Generalized Davenport transforms and the Cauchy-Davenport theorem

Salvatore TRINGALI

Starting from a generalization of the Davenport transform, we show how this can be used to prove that, for $\mathfrak{A} = (A, +, 0)$ a cancellative monoid and X, Y subsets of $\mathfrak{A}$ such that the smallest subsemigroup generated by Y is commutative, one has that

$$|X + Y| \ge \min\bigl(\sup_{y_0 \in Y^\times} \min_{y \in Y \setminus \{y_0\}} \mathrm{ord}(y - y_0), |X| + |Y| - 1\bigr)$$

if $2 \le |X|, |Y| < \infty$. The result extends the classical Cauchy-Davenport theorem to the broader and abstract setting of (possibly non-commutative) semigroups, while being a strengthening of a previous generalization by G. Károlyi relating to sum-sets in commutative groups, where the right-hand side in the above estimate is actually replaced by $\min(p, |X| + |Y| - 1)$, with p the order of the smallest non-trivial subgroup of the ambient group. Additionally, we show through examples that Károlyi's bound is actually much weaker in significant situations, and we prove that the result implies at once an extension of I. Chowla's generalization of the Cauchy-Davenport theorem to arbitrary moduli.