MONOIDS OF MODULES AND ZERO-SUM THEORY

ALFRED GEROLDINGER

Let R be a noetherian ring. Then every finitely generated R-module can be written as a direct sum of finitely many indecomposable modules, say $M = N_1 \oplus \ldots \oplus N_s$. If $R = \mathbb{Z}$, then finitely generated R-modules are finitely generated abelian groups. For every such group G we have,

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}/q_1\mathbb{Z} \oplus \ldots \oplus \mathbb{Z}/q_s\mathbb{Z}\,, \quad \text{where} \quad q_1, \ldots, q_s \text{ are prime powers}\,,$$

and such a decomposition is unique.

In general, there is no uniqueness and it is a classical topic in module theory to study the variety of possible direct-sum decompositions. The past decade has seen a new semigroup-theoretical approach, pushed forward by A. Facchini and R. Wiegand. For a suitable class of R-modules $\mathcal{C}$, let $\mathcal{V}(\mathcal{C})$, denote the set of (isomorphism classes of) modules in $\mathcal{C}$. Then $(\mathcal{V}(\mathcal{C}), +)$ is a commutative semigroup with operation defined by $[M] + [N] = [M \oplus N]$, and all information about direct-sum decomposition of modules in $\mathcal{C}$ can be studied in terms of factorization of elements in the semigroup $\mathcal{V}(\mathcal{C})$.

If the endomorphism ring $\mathrm{End}_R(M)$ is semilocal for all modules M in $\mathcal{C}$, then $\mathcal{V}(\mathcal{C})$ is a Krull monoid. Suppose this holds, and let G denote the class group of $\mathcal{V}(\mathcal{C})$ and let $G_{\mathcal{P}} \subset G$ denote the set of classes containing prime divisors. Then most problems of direct-sum decompositions of modules in $\mathcal{C}$ translate into zero-sum problems over $G_{\mathcal{P}}$.

In this talk we give a friendly introduction to this area (in the style of [2]), and we present several combinatorial problems arising from this interplay ([1]).

References

[1] N.R. Baeth and A. Geroldinger, *Monoids of modules and arithmetic of direct-sum decompositions*, manuscript.
[2] N.R. Baeth and R. Wiegand, *Factorization theory and decomposition of modules*, Am. Math. Mon. **120** (2013), 3 – 34.

Institute of Mathematics and Scientific Computing, University of Graz, Heinrichstrasse 36, 8010 Graz, Austria
E-mail address: alfred.geroldinger@uni-graz.at, www.uni-graz.at/alfred.geroldinger