ZERO-SUM PROBLEMS IN $\mathbb{Z}^r$

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Let $G = (\mathbb{Z}^r, +)$, with $r \in \mathbb{N}$, and $G_0 \subset G$ a subset which generates G. A sequence over G_0 means a finite sequence of terms from G_0, which is unordered and repetition of terms is allowed. We say that a sequence has sum zero if its terms add up to zero. The set $\mathcal{B}(G_0)$ of zero-sum sequences over G_0 forms a monoid, where the operation is simply the juxtaposition of sequences. We denote by $\mathcal{A}(G_0) \subset \mathcal{B}(G_0)$ the set of minimal zero-sum sequences over G_0. The maximal length $\mathsf{D}(G_0)$ of a minimal zero-sum sequence is called the *Davenport constant* of G_0, so we have

$$\mathsf{D}(G_0) = \sup\{|S| : S \in \mathcal{A}(G_0)\} \in \mathbb{N} \cup \{\infty\}\,.$$

Let S be a zero-sum sequence. If $S = U_1 \cdot \ldots \cdot U_k$, where $U_1, \ldots, U_k$ are minimal zero-sum sequences, then k is called the length of this factorization. The *set of lengths* $\mathsf{L}(S)$ is defined as the set of all possible $k \in \mathbb{N}$. In particular, $\mathsf{L}(S) \subset \mathbb{N}$ is a finite nonempty subset of the positive integers.

In this talk, we discuss the Davenport constant, sets of lengths, and further arithmetical invariants of $\mathcal{B}(G_0)$. To give an explicit question, we recall the following well-known results:

(a) There are subsets G_0 such that $|\mathsf{L}(S)| = 1$ for all zero-sum sequences S over G_0.
(b) If G_0 is finite, then all sets of lengths $\mathsf{L}(S)$ are generalized arithmetical progressions with a uniform bound for all parameters.
(c) If $G_0 = G$, then for every finite set $L \subset \mathbb{N}_{\ge 2}$ there is a zero-sum sequence S over G_0 such that $L = \mathsf{L}(S)$.

Now the problem is to understand which subsets $G_0 \subset G$ have Property (a), which have Property (b), and which have Property (c).

These questions are motivated by recent work in module theory ([1]), and much is known for infinite cyclic groups ([2]).

References

[1] N.R. Baeth and A. Geroldinger, *Monoids of modules and arithmetic of direct-sum decompositions*, manuscript.
[2] A. Geroldinger, D.J. Grynkiewicz, G.J. Schaeffer, and W.A. Schmid, *On the arithmetic of Krull monoids with infinite cyclic class group*, J. Pure Appl. Algebra **214** (2010), 2219 – 2250.

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