of a Cartan subgroup H induces to all of H. When H is abelian, all irreducible representations must be one-dimensional, and hence the two-dimensional induced representation must be reducible. Obviously this can (and does) fail for nonabelian H, and it is not obvious that this failure can be controlled. Cayley transforms arise in the Hecht-Schmid character identities (which are basic ingredients in Vogan's theory), and so these kinds of issues are of paramount importance in developing a Kazhdan-Lustzig algorithm. They are the subject of Section 6 below. A different kind of complication in the nonlinear setting is the formulation of a parity condition that guarantees vanishing of certain cohomology groups. In the highest weight setting the condition is that two Verma modules $M(w)$ and $M(y)$ have no extensions between them if the difference in the length of w and y is odd. The theory developed in [RT1]–[RT2] points to a more general cohomological parity condition. We introduce the extended integral length in Definition 6.7 and prove that it exhibits the right properties for our class of groups in Theorem 8.1 below.

After assembling the technical details of the previous paragraph, we are able to imitate Vogan's theory and arrive at an algorithm to compute irreducible characters for groups in our class. This is the main result of Part I. Next we turn to the issue of character duality.

Let $\mathcal{B}$ be a block of Harish-Chandra modules for $G_\mathbb{R}$. By a character multiplicity duality for $\mathcal{B}$, we mean the following: there exists a block $\mathcal{B}'$ of Harish-Chandra modules for a group $G'_\mathbb{R}$ and a bijection $\mathcal{B} \to \mathcal{B}'$ (denoted $\gamma \mapsto \gamma'$) such that

$$\overline{X}(\gamma) = \sum_{\delta \in \mathcal{B}} M(\gamma, \delta) X(\delta) \tag{0.1}$$

if and only if

$$X(\delta') = \sum_{\gamma' \in \mathcal{B}'} \epsilon_{\gamma\delta} M\gamma\delta \overline{X}(\gamma'); \tag{0.2}$$

here $\epsilon_{\gamma\delta}$ is ± 1 according to the parity of the difference in the length of γ and δ. In other words, computing the coefficients $M(\gamma, \delta)$ for $\mathcal{B}$ amounts to the inverse transpose of computing the corresponding coefficients for $\mathcal{B}'$. Such a duality theory is thus encoded in a symmetry of the algorithm to compute irreducible characters. Vogan's monumental achievement ([V4]) establishes a character multiplicity duality theory for any block for a linear group in Harish-Chandra's class with abelian Cartans. A duality theory for the metaplectic group was constructed in [RT1] and [RT2].

In Part II, we establish a complete duality theory for the nonlinear double covers $\widetilde{\mathrm{GL}}(n,\mathbb{R})$ and $\widetilde{\mathrm{U}}(p,q)$. As a consequence (following the philosophy of Vogan mentioned above), we obtain a set of Langlands parameters for these groups. We find that there is a natural injection from the space of parameters for $\widetilde{\mathrm{GL}}(n,\mathbb{R})$ to those for $\mathrm{GL}(n,\mathbb{R})$. Using it, we can form a kind of pullback of representations from the linear group to the nonlinear one. We prove that this pullback coincides with the lifting defined by Kazhdan and Patterson in [KP]. Dualizing this picture gives a lifting from the linear group $\mathrm{U}(p,q)$ to its nonlinear double cover $\widetilde{\mathrm{U}}(p,q)$. This appears to be new, and subsequent work of Adams and Herb have suggested a character-theoretic interpretation of it. The results of Part II are a paradigm for all groups with simply-laced Lie algebras. This will be explained in future work with Adams.

Finally, in Part III, we apply our theory to give a counterexample to a conjecture of Kazhdan and Flicker: we find an irreducible Harish-Chandra module for $\mathrm{GL}(n,\mathbb{R})$ (in fact with $n = 4$) whose Kazhdan-Patterson lift is reducible.

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Part 1. Kazhdan-Lusztig algorithm for nonlinear groups

1. Notation and preliminaries

We begin by recalling some notation and material from [RT1]. Most of it was taken from [V1], [V2] and [Vgr]. Let $G_{\mathbb{R}}$ be real a reductive group in Harish-Chandra class. Let $\mathfrak{g}_{\mathbb{R}}$ be the Lie algebra of $G_{\mathbb{R}}$ and let $\mathfrak{g}$ be its complexification. We fix a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$ with corresponding Cartan involution θ and we denote by K its complexification. We also fix a maximally split θ-stable Cartan subgroup $H^a_{\mathbb{R}}$ of $G_{\mathbb{R}}$ with Cartan subalgebra $\mathfrak{h}^a_{\mathbb{R}}$ (with complexification $\mathfrak{h}^a$), and we fix a positive root system Δ^+_a of $\Delta_a := \Delta(\mathfrak{g}, \mathfrak{h}^a)$. We let W_a denote the Weyl group of the root system Δ_a.

In [RT1], $G_{\mathbb{R}}$ was assumed to be connected with abelian Cartan subgroups, and satisfying $\mathrm{rk}(G_{\mathbb{R}}) = \mathrm{rk}(K_{\mathbb{R}})$. In [V1], [V2], the group $G_{\mathbb{R}}$ was only assumed to be connected, and in [Vgr], it was assumed to be linear. The definitions and results taken from these references we use here are still valid in our more general context, sometimes with obvious and minor modifications.

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, and λ is a regular element in $\mathfrak{h}^*$, we will write $\Delta^+(\lambda)$ for the positive root system of $\Delta(\mathfrak{g},\mathfrak{h})$ making λ dominant i.e. for all $\alpha \in \Delta^+(\lambda)$,

$$\mathcal{R}e\langle\alpha,\lambda\rangle \geq 0 \text{ and if } \mathcal{R}e\langle\alpha,\lambda\rangle = 0 \text{ then } \mathcal{I}m\langle\alpha,\lambda\rangle \geq 0$$

We also need

$$R(\lambda) = \left\{\alpha \in \Delta(\mathfrak{g},\mathfrak{h}) \,|\, 2\frac{\langle\alpha,\lambda\rangle}{\langle\alpha,\alpha\rangle} \in \mathbb{Z}\right\},$$

the integral roots for λ, and

$$R^+(\lambda) = R(\lambda) \cap \Delta^+(\lambda) \quad \text{and} \quad W(\lambda) = W(R(\lambda)),$$

the positive integral roots and the integral Weyl group.

Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$. Through Harish-Chandra's isomorphism, an element $\lambda \in \mathfrak{h}^*$ determines an infinitesimal character for $\mathfrak{g}$. Fix $\lambda_a \in (\mathfrak{h}^a)^*$ regular and Δ_a^+-dominant. Let $\mathfrak{h}$ be any Cartan subalgebra of $\mathfrak{g}$, and suppose that $\lambda \in \mathfrak{h}^*$ defines the same infinitesimal character as λ_a; i.e. suppose that there exists an inner automorphism $i_{\lambda_a,\lambda}$ of $\mathfrak{g}$, sending $(\lambda_a,(\mathfrak{h}^a)^*)$ onto $(\lambda,\mathfrak{h}^*)$. If $\lambda^i \in (\mathfrak{h}^i)^*$, $i=1,2$ define the same infinitesimal character as λ_a, we set $i_{\lambda^1,\lambda^2} := i_{\lambda_a,\lambda^2} \circ (i_{\lambda_a,\lambda^1})^{-1}$. The restriction of $i_{\lambda_a,\lambda}$ to $(\mathfrak{h}^a)^*$ is unique.

Let $\mathcal{HC}(\mathfrak{g},K)$ denote the category of (finite-length) Harish-Chandra modules for $G_{\mathbb{R}}$. For any infinitesimal character $\lambda_a \in (\mathfrak{h}^a)^*$, $\mathcal{HC}(\mathfrak{g},K)_{\lambda_a}$ is the full subcategory of modules having infinitesimal character λ_a, The Grothendieck groups of these categories are denoted respectively by $\mathbb{K}(\mathfrak{g},K)$ and $\mathbb{K}(\mathfrak{g},K)_{\lambda_a}$. We will write $[X]$ for the image of a module $X \in \mathcal{HC}(\mathfrak{g},K)$.

We recall (from [Vgr], e.g., or [RT1, Section 1.2]) the definition of a pseudocharacter $(H_{\mathbb{R}},\gamma)$ of $G_{\mathbb{R}}$: $H_{\mathbb{R}}$ is a θ-stable Cartan subgroup of $G_{\mathbb{R}}$ with Cartan decomposition $H_{\mathbb{R}} = T_{\mathbb{R}}A_{\mathbb{R}}$, and $\gamma = (\Gamma,\overline{\gamma})$, consists of an irreducible representation Γ of $H_{\mathbb{R}}$ and an element $\gamma \in \mathfrak{h}^*$, with certain compatibility conditions that we don't recall here. We write $(\widehat{H}_{\mathbb{R}})'$ for the set of pseudocharacters having $H_{\mathbb{R}}$ as their first component.

We are interested in irreducible admissible Harish-Chandra modules of $G_{\mathbb{R}}$ with regular infinitesimal character $\lambda_a \in (\mathfrak{h}^a)^*$. We write $(\widehat{H}_{\mathbb{R}})'_{\lambda_a}$ for the subset of $(\widehat{H}_{\mathbb{R}})'$ of pseudocharacters γ such that $\overline{\gamma}$ and λ_a define the same infinitesimal character; in this case, we say that γ is a λ_a-pseudocharacter. Recall that a pseudocharacter $(H_{\mathbb{R}},\gamma)$ specifies a standard representation $X(\gamma)$, containing a unique irreducible submodule $\overline{X}(\gamma)$. The Langlands classification parametrizes irreducible modules in $\mathcal{HC}(\mathfrak{g},K)_{\lambda_a}$ by the set $\mathcal{P}_{\lambda_a}$ of $K_{\mathbb{R}}$-conjugacy classes of λ_a-pseudocharacters. Furthermore, the following two sets are bases of the Grothendieck group $\mathbb{K}(\mathfrak{g},K)_{\lambda_a}$:

$$\{\,[\overline{X}(\gamma)]\,\}_{\gamma\in\mathcal{P}_{\lambda_a}} \quad \text{and} \quad \{\,[X(\gamma)]\,\}_{\gamma\in\mathcal{P}_{\lambda_a}} \tag{1.1}$$

Hence we can define the change of basis matrix

$$[\overline{X}(\delta)] = \sum_{\gamma\in\mathcal{P}_{\lambda_a}} M(\gamma,\delta)[X(\gamma)], \tag{1.2}$$

and the inverse matrix

$$[X(\delta)] = \sum_{\gamma\in\mathcal{P}_{\lambda_a}} m(\gamma,\delta)[\overline{X}(\gamma)]. \tag{1.3}$$

The main result of Section 9 is an algorithm to compute $M(\gamma,\delta)$ in the setting of Section 6.

Remark 1.1. We slightly abuse notation by writing γ for the pseudocharacter $(H_{\mathbb{R}},\gamma)$, and often γ for its $K_{\mathbb{R}}$-conjugacy class. For more details, see [RT1], Section 1.

Definition 1.2. Let $(H_{\mathbb{R}}, \gamma)$ be a pseudocharacter of $G_{\mathbb{R}}$. The length of γ is

$$l(\gamma) = \frac{1}{2}|\{\alpha \in \Delta^+(\overline{\gamma}) \mid \theta(\alpha) \notin \Delta^+(\overline{\gamma})\}| + \frac{1}{2}\dim \mathfrak{a}_{\mathbb{R}}.$$

The integral length of γ is

$$l^I(\gamma) = \frac{1}{2}|\{\alpha \in R^+(\overline{\gamma}) \mid \theta(\alpha) \notin R^+(\overline{\gamma})\}| + \frac{1}{2}\dim \mathfrak{a}_{\mathbb{R}}.$$

For linear groups, the root system $R(\overline{\gamma})$ is θ-stable ([Vgr], Lemma 8.2.5). This need not be the case for nonlinear groups. In Section 6, we will introduce a root system $\widetilde{R}(\overline{\gamma})$, the extended integral root system, containing $R(\overline{\gamma})$, which is θ-stable. The corresponding "extended integral length" $\tilde{l}^I$ will give us the correct notion needed for the induction in our Kazhdan-Lusztig algorithm.

2. Translation functors

To get a nice theory of translation functors, coherent continuation and cross-action, we need a reasonably large supply of finite-dimensional representations of our group $G_{\mathbb{R}}$. More precisely, we would like every irreducible finite-dimensional representation of $\mathfrak{g}$ to exponentiate to a representation of $G_{\mathbb{R}}$. Of course, this might not be the case, and we will replace the group $G_{\mathbb{R}}$ by a finite central covering $\bar{G}_{\mathbb{R}}$ (i.e. the kernel of the projection $p : \bar{G}_{\mathbb{R}} \to G_{\mathbb{R}}$ is central in $\bar{G}_{\mathbb{R}}$), which satisfies the property we want. The category $\mathcal{HC}(\mathfrak{g}, K)$ is then equivalent to the full subcategory of $\mathcal{HC}(\mathfrak{g}, \bar{K})$ ($\bar{K} := p^{-1}(K)$ is a maximal compact subgroup of $\bar{G}_{\mathbb{R}}$) consisting of modules with trivial action of the kernel of the projection p. Thus, to study the representation theory of $G_{\mathbb{R}}$, we can replace it by $\bar{G}_{\mathbb{R}}$. Let us recall the classical construction of $\bar{G}_{\mathbb{R}}$. Let us denote by $\mathcal{G}^{ad}$ the adjoint group of $\mathfrak{g}$, and by $\mathcal{G}^{sc}$ its simply-connected cover, with projection $q : \mathcal{G}^{sc} \to \mathcal{G}^{ad}$. Since $G_{\mathbb{R}}$ is the Harish-Chandra class, the adjoint action of $G_{\mathbb{R}}$ on $\mathfrak{g}$ defines a morphism $\mathrm{Ad} : G_{\mathbb{R}} \to \mathcal{G}^{ad}$. Define $\bar{G}_{\mathbb{R}}$ to be the fibered product $G_{\mathbb{R}} \times_{\mathcal{G}^{ad}} \mathcal{G}^{sc}$ with respect to the maps q and Ad, i.e.

$$\bar{G}_{\mathbb{R}} = \{(g,h) \in G_{\mathbb{R}} \times \mathcal{G}^{sc} \mid \mathrm{Ad}(g) = q(h)\}.$$

The natural projection p_1 and p_2 on the first and second factors fit into a commutative diagram :

$$\begin{tikzcd} \bar{G}_{\mathbb{R}} \arrow[r, "p_2"] \arrow[d, "p_1"] & \mathcal{G}^{sc} \arrow[d, "q"] \\ G_{\mathbb{R}} \arrow[r, "\mathrm{Ad}"] & \mathcal{G}^{ad} \end{tikzcd}$$

Since every irreducible finite-dimensional representation of $\mathfrak{g}$ lifts to a representation of $\mathcal{G}^{sc}$, it also becomes a representation of $\bar{G}_{\mathbb{R}}$ via p_2. The kernel of the projection p_1 is central in $\bar{G}_{\mathbb{R}}$ and isomorphic to the kernel of q, which is the center of $\mathcal{G}^{sc}$.

Let $\mathfrak{h}_{\mathbb{R}}$ be Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$, and let μ be an integral weight in $\mathfrak{h}^*$. There is a unique irreducible finite-dimensional representation F_μ of $\mathfrak{g}$ with extremal weight μ. As explained above, F_μ is also representation $\bar{G}_{\mathbb{R}}$, which is easily seen to remain irreducible. Notice also that the weights of a Cartan subgroup $\bar{H}_{\mathbb{R}}$ in F_μ (i.e. irreducible subrepresentations of $\bar{H}_{\mathbb{R}}$ in F_μ) are one-dimensional, even if $\bar{H}_{\mathbb{R}}$ is not abelian, because the action of $\bar{H}_{\mathbb{R}}$ on F_μ factors through a Cartan subgroup in $\mathcal{G}^{sc}$. Furthermore, these weights are in one-to-one correspondence with the weights of $\mathfrak{h}$ in F_μ. We won't distinguish in the notation between a weight of $\mathfrak{h}$ in a finite-dimensional representation of $\mathfrak{g}$, and the corresponding weight of $\bar{H}_{\mathbb{R}}$ in the corresponding representation of $G_{\mathbb{R}}$.

Since we can replace $G_{\mathbb{R}}$ by $\bar{G}_{\mathbb{R}}$ if needed, in order to simplify notation, we will assume from now on that $G_{\mathbb{R}}$ satisfies the required property, i.e. every irreducible finite-dimensional representation of $\mathfrak{g}$ to exponentiate to a representation of $G_{\mathbb{R}}$, and that weights of Cartan subgroups of $G_{\mathbb{R}}$ are one-dimensional.

The theory of translation functors we recall briefly here is taken from [V1]. Let $H_{\mathbb{R}}$ be a Cartan subgroup of $G_{\mathbb{R}}$, $\lambda \in \mathfrak{h}^*$ be regular and let F_μ be the finite-dimensional irreducible representation of $\mathfrak{g}$ with highest weight μ in $\mathfrak{h}^*$, with respect to the positive root system $\Delta^+(\lambda)$.

Since, by assumption, F_μ exponentiate to a representation of $G_{\mathbb{R}}$, we have then a translation functor $\psi_\lambda^{\lambda+\mu} : \mathcal{HC}(\mathfrak{g}, K)_\lambda \longrightarrow \mathcal{HC}(\mathfrak{g}, K)_{\lambda+\mu}$ (see [V1] for details). From this, we can define functors ψ_α and ϕ_α that push-to and push-off walls with respect to integral roots α which are simple for Δ_a^+. The functors ψ_α and ϕ_α are adjoint (e.g. [V1], Lemma 3.4).

Let us remark that for linear groups, one usually uses more general translations functors ψ_α and ϕ_α, namely one allows α to be a simple root for $R^+(\lambda_a)$. This is because translation across a non-integral simple wall is essentially trivial. In our context, some nontrivial phenomenon occurs when we cross non-integral walls. So we will need to recall properties of wall-crossing functors with respect to a non-integral simple root in detail.

In the same setting as above, we have the follow.

Theorem 2.1 ([Vgr], Proposition 7.3.3)**.** *Let $\alpha \in \Delta^+(\lambda)$ be a simple non-integral root, and fix an integral weight μ_α in $\mathfrak{h}^*$ such that $\lambda + \mu_\alpha$ is dominant and regular for $s_\alpha \cdot \Delta^+(\lambda)$. If $X \in \mathcal{HC}(\mathfrak{g}, K)_\lambda$ define*

$$\psi_\alpha(X) := \psi_\lambda^{\lambda+\mu_\alpha}(X), \quad \phi_\alpha(X) := \psi_{\lambda+\mu_\alpha}^{\lambda}(X).$$

The functor ψ_α realizes an equivalence of categories between $\mathcal{HC}(\mathfrak{g}, K)_\lambda$ and $\mathcal{HC}(\mathfrak{g}, K)_{\lambda+\mu_\alpha}$; its inverse is ϕ_α.

The notation ψ_α across a nonintegral wall depends (in an inessential way) on the choice of μ_α, but we find it convenient to adhere to the following convention.

Convention 2.2. If $s_\alpha \cdot \lambda - \lambda = -2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle}\alpha$ in the theorem above is an integral weight, then we choose $\mu_\alpha = -2\frac{\langle \alpha, \lambda \rangle}{\langle \alpha, \alpha \rangle}\alpha$.

Recall that we have fixed a maximally split θ-stable Cartan subgroup $H_{\mathbb{R}}^a$ of $G_{\mathbb{R}}$ and a regular dominant element $\lambda_a \in \mathfrak{h}^a$. The τ-invariant $\tau^a(X)$ of $X \in \mathcal{HC}(\mathfrak{g}, K)$ is a subset of the simple roots in $R^+(\lambda_a)$ defined in [Vgr], Definition 7.3.8. Sometimes it will be convenient to transport this to another Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. Suppose we have fixed $\lambda \in \mathfrak{h}^*$ such that λ is inner conjugate to λ_a in $\mathfrak{g}$. The τ-invariant of X with respect to $(\mathfrak{h}, \lambda)$ is the subset $\tau(X) = i_\lambda(\tau^a(X))$ of simple roots in $R^+(\lambda)$. Although not explicit in the notation, the choice of $(\mathfrak{h}, \lambda)$ is usually clear from the context.

Theorem 2.3. *Let $X \in \mathcal{HC}(\mathfrak{g}, K)_{\lambda_a}$ be an irreducible module, and let α be a simple integral root in Δ_a^+ not in $\tau^a(X)$. Then $\phi_\alpha\psi_\alpha(X)$ has X has its unique irreducible submodule and irreducible quotient, and the following sequence*

$$0 \to X \to \phi_\alpha\psi_\alpha(X) \to X \to 0,$$

defined by the adjointness of the two functors ψ_α and ϕ_α, is a chain complex. Define $U_\alpha(X)$ to be its cohomology. Then the module $U_\alpha(X)$ has finite composition series, and $\alpha \in \tau(U_\alpha(X))$.

Proof. See Section 7.3 of [Vgr].

3. A family of infinitesimal characters

The Kazhdan-Lusztig algorithm (at regular infinitesimal character λ_a) for linear groups keeps λ_a fixed. Fixing the infinitesimal character can be interpreted as choosing a representative of the single coset of $W(\lambda_a) \cdot \lambda_a + \mathcal{P}$ modulo $\mathcal{P}$, where $\mathcal{P}$ is the integral weight lattice in $(\mathfrak{h}^a)^*$. This is not sufficient for the purposes of computing characters for nonlinear groups. Instead one must fix a set of coset representatives for

$$(W_a \cdot \lambda_a + \mathcal{P})/\mathcal{P}.$$

We define a *family of infinitesimal characters containing* λ_a to be any set of coset representatives that consists of Δ_a^+-dominant weights (and which contains λ_a).

Fix λ_a dominant and regular, and fix a family $\mathcal{F}(\lambda_a)$ containing λ_a. It will be convenient to introduce a labeling of the elements of a family. Since $\lambda_a \in \mathcal{F}(\lambda_a)$, the members of $\mathcal{F}(\lambda_a)$ are clearly indexed by $W_a/W_{\mathcal{P}}(\lambda_a)$, where $W_{\mathcal{P}}(\lambda_a)$ consists of those elements which are weight-integral in the sense that $w\lambda_a - \lambda_a \in \mathcal{P}$: if $\nu_a \in \mathcal{F}(\lambda_a)$, then $\nu_a = y\lambda_a$ modulo $\mathcal{P}$, for some $y \in W_a$ which is unique modulo $W_{\mathcal{P}}(\lambda_a)$. In this case, we write $\nu_a = \nu_y$; for example $\nu_x = \lambda_a$ for any $x \in W_{\mathcal{P}}(\lambda_a)$.

Because the elements of $\mathcal{F}(\lambda_a)$ are indexed by cosets in W_a, there is an obvious action of W_a on $\mathcal{F}(\lambda_a)$. It will be important implement this action on the level of Harish-Chandra module using translation functors, and we need to introduce some weights to define the relevant functors. Define elements $\mu(y,w) \in \mathcal{P}$ by the requirement

$$w^{-1}(\nu_y + \mu(y,w)) \in \mathcal{F}(\lambda_a)$$

For instance, if $w \in W_{\mathcal{P}}(\nu_y)$, then $\mu(y,w) = w\nu_y - \nu_y$, and

$$w^{-1}(\nu_y + \mu(y,w)) = \nu_y.$$

In general, we have $w^{-1}(\nu_y + \mu(y,w)) = \nu_{yw}$.

We fix once and for all integral weights $\mu(y,w) \in \mathfrak{h}^*$ satisfying the above conditions, and use them to define a translation functor as follows. Fix $\nu_y \in \mathcal{F}(\lambda_a)$ and a simple root $\alpha \in \Delta_a^+$. Let s denote the corresponding reflection. If s is integral for λ_a, we have defined the pushing-to and pushing-off translation functors ϕ_α and ψ_α in Section 2. If s is not integral, we use $\mu(y,s)$ to define (nonintegral) wall crossing functor across the α wall,

$$\psi_\alpha = \psi_{\nu_y}^{\nu_{ys}} : \mathcal{HC}(\mathfrak{g},K)_{\nu_y} \to \mathcal{HC}(\mathfrak{g},K)_{\nu_{ys}}$$

as discussed in Section 2. Note that when $s \in W_{\mathcal{P}}(\nu_y)$, we recover Convention 2.2.

4. Cross-action

Let us make some remarks in connection with translation functors. Suppose that μ_a is an integral weight in $(\mathfrak{h}^a)^*$, and that F is the finite-dimensional representation of $\mathfrak{g}$ with extremal weight μ_a. Let $H_{\mathbb{R}}$ be an arbitrary Cartan subgroup of $G_{\mathbb{R}}$ and let i be one of the inner automorphism of $\mathfrak{g}$ carrying $\mathfrak{h}^a$ to $\mathfrak{h}$ of Section 1. Set $\mu = i(\mu_a)$. Suppose that μ_a was the integral weight used to define the translation functor $\psi_\beta = \psi_{\nu_a}^{\nu_a+\mu_a}$ with respect to a simple root β in Δ_a^+. We get an integral weight $i(\mu_\beta)$ in $\mathfrak{h}^*$. Let α be the transport of β by i and $\nu := i(\nu_a)$. Then μ can be used to define translation functors $\psi_\alpha = \psi_\nu^{\nu+\mu}$, $\phi_\alpha = \psi_{\nu+\mu}^{\nu}$ and obviously we have $\psi_\alpha = \psi_\beta$, $\phi_\alpha = \phi_\beta$. This will be used without further comment.

We will define the action of W_a on pseudocharacters, thereby extending the definitions of [Vgr, Definition 8.3.1] for the cross-action of $W(\lambda_a)$.

Definition 4.1. (i) Let $\nu_a = \nu_y \in \mathcal{F}(\lambda_a)$, and let $(H_\mathbb{R}, \gamma)$ be a ν_a-pseudocharacter. Let w be $W(\mathfrak{g}, \mathfrak{h})$ and let w_a in W_a be its image through the isomorphism $i := i_{\nu_a, \overline{\gamma}}$. We define the pseudocharacter $(H_\mathbb{R}, w \times \gamma)$ by $w \times \gamma = (w \times \Gamma, w \times \overline{\gamma})$ with

$$w \times \overline{\gamma} = \overline{\gamma} + i(\mu(y, w_a)) \text{ and } w \times \Gamma = \Gamma \otimes i(\mu(y, w_a)) \otimes \partial\rho(w)$$

where ρ_I (resp. ρ_c) denotes the half-sum of positive imaginary (resp. compact) roots in $\Delta^+(\overline{\gamma})$, $\partial\rho(w) := w \cdot (\rho_I - 2\rho_c) - (\rho_I - 2\rho_c)$ is a sum of root, and thus is well-defined as a one-dimensional representation of $H_\mathbb{R}$, and the weight $\mu(y, w_a)$ is fixed as in Section 3. The weight $i(\mu(y, w_a))$ is an integral weight in $\mathfrak{h}^*$, and as we have seen in Section 2, it determines uniquely a one-dimensional representation of $H_\mathbb{R}$, that we still denote by $i(\mu(y, w_a))$.
(ii) Suppose we are in the same setting as in (i), but we start with $w_a \in W_a$ instead of $w \in W(\mathfrak{g}, \mathfrak{h})$. We define the "abstract cross-action" of W_a on pseudocharacters by

$$w_a \times^a \gamma := w^{-1} \times \gamma.$$

For the reason of the appearance of the power -1 in (ii), we refer to [V4], Equation (2.8) and Definition 4.2. (The cross action is a right action.) Notice that the domain of the action of $W(\mathfrak{g}, \mathfrak{h})$ or W_a is the set of ν_a-pseudocharacter of $H_\mathbb{R}$ for all ν_a in $\mathcal{F}(\lambda_a)$.

We refer to [RT1], Section 1.2 for the role that ρ_I and ρ_c play in the compatibility conditions defining pseudocharacters. What follows easily from the definitions and the choices of the $\mu(y, w_a)$ is that $(H_\mathbb{R}, w \times \gamma)$ is indeed a $\nu_{yw} = \nu_y + \mu(y, w_a)$-pseudocharacter. If γ is a ν_a-pseudocharacter and w is integral for ν_a, then because of Convention 2.2, $(H_\mathbb{R}, w \times \gamma)$ is again a ν_a-pseudocharacter, and the definition coincides with Vogan's.

Remark 4.2. This definition of the cross-action depends on the choice of the weights $\mu(y, w)$ fixed in Section 3.

5. Cayley transforms

In this section, we recall basic facts about Cayley and inverse Cayley transforms. The results will be complemented in Section 6 for the more restrictive class of groups defined there.

Cayley transforms. Fix a Borel $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$, with $\mathfrak{h}$ defined over $\mathbb{R}$ and θ-stable and assume that α is a noncompact imaginary root for $\mathfrak{n}$. Choose a root vector X_α in $\mathfrak{g}$ such that $[X_\alpha, \overline{X_\alpha}] = h_\alpha$, where $h_\alpha \in \mathfrak{h}$ is the coroot of α. Let $c^\alpha = \mathrm{Ad}(\xi^\alpha)$, where $\xi^\alpha = \exp(\frac{\pi}{4}(\overline{X_\alpha} - X_\alpha))$ is an element of the adjoint group of $\mathfrak{g}$. Then $\mathfrak{h}^\alpha := c^\alpha(\mathfrak{h})$ is called the Cayley transform of $\mathfrak{h}$. Note that $\mathfrak{h}^\alpha$ is a θ-stable Cartan subalgebra of $\mathfrak{g}$ defined over $\mathbb{R}$, and $\beta = ({}^{tr}c^\alpha)^{-1}(\alpha)$ (say $\beta = c^\alpha(\alpha)$ for short) is a real root in $\Delta(\mathfrak{g}, \mathfrak{h}^\alpha)$, called the Cayley transform of α.

Let $(H_\mathbb{R}, \gamma)$ be a ν_a-pseudocharacter of $G_\mathbb{R}$, and suppose α is a noncompact imaginary root for γ. Let $H_\mathbb{R}^\alpha$ be the centralizer in $G_\mathbb{R}$ of $\mathfrak{h}^\alpha$. The type (I or II) of α and the Cayley transform $c^\alpha(\gamma)$ of γ by α are defined in Section 4 of [V1], with $c^\alpha(\gamma) = \{\gamma^\alpha\}$ or $c^\alpha(\gamma) = \{\gamma_+^\alpha, \gamma_-^\alpha\}$. If α is type I, or type II nonintegral then $c^\alpha(\gamma) = \{\gamma^\alpha\}$. If α is type II integral, and $G_\mathbb{R}$

has abelian Cartan subgroups, then $c^\alpha(\gamma) = \{\gamma_+^\alpha, \gamma_-^\alpha\}$. Without the assumption of abelian Cartan subgroup, if α is type II integral $c^\alpha(\gamma)$ could be single valued, with γ_α of twice the dimension of γ, or could be two pseudocharacters $\gamma_+^\alpha, \gamma_-^\alpha$ of the same dimension as γ.

Inverse Cayley transforms. Fix a Borel $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$ with $\mathfrak{h}$ defined over $\mathbb{R}$ and θ-stable, and assume that α is a real root for $\mathfrak{n}$. Choose a root vector X_α in $\mathfrak{g}_\mathbb{R}$ such that $[X_\alpha, \theta(X_\alpha)] = h_\alpha$, where $h_\alpha \in \mathfrak{h}_\mathbb{R}$ is the coroot of α. Let $c_\alpha = \mathrm{Ad}(\xi_\alpha)$, where $\xi_\alpha = \exp(\frac{i\pi}{4}(\theta(X_\alpha) - X_\alpha))$ is an element of the adjoint group of $\mathfrak{g}$. Define $\mathfrak{h}_\alpha := c_\alpha(\mathfrak{h})$, the inverse Cayley transforms of $\mathfrak{h}$. Then $\mathfrak{h}_\alpha$ is θ-stable and defined over $\mathbb{R}$, and $\beta = {}^{tr}c_\alpha^{-1}(\alpha)$ is a noncompact imaginary root in $\Delta(\mathfrak{g}, \mathfrak{h}_\alpha)$, called the inverse Cayley transform of α.

Let $\gamma = (\Gamma, \overline{\gamma})$ be a ν_a-pseudocharacter for $H_\mathbb{R}$, let α be a real root $\Delta(\mathfrak{h}, \mathfrak{g})$, and let $(H_\alpha)_\mathbb{R}$ be the centralizer of $\mathfrak{h}_\alpha$ in $G_\mathbb{R}$. Define $m_\alpha = \exp_{G_\mathbb{R}}(i\pi h_\beta) \in G_\mathbb{R}$; here h_β is the coroot of the inverse Cayley transform β of α. One can check that m_α is an element of $H_\mathbb{R} \cap (H_\alpha)_\mathbb{R}$, which depends on the choices only up to a replacement of m_α by m_α^{-1}.

Definition 5.1. In the setting above, let $\epsilon_\alpha = \pm 1$ be the sign defined in [Vgr, Definition 8.3.11]. We say that α satisfies the parity condition with respect to γ if and only if the eigenvalues of $\Gamma(m_\alpha)$ are of the form $\epsilon_\alpha \exp\left(\pm i\pi 2\frac{\langle \alpha, \overline{\gamma}\rangle}{\langle \alpha, \alpha\rangle}\right)$.

Let $(H_\mathbb{R}, \gamma)$ be a ν_a-pseudocharacter of $G_\mathbb{R}$, and let α be a real root for γ satisfying that parity condition. Let $(H_\alpha)_\mathbb{R}$ be the centralizer in $G_\mathbb{R}$ of $\mathfrak{h}_\alpha$. Then γ occurs in the right-hand side of Hecht-Schmid characters identity, and the left-hand side is a sum of two terms: one is a standard representation $X(\gamma')$, the other is a coherent continuation of a standard representation $X(\gamma'')$ (the role of γ' and γ'' can be exchanged). Then the inverse Cayley transform $c_\alpha(\gamma)$ is $\{\gamma_\alpha\} = \{\gamma'\}$ if $\gamma' = \gamma''$ or $c_\alpha(\gamma) = \{\gamma_\alpha^+, \gamma_\alpha^-\} = \{\gamma', \gamma''\}$ if $\gamma' \neq \gamma''$. If α is nonintegral, $c_\alpha(\gamma) = \{\gamma_\alpha\}$. If α is type I integral, $c_\alpha(\gamma) = \{\gamma_\alpha^+, \gamma_\alpha^-\}$. If α is type II integral, $c_\alpha(\gamma)$ can be either single or double valued. We refer to [Vgr, Section 8.3] or [V1] for omitted details.

Possible eigenvalues of $\Gamma(m_\alpha)$. Since the Cartan subgroups of $G_\mathbb{R}$ may not be abelian, the element m_α is not necessarily central in $H_\mathbb{R}$. But for all $h \in H_\mathbb{R}$, we have $hm_\alpha h^{-1} = m_\alpha^{\pm 1}$. Thus, $\Gamma(m_\alpha)$ has at most two distinct eigenvalues. When this happens, they are inverse to each other, with the same multiplicity. If the eigenvalues of $\Gamma(m_\alpha)$ are all in $\{\pm 1\}$, then $\Gamma(m_\alpha)$ is $\pm\mathrm{Id}$ on the representation space of Γ: indeed if the eigenvalue -1 (or 1) occurs, then the corresponding eigenspace is easily seen to be stable under Γ. Since Γ is irreducible, this proves the claim. If $G_\mathbb{R}$ is linear and has abelian Cartan subgroups, then $m_\alpha^2 = 1$ and it follows that $\Gamma(m_\alpha) = \pm 1$ (in this case Γ is a one-dimensional representation).

Assume that the root system $\Delta = \Delta(\mathfrak{g}, \mathfrak{h}^a)$ is simple and fix a pseudocharacter $(H_\mathbb{R}, \gamma)$ as above. Retain the notation above. According to [V2], proof of Lemma 6.18, we are in one of the following cases:

Case I. $\Gamma(m_\alpha) \in \{\pm\mathrm{Id}\}$ for all real roots α in $\Delta(\mathfrak{g}, \mathfrak{h})$. (This is always the case if $G_\mathbb{R}$ is linear.)

Case II. $\Gamma(m_\alpha^2) \in \{\pm\mathrm{Id}\}$ for all real roots α in $\Delta(\mathfrak{g}, \mathfrak{h})$ but $\Gamma(m_\alpha)$ is not always in $\{\pm\mathrm{Id}\}$. Then $\Gamma(m_\alpha) \in \{\pm\mathrm{Id}\}$ if α is short in type B_n, C_n, F_4, and the eigenvalues of $\Gamma(m_\alpha)$ are $\pm i$ otherwise (each with the same multiplicity if both appear, and $\Gamma(m_\alpha) = \pm i\,\mathrm{Id}$ if there is only one eigenvalue).

Case III. $\Gamma(m_\alpha^2)$ is not always in $\{\pm\mathrm{Id}\}$.

Convention 5.2. It will be convenient to refer to all roots in type A_n, D_n, E_n and G_2 as long roots. (This convention is unusual for G_2.)

Nonintegral wall-crosses. We need a result of Vogan describing translation functors across a nonintegral wall.

Theorem 5.3 ([V1], Corollary 4.8 and Lemma 4.9)**.** *Let γ be a genuine ν_a-pseudocharacter of $G_{\mathbb{R}}$. Suppose α is a nonintegral simple root in $\Delta^+(\overline{\gamma})$. Then, with the translation functor ψ_α defined by the weight μ_α fixed in Section 3, we have*

$$\psi_\alpha(\overline{X}(\gamma)) = \begin{cases} \overline{X}((\gamma+\mu_\alpha)^\alpha) = \overline{X}((s_\alpha \times \gamma)^\alpha) & \textit{if } \alpha \textit{ is noncompact imaginary,} \\ \overline{X}((\gamma+\mu_\alpha)_\alpha) = \overline{X}((s_\alpha \times \gamma)_\alpha) & \textit{if } \alpha \textit{ is real satisfying the parity condition} \\ \overline{X}(\gamma+\mu_\alpha) = \overline{X}(s_\alpha \times \gamma) & \textit{otherwise.} \end{cases}$$

Note that in the first case, $\gamma+\mu_\alpha = (\Gamma\otimes\mu_\alpha, \bar\gamma+\mu_\alpha)$ is not a pseudocharacter. Nevertheless, its Cayley transform is well defined ([V1], Section 4), and it is a pseudocharacter (namely $(s_\alpha \times \gamma)^\alpha$).

6. Nonlinear double covers

We will now change our notation slightly and consider the following situation: $\widetilde{G}_{\mathbb{R}}$ will be a double cover of a real reductive linear group $G_{\mathbb{R}}$ in the class defined in [Vgr], i.e. we have a central extension

$$1 \longrightarrow \{\mathbf{e},\mathbf{z}\} \longrightarrow \widetilde{G}_{\mathbb{R}} \xrightarrow{\mathbf{pr}} G_{\mathbb{R}} \longrightarrow 1$$

where $\{\mathbf{e},\mathbf{z}\}$ is a two-elements central subgroup of $\widetilde{G}_{\mathbb{R}}$; here $\mathbf{e}$ is the trivial element in $\widetilde{G}_{\mathbb{R}}$. (The trivial element in $G_{\mathbb{R}}$ will be simply denoted by 1.)

As a set, $\widetilde{G}_{\mathbb{R}} \simeq G_{\mathbb{R}} \times \{\mathbf{e},\mathbf{z}\}$ and the multiplication law on the right hand side is given by

$$(g,\epsilon)(g',\epsilon') = (gg', \epsilon\epsilon' c(g,g')),$$

c being a cocycle with values in $\{\mathbf{e},\mathbf{z}\}$.

If $M_{\mathbb{R}}$ is a subgroup of $G_{\mathbb{R}}$, we let $\widetilde{M_{\mathbb{R}}}$ denote its inverse image in $\widetilde{G}_{\mathbb{R}}$. It is clear that if $K_{\mathbb{R}}$ is a maximal compact subgroup of $G_{\mathbb{R}}$, then $\widetilde{K}_{\mathbb{R}}$ is a maximal compact subgroup of $\widetilde{G}_{\mathbb{R}}$. Since the adjoint action of $\widetilde{G}_{\mathbb{R}}$ on $\mathfrak{g}_{\mathbb{R}}$ factors through $G_{\mathbb{R}}$, the inverse image $\widetilde{H}_{\mathbb{R}}$ of a Cartan subgroup $H_{\mathbb{R}}$ of $G_{\mathbb{R}}$ is again a Cartan subgroup of $\widetilde{G}_{\mathbb{R}}$. Suppose some choices of a maximal compact subgroup $K_{\mathbb{R}}$ and a maximally θ-split Cartan subgroup $H^a_{\mathbb{R}}$ have been made for $G_{\mathbb{R}}$ as in Section 1. Then we get corresponding choices $\widetilde{K}_{\mathbb{R}}$ and $\widetilde{H}^a_{\mathbb{R}}$ for $\widetilde{G}_{\mathbb{R}}$.

In Section 2 we showed how to replace the group we are interested in by a finite central covering with the property that every finite-dimensional representation of $\mathfrak{g}$ exponentiates. We used a well-known fibered product construction to construct this cover. Let us see briefly how this applies to our situation. The projection $\widetilde{G}_{\mathbb{R}} \xrightarrow{\mathbf{pr}} G_{\mathbb{R}}$, the inclusion $G_{\mathbb{R}} \xrightarrow{\mathbf{i}} G_{\mathbb{C}}$, the adjoint morphism $G_{\mathbb{C}} \xrightarrow{\mathrm{Ad}} \mathcal{G}^{ad}$, and the projection $\mathcal{G}^{sc} \xrightarrow{\mathbf{q}} \mathcal{G}^{ad}$ fit into a commutative diagram :

$$\begin{tikzcd}
\overline{\widetilde{G}_{\mathbb{R}}} \arrow[r] \arrow[d] & \bar{G}_{\mathbb{R}} \arrow[r] \arrow[d] & \bar{G}_{\mathbb{C}} \arrow[r] \arrow[d] & \mathcal{G}^{sc} \arrow[d, "q"] \\
\widetilde{G}_{\mathbb{R}} \arrow[r, "\mathbf{pr}"] & G_{\mathbb{R}} \arrow[r, "\mathbf{i}"] & G_{\mathbb{C}} \arrow[r, "\mathrm{Ad}"] & \mathcal{G}^{ad}
\end{tikzcd}$$

If finite-dimensional representations of $\mathfrak{g}$ don't exponentiate to $\widetilde{G}_{\mathbb{R}}$, we replace the triplet $(\widetilde{G}_{\mathbb{R}}, G_{\mathbb{R}}, G_{\mathbb{C}})$ by the triplet $(\overline{\widetilde{G}_{\mathbb{R}}}, \bar{G}_{\mathbb{R}}, G_{\mathbb{C}})$. This new triplet satisfies the same hypotheses as the old one, every finite-dimensional representation of $\mathfrak{g}$ exponentiates to $\overline{\widetilde{G}_{\mathbb{R}}}$, with one-dimensional weights spaces with respect to Cartan subgroups, and the representation theory of $\widetilde{G}_{\mathbb{R}}$ can be deduced from the representation theory of $\overline{\widetilde{G}_{\mathbb{R}}}$.

Genuine modules. Let X be a module in $\mathcal{HC}(\mathfrak{g}, \widetilde{K})$, i.e. a Harish-Chandra module for $\widetilde{G}_{\mathbb{R}}$. Suppose that X admits a central character $\chi_X : Z(\widetilde{G}_{\mathbb{R}}) \to \mathbb{C}^*$. Since $\mathbf{z} \in Z(\widetilde{G}_{\mathbb{R}})$ has order two, we have $\chi_X(\mathbf{z}) = \pm 1$ and $\chi_X(\mathbf{z}) = 1$ if and only if X is the lift of a Harish-Chandra module for $G_{\mathbb{R}}$. If $\chi_X(\mathbf{z}) = -1$, we call X genuine. We denote by $\mathcal{HC}(\mathfrak{g}, \widetilde{K})^{\mathrm{gen}}$ the full subcategory of $\mathcal{HC}(\mathfrak{g}, \widetilde{K})$ generated by irreducible genuine modules.

Now let $\widetilde{H}_{\mathbb{R}}$ be a θ-stable Cartan subgroup of $\widetilde{G}_{\mathbb{R}}$, and let $\gamma = (\Gamma, \overline{\gamma})$ be a ν_a-pseudocharacter for $\widetilde{H}_{\mathbb{R}}$. We say that γ is genuine if $\Gamma(\mathbf{z}) = -1$. The following result is immediate.

Proposition 6.1. *a) The standard and irreducible modules $X(\gamma)$ and $\overline{X}(\gamma)$ are genuine if and only if γ is. The $\widetilde{K}_{\mathbb{R}}$-conjugacy classes of genuine ν_a-pseudocharacters parameterize the irreducible objects in $\mathcal{HC}(\mathfrak{g}, \widetilde{K})^{\mathrm{gen}}_{\nu_a}$.*

b) If $\widetilde{G}_{\mathbb{R}}$ is a trivial cover of $G_{\mathbb{R}}$, i.e. $\widetilde{G}_{\mathbb{R}} \simeq \mathbb{Z}/2\mathbb{Z} \times G_{\mathbb{R}}$, there is an obvious one-to-one correspondence between genuine objects for $\widetilde{G}_{\mathbb{R}}$ and objects for the linear group $G_{\mathbb{R}}$. Thus the genuine representation theory of $\widetilde{G}_{\mathbb{R}}$ reduces to the representation theory of the linear group $G_{\mathbb{R}}$.

Metaplectic roots. Suppose that $\mathfrak{h}_{\mathbb{R}}$ is a Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$ and that α is a real root in $\Delta(\mathfrak{g}, \mathfrak{h})$. Choose an $\mathfrak{sl}_2$-triple $X_\alpha, X_{-\alpha}, h_\alpha$ in $\mathfrak{g}_{\mathbb{R}}$ as in Section 5. Thus we get an embedding $\phi : \mathfrak{sl}(2, \mathbb{R}) \to \mathfrak{g}_{\mathbb{R}}$. Since $G_{\mathbb{R}}$ is linear, ϕ lifts to a homomorphism denoted again ϕ from $SL(2, \mathbb{R})$ to $G_{\mathbb{R}}$. The element m_α in $G_{\mathbb{R}}$ defined in Section 5 is the image of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in SL(2, \mathbb{R})$. On the other hand, since $\widetilde{G}_{\mathbb{R}}$ is not linear, ϕ doesn't necessarily lift to an homomorphism from $SL(2, \mathbb{R})$ to $\widetilde{G}_{\mathbb{R}}$: one has to consider the metaplectic cover $\widetilde{SL}(2, \mathbb{R})$ instead. Then ϕ lifts to a homomorphism $\tilde{\phi} : \widetilde{SL}(2, \mathbb{R}) \to \widetilde{G}_{\mathbb{R}}$. We have a commutative diagram:

$$\begin{array}{ccc} \widetilde{SL}(2,\mathbb{R}) & \xrightarrow{\mathbf{pr}} & SL(2,\mathbb{R}) \\ \Big\downarrow{\scriptstyle \tilde{\phi}} & & \Big\downarrow{\scriptstyle \phi} \\ \widetilde{G}_{\mathbb{R}} & \xrightarrow{\mathbf{pr}} & G_{\mathbb{R}} \end{array}$$

Definition 6.2. In case $\tilde{\phi}$ doesn't factor through $SL(2, \mathbb{R})$, we will call the real root α metaplectic. We will apply also this terminology to noncompact imaginary roots: if β is such a root, we call it metaplectic if and only if its Cayley transform is metaplectic.

Definition 6.3. We say that the covering $\widetilde{G}_{\mathbb{R}}$ is genuine if, in the setting above, a real or imaginary noncompact root is metaplectic if and only if it is a long root (with Convention (5.2)).

In the rest of the paper, we assume that the covering $\widetilde{G}_{\mathbb{R}}$ is genuine. Together with the assumptions we made on $\widetilde{G}_{\mathbb{R}}$ at the beginning of this section and the technical assumption made in 6.15 define the class of groups for which we shall establish a Kazhdan-Lusztig algorithm. This class is far from being empty; some examples are given at the end of this section.

Let $\tilde{m}_\alpha$ to be the element constructed in Section 5, but for $\widetilde{G}_\mathbb{R}$ instead of $G_\mathbb{R}$. Recall that the construction of m_α and $\tilde{m}_\alpha$ depends on the choice of a Cayley transform c_α. (A different choice would lead at worst to a replacement of m_α (resp. $\tilde{m}_\alpha$) by m_α^{-1} (resp. $\tilde{m}_\alpha^{-1}$).) Using the same Cayley transform to construct both m_α and $\tilde{m}_\alpha$, we get $m_\alpha = \mathbf{pr}(\tilde{m}_\alpha)$. Note that we always have $m_\alpha^2 = 1$ and thus $\tilde{m}_\alpha^2 = \mathbf{e}$ or $\mathbf{z}$.

Since $\phi : \mathfrak{sl}(2,\mathbb{R}) \to \mathfrak{g}_\mathbb{R}$ is injective, the image of $\phi : SL(2,\mathbb{R}) \to G_\mathbb{R}$ is either isomorphic to $SL(2,\mathbb{R})$, and $m_\alpha \neq 1$, or to $SL(2,\mathbb{R})/\{\pm \mathrm{Id}\}$ and $m_\alpha = 1$. Analogously, the image of $\tilde{\phi} : \widetilde{SL}(2,\mathbb{R}) \to \widetilde{G}_\mathbb{R}$ is isomorphic to $\widetilde{SL}(2,\mathbb{R})$, and $\tilde{m}_\alpha^2 = \mathbf{z}$, or to $SL(2,\mathbb{R})$ and $\tilde{m}_\alpha^2 = \mathbf{e}$, $\tilde{m}_\alpha \neq \mathbf{e}$, or to $SL(2,\mathbb{R})/\{\pm \mathrm{Id}\}$ and $\tilde{m}_\alpha = \mathbf{e}$. It is clear that $\tilde{m}_\alpha$ has order four exactly when the image of $\tilde{\phi} : \widetilde{SL}(2,\mathbb{R}) \to \widetilde{G}_\mathbb{R}$ is isomorphic to $\widetilde{SL}(2,\mathbb{R})$, i.e. when α is metaplectic.

Non-metaplectic roots. In the setting of the previous paragraph, we get the following result about non-metaplectic noncompact imaginary or real roots :

Lemma 6.4. *let us consider the situation where α is a short real root (and thus non-metaplectic). Then the cover $\widetilde{G}_\mathbb{R}$ splits over $\phi(SL(2,\mathbb{R})) \subset G_\mathbb{R}$.*

Proof. Let us consider a root subsystem of $\Delta(\mathfrak{g},\mathfrak{h})$ of type B_2 containing α. Using standard notation $\{\pm\epsilon_1 \pm \epsilon_2, \pm 2\epsilon_1, \pm 2\epsilon_2\}$ for a root system of type B_2, we may assume that $\alpha = \epsilon_1 + \epsilon_2$. Since α is real $\theta(\epsilon_1+\epsilon_2) = -\epsilon_1 - \epsilon_2$, which gives $\theta(\epsilon_i) = -\epsilon_i$ or $\theta(\epsilon_i) = -\epsilon_{3-i}$ for $i = 1, 2$. In the first case, all the roots in our B_2 subsystem are real, in the second case, $\epsilon_1 - \epsilon_2$ is imaginary, and the two long roots are complex, and exchanged by θ. In any case, the Lie subalgebra of $\mathfrak{g}$ of type B_2 we consider is defined over $\mathbb{R}$. Let S be the corresponding analytic subgroup of $\widetilde{G}_\mathbb{R}$. Listing the real connected groups with Lie algebra of type B_2, we see that the only one satisfying the property that long real roots are metaplectic is $\mathrm{Mp}(4,\mathbb{R})$ ($\mathrm{Sp}(1,1)$ has no nonlinear double cover). Now, in $\mathrm{Mp}(4,\mathbb{R})$, the property we want to establish holds, namely, the cover splits over $\phi(SL(2,\mathbb{R})) \subset \mathrm{Sp}(4,\mathbb{R})$, and thus it holds in $\widetilde{G}_\mathbb{R}$. $\square$

Extended integral root system and extended integral length. We introduce the following definition.

Definition 6.5. Let $\mathfrak{h}_\mathbb{R}$ be a θ-stable Cartan subalgebra of $\mathfrak{g}_\mathbb{R}$, and let $\lambda \in \mathfrak{h}^*$ be a regular element. Let $\widetilde{R}(\lambda)$ be the set of roots in $\Delta(\mathfrak{g},\mathfrak{h})$ which are short integral or long and half-integral. (Short and long roots are defined as in Section 5.2 and a root α is half-integral if α is integral for 2λ.)

We also set
$$\widetilde{R}(\lambda)^+ := \widetilde{R}(\lambda) \cap \Delta(\lambda)^+, \qquad \widetilde{W}(\lambda) := W(\widetilde{R}(\lambda)).$$

It is easily seen that $\widetilde{R}(\lambda)$ is a root system, which we shall call the extended integral root system for λ. We call respectively $\widetilde{R}(\lambda)^+$ and $\widetilde{W}(\lambda)$ the set of positive extended integral roots and the extended integral Weyl group.

Lemma 6.6. *Let $(\widetilde{H}_\mathbb{R}, \gamma)$ be a genuine pseudocharacter of $\widetilde{G}_\mathbb{R}$. Then $\widetilde{R}(\overline{\gamma})$ is θ-stable.*

It is important to note that $R(\overline{\gamma})$ need *not* be θ-stable in general for nonlinear groups.

Proof. Suppose that α is a root in $\widetilde{R}(\overline{\gamma})$. If α is imaginary or real, obviously $\theta\alpha$ belongs to $\widetilde{R}(\overline{\gamma})$. Thus, suppose that α is complex and let H_α denotes the corresponding coroot. First, assume that α is a long root (Convention 5.2). Since $\overline{\gamma}(H_\alpha) \in \mathbb{Z}/2$, we have $\overline{\gamma}(H_{\theta\alpha}) \in \mathbb{Z}/2$ iff $\overline{\gamma}(H_\alpha + H_{\theta\alpha}) \in \mathbb{Z}/2$. The Cartan decomposition of $\widetilde{H}_\mathbb{R}$ can be written $\widetilde{H}_\mathbb{R} = \widetilde{T}_\mathbb{R} \exp \mathfrak{a}_\mathbb{R}$. Recall that $\gamma = (\Gamma, \overline{\gamma})$ is a pseudocharacter of $\widetilde{G}_\mathbb{R}$, and $d\Gamma = \overline{\gamma} - \rho_I + 2\rho_c$. Since $2\rho_c$ is a sum of roots, $2\rho_c(H_\alpha)$ and $2\rho_c(H_{\theta\alpha})$ are in $\mathbb{Z}$. Furthermore $\rho_I(H_{\theta\alpha}) = \theta(\rho_I)(H_\alpha) = \rho_I(H_\alpha)$, and thus $\overline{\gamma}(H_\alpha + H_{\theta\alpha}) \in \mathbb{Z}/2$ iff $d\Gamma(H_\alpha + H_{\theta\alpha}) \in \mathbb{Z}/2$. Let $(T_\mathbb{R})_0$ (resp. $(\widetilde{T}_\mathbb{R})_0$) be the connected component of the identity in $T_\mathbb{R}$(resp. $\widetilde{T}_\mathbb{R}$), and T (resp. $\widetilde{T}$) its complexification. Now Γ is an irreducible representation of $\widetilde{H}_\mathbb{R}$, and its restriction to $(\widetilde{T}_\mathbb{R})_0$ splits into a direct sum of one-dimensional representations, which, by differentiation, give all the same element of $X^*(\widetilde{T})$, namely $d\Gamma$. Notice that $H_\alpha + H_{\theta\alpha} \in i\mathfrak{t}^* \cap Q\check{}$, where $Q\check{}$ is the coroot lattice of $\mathfrak{h}$ in $\mathfrak{g}$. Thus we have

$$H_\alpha + H_{\theta(\alpha)} \in \mathfrak{t}^* \cap Q\check{} \subset \mathfrak{t}^* \cap X_*(H) = X_*(T).$$

Since $\widetilde{T}$ is at most a double cover of T, we have $[X^*(T) : X^*(\widetilde{T})] = 1$ or 2, and we see that if $\Lambda \in X^*(\widetilde{T})$ and $X \in X_*(T)$, we have $\Lambda(X) \in \mathbb{Z}/2$, and we get the desired conclusion. Assume now that α is short. By a similar argument, it is enough to show that $d\Gamma(H_\alpha + H_{\theta\alpha}) \in \mathbb{Z}$. Since α and $\theta\alpha$ have the same length,

$$N_\alpha := \theta\alpha(H_\alpha) = \alpha(\theta(H_\alpha)) = \alpha(H_{\theta\alpha})$$

takes values in $\{0, -1, +1\}$. Let R_α be the root subsystem of $\Delta(\mathfrak{g}, \mathfrak{h})$ generated by α and $\theta\alpha$: it's a rank 2 root system, and thus it is of type $A_1 \times A_1$ or B_2 if $N_\alpha = 0$, or of type A_2 if $N_\alpha = \pm 1$ (type G_2 is excluded because, according to our convention, there are no short roots in type G_2). Suppose R_α is of type $A_1 \times A_1$. Let X_α and $X_{-\alpha}$ be root vectors in $\mathfrak{g}$ such that $\{X_\alpha, H_\alpha, X_{-\alpha}\}$ is an $\mathfrak{sl}_2$-triple, and take root vectors $X_{\theta\alpha} = \theta(X_\alpha)$ and $X_{-\theta\alpha} = \theta(X_{-\alpha})$ as root vectors for $\theta\alpha$ and $-\theta\alpha$. A straightforward computation, using the fact that $N_\alpha = 0$ and that $\alpha - \theta\alpha$ is not a root, shows that $H_\beta := H_\alpha + H_{\theta_\alpha} \in \mathfrak{t}$, $X_\beta := X_\alpha + X_{\theta_\alpha}$ and $X_{-\beta} := X_{-\alpha} + X_{-\theta_\alpha}$ forms an $\mathfrak{sl}_2$-triple in $\mathfrak{k}$. This lifts to a map from $SU(2)$ to $\widetilde{K}_\mathbb{R}$. It is well-known this implies that $d\Gamma(H_\beta) \in \mathbb{Z}$. Suppose now that R_α is of type B_2. Then $\alpha + \theta\alpha$ is a long root imaginary root β, and thus $H_\alpha + H_{\theta\alpha} = 2H_\beta$. Since $d\Gamma(H_\beta) \in \mathbb{Z}/2$, we get the result. Suppose now that R_α is of type A_2. If $N_\alpha = -1$, then $\beta := \alpha + \theta\alpha$ is a short imaginary root, $H_\alpha + H_{\theta\alpha} = H_\beta$ and $d\Gamma(H_\beta) \in \mathbb{Z}$. If $N_\alpha = 1$, notice that R_α is in an irreducible component of $\Delta(\mathfrak{g}, \mathfrak{h})$ of type C_n or F_4, because all short roots are orthogonal in type B_n. Take the standard roots in type C_n. We may assume without lost of generality that $\alpha = \epsilon_1 - \epsilon_2$, $\theta\alpha = \epsilon_1 - \epsilon_3$, which implies that $\theta(\epsilon_1) = \epsilon_1$, $\theta(\epsilon_2) = \epsilon_3$. Thus the roots $2\epsilon_1$ and $\epsilon_2 + \epsilon_3$ are imaginary, and $H_{\epsilon_1 - \epsilon_2} + H_{\epsilon_1 - \epsilon_3} = 2H_{2\epsilon_1} - H_{\epsilon_2 + \epsilon_3}$. We can conclude from here as above. Since no new ideas are required, we we omit the computation in type F_4. □

The following is the key definition needed for the inductive computation of Kazhdan-Lusztig polynomials.

Definition 6.7. Let $(\widetilde{H}_\mathbb{R}, \gamma)$ be a genuine pseudocharacter of $\widetilde{G}_\mathbb{R}$, where $\mathfrak{h}_\mathbb{R}$ has Cartan decomposition $\mathfrak{h}_\mathbb{R} = \mathfrak{t}_\mathbb{R} + \mathfrak{a}_\mathbb{R}$. The extended integral length of γ is

$$\widetilde{l}^I(\gamma) = \frac{1}{2}|\{\alpha \in \widetilde{R}^+(\overline{\gamma}) \mid \theta(\alpha) \notin \widetilde{R}^+(\overline{\gamma})\}| + \frac{1}{2}\dim \mathfrak{a}_\mathbb{R}.$$

Some key lemmas. We will now state and prove the key structural results we need for the Kazhdan-Lusztig algorithm.

Lemma 6.8. *Let $(\widetilde{H}_{\mathbb{R}}, \gamma)$ be a genuine pseudocharacter of $\widetilde{G}_{\mathbb{R}}$, and let α be a real root in $\Delta(\mathfrak{g}, \mathfrak{h})$. Then we are always in Case II of Section 5, i.e. we have $\Gamma(\tilde{m}_\alpha) \in \{\pm Id\}$ if α is short and the eigenvalues of $\Gamma(\tilde{m}_\alpha)$ are $\pm i$ if α is long. (Recall Convention 5.2: all roots in type A_n, D_n, E_n and G_2 are long.)*

Proof. Suppose α is long, and hence metaplectic. Then $\tilde{m}_\alpha$ has order 4, with $\tilde{m}_\alpha^2 = \mathbf{z}$. Since γ is genuine, $\Gamma(\mathbf{z}) = -\operatorname{Id}$ and thus the eigenvalues of $\Gamma(\tilde{m}_\alpha)$ are $\pm i$. ☐

Corollary 6.9. *Let $(\widetilde{H}_{\mathbb{R}}, \gamma)$ be a genuine pseudocharacter of $\widetilde{G}_{\mathbb{R}}$, and let α be a real root in $\Delta^+(\overline{\gamma})$.*

1. *If α is not an element of the extended integral roots system $\widetilde{R}(\nu_a)$ (Definition 6.5), then α does not satisfy the parity condition with respect to γ.*
2. *If we assume that α is long and in $\widetilde{R}(\nu_a)$, then α satisfies the parity condition if and only if α is half-integral but not actually integral.*

In particular, all long half-integral (but not integral) real roots satisfy the parity condition. If α is short and integral, it may or may not satisfy the parity conditions.

Proof. Suppose first that $\alpha \in \widetilde{R}(\nu_a)$ is long and real. By Lemma 6.8, the eigenvalues of $\Gamma(\tilde{m}_\alpha)$ are $\pm i$. Moreover $\exp\left(\pm 2i\pi \frac{\langle \alpha, \overline{\gamma} \rangle}{\langle \alpha, \alpha \rangle}\right) = \pm i$ if and only if α is half-integral (but not integral). This proves (2). The first assertion is clear.

We have a corresponding dual result for imaginary roots.

Corollary 6.10. *Retain the hypothesis of Corollary 6.9, but now let α be an imaginary root in $\Delta^+(\overline{\gamma})$. Then α always belongs to the extended integral roots system $\widetilde{R}(\overline{\gamma})$. Moreover,*

1. *If α is compact, then α is integral.*
2. *If α is noncompact, then α is half-integral (but possibly integral).*
3. *If we further assume that α is long, then α is compact if and only if α is integral.*

In particular, all long half-integral (but not integral) imaginary roots are noncompact.

Proof. The first assertion is a consequence of (1)–(2), and (1)–(2) follow exactly in the same way as the corresponding integrality conditions for linear groups; see, for instance, [V1], Section 4. To prove the final assertion, we must show that if α is noncompact imaginary and long, α cannot be integral. Suppose it were. We would then obtain a contradiction with Corollary 6.9(2) by taking the Cayley transform with respect to α. ☐

Remark 6.11. The previous corollaries fail for higher covers than double covers. Since they are essential in the proof of Theorem 8.1, we see that the extended integral length is not the right thing to consider in the case of these higher covers. We do not know at the present time how to modify the definitions of extended integral length, extended integral root system, etc, to recover the vanishing result of Theorem 8.1.

Cayley transforms and cross-action. In Section 5, we saw that for groups having non-abelian Cartan subgroups, the nature (single or double valued) of the Cayley transform of a genuine pseudocharacter with respect to a short integral type II real or imaginary noncompact root is not determined. We give more details on this. This will lead us to slightly change the terminology : sometimes short integral type II real or imaginary noncompact root behave like type I roots, so it will be notationally easier to simply redefine them as type I roots. (See the paragraph immediately before Lemma 6.14 below.)

Let $(\gamma, \widetilde{H}_{\mathbb{R}})$ be a genuine pseudocharacter of $\widetilde{G_{\mathbb{R}}}$, and let α be a short (and thus integral) noncompact imaginary root for γ. Consider the Cayley transform c^α with respect to α, denote by $\widetilde{H}^\alpha_{\mathbb{R}}$ the corresponding Cartan subgroup. Write $\widetilde{H}_{\mathbb{R}} = \widetilde{T}_{\mathbb{R}} \exp \mathfrak{a}_{\mathbb{R}}$ and $\widetilde{H}^\alpha_{\mathbb{R}} = \widetilde{T}^\alpha_{\mathbb{R}} \exp \mathfrak{a}^\alpha_{\mathbb{R}}$ for the Cartan decompositions of $\widetilde{H}_{\mathbb{R}}$ and $\widetilde{H}^\alpha_{\mathbb{R}}$. Let us write $T^1 = \widetilde{T}_{\mathbb{R}} \cap \widetilde{T}^\alpha_{\mathbb{R}}$. Notice that T^1 has the same Lie algebra as $\widetilde{T}^\alpha_{\mathbb{R}}$. Denote by α_1 the Cayley transform of α by c^α. Thus α_1 is a real root in $\Delta(\mathfrak{g}, \mathfrak{h}^\alpha)$.

Recall that α is type II if one of the following equivalent condition is satisfied :

- the reflection s_α with respect to the root α is realized in $W(\widetilde{G_{\mathbb{R}}}, \mathfrak{h})$
- T^1 is of index two in $\widetilde{T}^\alpha_{\mathbb{R}}$.
- there exists an element $t \in \widetilde{T}^\alpha_{\mathbb{R}}$ such that $\alpha_1(t) = -1$.

Proposition 6.12. *Suppose α is type II. Then the Cayley transform $c^\alpha(\gamma)$ is single valued if and only if $s_\alpha \times \gamma$ is not equivalent to γ (i.e. not conjugate under $\widetilde{K}_{\mathbb{R}}$).*

Proof. Let $m \in \widetilde{T}^\alpha_{\mathbb{R}} \backslash T^1$. Then $c^\alpha(\gamma)$ is single valued if and only if $\Gamma^m_{|T^1}$ is not equivalent to $\Gamma_{|T^1}$ (this is a simple application of Mackey theory). Suppose that $s_\alpha \times \gamma$ is $\widetilde{K}_{\mathbb{R}}$-conjugate to $m \cdot \gamma$. (Notice that we cannot write $s_\alpha \cdot \gamma$ since the action of s_α is not well-defined on $\widetilde{H}_{\mathbb{R}}$; indeed, since $\widetilde{H}_{\mathbb{R}}$ is not abelian, the conjugation action of a representative in $\widetilde{K}_{\mathbb{R}}$ of s_α on $\widetilde{H}_{\mathbb{R}}$ will depend on the choice of this representative. Of course the two pseudocharacters obtained by different choices of representative are equivalent, so we may denote by $s_\alpha \cdot \gamma$ their equivalence class. Anyway, here we choose a representative m.) Since these two pseudocharacters have the same infinitesimal component $s_\alpha \cdot \bar{\gamma}$, which is a regular element in $\mathfrak{h}$, if they are $\widetilde{K}_{\mathbb{R}}$-conjugate it must be by an element h of $\widetilde{H}_{\mathbb{R}}$. For all $t \in \widetilde{H}_{\mathbb{R}}$, $(s_\alpha \times \Gamma)(t) = \Gamma(t)\, \alpha(t)^{n+1}$, and $(m \cdot \Gamma)(t) = \Gamma(mtm^{-1})$. For $t \in T^1$, we have $\alpha(t) = 1$ and thus we get

$$\Gamma(hth^{-1}) = \Gamma(mtm^{-1}), \qquad (t \in T^1).$$

Since $(\widetilde{H}_{\mathbb{R}})_0$ is central in $\widetilde{H}_{\mathbb{R}}$, we can multiply the element h in the left hand side of the above equation by an element $h_0 \in (\widetilde{H}_{\mathbb{R}})_0$ to get an element $h_1 = hh_0$ in T^1 still satisfying the above equation. Thus, we see that $\Gamma^m_{|T^1}$ is equivalent to $\Gamma_{|T^1}$, so $c^\alpha(\gamma)$ is double valued.

Let us prove the other implication. Suppose now that $c^\alpha(\gamma)$ is double valued. Take m as above. Our assumption is that there exists an intertwining operator B such that

$$B\Gamma(t) = \Gamma(mtm^{-1})B, \qquad (t \in T^1).$$

If t and m commute, then B and $\Gamma(t)$ commute; if they don't, then $mtm^{-1} = \mathbf{z}t$, so $\Gamma(mtm^{-1}) = \Gamma(\mathbf{z}t) = -\Gamma(t)$, and $\Gamma(t)$ anticommute with B. Then m fixes pointwise the center of T^1. Indeed $\Gamma_{|T^1}$ is irreducible, so elements of the center acts by scalars, and they commute with B. We need the following result, which was indicated to us by J. Adams.

Lemma 6.13. *Suppose A is an abelian group, and*

$$1 \to \mathbb{Z}/2\mathbb{Z} \to \widetilde{A} \to A \to 1$$

is a central extension of A. Denote by $\widetilde{Z}$ the center of $\widetilde{A}$. Then, for every genuine irreducible representation χ of $\widetilde{A}$ there is a unique genuine representation $\pi(\chi)$ of $\widetilde{A}$ for which $\pi(\chi)_{|\widetilde{Z}}$ is a multiple of χ. The map $\chi \mapsto \pi(\chi)$ is a bijection between the set of classes of genuine irreducible representations of $\widetilde{Z}$ and the set of classes of genuine irreducible representations of $\widetilde{A}$. The dimension of $\pi(\chi)$ is $n = |\widetilde{A}/\widetilde{Z}|^{1/2}$ and $\mathrm{Ind}_{\widetilde{Z}}^{\widetilde{A}}\chi = n\pi(\chi)$.

This is elementary representation theory, and the proof is left to the reader. Let us now finish the proof of the proposition. Since $\Gamma_{|T^1}$ and $\Gamma^m_{|T^1}$ have the same central character, say χ, we see that both are embedded in $\mathrm{Ind}_{\widetilde{Z}(T^1)}^{T^1}\chi$. Thus they must be equal up to conjugacy by an element t_1 of T^1. It is then clear than $s_\alpha \times \Gamma$ and $m \cdot \Gamma$ are conjugate by t_1. ☐

Each Cayley transform of a pseudocharacter γ gives rise to a corresponding Hecht-Schmid identity. From the point of view of the Kazhdan-Lusztig algorithm, it is the form of this identity which matters. In the setting above, the Hecht-Schmid identity reads

$$[X(\gamma)] + [\phi_\alpha(s_\alpha \times \gamma)] = [X(\gamma^\alpha)] \text{ or } [X(\gamma^\alpha_+)] + [X(\gamma^\alpha_-)]$$

depending on $c^\alpha(\gamma) = \{\gamma^\alpha\}$ or $c^\alpha(\gamma) = \{\gamma^\alpha_+, \gamma^\alpha_-\}$.

Thus, with respect to the Kazhdan-Lusztig algorithm, single valued Cayley transforms with respect to a short integral noncompact imaginary type II root really behave exactly as a type I root. So, from now on, *we will call these roots type I.* The same change in terminology will apply to their Cayley transforms. With this convention, we get the following analog of Proposition 8.3.18 in [Vgr].

Lemma 6.14. *Recall the revised terminology for type I roots explained in the paragraph preceding the lemma. Let $(\widetilde{H}_{\mathbb{R}}, \gamma)$ be a genuine pseudocharacter of $\widetilde{G}_{\mathbb{R}}$, and let α be a root in $\widetilde{R}(\bar{\gamma})$.*

a) If α is compact imaginary (and thus integral), then

$$s_\alpha \times \gamma = s_\alpha \cdot \gamma.$$

b) If α is type I noncompact imaginary integral short, then $s_\alpha \times \gamma$ is not conjugate to γ. Then the Cayley transform is single valued, $c^\alpha(\gamma) = \{\gamma^\alpha\}$, and if we denote by α_1 the Cayley transform of α, then, up to equivalence,

$$c_{\alpha_1}(\gamma^\alpha) = \{\gamma, s_\alpha \times \gamma\}, \quad (s_\alpha \times \gamma)^\alpha = s_{\alpha_1} \times \gamma^\alpha = s_{\alpha_1} \cdot \gamma^\alpha.$$

c) If α is type II noncompact imaginary integral short, then

$$s_\alpha \times \gamma = s_\alpha \cdot \gamma.$$

The Cayley transform is double valued, $c^\alpha(\gamma) = \{\gamma^\alpha_\pm\}$,

$$c_{\alpha_1}(\gamma^\alpha_\pm) = \{\gamma\}, \quad s_{\alpha_1} \times \gamma^\alpha_+ = \gamma^\alpha_-.$$

d) If α is real type I satisfying the parity condition and short , then

$$s_\alpha \times \gamma = s_\alpha \cdot \gamma.$$

The Cayley transform is double valued, $c_\alpha(\gamma) = \{\gamma^\pm_\alpha\}$ and if we denote by α_1 the Cayley transform of α, then, up to equivalence,

$$c^{\alpha_1}(\gamma^\pm_\alpha) = \{\gamma\}, \quad s_{\alpha_1} \times \gamma^+_\alpha = \gamma^-_\alpha.$$

e) If α is real type II satisfying the parity condition and short , then $s_\alpha \times \gamma$ is not conjugate to γ. Then the Cayley transform is single valued, $c_\alpha(\gamma) = \{\gamma_\alpha\}$ and up to equivalence,

$$c^{\alpha_1}(\gamma_\alpha) = \{\gamma, s_\alpha \times \gamma\}, \quad (s_\alpha \times \gamma)_\alpha = s_{\alpha_1} \times \gamma^\alpha = s_{\alpha_1} \cdot \gamma^\alpha.$$

f) If α is real and does not satisfy the parity condition then

$$s_\alpha \times \gamma = s_\alpha \cdot \gamma.$$

g) If α is noncompact imaginary long (metaplectic), then α is not integral, the Cayley transform is single valued, $c^\alpha(\gamma) = \{\gamma^\alpha\}$, and if we denote by α_1 the Cayley transform of α, then, up to equivalence,

$$c_{\alpha_1}(\gamma^\alpha) = \{\gamma\}, \quad s_{\alpha_1} \times \gamma^\alpha = (s_\alpha \times \gamma)^\alpha.$$

and $(s_\alpha \times \gamma)^\alpha$ is not equivalent to γ^α.

h) If α is real satisfying the parity condition and long (metaplectic), then α is not integral, the Cayley transform is single valued, $c_\alpha(\gamma) = \{\gamma_\alpha\}$, and if we denote by α_1 the Cayley transform of α, then, up to equivalence,

$$c^{\alpha_1}(\gamma_\alpha) = \{\gamma\}, \quad s_{\alpha_1} \times \gamma_\alpha = (s_\alpha \times \gamma)_\alpha.$$

and $(s_\alpha \times \gamma)_\alpha$ is not equivalent to γ_α.

We can picture the different possibilities as follows : α is an imaginary noncompact root,

$$\begin{tikzcd} & \gamma^\alpha & \\ \gamma \arrow[ur] \arrow[rr, leftrightarrow, "\times s_\alpha"] & & s_\alpha \times \gamma \arrow[ul] \end{tikzcd}$$

α short, Type I

$$\begin{tikzcd} \gamma^\alpha_+ \arrow[rr, leftrightarrow, "\times s_{\alpha_1}"] & & \gamma^\alpha_- \\ & \gamma \arrow[ul] \arrow[ur] & \end{tikzcd}$$

α short, Type II

$$\begin{tikzcd} \gamma^\alpha \arrow[r, leftrightarrow, "\times s_{\alpha_1}"] & (s_\alpha \times \gamma)^\alpha \\ \gamma \arrow[u] \arrow[r, leftrightarrow, "\times s_\alpha"] & s_\alpha \times \gamma \arrow[u] \end{tikzcd}$$

α metaplectic

Proof. Parts (b), (c), (d) and (e) have been proved above. For (a), we notice that the computation can be reduced to the compact group generated by $\widetilde{T}_\mathbb{R}$ and the $SU(2)$ associated to the compact root α. The arguments given in [Vgr] then still apply. The proof of (f) is the same as in [Vgr]. Everything but the last assertions of (g) and (h) have also already been established. Notice that the infinitesimal parts of $(s_\alpha \times \gamma)^\alpha$ and γ^α in (g) are not necessarily equal. In the case they are equal, according to Convention 2.2, we have $\mu_\alpha = n\alpha$. Following the proofs in [Vgr] we see that the two pseudocharacters could be equivalent only if $\Gamma(m_\alpha) = \pm 1$, but since α is metaplectic, $\Gamma(m_\alpha) = \pm i$, and we get the conclusion. The proof for (h) is the same. □

Lowest K-types. To be able to use some the material of [Vgr] in our context, we need to restrict a little bit the class of groups we consider. Indeed for the groups we have defined so far, the lowest K-types of an irreducible (genuine) Harish-Chandra module might not have multiplicity one. We thank J. Adams for bringing to our attention the following example : $G_\mathbb{R}$ is $GSp(4,\mathbb{R}) \times GSp(4,\mathbb{R})$, and the inverse images in $\widetilde{G_\mathbb{R}}$ of the two factors don't commute. There, an irreducible subrepresentation of an ordinary genuine principal series representation admits lowest (fine) $\widetilde{K}_\mathbb{R}$-types with multiplicity two. The multiplicity one result is used in [Vgr] to obtain the classification of irreducible Harish-Chandra modules using cohomological induction (Vogan-Zuckerman classification), and part of [Vgr] and subsequent works on Kazhdan-Lusztig algorithm rely on computations in cohomology based on this classification.

Of course, other classifications of irreducible Harish-Chandra modules (Langlands, Beilinson-Bernstein) are established for the whole Harish-Chandra class, but it is not clear to us how the computations in cohomology alluded above can be rephrased in terms of these classifications. There are two subclasses of groups for which multiplicity one of lowest K-types holds however. The first one is connected groups ([VLKT]). The second is groups with only one root length (still with the convention that roots in G_2 are long). We will give a short argument for this last claim below.

If we give a closer look at the example given above, we see that if we require the inverse images in $\widetilde{G_{\mathbb{R}}}$ of the two $GSp(4,\mathbb{R})$ factors to commute, then we get multiplicity one. In general, it seems that sufficient conditions for multiplicity one can be obtained by requiring certain elements in different connected components of the group to commute. Unfortunately, we were unable to turn this into a simple statement. Thus instead of stating complicated conditions and giving complicated proofs that they imply multiplicity one, we prefer simply to assume the result.

Hypothesis 6.15. Genuine irreducible Harish-Chandra modules of the group $\widetilde{G_{\mathbb{R}}}$ have lowest K-types occurring with multiplicity one.

The groups appearing in Part II of this paper satisfy this hypothesis, because of the following result.

Proposition 6.16. *Assume that there is only one root length in $\mathfrak{g}$. Then lowest K-types of genuine irreducible Harish-Chandra modules of $\widetilde{G_{\mathbb{R}}}$ occur with multiplicity one.*

Proof. We give only a quick sketch. We consider first the case of principal series in quasisplit group (here we follow [VLKT], Section 6, without assuming that the group is connected). Suppose that, with the notation of this paper $\delta \in \hat{M}$ is fine and genuine. Let S be the set of root defined just before Lemma 6.19 (*loc. cit.*). This lemma asserts that if $\alpha \in S$, then $\delta(m_\alpha) = -\mathrm{Id}$. Suppose that α is long. Since δ is genuine, $\delta(m_\alpha)^2 = \delta(\mathbf{z}) = -\mathrm{Id}$, and we get a contradiction. Thus S consists only of short roots. If we assume that there are no short roots, then S is empty. This proves that the R-group R_δ is trivial. Therefore, the other results in Section 6 of *loc. cit.* hold trivially. Dually, we now want to prove that another R-group is trivial, namely the group R_μ in Section 5.1 of [Vgr]. Suppose that μ is a highest weight of a genuine irreducible representation of $\widetilde{K}_{\mathbb{R}}$, and that α is a long noncompact imaginary root. Again, we use that fact that $m_\alpha^2 = z$, so for an highest weight vector v, $m_\alpha^2 \cdot v = -v$. But $m_\alpha \cdot v = \exp i\pi H_\alpha \cdot v = e^{i\pi\mu(H_\alpha)} v$. Thus we get $\mu(H_\alpha) \in \mathbb{Z} + \frac{1}{2}$, and so $\langle \alpha, \mu \rangle$ is not zero. This proves again that R_μ is trivial if there is no short roots. These two facts easily implies the proposition. In fact, when there is only one root length, it is possible to show that the genuine representation theory of $\widetilde{G_{\mathbb{R}}}$ essentially reduces to the genuine representation theory of its identity connected component. ☐

Examples. Suppose that G is a simple, simply connected complex group, and that $G_{\mathbb{R}}$ is a split real form of G. The fundamental group of $G_{\mathbb{R}}$ is isomorphic to $\mathbb{Z}$ if $G_{\mathbb{R}}/K_{\mathbb{R}}$ is hermitian symmetric and $\mathbb{Z}/2\mathbb{Z}$ otherwise (see [Sek]). Thus $G_{\mathbb{R}}$ always admits a nonlinear covering. For a determination of the cocycle of this double cover, see [BD]. For classical groups, we get double covers of $\mathrm{SL}(n,\mathbb{R})$ (type A_n), $\mathrm{Spin}(n,n+1)$ (type B_n), $\mathrm{Sp}(2n,\mathbb{R})$ (type C_n) and $\mathrm{Spin}(n,n)$ (type D_n).

If we assume that $G_{\mathbb{R}}$ is a real form of a simple, simply connected group G, but not necessarily a split one, then in some (but not all) cases, $G_{\mathbb{R}}$ admits a nontrivial double cover (see [Sek]); for example $\mathrm{SU}(p,q)$, $(p>0,\ q>0)$, $\mathrm{Spin}(p,q)$, $(p>1,\ q>1)$.

In part II of this paper, the examples of double covers of $\mathrm{GL}(n,\mathbb{R})$ and $\mathrm{U}(p,q)$ (with $pq \neq 0$) are studied in detail.

7. Reducibility of standard modules and blocks.

Reducibility. When $X(\gamma)$ is irreducible, we say that γ is minimal. For linear groups, [Vgr, Theorem 8.6.4] gives a necessary and sufficient condition for a pseudocharacter γ to be minimal. For nonlinear groups the following generalization of this result can be found in [Mi1, Theorem 2.1].

Theorem 7.1. *The standard module $X(\gamma)$ is irreducible if and only if*

i) for all complex integral root α in $\Delta^+(\overline{\gamma})$, either $\theta(\alpha) \in \Delta^+(\overline{\gamma})$ or $\theta(\alpha) \notin \Delta^+(\overline{\gamma})$ and α is not minimal in $\{\alpha, -\theta(\alpha)\}$,

ii) for all real root α in $\Delta^+(\overline{\gamma})$, α does not satisfy the parity condition with respect to γ.

In *i*), "α not minimal in $\{\alpha, -\theta(\alpha)\}$" is with respect to the standard ordering of Σ_α^+, where Σ_α is the smallest θ-stable root system containing $\alpha, -\theta(\alpha)$ and $\Sigma_\alpha^+ = \Sigma_\alpha \cap \Delta^+(\overline{\gamma})$.

We deduce immediately from this a simple necessary condition for $X(\gamma)$ to be reducible.

Corollary 7.2. *The standard module $X(\gamma)$ is reducible only if there exists a root $\alpha \in \Delta^+(\overline{\gamma})$ such that either:*

1. *α is complex and $\theta(\alpha) \notin \Delta^+(\overline{\gamma})$ or,*
2. *α is real, and α satisfies the parity condition with respect to γ.*

Let us now recall three lemmas of Vogan ([Vgr, Lemmas 8.6.1–3]).

Lemma 7.3. *There exists a complex root $\alpha \in \Delta^+(\overline{\gamma})$ such that $\theta(\alpha) \notin \Delta^+(\overline{\gamma})$ if and only if there exists a simple complex root $\alpha \in \Delta^+(\overline{\gamma})$ such that $\theta(\alpha) \notin \Delta^+(\overline{\gamma})$.*

Lemma 7.4. *Suppose that no complex root $\alpha \in \Delta^+(\overline{\gamma})$ satisfies $\theta(\alpha) \notin \Delta^+(\overline{\gamma})$. Then the real roots in $\Delta(\overline{\gamma})$ are spanned by simple real roots.*

Lemma 7.5. *Suppose that no complex root $\alpha \in \Delta^+(\overline{\gamma})$ satisfies $\theta(\alpha) \notin \Delta^+(\overline{\gamma})$. Then there exists a real root $\alpha \in \Delta^+(\overline{\gamma})$ satisfying the parity condition with respect to γ if and only if there exists a simple real root $\alpha \in \Delta^+(\overline{\gamma})$ satisfying the parity condition with respect to γ.*

From these lemmas we now deduce the following result.

Proposition 7.6. *The standard module $X(\gamma)$ is reducible only if there exists a simple root $\alpha \in \Delta^+(\overline{\gamma})$ such that either:*

1. *α is complex and $\theta(\alpha) \notin \Delta^+(\overline{\gamma})$; or*
2. *α is real, and α satisfies the parity condition with respect to γ.*

The point of this result is that it is stated only in terms of simple roots, and this will be crucial for arguments based on induction on the length of pseudocharacters.

Remark 7.7. A real root α satisfying the parity condition with respect to γ is necessarily in $\widetilde{R}(\overline{\gamma})$. Furthermore in Theorem 7.1, the condition on complex roots involves only integral complex root. Thus we could apply the previous lemmas to the root system $\widetilde{R}(\overline{\gamma})$ instead of $\Delta(\overline{\gamma})$ and get:

- $X(\gamma)$ is reducible only if there exists a simple root $\alpha \in \widetilde{R}^+(\overline{\gamma})$ such that either:
  (1) α is complex and $\theta(\alpha) \notin \widetilde{R}^+(\overline{\gamma})$; or
  (2) α is real, and α satisfies the parity condition with respect to γ.

Blocks. For applications in Part II, we will need some results about blocks. These are also of independent interest. Let us recall that block equivalence on irreducible Harish-Chandra modules (in our setting, genuine modules for $\widetilde{G}_{\mathbb{R}}$) is the equivalence relation generated by

$$X \sim Y \quad \text{iff} \quad \mathrm{Ext}^1_{\mathfrak{g},\widetilde{K}}(X,Y) \neq 0.$$

A necessary condition for this to hold is that X and Y have same infinitesimal characters. Recall also the standard fact (for instance, [Vgr], Lemma 9.2.2) that $\mathrm{Ext}^1_{\mathfrak{g},\widetilde{K}}(X,Y) \neq 0$ if and only if there is a Harish-Chandra module Z, not equivalent to $X \oplus Y$, and a short exact sequence

$$0 \to Y \to Z \to X \to 0 \tag{7.1}$$

We note that the nonintegral wall-crossing functors ϕ_α and ψ_α of Section 2 preserve block equivalence: if T is such a functor, then $X \sim Y$ if and only if $T(X) \sim T(Y)$. This follows easily from the interpretation of the block equivalence given in Equation (7.1), and the fact that T is exact and maps irreducibles to irreducibles.

From the Kazhdan-Lusztig algorithm perspective, the key result (see [Vgr], Proposition 9.2.10) is that block equivalence of Harish-Chandra modules with nonsingular infinitesimal character is generated by

$$X \sim Y \quad \text{iff } X \text{ and } Y \text{ occur in a common standard representation}$$

Thus, if we fix a nonsingular infinitesimal character λ_a, block equivalence induces a equivalence relation (and we will call the equivalence classes also "blocks") on $\mathcal{P}_{\lambda_a}$, such that the change-of-bases matrices between

$$\{\ [\overline{X}(\gamma)]\ \}_{\gamma\in\mathcal{P}_{\lambda_a}} \quad \text{and} \quad \{\ [X(\gamma)]\ \}_{\gamma\in\mathcal{P}_{\lambda_a}}$$

of the Grothendieck group $\mathbb{K}(\mathfrak{g},K)_{\lambda_a}$ are block-diagonal.

What we aim for now is a characterization of blocks in terms of Cayley transforms and cross-action. The result is the following (compare [Vgr], Theorem 9.2.11).

Theorem 7.8. *Consider the equivalence relation $X \leftrightarrow Y$ on genuine irreducible Harish-Chandra modules for $\widetilde{G}_{\mathbb{R}}$ with nonsingular infinitesimal character generated by*

1. *Cayley transforms with respect to noncompact imaginary roots, i.e. $\overline{X}(\gamma) \leftrightarrow \overline{X}(\delta)$ if there exists a noncompact imaginary root α in $\widetilde{R}(\overline{\gamma})$ such that $\delta \in c^\alpha(\gamma)$.*
2. *Cross-action with respect to integral complex roots, i.e. $\overline{X}(\gamma) \leftrightarrow \overline{X}(\delta)$ if there exists a complex integral root α in $R(\overline{\gamma})$ such that $\delta = s_\alpha \times \gamma$.*

Then $\leftrightarrow$ coincides with block equivalence.

The proof is easily adapted from the one of [Vgr], Theorem 9.2.11, by induction on $\tilde{l}^I$ but we need to consider all irreducible modules with infinitesimal character in the family $\mathcal{F}(\lambda_a)$, because of the use of nonintegral wall-crossing functors. For the induction step, we use Theorem 7.6, and we reduce the length by using integral roots, in which case we apply arguments given in [Vgr], or using nonintegral wall-crossing functors, in which case we use the fact, noted above, that nonintegral wall crosses preserve block equivalence.

8. Representation theoretic algorithm

Our aim is now to establish an analog of the algorithm for Harish-Chandra modules described for linear groups in [V2] or [Vgr]. For the metaplectic group at half-integral infinitesimal character, the result is [RT1], Theorem 1.13 and Section 5. It proceeds by induction on the length, $l(\gamma)$, of a genuine pseudocharacter $(\widetilde{H}_{\mathbb{R}}, \gamma)$ of $\widetilde{G_{\mathbb{R}}}$ and computes:

(a) The composition series of $X(\gamma)$;

(b) The cohomology groups $H^i(\mathfrak{u}, \overline{X}(\gamma))$ as a $(\mathfrak{l}, \widetilde{L} \cap \widetilde{K})$-module, for each θ-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ of $\mathfrak{g}$; and

(c) For each simple integral root $\alpha \in \Delta^+(\overline{\gamma})$ such that $\alpha \notin \tau(\overline{X}(\gamma))$, the composition series of $U_\alpha(\overline{X}(\gamma))$.

All the pseudocharacters we consider in the sequel are ν_a-pseudocharacters for some $\nu_a \in \mathcal{F}(\lambda_a)$.

For a minimal pseudocharacter γ, step (a) in the algorithm is trivial by definition. Part (b) is Theorem 6.13 of [V2] which computes the cohomology of standard irreducible modules. Part (c) is obtained by observing that the only constituents of $U_\alpha(\overline{X}(\gamma))$ are the 'special constituents' of [V1, Theorem 4.12]. For nonminimal γ, it is possible to find a pseudocharacter γ' of length $l(\gamma') = l(\gamma) - 1$ obtained from γ either by Cayley transform with respect to a simple real root satisfying the parity conditions, or coherent continuation across a simple complex wall. As we will see below, steps (a), (b), and (c) for γ are computable from the data corresponding to γ' and other pseudocharacters of smaller length.

A vanishing result. We need an important result, namely a vanishing theorem in cohomology. The main difference here with Vogan's treatment is the replacement of the integral length by the extended integral length in the statement. For the metaplectic group at half-infinitesimal character, the extended integral length coincide with the length (see [RT1]). To state the result, we need notation and results related to cohomology of Harish-Chandra modules. For this, we refer to Section 1.3 of [RT1] or [Vgr].

Theorem 8.1 (see [V2], Theorem 7.2, [RT1], Theorem 1.13)**.** *Let $\nu_a \in \mathcal{F}(\lambda_a)$ and let $(\gamma^i, \widetilde{H}^i_{\mathbb{R}})$, $i = 1, 2$ be two genuine ν_a-pseudocharacters of $\widetilde{G}_{\mathbb{R}}$. Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ containing $\mathfrak{h}^2$. Then*

(a) $H^i(\mathfrak{u}, \overline{X}(\gamma^1))$ contains $\overline{X}^{\tilde{L}}(\gamma^2_{\mathfrak{q}})$ as a composition factor only if $(\tilde{l}^I(\gamma^1) - \tilde{l}^I(\gamma^2)) - (l_{\mathfrak{q}}(\gamma^2) - i)$ is even.

(b) If $\overline{X}(\gamma^1)$ and $\overline{X}(\gamma^2)$ are distinct, $l(\gamma^1) \geq l(\gamma^2)$, and

$$\mathrm{Ext}^1_{(\mathfrak{g},K)}(\overline{X}(\gamma^1), \overline{X}(\gamma^2)) \neq 0,$$

then $(\tilde{l}^I(\gamma^1) - \tilde{l}^I(\gamma^2))$ is odd.

(c) $H^i(\mathfrak{u}, \overline{X}(\gamma^1))$ is completely reducible as an $\mathfrak{l}$-module.

(d) Suppose that $\alpha \in \Delta^+(\overline{\gamma}^2)$ is a simple integral root not in $\tau(\overline{X}(\gamma^1))$, with $m = 2\frac{\langle \alpha, \overline{\gamma}^2 \rangle}{\langle \alpha, \alpha \rangle}$.

(d1) If $\alpha \in \Delta(\mathfrak{h}^2, \mathfrak{u})$ then the multiplicity of $\overline{X}^{\tilde{L}}(\gamma^2_{\mathfrak{q}})$ in $H^i(\mathfrak{u}, U_\alpha(\overline{X}(\gamma^1))$ is its multiplicity in $H^{i+1}(\mathfrak{u}, \overline{X}(\gamma^1))$ plus the multiplicity of $\overline{X}^{\tilde{L}}(\gamma^2_{\mathfrak{q}} - m\alpha)$ in $H^i(\mathfrak{u}, \overline{X}(\gamma^1))$.

(d2) If $\alpha \in \Delta(\mathfrak{h}^2, \mathfrak{l})$ *then the multiplicity of* $\overline{X}^{\tilde{L}}(\gamma^2_{\mathfrak{q}})$ *in* $H^i(\mathfrak{u}, U_\alpha(\overline{X}(\gamma^1))$ *is zero unless* $\psi^{\mathfrak{l}}_\alpha(\overline{X}^{\tilde{L}}(\gamma^2_{\mathfrak{q}})) = 0$ *(i.e.* α *lies in the* τ*-invariant with respect to* $\mathfrak{l}$*) and in that case it is the multiplicity of* $\overline{X}^{\tilde{L}}(\gamma^2_{\mathfrak{q}})$ *in*

$$\phi^{\mathfrak{l}}_\alpha\psi^{\mathfrak{l}}_\alpha(H^i(\mathfrak{u}, \overline{X}(\gamma^1)) \oplus H^{i-1}(\mathfrak{u}, \overline{X}(\gamma^1)) \oplus H^{i+1}(\mathfrak{u}, \overline{X}(\gamma^1)).$$

(d3) If $-\alpha \in \Delta(\mathfrak{h}^2, \mathfrak{u})$ *then the multiplicity of* $\overline{X}^{\tilde{L}}(\gamma^2_{\mathfrak{q}})$ *in* $H^i(\mathfrak{u}, U_\alpha(\overline{X}(\gamma^1))$ *is its multiplicity in* $H^{i-1}(\mathfrak{u}, \overline{X}(\gamma^1))$ *plus the multiplicity of* $\overline{X}^{\tilde{L}}(\gamma^2_{\mathfrak{q}} - m\alpha)$ *in* $H^i(\mathfrak{u}, \overline{X}(\gamma^1))$*.*

(e) In the setting of (d), $\overline{X}(\gamma^3)$ *occurs in* $U_\alpha(\overline{X}(\gamma^1))$ *only if* $(\tilde{l}^I(\gamma^1) - \tilde{l}^I(\gamma^3))$ *is odd.*

The proof is similar to the one of Proposition 7.2 in [V2], with adjustments coming from our more general setting. The main ideas for these modifications are already in [RT1], the novelty here is the introduction of the extended integral length, which makes things work in general.

The proof proceeds by induction on the dimension of $\mathfrak{g}$, and then by induction on pseudocharacter length. The inductive step reducing the dimension of $\mathfrak{g}$ is passing from $\mathfrak{g}$ to $\mathfrak{l}$, where $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$, where $\mathfrak{q}$ is a θ-stable parabolic subalgebra. The group $\widetilde{L}_{\mathbb{R}} = \mathrm{Norm}(\widetilde{G_{\mathbb{R}}}, \mathfrak{l})$ is a double cover of the linear reductive group $L_{\mathbb{R}}$, but notice that the covering $\mathbf{pr} : \widetilde{L}_{\mathbb{R}} \to L_{\mathbb{R}}$ can be trivial, i.e. $\widetilde{L}_{\mathbb{R}} \simeq L_{\mathbb{R}} \times \{\mathbf{e}, \mathbf{z}\}$. We illustrate this last remark by studying the case where $\dim \mathfrak{l}$ is minimal, which is the starting point of our induction.

Lemma 8.2. *Suppose* $\mathfrak{l}$ *is of the form*

$$\mathfrak{g} = \mathfrak{h}^a \oplus \mathfrak{g}^\alpha \oplus \mathfrak{g}^{-\alpha},$$

where α *is a root in* Δ_a*. If* α *is a short root, the length function* $\tilde{l}$ *(in* $\widetilde{L}_{\mathbb{R}}$*) coincide with the integral length* l^I*. Then Theorem 8.1 holds for* $\widetilde{L}_{\mathbb{R}}$*. If* α *is long,* $L_{\mathbb{R}} \simeq \mathrm{SL}(2, \mathbb{R})$ *modulo the center, and the cover over* $\mathrm{SL}(2, \mathbb{R})$ *is isomorphic to* $\widetilde{\mathrm{SL}}(2, \mathbb{R})$*. Then Theorem 8.1 holds for* $\widetilde{L}_{\mathbb{R}}$*.*

Proof. Using Lemma 6.4, the first case is a special case of [V2, Proposition 7.2] and in the second case, the 2-fold cover of $\mathrm{SL}(2, \mathbb{R})$ is studied in Section 4 of [RT1] and in [Mi1]. Theorem 8.1 for this cover follows easily from the material in these references. □

We remark that the induction step reducing the length of a pseudocharacter in the proof differs from the one in [V2] because the root α that reduces the length of a nonminimal pseudocharacter need not be integral. In that case, the induction makes use of nonintegral wall-crossing translation functors. The key point that we have to check is these functors preserve (in a suitable sense) the parity conditions in the statement of the theorem (in contrast, when α is in fact integral, it doesn't affect the arguments in [V2] for purely formal reasons). This is done in [RT1] for the metaplectic group at half-integral infinitesimal character in Section 5, after Lemma 5.1. The proof goes without change in our setting (with length replaced by extended integral length), except that in the discussion of cases p.269 of [RT1], we must add:

Case 4: α^2 is real not satisfying the parity condition. Then $\alpha^2 \notin \widetilde{R}(\overline{\gamma}^2)$ because of Corollary 6.9 (recall that α is not integral). Since $\tilde{l}^I(\gamma^2 - \mu_{\alpha^2}) = \tilde{l}^I(\gamma^2)$ and $l_{\mathfrak{q}}(\gamma^2 - \mu_{\alpha^2}) = l_{\mathfrak{q}}(\gamma^2)$, we get (a) from the inductive hypothesis.

This vanishing result allows one to argue as in [Vgr, Chapter 9] and [RT1] to inductively reduce the length in each of the steps (a)–(c) above.

Corollary 8.3. *There is an effective algorithm for computing composition series of genuine standard Harish-Chandra modules for $\widetilde{G}_{\mathbb{R}}$ with infinitesimal character in $\mathcal{F}(\lambda_a)$.*

9. Kazhdan-Lusztig algorithm for $\widetilde{G_{\mathbb{R}}}$

Let us start with $\lambda_a \in (\mathfrak{h}^a)^*$ regular and dominant as in Section 3. We will now use translation functors across walls with respect to simple roots in $\widetilde{R}(\lambda_a)$, not necessarily simple in Δ_a^+. Such a translation functor is naturally obtained as a composition of functors of the previous type. Note that in this process, we cross at most one wall with respect to a non-complex root, and thus the results of Theorem 5.3 still holds with α nonintegral simple in $\widetilde{R}(\lambda_a)$. Notice also that cross-action defines an action of $\widetilde{W}_a$ on $\mathcal{P}_{\mathcal{F}}$. We can restrict the family of infinitesimal characters introduced in Section 3 to the ones obtained only by wall crossing with respect to simple roots in $\widetilde{R}(\lambda_a)$. This change of perspective is harmless by the above remarks. We denote again by $\mathcal{F}(\lambda_a)$ this family. Notice now that $\widetilde{R}(\nu_a) = \widetilde{R}(\lambda_a)$ for all $\nu_a \in \mathcal{F}(\lambda_a)$. Let us denote $\mathcal{P}_{\mathcal{F}} := \coprod_{\nu_a \in \mathcal{F}(\lambda_a)} \mathcal{P}_{\nu_a}$ and let $\tilde{S}$ be the set of reflections with respect to simple roots in $\widetilde{R}(\lambda_a)$.

Bruhat $\mathcal{G}$-order. Due to the presence of nonintegral noncompact imaginary roots, the definition of the Bruhat $\mathcal{G}$-order is more natural on the set of $\mathcal{P}_{\mathcal{F}}$ than on individual $\mathcal{P}_{\nu_a}$, even if two comparable elements are in the same $\mathcal{P}_{\nu_a}$. Recall that we do not distinguish in the notation a pseudocharacter γ and its $K_{\mathbb{R}}$ conjugacy class in $\mathcal{P}_{\nu_a}$.

Definition 9.1. Let γ and γ' be two elements of $\mathcal{P}_{\mathcal{F}}$, and let $s \in \tilde{S}$. We write $\gamma' \xrightarrow{s} \gamma$ in the following cases:

(a) The simple root α in $\widetilde{R}^+(\overline{\gamma})$ corresponding to $s \in \tilde{S}$, s is noncompact imaginary and integral, and $\gamma' \in c^\alpha(\gamma)$

(b) The simple root α in $\widetilde{R}^+(\overline{\gamma})$ corresponding to $s \in \tilde{S}$ is noncompact imaginary and nonintegral, and $\gamma' = (s \times \gamma)^\alpha$.

(c) The simple root α in $\widetilde{R}^+(\overline{\gamma})$ corresponding to $s \in \tilde{S}$ is complex such that $\theta(\alpha) \in \widetilde{R}^+(\overline{\gamma})$ and $\gamma' = s \times \gamma$.

Definition 9.2. The Bruhat $\mathcal{G}$-order is the smallest order relation on $\mathcal{P}_{\mathcal{F}}$ having the following properties:

(o) If γ and γ' are comparable, they are in the same $\mathcal{P}_{\nu_a}$.

(i) If $\gamma \in \mathcal{P}_{\mathcal{F}}$, and α is a noncompact imaginary simple root in $\widetilde{R}^+(\overline{\gamma})$, then for all $\gamma' \in c^\alpha(\gamma)$, we have $\gamma < \gamma'$.

(ii) If $\gamma \in \mathcal{P}_{\mathcal{F}}$, and α is a complex simple *integral* root in $\widetilde{R}^+(\overline{\gamma})$, such that $\theta(\alpha) \in \widetilde{R}^+(\overline{\gamma})$, then $\gamma < s \times \gamma$.

(iii)(exchange condition) If $\gamma' \leq \delta'$, $\gamma \xrightarrow{s} \gamma'$ and $\delta \xrightarrow{s} \delta'$, then $\gamma \leq \delta$.

If s is a reflection with respect a nonintegral root, $\gamma' \leq \delta'$, $\delta \xrightarrow{s} \delta'$, and $\gamma' \xrightarrow{s} \gamma$, then $\gamma \leq \delta$.

This last line was not correctly typed in [RT1], Definition 7.2, were we wrote $\gamma \xrightarrow{s} \gamma'$ instead of $\gamma' \xrightarrow{s} \gamma$.

As a motivation for this complicated definition we state the following result.

Theorem 9.3. *Consider* $\gamma, \delta \in \mathcal{P}_{\mathcal{F}}$*.*

(1) *If* $\gamma < \delta$ *in the Bruhat* $\mathcal{G}$*-order, then* $l(\gamma) < l(\delta)$ *and* $\tilde{l}^I(\gamma) < \tilde{l}^I(\gamma')$*. Moreover, if* $(Q_\gamma, \mathcal{L}_\gamma)$ *and* $(Q_\delta, \mathcal{L}_\delta)$ *are the Beilinson-Bernstein parameters corresponding to* γ *and* δ*, then* $Q_\gamma \subset \overline{Q}_\delta$*.*

(2) *Suppose* γ*,* δ *in* $\mathcal{P}_{\mathcal{F}}$ *and* $\overline{X}(\gamma)$ *occurs as a composition factor in* $X(\delta)$*. Then* $\gamma \leq \delta$*.*

The proof is similar to the one of Theorem 7.3 of [RT1].

T_s **operators.** Let B be an abelian group containing an element u of infinite order. Let $\mathcal{M}$ (respectively $\mathcal{M}'$) be the free $\mathbb{Z}[u, u^{-1}]$-module (respectively B-module) with basis $\mathcal{P}_{\mathcal{F}}$. By analogy with [V3, Definition 6.4], we now define operators T_s, on the basis elements γ. (In the definition below, we denote by α the simple root in $\widetilde{R}^+(\overline{\gamma})$ corresponding to s.) Note that if α is an integral root, the formulas below are the ones given in [V3], and that if α is not integral, the formulas are essentially the ones given in [RT1], Definition 7.4.

Definition 9.4. (a1) If α is compact imaginary, then α is integral and $T_s\gamma = u\gamma$.

(a2) If α is real not satisfying the parity condition, then $T_s\gamma = -\gamma$.

(b1) If α is complex and $\theta(\alpha) \in \widetilde{R}^+(\overline{\gamma})$, then $T_s\gamma = s \times \gamma$.

(b2 integral) If α is integral, complex and $\theta(\alpha) \notin \widetilde{R}^+(\overline{\gamma})$, then

$$T_s\gamma = u(s \times \gamma) + (u-1)\gamma.$$

(b2 nonintegral) If α is nonintegral, complex and $\theta(\alpha) \notin \widetilde{R}^+(\overline{\gamma})$, then

$$T_s\gamma = u(s \times \gamma).$$

(c1 integral) If α is type II noncompact imaginary integral, then

$$T_s\gamma = \gamma + \gamma^\alpha_+ + \gamma^\alpha_-.$$

(c1 nonintegral) If α is type II noncompact imaginary nonintegral, then

$$T_s\gamma = (s \times \gamma)^\alpha + (s \times \gamma).$$

(c2 integral) If α is integral and real type II satisfying the parity condition, then

$$T_s\gamma = (u-1)\gamma - (s \times \gamma) + (u-1)\gamma_\alpha.$$

(c2 nonintegral) If α is nonintegral and real type II satisfying the parity condition, then

$$T_s\gamma = -(s \times \gamma) + (u-1)(s \times \gamma)_\alpha.$$

(d1 integral) If α is type I noncompact imaginary integral, then

$$T_s\gamma = s \times \gamma + \gamma^\alpha.$$

(d1 nonintegral) If α is type I noncompact imaginary nonintegral, then

$$T_s\gamma = (s \times \gamma)^\alpha + (s \times \gamma).$$

(d2 integral) If α is integral and real type I, satisfying the parity condition, then

$$T_s\gamma = (u-2)\gamma + (u-1)(\gamma^+_\alpha + \gamma^-_\alpha).$$

(d2 nonintegral) If α is nonintegral and real type I, satisfying the parity condition, then

$$T_s\gamma = -(s \times \gamma) + (u-1)(s \times \gamma)_\alpha.$$

Notice that in case (c1 integral), the Cayley transform is indeed double valued because of our shift in terminology for Type I short roots discussed in the paragraph before Lemma 6.14). (This case doesn't occur for long roots because of Lemma 6.10.)

By analogy with the linear case, one might expect that the $\mathbb{Z}[u, u^{-1}]$ algebra $\mathcal{H}$ generated by $\langle T_s \mid s \in \tilde{S}\rangle$ is isomorphic to $\mathcal{H}(\widetilde{W})$, the Hecke algebra of the extended Weyl group. This isn't quite true. What is true, however, is that $\mathcal{H}$ contains the Hecke algebra of the integral Weyl group. More precisely, we have the following result.

Proposition 9.5. *Extend the definitions of the various T_s (in Definition 9.4) to $\mathbb{Z}[u, u^{-1}]$-linear endomorphisms of $\mathcal{M}$, and write $\mathcal{H}$ for the algebra that they generate. We have the following conclusions: with $\mathfrak{h}$ defined over $\mathbb{R}$ and θ-stable.*

(1) *If $s \in \tilde{S}$ is integral, then operator T_s satisfies*
$$(T_s + 1)(T_s - u) = 0$$
with $\mathfrak{h}$ defined over $\mathbb{R}$ and θ-stable.

(2) *If $s \in \tilde{S}$ is nonintegral, the operator T_s satisfies*
$$T_s^2 = u\, Id.$$
In particular $\mathcal{H}$ is not isomorphic to the Hecke algebra of the complex Weyl group $\widetilde{W} := W(\widetilde{R}(\lambda_a))$.

Proof. To prove the proposition, one need only appeal to the definitions and check the relevant relations. For (a) and (b), this is quite easy. The final assertion involves a more complicated check, but it is also elementary. □

Verdier duality. We need now the result giving the unicity of a Verdier duality $\mathbb{D}$ on $\mathcal{M}'$ satisfying certain properties. This is obtained as in [V3] and [RT1], Proposition 7.7. The reader is invited to consult this last reference for a statement.

Kazhdan-Lusztig polynomials. We now give the definition of Kazhdan-Lusztig polynomials in our context.

Corollary 9.6. *Suppose $\mathbb{D}$ exists, and suppose that for some $\delta \in \mathcal{P}_{\mathcal{F}}$ there is an element*
$$C_\delta = \sum_{\gamma \leq \delta} P_{\gamma,\delta}(u)\gamma$$
with the following properties.

(a) $\mathbb{D}(C_\delta) = u^{-\tilde{l}^I(\delta)} C_\delta$

(b) $P_{\delta,\delta} = 1$

(c) If $\gamma \neq \delta$, then $P_{\gamma,\delta}$ is a polynomial in u of degree at most $\frac{1}{2}(\tilde{l}^I(\delta) - \tilde{l}^I(\gamma) - 1)$.

Then $P_{\gamma,\delta}$ is computable. (In particular, C_δ is unique).

The proof, which we omit, is similar to the one in [V3].

We see that from a combinatorial point of view, the presence of nonintegral simple roots is not a bad thing, as the formulas tend to be simpler. Furthermore, we have the following result.

Lemma 9.7. *Suppose that $\delta \in \mathcal{P}_{\mathcal{F}}$, let $s \in \tilde{S}$ be a reflection with respect to a nonintegral root and $\alpha \in \widetilde{R}^{+}(\overline{\delta})$ the corresponding simple root. Suppose the elements C_δ, $C_{s\times\delta}$, etc., of the previous corollary exist.*

(a) *If α is complex, $\theta(\alpha) \in \widetilde{R}^{+}(\overline{\delta})$ then $T_s C_\delta = C_{s\times\delta}$*
(b) *If α is complex, $\theta(\alpha) \notin \widetilde{R}^{+}(\overline{\delta})$ then $T_s C_\delta = uC_{s\times\delta}$*
(c) *If α is imaginary, then $T_s C_\delta = C_{(s\times\delta)^\alpha}$*
(c) *If α is real, then $T_s C_\delta = uC_{(s\times\delta)_\alpha}$*

The proof is similar to the one of Lemma 7.9 in [RT1]

The Kazhdan-Lusztig polynomials are characterized recursively by certain identities. The results we need are Proposition 7.10 and Corollary 7.11 of [RT1]. The proofs extend easily to our setting. We summarize this by

Proposition 9.8. *If $P_{\gamma',\delta'}$ is known when $\tilde{l}^I(\delta') < \tilde{l}^I(\delta)$, or $\tilde{l}^I(\delta') = \tilde{l}^I(\delta)$ and $\tilde{l}^I(\gamma') > \tilde{l}^I(\gamma)$, then the formulas in Proposition 7.10 and Corollary 7.11 of* [RT1] *determine $P_{\gamma,\delta}$.*

To interpret the polynomials $P_{\gamma\delta}$ just defined, we will use some results of [ABV], Section 17. The Langlands parameters there are not the one we consider here, unfortunately. They are what the authors call equivalence classes of final limit characters, and we will refer to them as ABV-parameters. Of course, the two parameterizations are equivalent, and a procedure to obtain a pseudocharacter from a final limit character, or conversely, is described in section 11 of [ABV]. If $\gamma \in \mathcal{P}_{\nu_a}$, we will denote by $\tilde{\gamma}$ the corresponding ABV-parameter.

Let γ, δ in $\mathcal{P}_{\nu_a}$, $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$ a representative of the K-orbit on the flag manifold associated to γ, with $\mathfrak{h}$ defined over $\mathbb{R}$ and θ-stable. Write d for the codimension of the orbit corresponding to δ in the flag manifold. Define

$$Q_{\gamma,\delta}(u) = \sum_{q\in\mathbb{Z}} u^{\frac{1}{2}(q-d)} \mathrm{mult}[\tilde{\gamma} \otimes \rho(\mathfrak{n}); H_q(\mathfrak{n}, \overline{X}(\delta))] \tag{9.1}$$

Let $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$, chosen for γ as in [V3], (A.2). Set

$$r = \tilde{l}^I(\delta) - \tilde{l}^I(\gamma) - \dim \mathfrak{u} \cap \mathfrak{p}$$

Then, by Corollary A.10 and Proposition 4.3 of [V3]

$$Q_{\gamma,\delta}(u) = \sum_{q\in\mathbb{Z}} u^{\frac{1}{2}(q+r)} \mathrm{mult}[\overline{X}^L(\gamma_{\mathfrak{q}}), H_q(\mathfrak{u}, \overline{X}(\delta))]$$

Theorem 9.9. *The polynomials $Q_{\gamma,\delta}$ defined above are the Kazhdan-Lusztig polynomials $P_{\gamma,\delta}$ of the previous section.*

The proof is the same as the corresponding result in [RT1], namely, Theorem 7.12.

Finally, we can state the main result of this section.

Theorem 9.10. *The value at* 1 *of the Kazhdan-Lusztig polynomials $P_{\gamma,\delta}$ gives the multiplicity of $X(\gamma)$ in $\overline{X}(\delta)$. More precisely, with the notation of Equation 1.2,*

$$M(\gamma, \delta) = (-1)^{l(\delta)-l(\gamma)} P_{\gamma,\delta}(1).$$

Proof. We will only sketch the proof, since all the necessary arguments are already in [ABV], Section 17, and in [RT1].

First of all, recall that Beilinson-Bernstein localization theory gives an equivalence of category between $\mathcal{HC}(\mathfrak{g},\widetilde{K})_{\nu_a}$ and $\mathcal{D}_{\nu_a}(X,\widetilde{K})$, the category of $\widetilde{K}$-equivariant $\mathcal{D}_{\nu_a}$-modules on X (the flag manifold of $\mathfrak{g}$), where $\mathcal{D}_{\nu_a}$ denotes the twisted sheaf of differential operators on X with twist given by $\nu_a \in (\mathfrak{h}^a)^*$. In order to apply Riemann-Hilbert correspondence, we need to introduce another category of $\mathcal{D}$-modules. The theory of intertwining operators shows that the multiplicity matrix $M(\gamma,\delta)_{\gamma,\delta}$ is unchanged by a small modification of the infinitesimal character that does not affect the set of integral roots. Thus, we may assume that ν_a is rational, i.e. there exist an integer $n \in \mathbb{N}^*$ and a weight $\mu \in X^*(H^a)$ such that $n(\nu_a - \rho) = \mu_a$. The weight μ_a defines a line bundle $\mathcal{L}$ on X, and $H := \mathbb{C}^* \times \widetilde{K}$ acts on $\mathcal{L}^*$ with $\mathbb{C}^*$ acting on the fibers of $\mathcal{L}^* \to X$ by

$$z \cdot \xi = z^n \xi \qquad (z \in \mathbb{C}^*, \xi \in \mathcal{L}^*).$$

This action of $\mathbb{C}^*$ on $\mathcal{L}^*$ allows to define "genuine" $\mathbb{C}^*$-equivariant object on $\mathcal{L}^*$, i.e. objects with the required monodromy, and we have (cf. [ABV], Proposition 17.5) an equivalence of category between $\mathcal{D}_{\nu_a}(X,\widetilde{K})$ and $\mathcal{D}^{gen}(\mathcal{L}^*,H)$ of H-equivariant genuine $\mathcal{D}_{\mathcal{L}*}$-modules on $\mathcal{L}^*$. Notice that now $\mathcal{D}_{\mathcal{L}*}$ is a sheaf of differential operators on $\mathcal{O}_{\mathcal{L}^*}$. We can now take the Riemann-Hilbert functor $RHom_{\mathcal{D}_{\mathcal{L}*}}(\ .\ ,\mathcal{O}_{\mathcal{L}^*})$ from $\mathcal{D}^{gen}(\mathcal{L}^*,H)$ to $\mathcal{A}(\mathcal{L}^*,H)$, the category of H-equivariant genuine perverse sheaves on $\mathcal{L}^*$. To summarize, what we have obtained so far is an equivalence of category between $\mathcal{HC}(\mathfrak{g},\widetilde{K})_{\nu_a}$ and $\mathcal{A}(\mathcal{L}^*,H)$. Recall that irreducible objects in $\mathcal{HC}(\mathfrak{g},\widetilde{K})_{\nu_a}$ are parameterized by the set $\mathcal{P}_{\nu_a}$, which, in the Beilinson-Bernstein picture, can be viewed as the set of irreducible $\widetilde{K}$-equivariant local systems on $\widetilde{K}$-orbit on X. Lemma 17.9 of [ABV] gives a bijection between $\mathcal{P}_{\nu_a}$ and the set of irreducible H-equivariant genuine local systems on H-orbit on $\mathcal{L}^*$ (which parameterizes irreducible objects in $\mathcal{A}(\mathcal{L}^*,H)$).

Let V be an object in $\mathcal{HC}(\mathfrak{g},\widetilde{K})_{\lambda_a}$ and let $P \in \mathcal{A}(\mathcal{L}^*,H)$ the corresponding perverse sheaf on $\mathcal{L}^*$. The Lie algebra homology of V can be computed from the decomposition of various $H^iP_{|S}$ in terms of irreducible H-equivariant genuine local systems on H-orbits S on $\mathcal{L}^*$.

We are thus reduced to a geometric problem, which is roughly speaking, computing the intersection cohomology sheaf on closure of H-orbits on $\mathcal{L}^*$. We are to prove that it is exactly what the KL-polynomials introduced in Corollary 9.6 do. Notice first that the uniqueness assertions about Verdier duality and in Proposition 9.6 are purely combinatorial statements. By a general argument (see [BBD], Section 6), the study of intersection complexes is reduced to the same algebro-geometric problem for varieties defined over the algebraic closure of a finite field. In this context, the modules $\mathcal{M}$ and $\mathcal{M}'$ have a geometric interpretation, where the variable u keeps track of eigenvalues of Frobenius action. Existence of Verdier duality and KL polynomials follows. Recall that the computation of Lie algebra homology groups is carried out in Theorem 8.1, under the semi-simplicity hypothesis of the U_α functors (see [RT1], Section 1). The idea is more or less to prove simultaneously this semi-simplicity hypothesis and the equality of the KL-polynomials with their representation theoretic analogs ([RT1], Section 7.3). The argument are given in [ABV], proof of Theorem 17.12 and [RT1], section 7.3. The fundamental difference between our algorithm and the one described in [V3] or [ABV] for linear groups is the use of Hecke operators corresponding to simple nonintegral roots in $\widetilde{R}_{\nu_a}$.

Part 2. Geometric theory of Kazhdan-Patterson lifting over $\mathbb{R}$

Let $G_{\mathbb{R}} = \mathrm{GL}(n, \mathbb{R})$, let $\widetilde{G}_{\mathbb{R}}$ denote its nonlinear double cover, and write **pr** for the projection. The square of any section $s : G_{\mathbb{R}} \to \widetilde{G}_{\mathbb{R}}$ defines a map from the conjugacy classes of $G_{\mathbb{R}}$ to those of $\widetilde{G}_{\mathbb{R}}$; this map is independent of the choice of s. It is natural to introduce a slight twist $t(g) = s^2(g)\eta(g)$, where $\eta : G \to \ker(\mathbf{pr})$; see [KP]. Following [KP], define a lifting of characters for $G_{\mathbb{R}}$ to characters of $\widetilde{G}_{\mathbb{R}}$ as follows. For a virtual representation π with global character Θ, the lifting T is defined on a semisimple element g in the image of t as

$$[\mathrm{T}(\Theta)](g) = 2^{[n/2]-n} \sum_{\{h \in G \mid t(h)=g\}} \frac{\Delta(h)}{\Delta(g)} \Theta(h),$$

where Δ is the Weyl denominator. Let $\widetilde{Z}_0$ denote the central subgroup of $\widetilde{G}_{\mathbb{R}}$ defined in Remark 10.5, and fix a character χ of $\widetilde{Z}_0$; there are at most two such choices. For $g \in t(G)$ and $z \in \widetilde{Z}_0$ define

$$[\mathrm{T}(\Theta)](gz) = [\mathrm{T}(\Theta)](g)\chi(z),$$

and finally, to complete the definition of the lifting, set $[\mathrm{T}(\Theta)](g) = 0$ if $g \notin t(G)\widetilde{Z}_0$.

The purpose of this part is to give a geometric definition of this lifting and, using the geometry, define an analog of this lifting for the indefinite unitary groups on the level of virtual representations.

10. A standard form for blocks

The main results of this section are Theorems 10.4 and 10.8 detailing a simple form for blocks (Section 7) of Harish-Chandra modules for $\widetilde{\mathrm{GL}}(n, \mathbb{R})$ and $\widetilde{\mathrm{U}}(p, q)$.

We recall some ideas from [V4, Section 5] adapted to our context. For an arbitrary group $G_{\mathbb{R}}$ and λ-pseudocharacter γ, an ordered set $\{\alpha_1, \ldots, \alpha_k\}$ of noncompact imaginary roots is called *admissible* if the iterated Cayley transform

$$c^{\alpha_k} \circ \cdots \circ c^{\alpha_1}(\gamma)$$

is well-defined. Similarly we say a sequence of real roots is admissible if the corresponding iterated inverse Cayley transform is well-defined. The next proposition is a simple complement to [V4, Corollary 5.9].

Proposition 10.1. *Let $\widetilde{G}_{\mathbb{R}}$ be a nonlinear double cover as in Section 6, and fix a genuine λ-pseudocharacter γ. Assume that $\mathfrak{g}$ is simple and admits only one root length in the sense of Convention 5.2; so $\mathfrak{g}$ is simply laced or of type G_2. Let $S = \{\alpha_1, \ldots, \alpha_k\}$ and $S' = \{\alpha'_1, \ldots, \alpha'_k\}$ be admissible sequences of noncompact imaginary roots for γ such that each α_i and α'_i is half-integral but not integral. Suppose the span of S coincides with that of S', and write $\mathfrak{s}$ for the common span. Then*

$$c^{\alpha_k} \circ \cdots \circ c^{\alpha_1}(\gamma) = c^{\alpha'_k} \circ \cdots \circ c^{\alpha'_1}(\gamma).$$

As a matter of notation, we write $c^{\mathfrak{s}}(\gamma)$ for this pseudocharacter. (Recall that Cayley transforms in half-integral roots are single-valued.)

Proof. Exactly as in the proof of [V4, Corollary 5.9], the current proposition reduces to the assertion that there exists an element $w \in W(\lambda)$ such that

$$w\{\pm\alpha_1, \ldots, \pm\alpha_k\} = \{\pm\alpha'_1, \ldots, \pm\alpha'_k\}, \tag{10.1}$$

as an equality of unordered sets. Since admissible sequences are always orthogonal and since the sum of two element $\alpha, \beta \in R(2\lambda) \setminus R(\lambda)$ can never be in $R(2\lambda) \setminus R(\lambda)$, it follows that both sets $\{\alpha_i\}$ and $\{\alpha_i'\}$ are automatically strongly orthogonal. Thus the assertion of Equation (10.1) follows from applying [V4, Proposition 5.8] to the system of imaginary roots integral for 2λ (with the grading supplied, of course, by compact and noncompact roots). The proposition follows. $\square$

Clearly one immediately obtains a version of Proposition 10.1 for real roots.

Proposition 10.2. *Retain the notation and hypotheses of Proposition 10.1, but instead suppose that S and S' are admissible sequences of real roots which again are half-integral but not integral. Suppose the span of S coincides with that of S', and write $\mathfrak{s}$ for the common span. Then*

$$c_{\alpha_k} \circ \cdots \circ c_{\alpha_1}(\gamma) = c_{\alpha_k'} \circ \cdots \circ c_{\alpha_1'}(\gamma).$$

As a matter of notation we write $c_{\mathfrak{s}}(\gamma)$ for this pseudocharacter. (Recall that inverse Cayley transforms in half-integral roots are single-valued.)

Remark 10.3. As mentioned in the proof of Proposition 10.1, any orthogonal sequence of half-integral (but not integral) compact imaginary roots is strongly orthogonal, and hence admissible (as is clear, for instance, from the discussion following Definition 5.2 in [V4]). Conversely, any admissible sequence is orthogonal. Thus a set of half-integral (but not integral) noncompact imaginary roots is orthogonal if and only if it is admissible (with respect to any ordering). The same conclusion holds for real roots.

Now we turn to our two groups of interest.

Theorem 10.4. *Let $\widetilde{G}_{\mathbb{R}} = \widetilde{\mathrm{GL}}(n, \mathbb{R})$, and fix arbitrary infinitesimal character λ. If n is even, there is a unique block of genuine representations of $\widetilde{G}_{\mathbb{R}}$ at fixed infinitesimal character λ; if n is odd, there are two such blocks. Each such block $\mathcal{B}$ contains a unique principal series, say γ_{ps}, and the representations in $\mathcal{B}$ are parameterized by subspaces of the span of the real roots for γ_{ps} in $R(2\lambda) \setminus R(\lambda)$ which are spanned by admissible subsets. More precisely, fix such a subspace $\mathfrak{s}$. Then the assignment*

$$\mathfrak{s} \mapsto c_{\mathfrak{s}}(\gamma_{\mathrm{ps}})$$

is bijective.

In particular, the two blocks in the case when n is odd are isomorphic. They are distinguished by the central character of the unique principal series in each of them. (See Remark 10.5.)

Proof. We begin by recalling some elementary facts about the preimage, say $\widetilde{M}$, of the diagonal $(\mathbb{Z}/2)^n$ in the Pin double cover of $\mathrm{O}(n)$. When n is even, $\widetilde{M}$ has two genuine representations, each of dimension 2^{n-1}; when n is odd $\widetilde{M}$ has a unique genuine representation of dimension 2^n. In the even case, the two representations are distinguished by the action of the nontrivial element in $\widetilde{M}$ projecting to the identity. Consequently, we conclude that if n is odd, $\widetilde{\mathrm{GL}}(n, \mathbb{R})$ has a unique principal series at fixed infinitesimal character, while if n is even there are two such principal series and they are distinguished by their central characters.

Now fix a genuine block $\mathcal{B}$ as in the theorem and fix $\gamma \in \mathcal{B}$. Let S denote the set of half-integral (but not integral) noncompact imaginary roots for γ. The set S clearly consists of orthogonal roots, and hence (by Remark 10.3, S (with any ordering) is admissible. Let $\mathfrak{s}$ denote the span of S. Proposition 10.1 and the unicity of genuine principal series (with a fixed central and infinitesimal character) reduce the proof of the current theorem to establishing that $\gamma' := c^{\mathfrak{s}}(\gamma)$ is indeed a principal series. To see this, note that by the definition of S, there are no half-integral (but not integral) noncompact imaginary roots for γ'. Since γ' is genuine, Lemma 6.10 implies that there are no noncompact imaginary integral roots for γ'. It follows that there are no noncompact imaginary roots for γ'. For representations of a split group, this implies γ' is a principal series. The theorem follows. $\square$

Remark 10.5. We will need to make the discussion of central character a little more precise for applications below. Define a subgroup Z_0 of $\mathrm{GL}(n,\mathbb{R})$ as follows:

$$Z_0 = \begin{cases} \mathrm{Id} & \text{if } n \text{ is even;} \\ \pm\mathrm{Id} & \text{if } n \text{ is odd,} \end{cases}$$

and set let $\widetilde{Z}_0$ denote the preimage of Z_0 in $\widetilde{\mathrm{GL}}(n,\mathbb{R})$. For each representation in a fixed block $\mathcal{B}$ for $\widetilde{\mathrm{GL}}(n,\mathbb{R})$, $\widetilde{Z}_0$ acts by a genuine character. There exists a unique such character if n is even, but if n is odd there are two. In the latter case, the two blocks of genuine representations at fixed infinitesimal character are completely distinguished by the action of $\widetilde{Z}_0$.

Let $\mathrm{U}(p,q)$ denote the isometry group of an indefinite Hermitian on $\mathbb{C}^{p+q}$ of signature (p,q). Its maximal compact subgroup is isomorphic to $\mathrm{U}(p)\times\mathrm{U}(q)$. Let $\mathrm{U}(\widetilde{p},q)$ denote the double cover of $\mathrm{U}(p,q)$ which restricts to the $\det^{1/2}\otimes\mathbb{1}$ cover of $\mathrm{U}(p)\times\mathrm{U}(q)$. $\widetilde{G}_{\mathbb{R}}$ is nonlinear.

Proposition 10.6. *Fix regular infinitesimal character λ for $\mathfrak{gl}(n)$. Write p (resp. q) for the number of integer (resp. half-integer) coordinates of λ. Fix $p'+q'=n$. Then $\mathrm{U}(\widetilde{p',q'})$ has no genuine discrete series representations with infinitesimal character λ unless*

$$(p',q') = \begin{cases} (p,q) & \text{if } n \text{ is odd;} \\ (q,p) & \text{if } n \text{ is even.} \end{cases}$$

In this case, $\mathrm{U}(\widetilde{p',q'})$ has a unique discrete series.

Proof. This follows from Harish-Chandra's classification of the discrete series. $\square$

The proposition motivates the following notation.

Notation 10.7. Fix $p+q=n$. Define

$$\widetilde{\mathrm{U}}(p,q) = \begin{cases} \mathrm{U}(\widetilde{p},q) & \text{if } n \text{ is even;} \\ \mathrm{U}(\widetilde{q},p) & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 10.8. *Fix an infinitesimal character λ for $\mathfrak{gl}(n)$ consisting of p distinct integers and q distinct half-integers with $n=p+q$. Define $\widetilde{G}_{\mathbb{R}} = \widetilde{\mathrm{U}}(p,q)$ as in Notation 10.7. Let γ_{ds} denote the unique discrete series in the unique block $\mathcal{B}$ of representation of $\widetilde{G}_{\mathbb{R}}$ at λ (Proposition 10.6). Then the representations in $\mathcal{B}$ are parameterized by subspaces of the span of the noncompact imaginary roots for γ_{ds} in $R(2\lambda)\setminus R(\lambda)$ which are spanned by admissible subsets. More precisely, fix such a subspace $\mathfrak{s}$. Then the assignment*

$$\mathfrak{s} \mapsto c^{\mathfrak{s}}(\gamma_{\mathrm{ds}})$$

is bijective.

Proof. Fix λ as in the theorem, and let $\mathcal{B}$ denote a block of genuine representations. Fix $\gamma \in \mathcal{B}$, and let S denote the set of real half-integral (but not integral) roots for γ. S consists of orthogonal roots and hence (with any ordering) is admissible by Remark 10.3. Write $\mathfrak{s}$ for the span of S, and set $\gamma' = c_{\mathfrak{s}}(\gamma)$. The theorem reduces to establishing that $\gamma' = \gamma_{\text{ds}}$. By definition there are no real roots for γ'. But for genuine representations of $\mathrm{U}(\widetilde{p}, q)$ at λ, this means that γ' is a discrete series, and hence the theorem follows from Proposition 10.6. $\square$

11. Duality for $\widetilde{\mathrm{GL}}(n, \mathbb{R})$

Let $G_{\mathbb{R}} = \widetilde{\mathrm{GL}}(n, \mathbb{R})$ and fix regular infinitesimal character $\lambda \in \mathfrak{h}^*$. Let $\mathfrak{g}^\vee$ denote the Langlands dual of $\mathfrak{g}$; so $\mathfrak{h}^*$ is a Cartan in $\mathfrak{g}^\vee$, and we can consider the centralizer, say $\mathfrak{g}'$, of 2λ in $\mathfrak{g}^\vee$. Then $\mathfrak{g}' \simeq \mathfrak{g}'_1 \oplus \cdots \oplus \mathfrak{g}'_k$ with each $\mathfrak{g}'_i \simeq \mathfrak{gl}(n_i)$ for integers n_i with $\sum_i n_i = n$. Let λ_i denote the restriction of λ to an infinitesimal character for $\mathfrak{gl}(n_i, \mathbb{C})$. Up to a central shift by a power of the determinant, λ_i consists of p_i distinct half-integers and q_i distinct integers. Let $(\lambda')_i$ denote the shift consisting of p_i integers and q_i half-integers. Let $\mathfrak{k}'_i$ denote the centralizer of λ_i in $\mathfrak{g}'_i$; so $\mathfrak{k}'_i \simeq \mathfrak{gl}(p_i, \mathbb{C}) \times \mathfrak{gl}(q_i, \mathbb{C})$. Let $(\mathfrak{g}'_i)_{\mathbb{R}} \simeq \mathfrak{u}(p_i, q_i)$ denote the real form of $\mathfrak{g}_i$ with Cartan involution θ_i and complexified θ_i-fixed points equal to $\mathfrak{k}_i$. Let $(\widetilde{G}'_i)_{\mathbb{R}} \simeq \widetilde{\mathrm{U}}(p_i, q_i)$ (Notation 10.7) denote the corresponding double cover, and let $\mathbf{z}_i$ denote the nontrivial element projecting to the identity. Set

$$H_{\mathbb{R}} = (\widetilde{G}'_1)_{\mathbb{R}} \times \cdots \times (\widetilde{G}'_k)_{\mathbb{R}}.$$

This is a 2^k-fold covering of a linear group whose genuine representation theory clearly coincides with that of the two-fold cover $\widetilde{G}'_{\mathbb{R}}$ obtained by quotienting $H_{\mathbb{R}}$ by all products $\prod_{j \in S} \mathbf{z}_j$ with S a proper subset of $\{1, \ldots, k\}$. Finally let λ' denote the concatenation of the λ'_i; this is an infinitesimal character for $\widetilde{G}'_{\mathbb{R}}$

Now fix a block $\mathcal{B}$ of genuine representations of $\widetilde{G}_{\mathbb{R}}$ at infinitesimal character λ. According to the discussion in Section 3, $\mathcal{B}$ is unique if n is even and depends only on a choice of central character if n is odd. Let $\gamma_{\text{ps}} = \gamma_{\text{ps}}(\lambda)$ denote the unique principal series in $\mathcal{B}$. According to Theorem 10.8, let $\mathcal{B}'$ denote the unique block of genuine representations of $\widetilde{G}'_{\mathbb{R}}$ at infinitesimal character λ', and let $\gamma'_{\text{ds}} = \gamma'_{\text{ds}}(\lambda')$ denote the unique discrete series in $\mathcal{B}'$. Note that by construction, there α is a half-integral (but not integral) and real for γ_{ps} if and only if α is half-integral (but not integral) and noncompact imaginary for γ'_{ps}. Thus Remark 10.3 implies that S is an admissible sequence of half-integral (but not integral) noncompact imaginary roots for γ'_{ds} if and only if S is an admissible sequence of half-integral (but not integral) real roots for γ_{ps}. By Theorems 10.4 and 10.8, we obtain a bijection

$$\mathcal{B} \longrightarrow \mathcal{B}' \tag{11.1}$$

$$c_{\mathfrak{s}}(\gamma_{\text{ps}}) \longrightarrow c^{\mathfrak{s}}(\gamma'_{\text{ds}}); \tag{11.2}$$

here $\mathfrak{s}$ is a subspace spanned by an admissible sequence of half-integral (but not integral) real roots for $\gamma_{\text{ps}}(\lambda)$; or, equivalently an admissible sequence of half-integral (but not integral) noncompact imaginary roots for $\gamma'_{\text{ds}}(\lambda')$.

Theorem 11.1. *Retain the notation of the previous paragraph. The bijection $\gamma \mapsto \gamma'$ is a character multiplicity duality.*

Proof. Like the proof of the main result in [V4], the current results is essentially a purely formal consequence of the standard form of blocks provided by Theorems 10.4 and 10.8. We briefly sketch the details.

We of course need to work within the formalism developed in Part 1. Let $\mathcal{F}$ and $\mathcal{F}'$ be choices of families containing λ and λ' (Section 3). According to Section 3, $\mathcal{F}$ is a system of representatives for $(W_a \cdot \lambda + \mathcal{P})/\mathcal{P}$, and likewise for $\mathcal{F}'$. Hence we obtain a bijection from $\mathcal{F}$ to $\mathcal{F}'$ which we shall denote $\nu \mapsto \nu'$ by requiring that $\nu + \mathcal{P}$ belong to $w\lambda + \mathcal{P}$ if and only if $\nu' + \mathcal{P}$ belongs to $w\lambda' + \mathcal{P}$.

To apply the machinery of Part 1, we need to extend the bijection of Theorem 11.1 to the family of infinitesimal characters $\mathcal{F}$. For $\nu \in \mathcal{F}$, let $\gamma_{\text{ps}}(\nu)$ denote a principal series representation of $\widetilde{G}_{\mathbb{R}}$ at infinitesimal character ν whose central character coincides with that of $\gamma_{\text{ps}} = \gamma_{\text{ps}}(\lambda)$ fixed above. Let $\mathcal{P}_\nu$ denote the set of genuine ν-pseudocharacters corresponding to the block containing $\gamma_{\text{ps}}(\nu)$. Similarly, let $\mathcal{P}'_{\nu'}$ denote the set of genuine ν'-pseudocharacters for $\widetilde{G}'_{\mathbb{R}}$. Let $\mathcal{P} := \mathcal{P}_{\mathcal{F}}(\widetilde{G}_{\mathbb{R}})$ denote the disjoint union of the sets $\mathcal{P}_\nu$ for each $\nu \in \mathcal{F}$ (and similarly for $\mathcal{P}'$). In Equation (11.1) we defined a bijection from $\mathcal{P}_\lambda$ to $\mathcal{P}'_{\lambda'}$. We claim that the same definition gives a bijection between $\mathcal{P}_\nu$ and $\mathcal{P}'_{\nu'}$. To see this, first note that since each $\nu' \in \mathcal{F}'$ differs from a W conjugate of λ' by a weight, $\widetilde{G}'_{\mathbb{R}}$ has a (unique) discrete series, say $\gamma'_{\text{ds}}(\nu')$ at each $\nu' \in \mathcal{F}'$; and second note that according to the definition of the bijection $\nu \mapsto \nu'$, α is half-integral (but not integral) for ν if and only if α is half-integral (but not integral) for ν'. Remark 10.3 allows us to identify the relevant admissible subspaces of real roots for $\gamma_{\text{ps}}(\nu)$ with the relevant admissible subspaces for $\gamma'_{\text{ds}}(\nu')$, which allows us to define a bijection between $\mathcal{P}_\nu$ and $\mathcal{P}'_{\nu'}$ exactly as in Equation (11.1). We thus obtain a bijection $\mathcal{P} \to \mathcal{P}'$ which we denote, as above, by $\gamma \mapsto \gamma'$.

Let $\mathcal{M}$ and $\mathcal{M}'$ denote the free $\mathbb{Z}[u, u^{-1}]$ module with basis indexed by $\mathcal{P}$ and $\mathcal{P}'$. Recall the algebra $\mathcal{H}$ defined in Proposition 9.5 and its $\mathbb{Z}[u, u^{-1}]$-linear action on $\mathcal{M}$ and $\mathcal{M}'$. Define the dual $\mathbb{Z}[u, u^{-1}]$ module

$$\mathcal{M}^* = \mathrm{Hom}_{\mathbb{Z}[u,u^{-1}]}(\mathcal{M}, \mathbb{Z}[u, u^{-1}]).$$

The transposed action of $\mathcal{H}$ on $\mathcal{M}$ defines an action of $\mathcal{H}^{\text{op}}$ on $\mathcal{M}^*$; choosing an antiautomorphism of $\mathcal{H}$ (identifying $\mathcal{H}$ with $\mathcal{H}^{\text{op}}$) makes $\mathcal{M}$ an $\mathcal{H}$ module. Concretely, for $\mu \in \mathcal{M}^*$ and for a simple reflection s we define

$$T_s \cdot \mu = [-u(T_s)^{-1}]^{tr} \cdot \mu; \tag{11.3}$$

here the invertibility of T_s follows directly from Proposition 9.5.

Write $\{\mu_\gamma \mid \gamma \in \mathcal{P}\}$ for the basis of $\mathcal{M}^*$ dual to the basis $\mathcal{P}$ of $\mathcal{M}$, and recall the bijection

$$\begin{aligned} \mathcal{P} &\longrightarrow \mathcal{P}' \\ \gamma &\longrightarrow \gamma'. \end{aligned}$$

defined above. Then combining the formalism of Part 1 with the formalism of [V4, Section 12], the current theorem follows from the following claim: the $\mathbb{Z}[u, u^{-1}]$ linear isomorphism

$$\begin{aligned} \varphi \;:\; \mathcal{M}^* &\longrightarrow \mathcal{M}' \\ \mu_\gamma &\longrightarrow (-1)^{\tilde{l}^I(\gamma)}\gamma' \end{aligned}$$

is an isomorphism of $\mathcal{H}$ modules; here $\tilde{l}^I$ is the extended integral length function of Section 6.

As in [V4], this assertion must be verified from the formulas defining the $\mathcal{H}$ action. As example, we perform the check for one case of an operator T_s corresponding to a nonintegral simple root. This is one of the most interesting cases and once again reveals the importance of the interaction (which was crucial in Part 1) between the parity of $\tilde{l}^I$ and the nonintegral wall crossing functors.

We are to establish the equivariance of φ. Suppose s corresponds to a root α which is half-integral but not integral for $\overline{\gamma}$. From Proposition 9.5(2), and the definition of the $\mathcal{H}$ action on $\mathcal{M}^*$, the equivariance amounts to the following. Write

$$-\mu_\gamma \circ T_s = \sum_{\delta \in \mathcal{P}} a_\delta \mu_\delta,$$

and

$$T_s \gamma' = \sum_{\phi \in \mathcal{P}'} b_\phi \phi.$$

then checking the equivariance $\varphi(T_s\gamma) = T_s\varphi(\gamma)$ amounts to

$$a_\delta = (-1)^{\tilde{l}^I(\delta)+\tilde{l}^I(\gamma)} b_{\delta'},$$

where $\delta \mapsto \delta'$ is the bijection defined above. In other words, one must check that the coefficient of γ in $-T_s\delta$ is $(-1)^{\tilde{l}^I(\delta)+\tilde{l}^I(\gamma)}$ times the coefficient of δ' in $T_s\gamma'$. We perform one instance of this check. Suppose α is noncompact imaginary for γ' and δ. By construction of the bijection $\phi \mapsto \phi'$, s is real (and necessarily satisfying the parity condition by Corollary 6.9(3)) for γ and δ'. Using Definition 9.4 (d1 nonintegral), we get

$$-T_s\delta = -(s \times \delta)^\alpha - (s \times \delta), \tag{11.4}$$

and

$$T_s\gamma' = (s \times \gamma')^\alpha + (s \times \gamma'). \tag{11.5}$$

We are interested in the coefficient of γ in $-T_s\delta$, and since s is real satisfying the parity condition for γ, examining the right-hand side of Equation (11.4) shows that this coefficient is zero unless

$$\gamma = (s \times \delta)^\alpha, \tag{11.6}$$

in which case it is -1. On the other hand, consider the coefficient of δ' in $T_s\gamma'$. Again since α is real satisfying the parity condition for δ', examining the right-hand side of Equation (11.5) shows that this coefficient is zero unless

$$\delta' = (s \times \gamma')^\alpha, \tag{11.7}$$

in which case it is 1.

We claim (11.6) holds if and only if (11.7) holds. This will prove $\varphi(T_s\gamma) = T_s\varphi(\gamma)$ up to sign. (We treat the sign in a moment.) If $\gamma = (s \times \delta)^\alpha$, then

$$\begin{aligned}\gamma' &= [(s \times \delta)^\alpha]' \quad \text{since } \gamma \mapsto \gamma' \text{ is a bijection}\\ &= [(s \times \delta)']_\alpha] \quad \text{by Equation (11.1)}\\ &= s \times (\delta')_\alpha \quad \text{again by Equation (11.1).}\end{aligned}$$

Taking the cross action of s with both sides we conclude $s \times \gamma' = (\delta')_\alpha$, and then taking a Cayley transform in α, we obtain $(s \times \gamma')^\alpha = \delta'$. Finally taking $'$ of both sides and arguing as above, we conclude $\delta = (s \times \gamma)_\alpha$, which is (11.7). These steps are all reversible and establish the equivalence of (11.6) with (11.7).

Finally note that since α is noncompact imaginary for δ, the extended integral length of δ differs from that of $(s \times \delta)^\alpha$ by 1, $(-1)^{\tilde{l}^I[(s\times\gamma)^\alpha]-\tilde{l}^I(\delta)} = -1$. (This is an instance of the general fact that nonintegral wall crosses always change the parity of integral length.) Hence the verification of the equivariance in this case is complete.

We leave the remaining verifications to the reader. □

Retain the notation of the above proof. Given γ and δ in $\mathcal{B}$, let $P_{\gamma\delta}$ denote the polynomial defined in Section 9.6 for the $\mathcal{H}$ submodule $\mathcal{M}$ spanned by basis elements corresponding to $\mathcal{B}$. For γ and δ in $\mathcal{B}'$, let $Q_{\gamma\delta}$ denote the corresponding polynomial. Section 9.6 and the proof of Theorem 11.1 immediately give the following corollary.

Corollary 11.2. *The inverse of the matrix*

$$(P_{\gamma\delta})_{\gamma,\delta\in\mathcal{B}}$$

is the matrix

$$(\eta_{\gamma\delta}Q_{\delta'\gamma'})_{\gamma,\delta\in\mathcal{B}},$$

where $\eta_{\gamma\delta} = (-1)^{\tilde{l}^I(\gamma)-\tilde{l}^I(\delta)}$*, the sign of the parity of the difference between the extended integral length of* γ *and* δ*.*

We briefly discuss the choices involved in the duality map $\gamma \mapsto \gamma'$. We may assume that λ is half-integral, i.e. 2λ is integral. (As is clear from the definitions, the duality map at general infinitesimal character is a product of duality maps for the half-integral case.) The main issue is that the central shift λ' of λ could consist of either p integers and q half-integers or (by shifting by all half-integers) q integers and p half-integers. There is no way to make this choice canonically. This ambiguity manifests itself in the appearance of the outer automorphism in Theorem 12.2.

12. Functorial description of Kazhdan-Patterson lifting

Recall the central subgroup $\widetilde{Z}_0$ of $\widetilde{\mathrm{GL}}(n,\mathbb{R})$ defined in Remark 10.5, and fix a genuine character χ^+ of $\widetilde{Z}_0$. (If n is even, this choice is unique; if n is odd, there are two such choices.) Let $\mathbb{K}\mathcal{HC}^{\mathrm{gen}}(\widetilde{\mathrm{GL}}(n,\mathbb{R}))^+$ denote the Grothendieck group of genuine virtual representation of $\widetilde{\mathrm{GL}}(n,\mathbb{R})$ on which $\widetilde{Z}_0$ acts by χ^+. (This the full Grothendieck if n is even, but only half of the full Grothendieck group in the odd case.) Given a choice of χ_0, recall the Kazhdan-Patterson lifting T defined above. On the level of virtual representations, it amounts to a map

$$\mathrm{T} \;:\; \mathbb{K}(\mathrm{GL}(n,\mathbb{R})) \longrightarrow \mathbb{K}^{\mathrm{gen}}(\widetilde{\mathrm{GL}}(n,\mathbb{R}))^+,$$

It has a simple effect on infinitesimal characters and factors to a map

$$\mathrm{T}_\lambda \;:\; \mathbb{K}(\mathrm{GL}(n,\mathbb{R}))_\lambda \longrightarrow \mathbb{K}^{\mathrm{gen}}(\widetilde{\mathrm{GL}}(n,\mathbb{R}))^+_{\lambda/2}.$$

The purpose of this section is to reinterpret this map in terms of the duality defined in Section 11.

Fix a regular infinitesimal character $\lambda/2$; the singular case is discussed below. Write (as in the beginning of Section 11 with $\lambda/2$ instead of λ) $\lambda/2$ as the concatenation of regular integral infinitesimal characters λ_i for $\mathfrak{gl}(n_i)$ where $n_1+\cdots+n_k = n$. Fix χ^+, and fix a choice of the the duality of the previous section at $\lambda/2$; this gives a bijection $\mathcal{P}^+_{\lambda/2}(\widetilde{\mathrm{GL}}(n)) \hookrightarrow \coprod_{i=1}^k \mathcal{P}_{\lambda_i}(\widetilde{\mathrm{U}}(p_i,q_i))$ for some integers p_i and q_i such that $p_i+q_i = n_i$. (The precise values of p_i, q_i, and λ_i depend on the choice of duality.) If we follow this bijection by taking the support of the appropriate $\mathcal{D}$-module localization, we obtain a map

$$\widetilde{\Phi}_{\lambda/2} \;:\; \mathcal{P}^+_{\lambda/2}(\widetilde{\mathrm{GL}}(n,\mathbb{R})) \longrightarrow \coprod_{i=1}^{k}(\mathrm{GL}(p_i,\mathbb{C}) \times \mathrm{GL}(q_i,\mathbb{C}))\backslash\mathfrak{B}_i,$$

where $\mathfrak{B}_i$ denotes the flag variety for $\mathfrak{g} = \mathfrak{gl}(n_i,\mathbb{C})$. Since every orbit of $\mathrm{GL}(p_i,\mathbb{C}) \times \mathrm{GL}(q_i,\mathbb{C})$ on $\mathfrak{B}_i$ admits at most one equivariant local system (since Cartan subgroups for $\widetilde{\mathrm{U}}(p_i,q_i)$ are connected), $\widetilde{\Phi}_{\lambda/2}$ is injective. The definition of $\widetilde{\Phi}_{\lambda/2}$ depends on a choice of the duality of Section 11.

We can make the same construction for the linear group $\mathrm{GL}(n,\mathbb{R})$. Once we fix of a choice of the duality of [V4] for $\mathrm{GL}(n,\mathbb{R})$ at λ, we obtain a bijection $\mathcal{P}_\lambda(\mathrm{GL}(n,\mathbb{R})) \to \coprod_i \coprod_{r_i+s_i=n_i} \mathcal{P}_\rho(\mathrm{U}(r_i,s_i))$. After taking supports we obtain a bijection

$$\Phi_\lambda \;:\; \mathcal{P}_\lambda(\mathrm{GL}(n,\mathbb{R})) \longrightarrow \coprod_{i=1}^{k} \coprod_{r_i+s_i=n_i} (\mathrm{GL}(r_i,\mathbb{C}) \times \mathrm{GL}(s_i,\mathbb{C}))\backslash\mathfrak{B}_i.$$

(Here it is important to note that image is the coproduct over all $r_i+s_i = n$. For instance, the unique orbits of $\mathrm{GL}(n_i,\mathbb{C}) \times \mathrm{GL}(0,\mathbb{C})$ and $\mathrm{GL}(0,\mathbb{C}) \times \mathrm{GL}(n_i,\mathbb{C})$ on $\mathfrak{B}_i$ clearly coincide as sets, but they are distinct in the image of Φ_λ.) Like the definition $\widetilde{\Phi}$, Φ depends on a choice of the duality of [V4] for $\mathrm{GL}(n,\mathbb{R})$.

Define a map

$$\mathrm{S}_\lambda \;:\; \mathcal{P}_\lambda(\mathrm{GL}(n,\mathbb{R})) \longrightarrow \mathcal{P}^+_{\lambda/2}(\widetilde{\mathrm{GL}}(n,\mathbb{R})) \cup \{0\},$$

as follows. (Here 0 is to be considered as a formal symbol.) Set

$$\mathrm{S}_\lambda(\gamma) = (\widetilde{\Phi}_{\lambda/2})^{-1} \circ \Phi_\lambda(\gamma),$$

if $\Phi(\gamma) \in \mathrm{Image}(\widetilde{\Phi}_{\lambda/2})$; otherwise set $\mathrm{S}(\gamma) = 0$. Write S for the resulting map from $\mathcal{P}(\mathrm{GL}(n,\mathbb{R}))$ to $\mathcal{P}(\widetilde{\mathrm{GL}}(n,\mathbb{R})) \cup \{0\}$.

Define $X_{\widetilde{\mathrm{GL}}}(0)$ to be the zero element in the Grothendieck group of genuine representations of $\widetilde{\mathrm{GL}}(n,\mathbb{R})$. Then S_λ induces a map (which we also call S_λ) of Grothendieck groups,

$$\begin{aligned} \mathrm{S}_\lambda \;:\; \mathbb{K}(\mathrm{GL}(n,\mathbb{R}))_\lambda &\longrightarrow \mathbb{K}^{\mathrm{gen}}(\widetilde{\mathrm{GL}}(n,\mathbb{R}))^+_{\lambda/2} \\ X_{\mathrm{GL}}(\gamma) &\longrightarrow X_{\widetilde{\mathrm{GL}}}(\mathrm{S}(\gamma)). \end{aligned}$$

Note that the lifting S_λ depends a choice of the duality of Section 11 for $\widetilde{\mathrm{GL}}$ at infinitesimal character $\lambda/2$, and a choice of the duality of [V4] for $\mathrm{GL}(n,\mathbb{R})$ at λ.

Example 12.1. Here we compute S_λ for $\mathrm{GL}(n,\mathbb{R})$ when $n = 1$ or $n = 2$. Fix χ^+ as always. When $n = 1$, there are two representations of $\mathbb{R}^\times$ with fixed infinitesimal character: the one which is trivial on the component group of $\mathbb{R}^+$ (which, to be consistent with notation introduced below, we denote $X(\gamma_0)$); and the one which is nontrivial (denoted $X(\gamma_1)$). There is only one only genuine representation (which we denote $X(\tilde{\gamma})$) of $\widetilde{\mathbb{R}}^\times$ with the prescribed action of $\widetilde{Z}_0$. Depending on the choices of duality defining S_λ, either $\mathrm{S}_\lambda(X(\gamma_0)) = X(\tilde{\gamma})$ and $\mathrm{S}_\lambda(X(\gamma_1)) = 0$, or vice versa.

Now consider the $n = 2$ case. If λ is not integral, there are only the four principal series for $\mathrm{GL}(2,\mathbb{R})$ (all of which are irreducible), and there is only the unique genuine (irreducible) principal series, say $X(\tilde{\gamma})$ of $\widetilde{\mathrm{GL}}(2,\mathbb{R})$ at $\lambda/2$. Then it is easy to check that S_λ will map three of the four principal series to 0, and one to the unique genuine principal series. Exactly which principal series has nonzero image depends on the choices of dualities. (For different choices, each principal series may have nonzero image.)

The case of λ integral is more interesting. Set $l = \langle \lambda, \alpha \rangle \in \mathbb{Z}$. Recall that a representation is called *spherical* if its lowest K-type is trivial. There is a unique discrete series representation, say $X(\delta)$, of $\mathrm{GL}(2,\mathbb{R})$ at λ; and there are again four principal series $X(\gamma_0), \ldots, X(\gamma_3)$. Arrange the notation so that $X(\gamma_0)$ is spherical , and $X(\gamma_1)$ is the outer automorphism conjugate of γ_0; i.e. $X(\gamma_1)$ has a one-dimensional nontrivial lowest K type. There is a unique genuine relative discrete series , say $X(\tilde{\delta})$, of $\widetilde{\mathrm{GL}}(2,\mathbb{R})$ at $\lambda/2$, if l is odd; and there are no such discrete series if l is even. On the other hand, there is always a unique genuine principal series, again say $X(\tilde{\gamma})$, for $\widetilde{\mathrm{GL}}(2,\mathbb{R})$ at $\lambda/2$.

Suppose first that l is odd. Independent of any choices the support of the duals of $X(\delta)$ and $X(\tilde{\delta})$ is the open orbit of $\mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ on $\mathfrak{B}$, the flag variety for $\mathfrak{gl}(2)$. Thus $\mathrm{S}_\lambda(X_{\mathrm{GL}(2)}(\delta)) = X_{\widetilde{\mathrm{GL}}(2)}(\tilde{\delta})$. Depending on the choices of the duality of Section 11, the support of the dual of $X(\tilde{\gamma})$ is one of the two closed orbits, say Q_+ or Q_-, of $\mathrm{GL}(1,\mathbb{C}) \times \mathrm{GL}(1,\mathbb{C})$ on $\mathfrak{B}$. On the other hand, since l is odd, the principal series $X(\gamma_2)$ and $X(\gamma_3)$ are irreducible and have duals which necessarily are supported on the orbits, say Q'_+ and Q'_-, of $\mathrm{GL}(2,\mathbb{C}) \times \mathrm{GL}(0,\mathbb{C})$ or $\mathrm{GL}(0,\mathbb{C}) \times \mathrm{GL}(2,\mathbb{C})$ on $\mathfrak{B}$. Meanwhile the reducible principal series $X(\gamma_0)$ and $X(\gamma_1)$ have irreducible quotients whose duals are supported on Q_+ and Q_-, and exactly which one depends on the choices. Hence we conclude that

$$
\begin{aligned}
&\mathrm{S}_\lambda(X(\gamma_2)) = \mathrm{S}_\lambda(X(\gamma_3)) = 0; \text{ and either}\\
&\mathrm{S}_\lambda(X(\gamma_0)) = X(\tilde{\gamma}) \text{ and } \mathrm{S}_\lambda(X(\gamma_1)) = 0; \text{ or}\\
&\mathrm{S}_\lambda(X(\gamma_1)) = X(\tilde{\gamma}) \text{ and } \mathrm{S}_\lambda(X(\gamma_0)) = 0.
\end{aligned}
$$

If, on other hand, l is even, then $X(\tilde{\gamma})$ is irreducible, and its dual is supported on either Q'_+ or Q'_-. This time, it is $X(\gamma_0)$ and $X(\gamma_1)$ which are irreducible and necessarily have duals supported on Q'_+ and Q'_-. Thus the conclusions of the previous displayed possibilities hold without change. Since an outer automorphism of $\mathrm{GL}(2,\mathbb{R})$ switches $X(\gamma_0)$ and $X(\gamma_1)$ (and also switches $X(\gamma_2)$ and $X(\gamma_3)$), we see that the two possibilities displayed above differ by twisting by an outer automorphism.

Thus we conclude that while there are a number of possible choices for the dualities for $\mathrm{GL}(2,\mathbb{R})$ and $\widetilde{\mathrm{GL}}(2,\mathbb{R})$, these choices lead to only two distinct maps S_λ. The two maps are distinguished by their value on the spherical principal series $X(\gamma_0)$: one maps $X(\gamma_0)$ to zero, the other does not. Suppose we fix choices so that we are in the latter case. For $n = 1$, this requirement implies $\mathrm{S}_\lambda(X(\gamma_0)) = X(\tilde{\gamma})$; for $n = 2$, S_λ maps the spherical principal series to $X(\tilde{\gamma})$, and takes all other principal series to 0. When relative discrete series of $\mathrm{GL}(2,\mathbb{R})$ exist at λ, they are unique, and are mapped to the unique genuine relative discrete series of $\widetilde{\mathrm{GL}}(2,\mathbb{R})$. It is straightforward to work directly from the definition of T_λ in [KP], to see that with these choices $\mathrm{S}_\lambda = \mathrm{T}_\lambda$ for $n = 1$ and 2.

In fact, since the character formulas (of expressing standard modules in terms of irreducibles) are easy and well-known for $\mathrm{GL}(2,\mathbb{R})$ and $\widetilde{\mathrm{GL}}(2,\mathbb{R})$, at $\lambda = \rho$ it is easy to see that these choices give

$$\begin{aligned}\mathrm{S}_\lambda(\overline{X}(\delta)) &= \overline{X}(\tilde{\delta});\\ \mathrm{S}_\lambda(\mathrm{triv}) &= -\overline{X}(\tilde{\gamma});\\ \mathrm{S}_\lambda(\overline{X}(\gamma_i)) &= 0, \text{ for } i \geq 1;\end{aligned}$$

here of course triv $= \overline{X}(\gamma_0)$. [1]

This example shows, at least for $\mathrm{GL}(2,\mathbb{R})$, that the different choices involved in the definition of S_λ, amount to (at most) a twist of T_λ by an outer automorphism. In fact, this is true in general.

Theorem 12.2. *Fix regular infinitesimal character λ, and let $\mathbf{A}$ denote an outer automorphism of $\mathrm{GL}(n,\mathbb{R})$. Fix choices of the duality for $\mathrm{GL}(n,\mathbb{R})$ at λ and $\widetilde{\mathrm{GL}}(n,\mathbb{R})$ at $\lambda/2$, and use these choices to define S_λ. Then either*

$$\mathrm{S}_\lambda = \mathrm{T}_\lambda \quad \text{or} \quad \mathrm{S}_\lambda \circ \mathbf{A} = \mathrm{T}_\lambda.$$

Both possibilities are distinct and each can occur for appropriate choices of the dualities involved.

Proof. It is clear from the definition of S_λ that it suffices to treat the case when λ is regular and integral. (The general case is simply a kind of direct product of integral cases.)

Let γ_0 parametrize the spherical principal series of $\mathrm{GL}(n)$ at λ, and let γ_1 denote its outer automorphism conjugate. (So $X(\gamma_1)$ has nontrivial one-dimensional lowest K type.) Let $\tilde{\gamma}$ parametrize the unique principal series of $\widetilde{\mathrm{GL}}(n)$ at λ with prescribed central character χ^+. We begin by showing that independent of any choices of the dualities defining S_λ, either

$$\mathrm{S}_\lambda(X(\gamma_0)) = X(\tilde{\gamma}) \text{ while } \mathrm{S}_\lambda(X(\gamma_1)) = 0; \tag{12.1}$$

or

$$\mathrm{S}_\lambda(X(\gamma_0)) = 0 \text{ while } \mathrm{S}_\lambda(X(\gamma_1)) = X(\tilde{\gamma}). \tag{12.2}$$

First observe that a (real) root α for γ_i satisfies the parity condition and gives reducibility if and only if $\langle\lambda,\alpha\rangle \in 2\mathbb{Z}+1$. Fix an arbitrary choice of the duality for $\mathrm{GL}(n,\mathbb{R})$ at λ, and let γ_i' parametrize the dual of $\overline{X}(\gamma_i)$; so $X(\gamma_0')$ is a representation of some $\mathrm{U}(p,q)$, while $X(\gamma_1')$ is a representation of $\mathrm{U}(q,p)$ (for the same integers p and q). To distinguish these cases set $(p_0,q_0) = (p,q)$ and $(p_1,q_1) = (q,p)$. Let Q_i' denote the support of γ_i. Each Q_i' is an orbit of $\mathrm{GL}(p_i,\mathbb{C}) \times \mathrm{GL}(q_i,\mathbb{C})$ on $\mathfrak{B}$. From the general properties of [V4], it follows that Q_i' is a closed orbit (since $\overline{X}(\gamma_i')$ must be a discrete series) and, moreover, α is noncompact for Q_i' if and only if $\langle\lambda,\alpha\rangle \in 2\mathbb{Z}+1$. It follows that each Q_i' supports a genuine discrete series representation, say $\overline{X}(\tilde{\delta}_i)$ for $\widetilde{\mathrm{U}}(p_i,q_i)$. The infinitesimal character of $\overline{X}(\tilde{\delta}_i)$ either consists of p_i integers and q_i half-integers, or vice-versa. The exact details do not concern us here, except for the fact that we may uniquely specify the duality for $\widetilde{\mathrm{GL}}(n,\mathbb{R})$ at $\lambda/2$ by requiring that the image of $\overline{X}(\tilde{\gamma})$ be $\overline{X}(\tilde{\delta}_0)$. In this case, the tracing through the definitions shows that Equation (12.1) holds. On the other hand, we may uniquely specify the duality for

[1]Note that the minus sign in the second displayed equation contradicts the main proposition in the introduction of [AH]. In fact, that proposition is stated incorrectly; in part (1), the right-hand side needs a factor of $(-1)^{n/2}$.

$\widetilde{\mathrm{GL}}(n,\mathbb{R})$ at $\lambda/2$ by requiring that the image of $\overline{X}(\tilde{\gamma})$ be $\overline{X}(\tilde{\delta}_1)$. In this case Equation (12.2) holds. These are the only two possibilities that can occur.

For future reference, we refer to choices of the dualities which lead to Equation (12.1) as *spherical choices*). Likewise, we call those that lead to Equation 12.2 *antispherical choices*.

We now give an alternative description of S_λ. In discussion that follows, let $i = 0$ if the choices of the dualities defining S_λ are spherical and let $i = 1$ if they are antispherical. As we saw above, α satisfies the parity condition for γ_i if and only if α is half-integral (but not integral) for $\lambda/2$. It is easy to see that a subspace $\mathfrak{s}$ of the span of the half-integral roots for $\mathrm{S}_\lambda(\gamma_i)$ is admissible if and only if $\mathfrak{s}$ is an admissible subspace of the span of the real roots satisfying the parity condition for γ_i. From the properties of the duality of [V4] together with the definition of the duality in Section 11, we conclude:

$$c_{\mathfrak{s}}\big(\mathrm{S}_\lambda(\gamma_i)\big) = \mathrm{S}_\lambda\big(c_{\mathfrak{s}}(\gamma_i)\big). \tag{12.3}$$

Moreover, if $\gamma \in \mathcal{P}_\lambda(\mathrm{GL}(n))$,

$$\mathrm{S}_\lambda(X(\gamma_i)) \neq 0 \text{ iff there exists } \mathfrak{s} \text{ such that } c_{\mathfrak{s}}(\gamma_0) = \gamma. \tag{12.4}$$

Now suppose for definiteness that we are in the spherical case. Equation (12.1) shows that different spherical choices of the dualities do not affect the value of $\mathrm{S}_\lambda(X(\gamma_0))$. Hence Equations (12.3) and (12.4) show that different spherical choices of the dualities do not change *any* values of the map S_λ. The same argument shows that different antispherical choices of the dualities do not affect the map S_λ. So, indeed, as we vary all of the choices of the dualities defining S_λ, we obtain just two different maps; they are distinguished by whether Equation (12.1) or (12.2) holds. Henceforth we will write $\mathrm{S}^{\mathrm{s}}_\lambda$ for the map for which Equation (12.1) holds, and $\mathrm{S}^{\mathrm{a}}_\lambda$ for the map for which Equation (12.2) holds.

Notice that since the iterated Cayley transform $c_{\mathfrak{s}}$ commutes with $\mathbf{A}$ and since $\mathbf{A}(X(\gamma_0)) = X(\gamma_1)$, Equation (12.3) implies that $\mathrm{S}^{\mathrm{a}}_\lambda \circ \mathbf{A} = \mathrm{S}^{\mathrm{s}}_\lambda$. Thus all the assertions of the present theorem will follow if we can show $\mathrm{S}^{\mathrm{s}}_\lambda = \mathrm{T}_\lambda$.

To continue, we need to recall the notion of induction product for $\widetilde{\mathrm{GL}}(n,\mathbb{R})$. For $\mathrm{GL}(n,\mathbb{R})$, recall that parabolic induction defines an associative unitary-preserving "induction product" which takes representations of $\mathrm{GL}(n_1,\mathbb{R}) \times \mathrm{GL}(n_2,\mathbb{R})$ to those of $\mathrm{GL}(n_1+n_2,\mathbb{R})$. The analog for $\widetilde{\mathrm{GL}}(n,\mathbb{R})$, which we now discuss, is a little subtle since $\widetilde{\mathrm{GL}}(n_1,\mathbb{R}) \times \widetilde{\mathrm{GL}}(n_2,\mathbb{R})$ is not the Levi factor of a parabolic subgroup of $\widetilde{\mathrm{GL}}(n,\mathbb{R})$.

Let $P_\mathbb{R} = M_\mathbb{R} N_\mathbb{R}$ be a parabolic subgroup of $G_\mathbb{R} = \mathrm{GL}(n,\mathbb{R})$ with Levi factor $M_\mathbb{R} = \mathrm{GL}(n_1,\mathbb{R}) \times \mathrm{GL}(n_2,\mathbb{R})$. Write $\widetilde{P}_\mathbb{R}$ for the preimage of $P_\mathbb{R}$ in $\widetilde{G_\mathbb{R}} = \widetilde{\mathrm{GL}}(n,\mathbb{R})$. Since the cover is obviously split over $N_\mathbb{R}$, we may write $\widetilde{P}_\mathbb{R} = \widetilde{M_R} N_\mathbb{R}$. Fix a genuine character χ of $\widetilde{Z}_o$ as in Remark 10.5. Let τ_i be a genuine representation of $\widetilde{M^i_\mathbb{R}} = \widetilde{\mathrm{GL}}(n_i,\mathbb{R})$. The restriction to the preimage, say $\widetilde{M^{i+}_\mathbb{R}}$, of the connected component of the identity in $\mathrm{GL}(n_i,\mathbb{R})$ is irreducible (if n is odd) and breaks into two inequivalent representations (if n is even). Choose irreducible constituents σ_i of the restriction. The representation $\sigma_1 \otimes \sigma_2 \otimes \chi$ factors to a representation, say τ, of $\widetilde{M^+_\mathbb{R}}\widetilde{Z}_o$; here $\widetilde{M^+_\mathbb{R}}$ is the preimage in $\widetilde{\mathrm{GL}}(n,\mathbb{R})$ of the identify component of $\mathrm{GL}(n_1,\mathbb{R}) \times \mathrm{GL}(n_2,\mathbb{R})$. Define the induction product of τ_1 and τ_2, denoted $\mathrm{Ind}^{\widetilde{G_\mathbb{R}}}_{\widetilde{M_\mathbb{R}}N_\mathbb{R}}(\tau_1 \otimes \tau_2)$, as

$$\mathrm{Ind}^{\widetilde{G_\mathbb{R}}}_{\widetilde{M_\mathbb{R}}N_\mathbb{R}}(\tau_1 \otimes \tau_2) = 2^{-[q/2]}\mathrm{Ind}^{\widetilde{G_\mathbb{R}}}_{\widetilde{M^+_\mathbb{R}}\widetilde{Z}_o N}(\tau),$$

where $q = \#\{n_i \mid n_i \text{ is odd}\}$. This depends only on the choice of χ (in the odd case) and not on the choice of the σ_i. It requires a little checking of the definitions to see that the induction product is associative, and the obvious induction in stages statement holds.

We now return to the proof of Theorem 12.2, and prove that $\mathrm{S}^{\mathrm{s}}_\lambda$ commutes with induction product. More precisely, fix a partition $n_1+n_2 = n$, let λ_i be regular integral infinitesimal characters for $\mathfrak{gl}(n_i)$, and fix $\gamma^i \in \mathcal{P}_{\lambda_i}(\mathrm{GL}(n_i,\mathbb{R}))$. Let $\mathrm{S}^{\mathrm{s}}_{\lambda_i}$ denote the map defined with respect to spherical choices of the duality (and the same fixed central character χ^+) for $\mathrm{GL}(n_i,\mathbb{R})$. To conserve notation we write S^{s} for $\mathrm{S}^{\mathrm{s}}_{\lambda_i}$; the context will allow for no confusion. Let $P_\mathbb{R}$ be a parabolic subgroup of $\mathrm{GL}(n,\mathbb{R})$ with Levi factor $\mathrm{GL}(n_1,\mathbb{R}) \times \mathrm{GL}(n_2,\mathbb{R})$. Write $\widetilde{P}_\mathbb{R}$ for the preimage of $P_\mathbb{R}$ in $\widetilde{G_\mathbb{R}} = \widetilde{\mathrm{GL}}(n,\mathbb{R})$. To conserve notation, write Ind for $\mathrm{Ind}_{P_\mathbb{R}}^{\mathrm{GL}(n)}$ and $\mathrm{Ind}^{\sim}$ for $\mathrm{Ind}_{\widetilde{P}_\mathbb{R}}^{\widetilde{\mathrm{GL}}(n)}$. We claim that

$$\mathrm{S}^{\mathrm{s}}_\lambda\big(\mathrm{Ind}(X(\gamma^1)\otimes X(\gamma^2))\big) = \mathrm{Ind}^{\sim}\big(X(\mathrm{S}^{\mathrm{s}}(\gamma^1))\otimes X(\mathrm{S}^{\mathrm{s}}(\gamma^2))\big). \tag{12.5}$$

Let γ_0^i parametrize the unique spherical principal series at λ_i for each $\mathrm{GL}(n_i,\mathbb{R})$, and similarly let γ_0 denote this element for $\mathrm{GL}(n,\mathbb{R})$ as above. Suppose first that there does not exist admissible subspace of real roots $\mathfrak{s}_i$ for γ_0^i such that $c_\mathfrak{s}(\gamma_0^i) = \gamma^i$. Then it is easy to see that there exists no admissible sequence $\mathfrak{s}$ for γ_0 such that $X(c_\mathfrak{s}(\gamma_0)) = \mathrm{Ind}(X(\gamma^1)\otimes X(\gamma^2))$. By (12.4) both sides of (12.5) are zero, and the claim holds.

Thus to establish (12.5), we may assume that there are admissible subspaces $\mathfrak{s}_i$ of the real roots for γ_0^i such that $c_{\mathfrak{s}_i}(\gamma_0^i) = \gamma^i$. Let $\mathfrak{s}$ denote the span of $\mathfrak{s}_1$ and $\mathfrak{s}_2$. It is easy to see that

$$\mathrm{Ind}\big(X(c_{\mathfrak{s}_1}(\gamma_0^1))\otimes X(c_{\mathfrak{s}_2}(\gamma_0^2))\big) = X(c_\mathfrak{s}(\gamma_0)).$$

Given this and the definition of the $\mathfrak{s}_i$, we can rewrite the claim of (12.5) as

$$\mathrm{S}^{\mathrm{s}}_\lambda(X(c_\mathfrak{s}(\gamma_0))) = \mathrm{Ind}^{\sim}\big(X(\mathrm{S}^{\mathrm{s}}(c_{\mathfrak{s}_1}(\gamma_0^1)))\otimes X(\mathrm{S}^{\mathrm{s}}(c_{\mathfrak{s}_2}(\gamma_0^2)))\big). \tag{12.6}$$

Using (12.3) on the right-hand side, the claim becomes

$$\mathrm{S}^{\mathrm{s}}_\lambda(X(c_\mathfrak{s}(\gamma_0))) = \mathrm{Ind}^{\sim}\big(X(c_{\mathfrak{s}_1}(\mathrm{S}^{\mathrm{s}}(\gamma_0^1)))\otimes X(c_{\mathfrak{s}_2}(\mathrm{S}^{\mathrm{s}}(\gamma_0^2)))\big). \tag{12.7}$$

According the choices made defining S^{s}, each $\mathrm{S}^{\mathrm{s}}(\gamma_0^i))$ is the unique genuine principal series of $\widetilde{\mathrm{GL}}(n_i,\mathbb{R})$ at infinitesimal character $\lambda_i/2$ (with prescribed central character χ^+). So by induction in stages,

$$\mathrm{Ind}^{\sim}\big(X(\mathrm{S}^{\mathrm{s}}(\gamma_0^1))\otimes X(\mathrm{S}^{\mathrm{s}}(\gamma_0^2))\big) = X\big(\mathrm{S}^{\mathrm{s}}_\lambda(\gamma_0)\big),$$

the unique genuine principal series of $\widetilde{\mathrm{GL}}(n,\mathbb{R})$ at infinitesimal character $\lambda/2$ (again with prescribed central character χ^+). Hence we conclude that the right-hand side of (12.7) is just $X(c_\mathfrak{s}(\mathrm{S}^{\mathrm{s}}_\lambda(\gamma_0)))$, and so (12.7) becomes

$$\mathrm{S}^{\mathrm{s}}_\lambda(X(c_\mathfrak{s}(\gamma_0))) = X(c_\mathfrak{s}(\mathrm{S}^{\mathrm{s}}_\lambda(\gamma_0))).$$

But this is just (12.3). This establishes (12.5).

In [KP, Proposition 26.2], the analog of (12.5) is established for the lifting T_λ. The Langlands classification for $\mathrm{GL}(n,\mathbb{R})$ at regular integral λ implies that the Grothendieck group $\mathbb{K}(\mathrm{GL}(n,\mathbb{R}))_\lambda$ is spanned by representations obtained as an induction product of representations (with regular integral infinitesimal character) for $\mathrm{GL}(1)$ and $\mathrm{GL}(2)$. Since $\mathrm{S}^{\mathrm{s}}_\lambda$ and T_λ both satisfy (12.5), the current theorem reduces to establishing $\mathrm{S}^{\mathrm{s}}_\lambda = \mathrm{T}_\lambda$ for $\mathrm{GL}(1,\mathbb{R})$ and $\mathrm{GL}(2,\mathbb{R})$ at regular integral infinitesimal character. This is the content of Example 12.1. The theorem follows. $\square$

Singular infinitesimal character. Since the above discussion was confined entirely to the case that λ was regular, we need to discuss singular infinitesimal character. We offer two equivalent approaches. Fix λ regular and let ν be an element of the weight lattice such that $\lambda + 2\nu$ is dominant and regular. Consider the translation functors $\psi = \psi_{\lambda}^{\lambda+2\nu}$ and $\psi' = \psi_{\lambda/2}^{\lambda/2+\nu}$. Since ψ maps spherical principal series to spherical principal series and commutes with the action of $\mathbf{A}$, it is clear that

$$\psi' \circ \mathrm{S}_\lambda = \mathrm{S}_{\lambda+2\nu} \circ \psi. \tag{12.8}$$

Now suppose λ is regular and that $\lambda+2\nu$ is dominant but potentially singular. (Of course every singular infinitesimal character arises in this way.) Since each irreducible representation at $\lambda + 2\nu$ is of the form $\psi(X)$ for an irreducible representation at the regular infinitesimal character λ (by the translation principle), we can we may simply define S at the singular infinitesimal $\lambda_s = \lambda + 2\nu$ by requiring that Equation (12.8) hold. It easy to establish that T also satisfies Equation (12.8) (see [AH, Proposition 4.3]), so our definition of S at singular infinitesimal character coincides with T (up to, of course, possibly twisting by $\mathbf{A}$).

The discussion of the previous paragraph may seem a little unsatisfactory: S was defined geometrically for λ regular, but we resorted to an algebraic definition for λ_s singular. In fact it essentially a formal exercise to translate the above description into a geometric one for λ_s singular. We briefly sketch the details. The crucial point is to define an analog of Φ_{λ_s} when λ_s is singular. Fix $\gamma_s \in \mathcal{P}_{\lambda_s}$. Using the translation principle as above, we may write $\overline{X}(\gamma_s) = \psi(\overline{X}(\gamma))$ where ψ is a dominance-preserving translation functor (a "push to multiple walls") from a regular infinitesimal character, say λ, to λ_s. The fact that $\psi(\overline{X}(\gamma)) \neq 0$ implies that the dual module $\overline{X}(\gamma')$ may be obtained by localization on the partial flag variety, say $\mathfrak{B}_s$, defined by the roots for which λ_s is singular. We define the support of this localization to be $\Phi_{\lambda_s}(\lambda)$. As long as the translation $\lambda - \lambda_s$ is *twice* a weight, we may make the analogous definition for $\widetilde{\Phi}_{\lambda_s/2}$ using a translation functor from $\lambda_s/2$ to $\lambda/2$, and we may proceed to define the lifting as in the case of λ regular. This lifting then coincides with T_{λ_s} (up to $\mathbf{A}$).

13. Duality for $\mathrm{U}(\tilde{p}, q)$

Theorem 11.1 provides a character multiplicity duality theory for $\mathrm{U}(\tilde{p}, q)$ when the infinitesimal character coincides with that of a discrete series. The purpose of this section is to establish a complete duality theory for $\mathrm{U}(\tilde{p}, q)$. We begin with the case of integral infinitesimal character which essentially reduces to the linear group $\mathrm{SU}(p,p)$.

Proposition 13.1. *Let $\widetilde{G}_{\mathbb{R}} = \widetilde{\mathrm{U}}(p,q)$ (Notation 10.7), assume $pq \neq 0$ and fix regular integral infinitesimal character $\lambda = (\lambda_1, \ldots, \lambda_n)$. Then $\widetilde{G}_{\mathbb{R}}$ has no genuine representations at λ unless: $p = q$; $4\lambda_i \in \mathbb{Z}$; and, finally, there is a fixed choices of sign ϵ so that $4\lambda_i = \epsilon \mod 4$ for all i. In this case there is a unique block, say $\mathcal{B}$ of representations and it is isomorphic to the unique block, say $\mathcal{B}_1$, of representations of $\mathrm{SU}(p,p)$ at trivial infinitesimal character not containing the trivial representation. More precisely, there is an isomorphism of posets (in the relevant Bruhat $\mathcal{G}$-orders),*

$$\mathcal{B} \simeq \mathcal{B}_1$$
$$\gamma \mapsto \gamma_1$$

Consequently this bijection preserves composition series in the sense that

$$X_{\widetilde{\mathrm{U}}(p,p)}(\delta) = \sum_{\gamma \in \mathcal{B}} m(\gamma,\delta)\overline{X}_{\widetilde{\mathrm{U}}(p,p)}(\gamma) \quad \Longleftrightarrow \quad X_{\mathrm{SU}(p,q)}(\delta_1) = \sum_{\gamma_1 \in \mathcal{B}_1} m(\gamma,\delta)\overline{X}_{\mathrm{SU}(p,p)}(\gamma_1).$$

Proof. Consider the first statement about the existence of genuine representations and, without loss of generality, assume $p \geq q$. It suffices to consider the existence of genuine principal series. Since $pq \neq 0$ by hypothesis, there exists at least one $\mathbb{C}^\times$ factor in the maximally split Cartan of $\mathrm{U}(p,q)$. This is covered nontrivially in $\widetilde{\mathrm{U}}(p,q)$, and consequently there must be a pair of coordinates in λ which sum to a half-integer and whose difference (by the integrality assumption) must be an integer. Thus there must be a pair of coordinates of the form $(a_1, a_1 + n_1)$ where $4a$ and n are integers and $4a_i = \pm 1 \mod 4$. The same analysis applies to other $q-1$ $\mathbb{C}^\times$ factors of the Cartan. We conclude that there are q pairs of coordinates of λ of the form $(a_i, a_i + n_i)$ with $4a_i, n_i \in \mathbb{Z}$ and, moreover, by the integrality assumption there is a choice of sign ϵ such that $4a_i = \epsilon \mod 4$ for all i. If $p > q$, there is at least one S^1 factor of the maximally split Cartan in $\mathrm{U}(p,q)$. It may or may not be covered nontrivially, but in each respective case to accommodate a genuine principal series, there must be an half-integer or integer coordinate of λ. This violates the integrality hypothesis on λ, and hence $p = q$, and the first assertion is proved.

The above discussion implies that when $p = q$ and λ satisfies the conditions of the proposition, there is a principal series, say γ_{ps}, supported on the open orbit of the flag variety. Moreover, it follows that each root is complex for γ_{ps} except the roots $e_i - e_{2p+1-i}$ which are real but which do not give reducibility. It follows that there is a unique block $\mathcal{B}$ of representations at λ; it is obtained as a $W = W(\lambda) \simeq S_{2n}$ orbit of γ_{ps} under the cross-action. The stabilizer of γ_{ps} is isomorphic to $W(C_n)$ embedded symmetrically in W. The same discussion applies without change to the unique block $\mathcal{B}_1$ of representations of $\mathrm{SU}(p,p)$ at trivial infinitesimal character which does not contain the trivial representation. Write $(\gamma_{\mathrm{ps}})_1$ for the maximal principal series in this block. The bijection of the theorem is obtained by sending γ_{ps} to $(\gamma_{\mathrm{ps}})_1$ and requiring that the map intertwine the cross action. It is easy to check that this is an isomorphism in the Bruhat $\mathcal{G}$-order. The final assertion of the proposition is thus purely formal. □

Corollary 13.2. *Let $\widetilde{G}_{\mathbb{R}} = \widetilde{\mathrm{U}}(p,q)$ (Notation 10.7) and fix regular infinitesimal character $\lambda = (\lambda_1, \ldots, \lambda_n)$. Let r (resp. s) denote the number of half-integral (resp. integral) coordinates of λ. Set*

$$l_+ = \#\{i \mid 4\lambda_i = 1 \mod 4\}$$
$$l_- = \#\{i \mid 4\lambda_i = -1 \mod 4\}.$$

Then $\mathrm{U}(\widetilde{p},q)$ has no genuine representations at infinitesimal character λ unless

(1) *Both l_+ and l_- are even; and*
(2) $(r,s) = (p - (l_+ + l_-)/2, q - (l_+ + l_-)/2)$.

In this case, there is a unique block, say $\mathcal{B}$, of genuine representations.

Moreover, write the centralizer of 2λ in $\mathfrak{g}^\vee$ as

$$\mathfrak{gl}(l_+ + l_-) \oplus \mathfrak{gl}(r+s),$$

and let $\lambda_\circ$ denote the infinitesimal character for $\mathfrak{gl}(r+s)$ obtained by restriction of λ. Let $\mathcal{B}_+$ denote the (unique) block of representations for $\mathrm{SU}(\frac{l_+}{2}, \frac{l_+}{2})$ at ρ which does not contain the trivial representation; similarly define $\mathcal{B}_-$; and finally let $\mathcal{B}_\circ$ denote the unique block of *genuine representations for $\widetilde{\mathrm{U}}(r,s)$ (Notation 10.7) at the infinitesimal character $\lambda_\circ$ of a discrete series. Then*

$$\mathcal{B} \simeq \mathcal{B}_+ \times \mathcal{B}_- \times \mathcal{B}_\circ$$

as posets equipped with the relevant Bruhat $\mathcal{G}$-orders.

Proof. This follows from a similar (and only slightly more detailed) analysis as found in the proof of Proposition 13.1. We omit the details ☐

Now [V4] provides a character multiplicity duality for $\mathcal{B}_+$ with the unique block of $\mathrm{GL}(l_+/2,\mathbb{H})$ at infinitesimal character ρ (and likewise for $\mathcal{B}_-$), and combined with the duality theory of Theorem 11.1 for $\mathcal{B}_\circ$, we obtain a full duality theory for $\widetilde{\mathrm{U}}(p,q)$.

Corollary 13.3. *Retain the notation and hypothesis of Corollary 13.2. Let*

$$\widetilde{G}'_{\mathbb{R}} = \mathrm{GL}(l_+/2,\mathbb{H}) \times \mathrm{GL}(l_-/2,\mathbb{H}) \times \widetilde{\mathrm{GL}}(r+s,\mathbb{R}),$$

and let λ' denote the infinitesimal character obtained by concatenating $\rho(\mathfrak{gl}(l_+))$, $\rho(\mathfrak{gl}(l_-))$, and $\lambda_\circ$. Fix a block $\mathcal{B}'$ of representations for $\widetilde{G}'_{\mathbb{R}}$ at λ'. (If $r+s$ is even, this block is unique; if $r+s$ is odd, there are two such blocks which are isomorphic.) Then there is a bijection between $\mathcal{B}$ and $\mathcal{B}'$ which is a character multiplicity duality.

14. Functorial lifting for $\mathrm{U}(p,q)$

With the full duality theory in hand, we may now pursue a geometric lifting theory for $\mathrm{U}(p,q)$ by analogy with the discussion of Kazhdan-Patterson lifting in Section 12. This is more delicate mostly because of the nontrivial role of stable conjugacy, but also because of certain central shifts in the infinitesimal character. We only sketch the essential details.

Fix a regular infinitesimal character λ for $\mathfrak{gl}(n)$, and fix integers $p+q=n$. If n is even (resp. odd), the linear group $\mathrm{U}(p,q)$ has no representations at λ unless there are an even number, say l, of integer (resp. half-integer) coordinates, while the remaining, say m, coordinates are all half-integers (resp. integers). In this case, [V4] gives a duality between the unique block of representations for $\mathrm{U}(p,q)$ at λ and a block of representations for $G'_{\mathbb{R}} = \mathrm{GL}(l/2,\mathbb{H}) \times \mathrm{GL}(m,\mathbb{R})$. Consequently, following [ABV], super L-packets of representations for inner forms of $\mathrm{U}(p,q)$ at infinitesimal character λ are parameterized by the orbits of $K' := \mathrm{Sp}(l,\mathbb{C}) \times \mathrm{O}(m,\mathbb{C})$ (the complexified maximal compact for $G'_{\mathbb{R}}$) on the product of flag varieties $\mathfrak{B}' := \mathfrak{B}_l \times \mathfrak{B}_m$. Let $\mathbb{K}(\mathrm{U}(\mathbf{n}))^{\mathrm{st}}_\lambda$ denote the space of superstable (in the sense of [ABV]) virtual representations of the inner forms of $\mathrm{U}(p,q)$ with infinitesimal character λ, and let $\mathcal{P}(\mathrm{U}(\mathbf{n}))^{\mathrm{st}}_\lambda$ denote the basis of this space parametrized by super L-packets (obtained by summing the standard representations in the super L-packet; see again [ABV]). Because super L-packets are parametrized by $K'\backslash\mathfrak{B}'$, we thus obtain an injection

$$\Psi_\lambda \;:\; \mathcal{P}_\lambda(\mathrm{U}(\mathbf{n}))^{\mathrm{st}} \hookrightarrow K'\backslash\mathfrak{B}'.$$

This is the analog of Φ_λ defined in Section 12.

On the other hand, suppose λ is of the form specified in the previous paragraph, and set

$$\tilde{\lambda} = \begin{cases} \lambda/2 & \text{if } n \text{ is odd; and} \\ \lambda/2 + (\frac{1}{4},\dots,\frac{1}{4}) & \text{if } n \text{ is even.} \end{cases}$$

Then, in the notation of Corollary 13.2, $l_+l_- = 0$. So the dual group of Corollary 13.3 is $\widetilde{G}'_{\mathbb{R}} = \mathrm{GL}(l/2,\mathbb{H}) \times \widetilde{\mathrm{GL}}(m,\mathbb{R})$, a double cover of the dual group $G_{\mathbb{R}}$ above. (This is the reason why the shift was introduced in the even case: without it $\widetilde{G}'_{\mathbb{R}}$ need not be a related in a simple way to $G_{\mathbb{R}}$.) The complexified maximal compact subgroup of $\widetilde{G}_{\mathbb{R}}$ is $\widetilde{K}' = \mathrm{Sp}(l, \mathbb{C}) \times \mathrm{Pin}(m, \mathbb{C})$, and the orbits of $\widetilde{K}'$ and K' on $\mathfrak{B}'$ coincide. Set

$$\mathcal{P}_{\tilde{\lambda}}(\widetilde{\mathrm{U}}(\mathbf{n})) = \coprod_{p+q=n} \mathcal{P}_{\tilde{\lambda}}(\widetilde{\mathrm{U}}(p, q)).$$

Now using the duality of Corollary 13.3 and then passing to supports, we obtain an injection

$$\widetilde{\Psi}_{\tilde{\lambda}} \ : \ \mathcal{P}_{\tilde{\lambda}}(\mathrm{U}(\mathbf{n})) \hookrightarrow K'\backslash\mathfrak{B}'.$$

(Here the role of stability (even superstability) is empty because at any given infinitesimal character, there is at most one inner form $\mathrm{U}(\widetilde{p}, q)$ with a discrete series and, moreover, each form has at most one discrete series.) This is the analog of $\widetilde{\Phi}_{\lambda/2}$ defined in Section 12.

It is important to notice that the correspondence of infinitesimal characters $\lambda \mapsto \tilde{\lambda}$ is more complicated than simply dividing by 2. In particular, there are some infinitesimal characters for which $\mathrm{U}(\widetilde{p}, q)$ has genuine representation but are not of the form $\tilde{\lambda}$ for a λ for which some $\mathrm{U}(p, q)$ has representations. (More precisely, this happens whenever $l_+ l_- \neq 0$ in the notation of Corollary 13.2.)

In any event, we now may simply copy the discussion of Section 12 (with the obvious modifications needed to take into account the role of stability) to obtain a map

$$\mathrm{S}_\lambda \ : \ \mathcal{P}(\mathrm{U}(\mathbf{n}))^{\mathrm{st}}_\lambda \longrightarrow \mathcal{P}(\widetilde{\mathrm{U}}(\mathbf{n}))_{\tilde{\lambda}} \cup \{0\},$$

defined by

$$\mathrm{S}_\lambda(\gamma) = (\widetilde{\Psi}_{\tilde{\lambda}})^{-1} \circ \Psi_\lambda(\gamma),$$

if $\Psi_\lambda(\gamma) \in \mathrm{Image}(\widetilde{\Phi}_{\tilde{\lambda}})$; otherwise set $\mathrm{S}_\lambda(\gamma) = 0$. As before, we may then linearize to obtain a lifting

$$\mathrm{S}_\lambda \ : \ \mathbb{K}\mathcal{HC}(\mathrm{U}(\mathbf{n}))^{\mathrm{st}}_\lambda \longrightarrow \mathbb{K}\mathcal{HC}(\widetilde{\mathrm{U}}(\mathbf{n}))_{\tilde{\lambda}} \tag{14.1}$$

of superstable virtual representations of inner forms of $\mathrm{U}(p, q)$. This is the geometric analog of Kazhdan-Patterson lifting for the indefinite unitary group.

Adams and Herb have defined a lifting as in Equation (14.1) on the level of formulas for global characters. Joint work with Adams suggests that it is likely that their lifting coincides with ours. We will return to this elsewhere (and in greater generality).

Part 3. A counterexample to the Kazhdan-Flicker conjecture

The purpose of this part is to compute character formulas for $\mathrm{GL}(4, \mathbb{R})$ and $\widetilde{\mathrm{GL}}(4, \mathbb{R})$ and, using them, produce an example of an irreducible admissible representation of $\mathrm{GL}(4, \mathbb{R})$ whose image under T is a *reducible* virtual representation. This disproves [A2, Conjecture 5.3] (attributed there to Kazhdan and Flicker).

15. Representations of U(2, 2)

Let $G_{\mathbb{R}} = \mathrm{U}(2, 2)$; so $K_{\mathbb{R}} \simeq \mathrm{U}(2) \times \mathrm{U}(2)$, and we may take $K = \mathrm{GL}(2, \mathbb{C}) \times \mathrm{GL}(2, \mathbb{C})$ embedded block diagonally in $G = \mathrm{GL}(4, \mathbb{C})$. Since all Cartans in $\mathrm{U}(2, 2)$ are connected, the Harish-Chandra modules for $\mathrm{U}(2, 2)$ at infinitesimal character ρ are parametrized by K orbits on $\mathfrak{B}$. There are six closed orbits, corresponding to discrete series representations, and in total, twenty-one orbits. The graph of the Bruhat order is represented in Figure 15, where dashed edges indicate cross action with respect to simple complex roots, and full edges indicate Cayley transforms with respect to simple noncompact imaginary roots.

The simple root $e_i - e_{i+1}$, $i = 1, 2, 3$ is indicated by the label $i\ i+1$. For convenience, we will denote the corresponding irreducible characters by letters $T_1, \dots T_6$ (discrete series), $S_1, \dots, S_6$, $R_1, \dots, R_4$, C, B_1, B_2, D and A, while the corresponding standard characters will be denoted by $t_1, \dots, t_6, s_1, \dots, s_6, r_1, \dots, r_4, c, b_1, b_2, d, a$. Applying the KL-algorithm of [V3] gives:

$$
\begin{aligned}
&T_i = t_i,\ i = 1, \dots, 6,\\
&S_1 = s_1 - t_1 - t_2, \quad S_2 = s_2 - t_2 - t_3, \quad S_3 = s_3 - t_2 - t_4,\\
&S_4 = s_4 - t_4 - t_5, \quad S_5 = s_5 - t_3 - t_5, \quad S_6 = s_6 - t_5 - t_6,\\
&R_1 = r_1 - s_1 - s_2 + t_1 + t_2 + t_3, \quad R_2 = r_2 - s_1 - s_3 + t_1 + t_2 + t_4,\\
&R_3 = r_3 - s_4 - s_6 + t_4 + t_5 + t_6, \quad R_4 = r_4 - s_5 - s_6 + t_3 + t_5 + t_6,\\
&C = c - s_2 - s_3 - s_4 - s_5 + t_2 + t_3 + t_4 + t_5,\\
&B_1 = b_1 - r_1 - r_2 - c + s_1 + s_2 + s_3 + s_4 + s_5 + 2t_1 + t_2 + t_3 + t_4 + t_5,\\
&B_2 = b_2 - r_3 - r_4 - c + s_2 + s_3 + s_4 + s_5 + s_6 + t_2 + t_3 + t_4 + t_5 + 2t_6,\\
&D = d - r_1 - r_2 - r_3 - r_4 - c + s_1 + s_2 + s_3 + s_4 + s_5 + s_6 - t_1 - t_2 - 2t_3 - 2t_4 - t_5 - t_6,\\
&A = a - b_1 - b_2 - d + c + r_1 + r_2 + r_3 + r_4 - s_1 - s_2 - s_3 - s_4 - s_5 - s_6\\
&\quad + t_1 + t_2 + t_3 + t_4 + t_5 + t_6.
\end{aligned}
$$

and inverting this matrix gives:

$$
\begin{aligned}
&t_i = T_i,\ i = 1, \dots, 6,\\
&s_1 = S_1 + T_1 + T_2, \quad s_2 = S_2 + T_2 + T_3, \quad s_3 = S_3 + T_2 + T_4,\\
&s_4 = S_4 + T_4 + T_5, \quad s_5 = S_5 + T_3 + T_5, \quad s_6 = S_6 + T_5 + T_6,\\
&r_1 = R_1 + S_1 + S_2 + T_2, \quad r_2 = R_2 + S_1 + S_3 + T_2,\\
&r_3 = R_3 + S_4 + S_6 + T_5, \quad r_4 = R_4 + S_5 + S_6 + T_5,\\
&c = C + S_2 + S_3 + S_4 + S_5 + T_2 + T_3 + T_4 + T_5,\\
&b_1 = B_1 + R_1 + R_2 + C + S_1 + S_2 + S_3 + T_1 + T_2,\\
&b_2 = B_2 + R_3 + R_4 + C + S_4 + S_5 + S_6 + T_5 + T_6,\\
&d = D + R_1 + R_2 + R_3 + R_4 + C + S_1 + S_2 + S_3 + S_4 + S_5 + S_6 + T_2 + T_3 + T_4 + T_5,\\
&a = A + B_1 + B_2 + D + 2C + R_1 + R_2 + R_3 + R_4 + S_1 + S_2 + S_3 + S_4 + S_5 + S_6\\
&\quad + T_1 + T_2 + T_3 + T_4 + T_5 + T_6.
\end{aligned}
$$

16. Representations of $\mathrm{GL}(4, \mathbb{R})$

There block containing the trivial representation of $\mathrm{GL}(4, \mathbb{R})$ is dual to block for $\mathrm{U}(2, 2)$ described in the previous section The graph of the Bruhat order for this block is the graph of Figure 15 turned upside-down. We will denote the corresponding characters of $\mathrm{GL}(4, \mathbb{R})$ by the same letters we used in the previous section to denote their duals. From duality, we

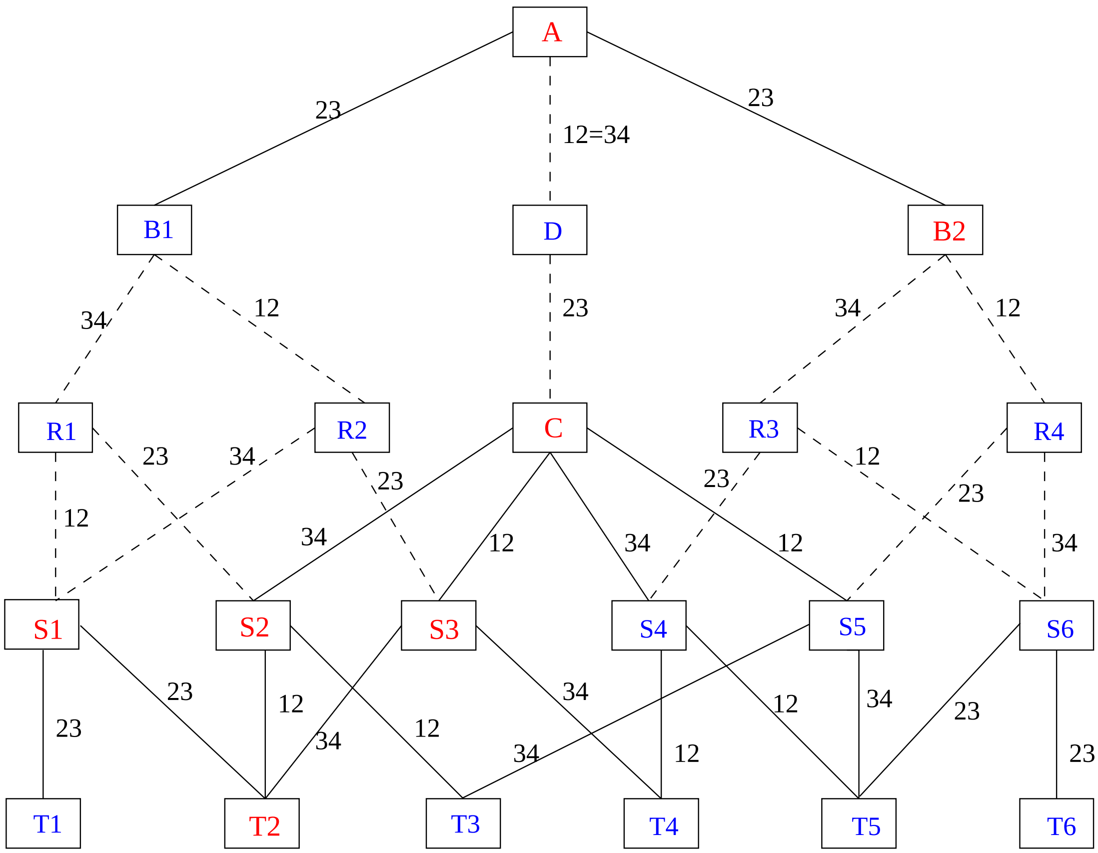

FIGURE 15.1. Representations of U(2, 2) at infinitesimal character ρ.

obtain the following character identities:

$$A = a, \quad B_1 = b_1 - a, \quad B_2 = b_2 - a, \quad D = d - a,$$
$$C = c - b_1 - b_2 - d + 2a$$
$$R_1 = r_1 - b_1 - d + a, \quad R_2 = r_2 - b_1 - d + a, \quad R_3 = r_3 - b_2 - d + a, \quad R_4 = r_4 - b_2 - d + a,$$
$$S_1 = s_1 - r_1 - r_2 + b_1 + d - a, \quad S_2 = s_2 - r_2 - c + b_1 + d - a,$$
$$S_3 = s_3 - r_2 - c + b_1 + d - a, \quad S_4 = s_4 - r_3 - c + b_2 + d - a,$$
$$S_5 = s_5 - r_4 - c + b_2 + d - a, \quad S_6 = s_6 - r_3 - r_4 + b_2 + d - a,$$
$$T_1 = t_1 - s_1 - b_1 + a$$
$$T_2 = t_2 - s_1 - s_2 - s_3 + r_1 + r_2 + c - b_1 - d + a$$
$$T_3 = t_3 - s_2 - s_5 + c - d + a$$
$$T_4 = t_4 - s_3 - s_4 + c - d + a$$
$$T_5 = t_5 - s_3 - s_4 - s_5 + r_3 + r_4 + c - b_2 - d + a$$
$$T_6 = t_6 - s_6 - b_6 + a$$

17. Representations of $\widetilde{\mathrm{GL}}(4,\mathbb{R})$

We now describe genuine representations of $\widetilde{\mathrm{GL}}(4,\mathbb{R})$ at infinitesimal character $\frac{1}{2}\rho = (3/4, 1/4, -1/4, -3/4)$ (in usual coordinates). To perform the KL-algorithm we need some nonintegral translation functors which will change the infinitesimal character. Let us denote F_{12} the translation functor with respect to the integral weight $(1,2,0,0)$ (resp. F_{23} w.r.t $(1,0,1,0)$ and F_{34} w.r.t. $(0,0,-2,-1)$). These are nonintegral wall-crossing functors with respect to the simple walls, and the infinitesimal character obtained are represented respectively by dominant weights $(9/4, 7/4, -1/4, -3/4)$, $(7/4, 3/4, 1/4, -1/4)$ and $(3/4, 1/4-7/4, -9/4)$. There are seven genuine irreducible representations of $\widetilde{\mathrm{GL}}(4,\mathbb{R})$ for each of these infinitesimal characters. The graph of the Bruhat order is shown in Figure 17. Conventions for Cayley transforms and integral cross-actions are the same as in Figure 15 and nonintegral wallcrossing functors are represented in dotted lines.

We will denote these irreducible characters by capital letters, and the corresponding standard characters by minuscule letters (see Figure 17). At infinitesimal character $\frac{1}{2}\rho = (3/4, 1/4, -1/4, -3/4)$, the characters are

$$\tilde{A}, \tilde{B}_2, \tilde{C}, \tilde{S}_1, \tilde{S}_2, \tilde{S}_3, \tilde{T}_2$$

At infinitesimal character $(9/4, 7/4, -1/4, -3/4)$ the characters are

$$\tilde{D}^l, \tilde{C}^l, \tilde{R}_1^l, \tilde{R}_4^l, \tilde{S}_2^l, \tilde{S}_5^l, \tilde{T}_3^l$$

At infinitesimal character $(7/4, 3/4, 1/4, -1/4)$ the characters are

$$\tilde{D}^r, \tilde{C}^r, \tilde{R}_2^r, \tilde{R}_3^r, \tilde{S}_3^r, \tilde{S}_4^r, \tilde{T}_4^l$$

At infinitesimal character $(3/4, 1/4-7/4, -9/4)$ the characters are

$$\tilde{A}^b, \tilde{B}_1^b, \tilde{D}^b, \tilde{R}_1^b, \tilde{R}_2^b, \tilde{S}_1^b, \tilde{T}_1^b$$

We will also use these letters to denote the corresponding Langlands parameters. Thus, in the KL-algorithm, we will consider the free $\mathbb{Z}[u,u^{-1}]$-module with free basis $\{\tilde{a}, \tilde{a}^b, \tilde{b}_1^b, \tilde{b}_2, \dots\}$.

From the Kazhdan-Lusztig algorithm developed in Part I, we get the following character formulas:

$$\tilde{A} = \tilde{a}, \qquad \tilde{B}_2 = \tilde{b}_2 - \tilde{a}, \qquad \tilde{C} = \tilde{c} - \tilde{b}_2 + \tilde{a}$$
$$\tilde{S}_1 = \tilde{s}_1 - \tilde{a}, \qquad \tilde{S}_2 = \tilde{s}_2 - \tilde{c} + \tilde{a}$$
$$\tilde{S}_3 = \tilde{s}_3 - \tilde{c} + \tilde{a}, \qquad \tilde{T}_2 = \tilde{t}_2 - \tilde{s}_1 - \tilde{s}_2 - \tilde{s}_3 + \tilde{c} + \tilde{a}$$

From the formulation of T given in Section 12, we easily calculate:

$$\mathrm{T}(a) = \tilde{a}, \quad \mathrm{T}(b_1) = \tilde{0}, \quad \mathrm{T}(b_2) = \tilde{b}_2, \quad \mathrm{T}(d) = 0$$
$$\mathrm{T}(c) = \tilde{c}, \quad \mathrm{T}(r_1) = \mathrm{T}(r_2) = \mathrm{T}(r_3) = \mathrm{T}(r_4) = 0$$
$$\mathrm{T}(s_1) = \tilde{s}_1, \quad \mathrm{T}(s_2) = \tilde{s}_2, \quad \mathrm{T}(s_3) = \tilde{s}_3$$
$$\mathrm{T}(s_4) = \mathrm{T}(s_5) = \mathrm{T}(s_6) = 0$$
$$\mathrm{T}(t_1) = \mathrm{T}(t_3) = \mathrm{T}(t_4) = \mathrm{T}(t_5) = \mathrm{T}(t_6) = 0$$
$$\mathrm{T}(t_2) = \tilde{t}_2$$

Thus, using character formulas for $\mathrm{GL}(n,\mathbb{R})$ at infinitesimal character ρ and and for $\widetilde{\mathrm{GL}}(n,\mathbb{R})$ at infinitesimal character $\frac{1}{2}\rho$, we obtain

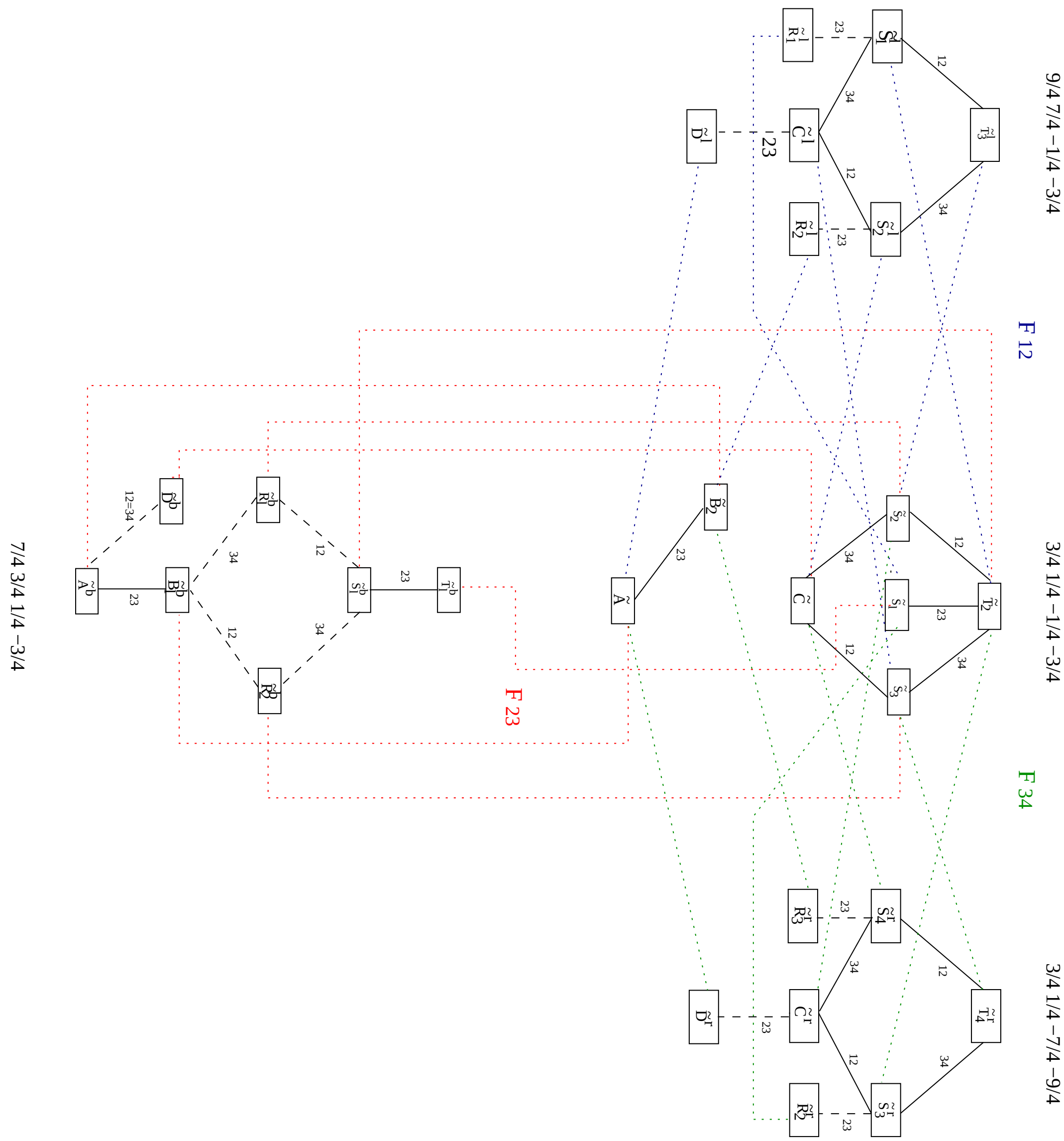

FIGURE 17.1. Representations of $\widetilde{\mathrm{GL}}(4,\mathbb{R})$ at infinitesimal character $\frac{1}{2}\rho$

$$\mathrm{T}(A) = \tilde{A}, \quad \mathrm{T}(D) = -\tilde{A}, \quad \mathrm{T}(B_1) = -\tilde{A}, \quad \mathrm{T}(B_2) = -\tilde{B}_2$$
$$\mathrm{T}(C) = \mathrm{T}(c - b_1 - b_2 - d + 2a) = \tilde{c} - \tilde{b}_2 - \tilde{d} + 2\tilde{a} = \tilde{C} + \tilde{A}$$
$$\mathrm{T}(S_1) = \tilde{S}_1, \quad \mathrm{T}(S_2) = \tilde{S}_2, \quad \mathrm{T}(S_3) = \tilde{S}_3$$
$$\mathrm{T}(S_4) = -\tilde{C}, \quad \mathrm{T}(S_5) = -\tilde{C}, \mathrm{T}(S_6) = \tilde{B}_2$$
$$\mathrm{T}(T_1) = \tilde{S}_1, \quad \mathrm{T}(T_2) = \tilde{T}_2, \quad \mathrm{T}(T_3) = \tilde{S}_2$$
$$\mathrm{T}(T_4) = \tilde{S}_3, \quad \mathrm{T}(T_5) = \tilde{C}, \quad \mathrm{T}(T_6) = -\tilde{B}_2$$

This shows that the image of irreducible character under the lifting T is, up to sign, an irreducible character, except in one instance: we have $\mathrm{T}(C) = \tilde{C} + \tilde{A}$.

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