YAHYA OULD HAMIDOUNE'S MATHEMATICAL JOURNEY: A CRITICAL REVIEW OF HIS WORK

A. PLAGNE, O. SERRA, AND G. ZÉMOR

Abstract. We present the mathematical work of Yahya Ould Hamidoune, emphasizing his main achievements, notably in graph theory and additive combinatorics.

1. Introduction

This paper aims to provide a brief account of the contributions to mathematics of Yahya Ould Hamidoune, and to give them perspective. The reader interested in his personal biography and other aspects of his otherwise profoundly humane personality is referred to [B, C, D].

This presentation of Yahya's mathematical work is structured into four sections which roughly reflect his evolution from his early interests in graph theory to the development of what became his favorite topic, the application of the isoperimetric method to problems in additive combinatorics.

His early works, covering the years 1977–1983, were mainly devoted to graph theory, particularly in the area of graph connectivity, which sets the basis of his later evolution. They are described in Section 2.

Yahya loved playing games. First of all, he had been a national champion of *Srand* (or *Dhamet*, the Mauritanian draughts), for which he was famous as the first modern theoretician (see the paper [A] which contains some problems and combinations created by Yahya). Moreover, he was a keen chess player and played bridge and backgammon honorably as well. It is therefore not very surprising that abstract game theory, to which Yahya was introduced by Las Vergnas, motivated some of his early work. Even when his attention focused increasingly on additive theory, his early pursuits never lost their appeal and he occasionally came back to problems in matroid theory, game theory and algorithmics. Section 3 describes his main contributions to these areas.

Yahya came to realize that many classical results in additive number theory could be rephrased in terms of connectivity of Cayley graphs. Inspired by this connection, he started to develop what he called the *isoperimetric method* in additive theory. Section 4 describes the evolution of the method and its direct applications to the estimation of the cardinality of sumsets. The isoperimetric method is of combinatorial nature and it is expressed in the language of graphs. Yahya occasionally wrote papers on applications of the isoperimetric method to graph theory, and some of his results in this area are also mentioned in this section. We may say that Yahya considered the isoperimetric method and its applications to additive number theory one of his more personal creations. His frequent *meditations* (a word he was very fond of and which he used frequently when mentioning mathematical thought processes) on this method was for him a way to achieve a better and deeper understanding of the classical addition theorems. He was happy to make subtle observations which made the whole scene fresher and brighter. This silent and discreet activity underlying an abundant scientific production was a prominent characteristic of Yahya's personality.

During his mathematical journey, Yahya became increasingly interested in problems in additive theory, a topic which he eventually identified with his main area of research. Section 5 presents his many results in the field known today as *additive combinatorics*. They contain what we think are his more substantial contributions, including, among many others, his proof with Dias da Silva of the Erdős–Heilbronn conjecture on the cardinality of restricted addition of sets or his proof with Gao of the Diderrich conjecture on the Olson constant of an abelian group.

The present paper includes a complete – to the best of our knowledge – bibliography of Yahya's works, including some of his last preprints published on `arxiv` (and not published in any other form) whose process of publication was interrupted by his sudden passing. When looking up Yahya's written material, one finds his name to sometimes take different forms. This is due to Mauritanian culture, for which people's names can be varying entities, and also to transliteration problems. In the present bibliography we uniformized it to the form "Y. O. Hamidoune", as he himself did most of time (but typically not in his theses), abbreviating the "Ould" (which means 'son of' and can also be written in lowercase) as if it were a second given name. References are listed in chronological order. We only included papers by Yahya, all scientific references by other authors are to be found in the reference lists of Yahya's papers. In the body of our text below, we do not mention systematically the coauthors of Yahya's papers but since we refer to published material, they can be easily identified by looking them up in the bibliography. We have also included a few additional references (referred [A] to [D]) which contains papers *on* (or partly on) Yahya.

Yahya was in full mathematical activity when he left us. With his numerous collaborators, he harboured many promising projects, some of which are reflected in the contributions to this volume. To those of us privileged to have worked with him, the memory of his friendly and generous nature lives on. His mathematical legacy, which we have done our best to summarize in this article, will no doubt continue to inspire many mathematicians in the future.

2. Beginning with graph theory (1977–1987)

In 1975, Yahya started his doctoral studies in combinatorics with Michel Las Vergnas at the University Pierre-et-Marie-Curie (also known as "Paris 6"). Yahya's first paper [1], an article in french entitled "Sur les atomes d'un graphe orienté" ("On the atoms of a directed graph") appeared in the *Comptes Rendus de l'Académie des Sciences (Paris)* in 1977. This first achievement was one of the main inspirations of Yahya throughout his mathematical career, and he quoted this paper in many of his later works throughout his life.

Mader had introduced the notion of an *atom* of a graph as the smallest set of vertices whose neighbourhood has minimum cardinality. The main property of such extremal sets is the fact that two distinct atoms are disjoint, a property which explains the terminology. Watkins had also used the same notion independently.

It was known by that time that the notion does not extend straightforwardly to the case of directed graphs, in the sense that disjointness of atoms may fail to hold. Yahya's observation was that the notion can be translated to directed graphs as long as one is allowed to move from a directed graph G to its inverse G^{-1}, obtained from G by reversing the orientations of all arcs. Atoms of a directed graph G and of its inverse graph G^{-1} do not necessarily have the same size (actually the value of the connectivity can even differ in infinite graphs.) The disjointness property, however, always holds in G or its inverse G^{-1}, whichever has the smaller atoms. This is one of the main theorems in [1]. Figure 1 illustrates this fact.

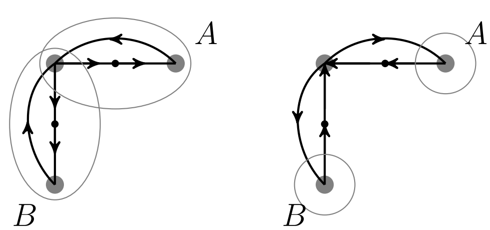

Figure 1. The graphs in the picture, one inverse of the other one, have both connectivity 1. Thick gray nodes represent cliques of size $m > 1$, and arrows mean 'from all vertices to all vertices'. The graph on the left has two atoms A, B which intersect nontrivially, whereas the ones on the right are disjoint.

Yahya obtained his PhD [3] in 1978 and was admitted as a C.N.R.S. full–time researcher in 1979 ; his *thèse d'État* [9] was defended in 1980. He belonged to the Paris 6 combinatorics group, to which he stayed faithfull throughout his career. This group was directed by Claude Berge, whose influence was quite strong, and graph theory was naturally at the forefront of their preoccupations. Yahya followed this trend and soon started to specialize in connectivity issues in graphs, becoming the group's expert on the topic. During his first decade as a researcher, he published about 30 papers in three main areas: studies on connectivity, its applications to graph theory and works with his former PhD advisor and then colleague and friend Michel Las Vergnas on game theory and matroid theory (see the following section).

Among his early papers on connectivity he proved results on the minimum degree of critically k–connected graphs [2, 7, 10, 11, 14]. A graph is critically k–connected if its number of vertices is minimal with respect to the property of being k–connected. The minimum degree $\delta(G)$ of a k–connected graph G is at least k, and these papers are devoted to showing that criticality implies that indeed $\delta(G) = k$ and that this minimum degree is attained in a number of vertices. In his elegant highly concise style, he derived new proofs of several theorems, solved several conjectures and extended several previous works of Mader [5], Chartrand [6], Jolivet [6], Lesniak [6], Chartrand, Kaugars and Lick [7], Maurer and Slater [11], or Entringer and Slater [14], and his papers always include a generalization or an extension of these known results. In [10] there is a section on vertex–transitive graphs which contain two key results: the arc–connectivity of a connected vertex–transitive graph equals the degree and its vertex–connectivity is at least half the degree, and the bound is tight. The former result was given by Mader for undirected graphs and the latter was an extension of results by Watkins for antisymmetric graphs and by Olson for Cayley graphs. This was one of the results which opened the way to later applications to additive theory. This trend is also reflected in his papers on connectivity problems related to groups and Cayley graphs [8, 12, 17, 18, 29] which was to become one of the central topics of his research. One of the key results in [17] states that the atom containing 1 of a Cayley graph $\mathrm{Cay}(G, S)$ or its inverse $\mathrm{Cay}(G, S^{-1})$ is a subgroup. In [17] there are some applications to the connectivity of circulant graphs (Cayley graphs on cyclic groups), Cayley graphs on minimal generating sets and of the assignment polytope, extending results of Boesch and Felzer, Godsil and Imrich respectively. Yahya also gave a version of the main result in [17] for abelian Cayley graphs [18] and a generalization to infinite Cayley graphs [29].

The papers [8, 12] deal with cycles in graphs, a topic that Yahya addressed several times from his early papers. He had obtained a tight bound on the length of a cycle passing through all vertices in a k–connected graph of order $n \geq 8k^3 + k$ [4] and started a series of papers on a version by Behzad, Chartrand and Wall of a longstanding conjecture of Cacceta and Häggkvist still wide open today. The conjecture states that a graph with minimum degree $\delta(G) = \min\{\delta^+(G), \delta^-(G)\}$ and girth $g(G)$ has at least

$$n \geq \delta(G)(g(G) - 1) + 1$$

vertices. Actually the conjecture of Behzad, Chartrand and Wall assumes the graph to be regular, and Cacceta and Häggkvist conjecture the same conclusion with the minimum out–degree $\delta^+(G)$ in place of $\delta(G)$. Yahya proved the conjecture for the class of vertex–transitive graphs [12], for graphs with minimum degree $\delta(G) = \min\{\delta^-(G), \delta^+(G)\} \leq 4$ (see [13]) and for graphs with minimum out–degree $\delta^+(G) \geq 3$ (see [22]), three results which are still central in this field. He also gave a formulation for groups [8, 12, 30] which states that every element in a group G of order n admits a factorization of length at most $\lceil n/s \rceil$ by elements of a set $S \subset G$ with cardinality $|S| = s$. Again, this result established a connection with classical addition theorems.

3. Games

During the early years of Yahya's career, Parisian combinatorics were under the influence of Berge, himself very interested in the graph theory problems posed by Shannon. Berge being besides a notorious amateur of the game of Hex, it was natural for those around him to become exposed to the related and more general *Shannon switching game*. The Shannon switching game is played on a graph, on which two vertices, a and b say, have been singled out. Two players, called *Short* and *Cut*, play alternately. Cut, when it is his turn, chooses an edge and deletes it from the graph. On Short's turn, he marks an edge so that it can't be deleted by Cut anymore. Short wins if he manages to mark a path joining a to b, otherwise he loses. The respective roles of Short and Cut are more similar than they may at first appear to be. If one imagines a fixed edge linking a to b, then Short is trying to mark a cycle passing through a and b, while Cut is trying to mark a cocycle (or cutset) separating a from b. When one generalizes the game from graphs to matroids, then both players are trying to mark circuits of matroids that are dual to each other. Matroid duality helps analysis and in 1964 Lehman gave a complete solution to the general Shannon switching game on graphs by solving the more general matroid problem.

We see that Short wins when a and b are connected in a very strong sense. Having become an expert on graph connectivity, Yahya must have seen a suitable terrain for him to attempt again generalizations to digraphs. Given the matroid connection to the problem it was also natural for him to pair up with his early mentor Las Vergnas who was keen to develop the theory of oriented matroids. Together they considered two variants of the Shannon switching game. The players, now called *White* and *Black*, start with an undirected graph. On White's turn he chooses a non–played edge and directs it. Black chooses a unplayed edge and marks it, so it can not be directed anymore. White wins if a directed path from a to b eventually appears. This version is called the directed switching game. In the other version, called the *One way game*, Black also directs edges instead of marking them. Again, White wins if a directed path (possibly with both white–directed and black–directed edges) appears. After having annonced their results in [16, 19] Hamidoune and Las Vergnas gave a comprehensive account of their study in [20] characterizing the winning positions for the directed variants on graphs by putting them in an oriented matroid context.

Shortly after, Yahya published a paper [21] giving a complete solution to the *Box game*, a game first introduced by Chvátal and Erdős. This game can be thought of as yet another variant of the Shannon switching game, where the graph is obtained from a single path relating a to b, and every edge has been transformed into a set of parallel edges (called a "box" in the original terminology). The game deviates from the standard switching game in that Short and Cut are allowed to mark and delete up to q edges and p edges respectively. The solution does not make use of matroid theory and is somewhat *ad hoc*. A little later, Yahya went back to the more classical Shannon switching game and published a paper [26] on its misère variant.

At about the same period of time, Yahya wrote two other papers on the topic of games and graphs. In [23] the game under study is played on a graph: the *Evader*, when it is his turn, moves from a vertex to a neighbouring one. The *Catchers*, there are p of them, simultaneously and in coordination move from the vertices they sit on to neighbouring vertices. For a given graph G, one wishes to determine the number $p(G)$ which is the minimum p such that the Catchers can be always guaranteed to catch the Evader. Previous work was mostly concerned with undirected graphs, and as often, Yahya tackled the directed case. He proved that for the class of strongly connected abelian Cayley digraphs of out–degree k, we have $p(G) \leq k+1$, the best possible result. Results by Frankl (who called the game *Cops and robbers*) on abelian Cayley graphs are also improved. The paper [24] does not study games *per se*, but is devoted to the study of kernels, or in this case quasi–kernels, in directed graphs, that are a key to winning strategies for many games: the influence of Henri Meyniel, who spent a lot of time studying kernels, can be felt in this paper.

Ten years later, Yahya came back to the Shannon switching game with a follow–up [60] to [20], even though by then most of his attention was focused on additive problems.

4. The isoperimetric method

After reading Mann's book "Addition Theorems", dealing with the theory of *Minkowski sumsets* defined as

$$A + B = \{a + b,\ a \in A,\ b \in B\},$$

for A and B two subsets of an additive monoid (if $A = B$, then $A + A$ is denoted $2A$ and hA is defined recursively as $(h-1)A + A$, for h a positive integer), Yahya realized that some of his results on graph connectivity generalize, in a disguised form, some classical combinatorial results of additive number theory (at that time, this body of work was not yet called additive combinatorics). The isoperimetric method was born. Yahya proceeded to investigate in a systematic way the classical theorems of the theory, those due to Cauchy–Davenport, Chowla, Olson, Mann, Shepherdson, Shatrowsky, Vosper, Kneser, Kemperman,...

Given two finite sets A, B of a group G, written additively, with $0 \in B$ one can write the sumset $A + B$ as

$$A + B = A \cup \partial A,$$

where $\partial A = (A + B) \setminus A$ is the boundary of A in the Cayley graph $\mathrm{Cay}(G, B)$. The estimation of the minimum cardinality of a sumset $A + B$ can be interpreted as the connectivity of the graph. As we have already mentioned, in his paper on the connectivity of Cayley graphs [17] from 1984 Yahya had already proved that the atom containing 0 of the Cayley graph $\mathrm{Cay}(G, S)$ is a subgroup, a result which corresponds to the classical theorem of Mann that the smallest cardinality of a sumset for a fixed B is attained when A is a subgroup. One advantage of the graph setting is that the result extends naturally to nonabelian groups and to infinite groups: the only requirement is that the graph has a regular transitive group of automorphisms.

The isoperimetric method deals with a generalization of connectivity, fragments and atoms to the parametrized notions of k–connectivity, k–fragments and k–atoms. The *k–connectivity* $\kappa_k(G)$ of a graph G is

$$\kappa_k(G) = \min\{|\partial X| : |X| \geq k, V(G) \setminus (X \cup \partial X)| \geq k\},$$

the smallest cardinality of the boundary of a set with cardinality at least k whose exterior, the complement of the set and its boundary, has also cardinality at least k (as long as such a set exists, namely, the graph is *k–separable*.) This explains the "isoperimetric" terminology. A *k–fragment* is a set which attains the k–connectivity, and a *k–atom* is a k–fragment with minimum cardinality.

The general properties of atoms and fragments extend to the corresponding isoperimetric analogues and the general theory became one of the main contributions of Yahya (see his survey article [95] or [106] for instance). One of the key results is that, if the k–atom of G is not larger than the k–atom of its inverse graph G^{-1}, then a k–atom and a k–fragment not containing it intersect in at most $k-1$ points. He developed a number of applications of the theory in a long series of papers to deal with the somewhat intricate structure of sets with small sumset. This topic has become one of the central parts of additive combinatorics, the *Inverse theory of set addition* which had been pioneered by Freiman. Yahya was always sensitive to exact rather than asymptotic or approximate results, and his contributions were focused in this direction. We next summarize some of his most substantial contributions in the area.

The Cauchy–Davenport theorem, giving the inequality

$$|A+B| \geq \min\big(p, |A|+|B|-1\big)$$

in a cyclic group of prime order p follows easily from the fact that an atom is a subgroup. Its corresponding inverse result, the theorem of Vosper, which states that non-degenerate sets achieving equality in the Cauchy–Davenport inequality are arithmetic progressions, found a natural explanation in the setting of the isoperimetric method, explaining in particular the nature of the degenerate cases of the theorem. The characterization of extremal sets for the Cauchy–Davenport inequality in different settings is what Kemperman called the *critical pair* problem. There are many extensions of these results which Yahya undertook successfully with the isoperimetric approach.

In the case of cyclic groups of prime order, the Hamidoune–Rødseth theorem [74] gives the precise structure of pairs of sets $\{A, B\}$ of cardinality at least 3 with

$$7 \leq |A+B| = |A|+|B| \leq p-4$$

and subsequent extensions can be found in [84]. For general cyclic groups, Chowla had extended the Cauchy–Davenport inequality by requiring that the elements of one set are invertible, and such extensions were considered by Yahya [43, 86]. He realized that Chowla's condition could be relaxed to the fact that every element of a set S has order at least $|S|+1$, and that this condition can be extended from cyclic groups to general groups.

For general abelian groups, the extension of the Cauchy–Davenport theorem is Kneser's theorem, which states that a pair of sets violating the inequality has a *periodic* sumset, meaning a union of cosets modulo some non-trivial subgroup. By revisiting the isoperimetric theory, Balandraud obtained in his thesis a new proof and a refinement of Kneser's theorem: the key observation is that the period of the sumset of such a pair of sets can be made to depend essentially on only one of the two sets, a result which was reobtained and generalized by Yahya [89] who managed also to give an isoperimetric proof of Kneser's theorem in [91].

The exact structural description of pairs of sets satisfying an inequality beyond Cauchy–Davenport is provided by the rather intricate description of the Kemperman structure theorem. Yahya had the view that the recursive description of Kemperman was not completely satisfactory, and he came back to the result on several occasions [39, 59, 73, 76, 81] until he was satisfied with his formulation in [105]. The formulations in the latter references are similar in spirit, and Yahya called them Vosper theorems for abelian groups. The most compact one can be found in [81] and reads as follows: for every generating set A containing 0 of a finite abelian group G such that $2 \leq |A| \leq |G|/2$, there is a subgroup $H < G$ satisfying

$$|A + H| < \min\big(|G|, |A| + |H|\big)$$

and either

(i) A/H is an arithmetic progression, or
(ii) $|A/H + B/H| \geq \min\big(|G/H| - 1, |A/H| + |B/H|\big)$ for every subset $B \subset G$ with $|B/H| \geq 2$.

Here X/H stands for the image of the set $X \subset G$ under the canonical homomorphism $G \to G/H$. The versions above are simple and useful but miss the joint structure of a critical pair provided by the Kemperman structure theorem. To avoid degenerate cases, a critical pair in a group written additively is defined to be a pair of finite sets $\{A, B\}$ with $|A|, |B| \geq 2$ and $|A + B| = |A| + |B| - 1 < |G| - 1$. Yahya persisted in his search of a compact description of critical pairs. Some simplified versions of Kemperman's theorem had been given by Lev and by Grynkiewicz and Yahya was searching for a different one. In one of his last preprints [105] he proposes the following formulation. Consider $\{A, B\}$ a critical pair in an abelian group G. For the structural characterization of the pair $\{A, B\}$, one may assume that $A \cup B$ generates G , $0 \in A \cap B$, $A + B$ is aperiodic, and A is not an arithmetic progression. Then there is a subgroup $H < G$ such that both A and B are *almost* H–periodic, that is, each set is a union of H–cosets but for one coset which is only partially filled, say $A_0 = A \cap (H + a) \nsubseteq H + a$ and $B_0 = B \cap (H + b) \nsubseteq H + b$, which satisfy $|A_0 + B_0| = |A_0| + |B_0| - 1$. Moreover one of the following conditions hold:

(i) $A/H = \{0\}$, or
(ii) $A/H + B/H = G/H$ and $A_0/H + B_0/H$ has a unique expression in $A/H + B/H$, or

(iii) both A/H and B/H are arithmetic progressions with the same difference; moreover A_0/H and B_0/H are initial elements of the progressions A/H and B/H respectively.

One may want to check the Kemperman structure theorem to appreciate the simplicity of the above formulation. Moreover, one of the drawbacks of Kemperman's theorem is that the characterization is recursive in the pair $\{A_0, B_0\}$ and in the pair $\{A/H, B/H\}$. The latter dependency, which leads to some technical difficulties, disappears in Yahya's formulation. Finally, the nature of the period H in his version is identified within the isoperimetric method. In [102], Yahya went one step further by characterizing the pairs $\{A, B\}$ in an abelian group such that $|A+B| = |A|+|B|$, by recovering a structural result of Grynkiewicz.

The inverse sumset problem has been the object of special attention in the case of the integers. This part of the theory was initiated by Freiman who gave the general structure of sets $A \subset \mathbb{Z}$ with $|2A| \leq c|A|$ for a positive number c. Freiman himself had developed more precise results for $c = 3$. The $(3k-4)$–theorem of Freiman (the name of this theorem comes from the historical notation of Freiman $|A| = k$) states that a set A of integers with $|2A| \leq 3|A|-4$ is a dense subset of an arithmetic progression. The value $3|A|-4$ is critical for this result and Freiman went a step further by establishing a $(3k-3)$–theorem giving the structure of sets A satisfying $|2A| = 3|A|-3$ which, except for some sporadic examples, are dense subsets of at most two arithmetic progressions. Yahya noticed that structural results on small sumsets in cyclic groups provide the means to prove inverse theorems in the integers, and he exploited this approach several times. He obtained [51] the extension to the sum of two distinct sets of Freiman's $(3k-4)$–theorem, a result independently given by Lev and Smeliansky that Yahya did not eventually publish. Yahya further made more precise and generalized the $(3k-3)$–theorem [78] by extending the result to cover also the case where $|A + \lambda \cdot A| = 3|A| - 3$ (λ an integer) and by giving the precise structure of the corresponding sets. Here, we use the notation

$$\lambda \cdot A = \{\lambda a,\ a \in A\}$$

for the λ–dilate of the set A (this paper contains in particular the first occurrence of the sum of dilates problem), a subject to which Yahya returned later (see Subsection 5.9 below). He also came back to the problem [81] by characterizing the extremal sets for a multiple addition version of the $(3k-3)$–theorem which had been formulated by Lev.

As has been already mentioned, the isoperimetric method is applicable to the sumset problem in nonabelian groups (in this case, people speak rather, multiplicatively, of *product set*). Yahya's 1977 paper [1] already contains a proof of a theorem by Olson, also from 1977, that the connectivity of a Cayley graph $\mathrm{Cay}(G, S)$ is at least $|S|/2$. This first result prompted Yahya to explore further the generalization of "additive" problems to nonabelian groups: he started with a sufficient condition for an extension of Olson's theorem [39, 45]. In [52] he formulated a Vosper–like theorem for nonabelian groups. The conclusion, that a critical $\{A, B\}$ pair consists of two progressions, is showed to hold under the condition that every element in $G\setminus\{1\}$ has order at least $|B|$. This condition clearly holds in cyclic groups of prime order, giving the original theorem of Vosper, or in torsion–free groups, providing a theorem by Brailovsky and Freiman. In torsion–free groups, Yahya [66] obtained also a proof of the Hamidoune–Rødseth theorem, namely, that pairs of sets with $|AB| = |A|+|B|$ are arithmetic progressions with one missing element.

There has been recently an increasing interest for sets with small doubling in nonabelian groups, objects called *approximate groups*, culminating in recent work by Breuillard, Green and Tao (their common paper in the present volume deals with this notion again). Olson had given an example which shows that the natural extension of Kneser's theorem does not hold in nonabelian groups: there are nonabelian groups containing pairs of sets $\{A, B\}$ with $|AB| < |A| + |B| - 1$ for which AB is not a union of left–cosets, right–cosets nor left–right–cosets xHy of a subgroup H. In 2009, Tao discussed in his blog a theorem of Freiman concerning the structure of a set A in a nonabelian group such that $|A^2| \le c|A|$. The theorem of Freiman can be seen as a nonabelian analogue of Kneser's theorem in that, for $c \le 1.6$, such a set A must be the union of right–cosets of a subgroup. Tao gave a simple proof of this structural result for sets with $|AA^{-1}|, |A^{-1}A| \le (1 + \sqrt{5})/2$ and he asked if this result could be extended to $c < 2 - \epsilon$, a question which was immediately answered by Yahya [111] in the affirmative by an application of the isoperimetric method. He showed that the condition $|A^{-1}A| \le 2|A| - 2 < |G|$, where G is the group generated by $A^{-1}A$, implies that

$$A^{-1}HA \setminus a^{-1}Ha \subset A^{-1}A,$$

for some proper subgroup H and some element $a \in A$, or the same conclusion by exchanging A by A^{-1}. In fact, the subgroup H is an atom of some right–translate of A. This can be seen as a nonabelian version of Kneser's theorem. If one insists on concluding that $A^{-1}A$ is a periodic set, Yahya [111] showed that if $|A^{-1}A| \le \min\{|G|, 5/A|/3\}$ then A is a union of at most 6 cosets of a normal subgroup, which again is closely related to an atom of a translate of A. The paper by Tao in this volume suggests a subtly different approach to the same question based on the ideas of Yahya for reaching the constant $c = 2 - \epsilon$.

The framework for the above applications of the isoperimetric method to the small sumset problem was progressively refined in several papers of Yahya [68, 73, 76, 102] where the method was presented in an increasingly compact form. Like a patient and conscientious craftsman, he came back again and again to polish and shine this tool till the very end of his life. The paper "Some additive applications of the isoperimetric approach" [90] and his preprint "On Minkowski product size: The Vosper's property" from 2011 [100] have become his last accounts of the method. The paper by Serra and Zémor included in this volume somewhat completes this last preprint.

The isoperimetric method is of combinatorial nature and it finds a natural expression in the language of graphs. Actually, Yahya never lost contact with his early works on graph connectivity. In parallel to his papers on applications of the isoperimetric method to additive problems, he used to publish closely related ones applied to networks, where the addition theorems are used to answer questions on connectivity of graphs. Examples of such works are an estimation of bisection cuts in abelian Cayley graphs by using the Plünnecke–Ruzsa inequalities [54], the characterization of undirected Cayley graphs for which the minimum cuts isolate a single vertex, in the abelian case [38] by using the Kemperman theorem, for quasiminimal Cayley graphs [55, 69], in vertex–transitive graphs [103], or in arc–transitive graphs [108]. The isoperimetric method applies to submodular functions other than the vertex–boundary of a set in a graph, particularly for the edge–boundary, leading to results on edge–connectivity. In [37], a characterization of vertex–transitive graphs in which the minimum edge–cuts isolate a single vertex is given. A stronger statement is given in [72] which shows that, in a vertex–transitive graph G with degree d and order n, the edge–boundary of a set X with $d/3 \le |X| \le n/2$ has at least $2d^2/9$ edges unless G has a strong structure.

5. Additive Theory

The development of the isoperimetric method placed Yahya in the area of additive number theory, where he studied many different problems, often treating them with his favorite tool. In this section we briefly describe his main contributions in the area. The reference [53] contains many related results (see also [48]).

5.1. **A theorem by Shepherdson.**

In one of his early papers Yahya [12] proved the Cacceta–Häggkvist conjecture on the smallest order of a directed graph with given out–degree and girth, for the class of vertex–transitive graphs. He later uncovered a theorem of Shepherdson from 1947 which states the same result when restricted to Cayley graphs on cyclic groups. The result by Shepherdson states that, for a given set $S \subset \mathbb{Z}/n\mathbb{Z}$ with cardinality $s = |S|$, there are elements $s_1, \ldots, s_k \in S$, possibly with repetitions, with $k \le \lfloor n/s \rfloor$ such that

$$s_1 + \cdots + s_k = 0.$$

The bound on k is best possible as shown by taking S the arithmetic progression $\{1, 2, \dots, s\}$. This is actually the example which shows that the Cacceta–Häggkvist conjecture is tight. By extending previous results by Diderrich and Kemperman, Yahya [41] gave the following nice generalization of the above results. Let G be a finite group (multiplicatively written) of order n and $S \subset G \setminus \{1\}$ with cardinality $s = |S|$. Then, for $k = \lfloor n/s \rfloor$, there is a subgroup $H < G$ contained in

$$H \subset S \cup S^2 \cup \cdots \cup S^k.$$

The extremal examples for Shepherdson's theorem were characterized in [49]. The bound for k can be rewritten as $n \geq (k-1)|S|+1$ and it is improved to $n \geq (k-1)(|S|+1)-1$ with the extremal exceptions, a bound which is again tight. The Cacceta–Häggkvist conjecture was rephrased by Seymour in the following way: given a graph G with girth $g(G)$ and out–degree r, there is a vertex x of G such that

$$|B_G^{g-2}(x)| \geq r(g-2),$$

where $B_G^k(x) = \{y \in G : d(x,y) \leq k\}$ denotes the ball centered on x with radius k in G. Yahya [85] proved Seymour's conjecture for locally finite vertex–transitive graphs, thus including Cayley graphs $\mathrm{Cay}(G, S)$ with S finite.

The paper by Lladó in this volume addresses Seymour's conjecture by using the isoperimetric method.

5.2. **Bases for finite and σ-finite groups.** If A is a subset of a group G which it generates, it is called a basis of order h if $hA = G$ (in additive notation). Hamidoune's paper [12] implies that if G is a finite abelian group and A is a subset of G, then A is a basis of the subgroup it generates, say H, of order at most

$$\max\left(2, \frac{2|H|}{|A|} - 1\right).$$

Rødseth obtained independently the same result. Yahya and Rødseth joined forces to extend this result to the more general situation of σ-finite groups. A group G is *σ-finite* if it is the increasing union of finite groups : $G = \cup_{i=1}^{+\infty} G_i$. The upper density of a subset A of G is then defined as the limit superior of the ratio $|A_i|/|G_i|$ when i tends to infinity. In [56], the extension of the previous theorem was finally obtained : if G is a σ-finite abelian group and A is a subset of G with a positive upper density δ, then A is a basis of the subgroup it generates, say H, of order at most

$$\max\left(2, 2\delta^{-1} - 1\right).$$

5.3. **Distinct sums.** Yahya's most famous result is certainly his 1991 proof with Dias da Silva of a conjecture due to Erdős and Heilbronn on restricted addition of sets modulo a prime p (see [61]). The paper [46] appears in the *Bulletin of the London Mathematical Society* (1994) under the slightly cryptic title "Cyclic spaces for Grassmann derivatives and additive theory". The main theorem states that

$$|h^\wedge A| \geq \min\left(h|A| - h^2 + 1, p\right)$$

where $h^\wedge A = \{a_1 + \cdots + a_h,\ a_1, \ldots, a_h \in A,\ a_i \neq a_j \text{ for all } 1 \leq i \neq j \leq h\}$ is the *h–fold restricted sumset*, the set of sums of h distinct elements of A. Here the method relies on exterior algebra, as the notation for distinct sums suggests, the set of distinct sums appearing as the spectra of a linear operator. The paper by Károlyi and Pauli in this volume gives a neat description of the method. Shortly after the publication of Yahya's paper, another proof of the same result was given by Alon, Nathanson and Ruzsa, using what has become known as the *polynomial method*, a fruitful methodology that is still attracting a lot of attention in the area of additive combinatorics. Yahya's method, however, has found other applications in the field and is also expected to yield exciting developments.

In [75], the same problem is studied in the framework of more general abelian groups like those of odd order, or cyclic groups.

This is the place to mention another, earlier, paper by Yahya [34] relating linear algebra to addition theorems. In this paper, a connection is made between the degree of the minimal polynomial of a sum of linear mappings and the ones of each of these mappings. This result is shown to imply the Cauchy–Davenport theorem.

5.4. **Subset sums.** Erdős and Heilbronn made additional conjectures that also aroused Yahya's interest. The distinct sums problem is connected to subset sum problems. Notably one may consider the minimum size of a set in the cyclic group of prime order p which guarantees that the set of subset sums

$$\Sigma(S) = \left\{ \sum_{x \in T} x : \emptyset \neq T \subset S \right\}$$

covers the whole group. The bound on $|h^\wedge A|$ obtained in [46] gives essentially the best answer: a set $S \subset \mathbb{Z}/p\mathbb{Z}$ with cardinality $|S| \geq \sqrt{4p-7}$ satisfies $\Sigma(S) = \mathbb{Z}/p\mathbb{Z}$. The problem is also related to the minimum size of a set which guarantees that $\Sigma(S)$ contains 0. Erdős and Heilbronn first showed this quantity to be not larger than $c\sqrt{p}$ and asked for the optimal constant c. Olson gave a proof with the constant $c = 2$, and Yahya came close to the optimal solution [57] by obtaining that

$$|S| \geq \sqrt{2p} + 5\log p \text{ implies } 0 \in \Sigma(S).$$

The logarithmic term has been recently removed by Nguyen, Szemerédi and Vu for sufficiently large primes, the whole conjecture being eventually settled by Balandraud.

Both conjectures, the *complete sets* conjecture (sets for which $\Sigma(S) = G$) and the *zero–subset sum* conjecture (sets for which $0 \in \Sigma(S)$) naturally extend to general abelian groups in different directions. The minimum size for which every set contains 0 in its set of subset sums in an abelian group of order n, a number called nowadays the *Olson constant* of the group, is again of order $c\sqrt{n}$, as was first proved by Szemerédi. Olson proved an upper bound of 3 for the constant c and again Yahya [57] gave the exact value of $c = \sqrt{2}$ up to an error term of order $O(n^{1/3}\log n)$.

The corresponding minimum size for complete sets, also called the *critical number* $c(G)$ of an abelian group G, was established for groups of order the product of one or two primes and gave rise to another conjecture by Diderrich in the case when the order of G has at least three prime factors, namely:

$$c(G) = \frac{n}{p} + p - 1,$$

where p is the smallest prime divisor of n. Yahya had an isoperimetric view of the problem which he developed in [47, 62, 70, 71]. When Gao announced his proof of the Diderrich conjecture for large primes, Yahya was prompted to join efforts with him to settle the complete conjecture [67], and extended the result by characterizing the extremal sets of the problem [79] well beyond the critical number.

More recently Yahya came back to the complete sets problem prompted by a conjecture by Vu. Vu had proved that every set $S \subset (\mathbb{Z}/n\mathbb{Z})^*$ of invertible elements in a cyclic group with cardinality $|S| \geq c\sqrt{n}$ is a complete set, and he conjectured that one can take $c = \sqrt{2}$. This conjecture is proved in [88], a result particularly praised by Endre Szemerédi in the talk he gave at the conference *Combinatoire Additive à Paris 2012 (Additive Combinatorics in Paris 2012)* held in the *Institut Henri Poincaré* in July 2012 in Yahya's memory.

Yahya also solved another problem of Diderrich on restricted addition by analyzing the extremal case $|h^\wedge A| = |A|, h = 2, \ldots, |A| - 2$ (see [107]). One of his last papers [109], which is concerned with the subset sum problem, is coauthored with Balandraud, Girard and Griffiths and included in the present volume. It contains a generalization of [88] which relies on the solution of Diderrich's problem [107] and on the isoperimetric method.

The paper by Hamidoune, López and Plagne [110], also in the volume, addresses a more general problem on restricted sums.

5.5. **Zero–sum sequences.** The zero–subsetsum problem is connected to another prolific area in additive theory through the Erdős–Ginzburg–Ziv theorem, the zero–subsequencesum. Here the question concerns the minimum length of a sequence of elements in an abelian group which guarantees the existence of a subsequence with sum zero. This minimum length is called the *Davenport constant* of the group. Yahya wrote several papers on this problem starting with the following weighted generalization [50] of a result by Olson. Let G be an abelian group of order n, let $x = (x_1, \dots, x_{2n-1})$ be a sequence of elements in G with length $2n-1$ and let $w = (w_1, \dots, w_{n-1})$ be a sequence of integer weights, each coprime with n. There is a nonzero subgroup $H < G$ and a permutation τ (depending on x) of $[1, 2n-1]$ for each element $x \in H$ such that

$$x = \sum_{1 \le i \le n-1} w_i (x_{\tau(i)} - x_{\tau(n)}). \tag{1}$$

This reduces to the theorem of Olson mentioned above when all weights are equal to 1. Yahya came back to such weighted extensions [58] by proving that if the length of the sequence is $n+k$ (k an integer) then equality (1) holds for $x = 0$ and $x_{\tau(n)}$ the most repeated element in the sequence. This version proves a conjecture by Caro for weights coprime with n, which had been proved by Alon and Dubiner for n prime. It also extends a theorem by Gao on zero–sum sequences of length $n + D(G) - 1$, where $D(G)$ is the Davenport constant of G, a result that Yahya revisited in [92]. A version for weighted subsequence products in a finite group is given in [82].

In [64] Yahya proved a conjecture of Bialostocki and Lotspeich on zero–sum sequences of length $2n - k^2/4 + k - 2$ assuming k distinct values in an abelian group of order $n \ge k^2 - 4k + 8$. A close conjecture by Bialostocki was proved in [65]. One of Yahya's last papers on the topic [96] contains a proof of a conjecture by Graham stating that a sequence of length n in the cyclic group $\mathbb{Z}/n\mathbb{Z}$ such that all nontrivial zero–sum subsequences have the same length, contains at most two different elements. The conjecture had been proved by Erdős and Szemerédi for sufficiently large primes and stated by Graham only for n prime.

In [77], Yahya studies zero–sum–free sequences in cyclic groups. These results are then applied to questions coming from algebraic number theory. Some information on the non-uniqueness of factorizations in a Krull monoid having a cyclic divisor class group is obtained.

Three papers in the present volume are related to this subject. The paper by Fan, Gao, Whang and Zhong addresses zero–sum sequence problems linked with Yahya's work on the subject. The paper by Ordaz, Plagne and Schmid is devoted to this topic through the notion of *barycentric* sequences defined as sequences satisfying (1) with $x = 0$ and all weights equal to 1. Finally, the paper by Baginski, Geroldinger, Grynkiewicz and Philipp deals with applications of zero-sum sequences to the non-uniqueness of factorizations in a Krull monoid.

5.6. **The range of diagonal forms.** It was already noticed by Cauchy that every element in a finite field of prime order p is the sum of k k–th powers. More generally, Chowla, Mann and Strauss considered the range of the diagonal form $f(x_1, \ldots, x_n) = a_1x_1^k + \cdots + a_nx_n^k$ over the finite field $\mathbb{F}_p$, where $a_1, \ldots, a_n$ are nonzero elements of $\mathbb{F}_p$. By an application of Vosper's theorem they proved that

$$|f(\mathbb{F}_p, \ldots, \mathbb{F}_p)| \geq \min\big(p, (2n-2)((|P_k|-1)+1)\big),$$

where $P_k = \{x^k : x \in \mathbb{F}_p\}$ is the multiplicative subgroup of k–th powers in the field. In particular, if $k \neq (p-1)/2$ then every element in $\mathbb{F}_p$ is the sum of $\lceil k/2 \rceil + 1$ k–th powers. The above estimation for the range of the diagonal form f was extended by Tietäväinen to any finite field with odd characteristic when P_k generates the whole field. Yahya [59, 68] used his version of Vosper's theorem for general abelian groups to give a proof of the Chowla–Mann–Strauss inequality which is valid for all finite fields.

5.7. **The Frobenius problem.** There are several problems in additive theory which are clearly amenable to the isoperimetric method. One of them is connected to the Frobenius problem: given a finite set A of positive integers with $\gcd(A) = 1$, compute the largest integer not lying in the semigroup of integers generated by A, the so–called *Frobenius number* of A. Graham and Erdős formulated the extremal problem of finding the largest Frobenius number $f(n, m)$ of sets of integers in $[1, n]$ with fixed cardinality m. They showed that

$$\frac{n^2}{m-1} - 5m \leq f(n, m) < \frac{2n^2}{m}$$

and conjectured that $f(n, m)$ is roughly $n^2/(m-1)$. The conjecture was proved by Dixmier in 1990 by using Kneser's theorem, and Lev reproved it in 1996 by using Freiman's $(3k-4)$–theorem.

Yahya came back to the problem by using first the simpler theorem of Mann [63], obtained later a simpler proof with the isoperimetric method [76] and, by establishing a multiple set version of the $(3k-3)$–Freiman theorem, he gave a third approach to the problem [83]. The advantage of these proofs is that they provide the structure of the extremal sets for the conjecture and the asymptotic value of $f(n, m)$ conjectured by Lewin.

5.8. **Sum–free sets.** Recall that a set A in an abelian group is *sum–free* if $2A \cap A = \emptyset$. The structure of maximal sum–free sets and the largest density of a sum–free set in an abelian group have been topics largely studied in the literature. The version of Vosper's theorem for general abelian groups obtained by Yahya [76] was subsequently refined in [81] to be applied to the study of extremal (k,l)–sum–free sets, where $k > l$ and $kA \cap lA = \emptyset$ (the standard sum–free sets correspond to $k = 2$ and $l = 1$). By extending previous results of Plagne for the cyclic groups of prime order, general upper and lower bounds for the cardinality of a maximal sum–free set in an abelian group are obtained in [81] which provide the exact value for cyclic groups of order n whenever $\gcd(n, k-l) = 1$. The structure of sum–free sets with cardinality at least one third that of its ambient abelian group is also specified, extending previous results by Yap.

5.9. **Sums of dilates.** Yahya entered the subject of sums of dilates twice. Basically, the question is to estimate quantities of the form

$$|\lambda \cdot A + \mu \cdot A|$$

where A is a set of, say, integers and λ and μ are two coprime integers. In fact, Yahya even introduced this problem in his paper [78], where the very first and simple estimates of this type are obtained. Then the subject developed without him : motivated by the study of sumset estimates for bilinear forms by Nathanson and, more generally, of the distribution of planar sets by Łaba and Konyagin, Bukh obtained the general lower bound for the cardinality of sums of dilates

$$|\lambda_1 \cdot A + \cdots + \lambda_k \cdot A| \geq \left(\sum_i |\lambda_i|\right)|A| - o(|A|),$$

where A is a finite set of integers and $\lambda_1, \ldots, \lambda_k$ are integers with $\gcd(\lambda_1, \ldots, \lambda_k) = 1$. The error term in the above estimate is believed to be a function of the λ_i's only. Reentering the subject once again, Yahya obtained such an estimate in the case of $A + p \cdot A$ with p a prime number [94],

$$|A + p \cdot A| \geq (p+1)|A| - \left\lceil \frac{p(p+2)}{4} \right\rceil,$$

for sufficiently large sets A. The bound is tight and the extremal sets are characterized. Yahya believed that an analogous approach could also provide the tight bound and the characterization of extremal sets, conjectured to be arithmetic progressions, for the general case. A first attempt with $2 \cdot A + p \cdot A$ when p is an odd prime was successfully solved [101]. This certainly counts as one of Yahya's unfinished projects.

The paper by Breuillard and Green in this volume suggests another approach to the estimation of sums of dilates.

5.10. **A theorem of Pollard.** Pollard gave a nice generalization of the Cauchy–Davenport inequality. Let $A +_k B$ denote the set of elements in the sumset which can be written in at least k different ways. Then, for $\min(|A|, |B|) \geq t$,

$$\sum_{i=1}^{t} |A +_k B| \geq t \min\big(p, |A| + |B| - t\big).$$

A version for cyclic groups when $(B - B) \setminus \{0\}$ consists only of invertible elements was also proved by Pollard, and Grynkiewicz extended the result to general abelian groups. Yahya became attracted to the problem when Dicks told him about the connection, for nonabelian groups, with an old standing conjecture by Neumann. Yahya [93] started work on the problem, but when he became acquainted with Grynkiewicz's result he decided not to publish it.

The paper by Dicks and Serra in this volume gives an account of Yahya's ideas on the problem.

5.11. **The Scherk–Kemperman theorem.** By extending a previous result by Scherk to nonabelian groups, Kemperman obtained the following version of the Cauchy–Davenport inequality. Let A, B be finite subsets of a group G. Then, for every $c \in AB$,

$$|AB| \geq |A| + |B| - |A \cap (cB^{-1})|,$$

where $|A \cap (cB^{-1})|$ is the number of representations of c in AB. This result has been widely used in the zero–sum problems mentioned before, particularly by Olson and Gao. Yahya [97] considered an extension of the inequality to vertex–transitive graphs, which provides as a corollary the Scherk–Kemperman theorem. In the same paper he applied this extension to generalize to infinite locally finite vertex–transitive graphs a theorem by Mader, on the existence of r cycles $C_1, \ldots, C_r$ such that $C_i \cap C_j = \{v\}$, $i < j$, where v is a fixed vertex of the graph [97]. He also used this result in a weighted generalization of a theorem by Gao on zero–sum sequences [92].

We mention a last application to the so–called *Wakeford–Fan–Losonczy pairings.* Let A, B be finite subsets of a group G with $|A| = |B|$. A Wakeford–Fan–Losonczy pairing from B to A is a bijection $f : B \to A$ such that $bf(b) \notin A$ for each $b \in B$. In other words, such a pairing is a rainbow matching between A and $BA \setminus A$ in the (left) Cayley graph $\mathrm{Cay}(G, B)$. By extending to nonabelian groups previous results by Losonczy, Eliahou and Lecouvey proved that the only groups for which every pair $\{A, B\}$ of sets with the same cardinality and $1 \notin B$ admit a pairing from B to A, are either torsion–free or have prime order. Yahya [104] gave a tight lower bound on the number of such pairings. This lower bound provides in particular a characterization of the sets which admit a unique such pairing, which are progressions. In passing, he gave a simple proof of the result by Eliahou and Lecouvey. The existence of Wakeford–Fan–Losonczy pairings is also an ingredient of Yahya's proof of Kneser's theorem [89] by using the isoperimetric method.

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References

[1] Y.O. Hamidoune, Sur les atomes d'un graphe orienté, C.R. Acad. Sc. Paris Sér. A-B 284 (1977), 1253–1256.
[2] Y.O. Hamidoune, Sur les sommets de demi-degré h d'un graphe fortement h-connexe minimal, C.R. Acad. Sc. Paris Sér. A-B 286 (1978), 863–865.
[3] Y.O. Hamidoune, Quelques problèmes de connexité dans les graphes orientés, Thèse de doctorat de 3ème cycle, Université Pierre-et-Marie-Curie (Paris 6), 16 février 1978.
[4] Y.O. Hamidoune, Sur les parcours hamiltoniens dans les graphes orientés, Discrete Math. 26 (1979), 227–234.
[5] Y.O. Hamidoune, On the decomposition of a minimally strongly h-connected digraph into $h+1$ acircuitic subgraphs, Discrete Math. 31 (1980), 89–90.
[6] Y.O. Hamidoune, A property of a-fragments of a digraph, Discrete Math. 31 (1980), 105–106.
[7] Y.O. Hamidoune, On critically h-connected simple graphs, Discrete Math. 32 (1980), 257–262.
[8] Y.O. Hamidoune, Connectivity of transitive digraphs and a combinatorial property of finite groups, Combinatorics 79 (Proc. Colloq., Univ. MOntréal, Montreal, Quebec, 1979), Part I, Ann. Discrete Math. 8 (1980), 61–64.
[9] Y.O. Hamidoune, Contribution à l'étude de la connectivité d'un graphe, Thèse de doctorat d'État, Université Pierre-et-Marie-Curie (Paris 6), 3 juin 1980.
[10] Y.O. Hamidoune, Quelques problèmes de connexité dans les graphes orientés, J. Combin. Theory Ser. B 30 (1981), 1–10.
[11] Y.O. Hamidoune, On multiply critically h-connected graphs, J. Combin. Theory Ser. B 30 (1981), 108–112.
[12] Y.O. Hamidoune, An application of connectivity theory in graphs to factorizations of elements in groups, European J. Combin. 2 (1981), 349–355.
[13] Y.O. Hamidoune, A note on the girth of digraphs, Combinatorica 2 (1982), 143–147.
[14] Y.O. Hamidoune, On a conjecture of Entringer and Slater, Discrete Math. 41 (1982), 323–326.

[15] P. Duchet, Y.O. Hamidoune, M. Las Vergnas, H. Meyniel, Representing a planar graph by vertical lines joining different levels, Discrete Math. 46 (1983), 319–321.

[16] Y.O. Hamidoune, M. Las Vergnas, Jeux de commutation orientés sur les graphes et les matroïdes, C.R. Acad. Sc. Paris Sér. I Math. 298 (1984), 497–499.

[17] Y.O. Hamidoune, On the connectivity of Cayley digraphs, European J. Combin. 5 (1984), 309–312.

[18] Y.O. Hamidoune, Sur la séparation dans les graphes de Cayley abéliens, Discrete Math. 55 (1985), 323–326.

[19] Y.O. Hamidoune, M. Las Vergnas, The directed Shannon switching game and the one-way game, Graph theory with applications to algorithms and computer science (Kalamazoo, Mich., 1984), 391–400, Wiley-Intersci. Publ., Wiley, New York, 1985.

[20] Y.O. Hamidoune, M. Las Vergnas, Directed switching games on graphs and matroids, J. Combin. Theory Ser. B 40 (1986), 237–269.

[21] Y.O. Hamidoune, M. Las Vergnas, A Solution to the Box Game, Discrete Math. 65 (1987), 157–171.

[22] Y.O. Hamidoune, A note on minimal directed graphs with given girth, J. Combin. Theory Ser. B 43 (1987), 343–348.

[23] Y.O. Hamidoune, On a pursuit game of Cayley digraphs, European J. Combin. 8 (1987), 289–295.

[24] P. Duchet, Y.O. Hamidoune, H. Meyniel, Sur les quasi-noyaux d'un graphe, Discrete Math. 65 (1987), 231–235.

[25] Y.O. Hamidoune, M. Las Vergnas, Local edge-connectivity in regular bipartite graphs, J. Combin. Theory Ser. B 44 (1988), 370–371.

[26] Y.O. Hamidoune, M. Las Vergnas, A Solution to the misère Shannon switching game, Proceedings of the First Japan Conference on Graph Theory and Applications (Hakone, 1986), Discrete Math. 72 (1988), 163–166.

[27] Y.O. Hamidoune, D. Roeder, S. Janke, T. Feil, R. Koo, The probability of splitters in a list, J. Algorithms 10 (1989), 151–154.

[28] Y.O. Hamidoune, I. Salaün, On the independence numbers of a matroid, J. Combin. Theory Ser. B 47 (1989), 146–152.

[29] Y.O. Hamidoune, Sur les atomes d'un graphe de Cayley infini, Discrete Math. 73 (1989), 297–300.

[30] Y.O. Hamidoune, Factorisations courtes dans un groupe fini, First Montreal Conference on Combinatorics and Computer Science, 1987, Discrete Appl. Math. 24 (1989), 153–165.

[31] Y.O. Hamidoune, On the numbers of independent k-sets in a claw free graph, J. Combin. Theory Ser. B 50 (1990), 241–244.

[32] Y.O. Hamidoune, Connectivité des graphes de Cayley abéliens sans K_4, Discrete Math. 83 (1990), 21–26.

[33] Y.O. Hamidoune, A note on the addition of residues, Graphs Combin. 6 (1990), 147–152.

[34] J. Dias da Silva, Y.O. Hamidoune, A note on the minimal polynomial of the Kronecker sum of two linear operators, Linear Algebra Appl. 141 (1990), 283–287.

[35] Y.O. Hamidoune, On some graphic aspects of addition theorems, Topics in combinatorics and graph theory (Oberwolfach, 1990), 311–318, Physica, Heidelberg, 1990.

[36] Y.O. Hamidoune, A graphic approach to addition theorems, Seminário, Centro de Álgebra da Universidade de Lisboa (1990), 23-29.

[37] Y.O. Hamidoune, R. Tindell, Vertex transitivity and super line connectedness, SIAM J. Discrete Math. 3 (1990), 524–530.

[38] Y.O. Hamidoune, A. Lladó, O. Serra, Vosperian and superconnected abelian Cayley digraphs, Graphs Combin. 7 (1991), 143–152.

[39] Y.O. Hamidoune, On the subsets product in finite groups, European J. Combin. 12 (1991), 211–221.

[40] Y.O. Hamidoune, Combinatorial additive number theory and Cayley graphs, Graph theory, combinatorics, and applications. Vol. 2 (Kalamazoo, MI, 1988), 615–620, Wiley-Intersci. Publ., Wiley, New York, 1991.

[41] Y.O. Hamidoune, On a subgroup contained in some words with a bounded length, Discrete Math. 103 (1992), 171–176.

[42] Y.O. Hamidoune, A. Lladó, O. Serra, The connectivity of hierarchical Cayley digraphs, Discrete Appl. Math. 37/38 (1992), 275–280.

[43] Y.O. Hamidoune, A generalization of an addition theorem of Shatrowsky, European J. Combin. 13 (1992), 249–255.

[44] Y.O. Hamidoune, The minimum order of a Cayley graph with given degree and diameter, Networks 23 (1993), 283–287.

[45] C. Delorme, Y.O. Hamidoune, On products of sets in groups, Graphs Combin. 10 (1994), 101–104.

[46] J. A. Dias da Silva, Y.O. Hamidoune, Cyclic spaces for Grassmann derivatives and additive theory, Bull. London Math. Soc. 26 (1994), 140–146.

[47] Y.O. Hamidoune, The representation of some integers as a subset sum, Bull. London Math. Soc. 26 (1994), 557–563.

[48] Y.O. Hamidoune, Méthodes isopérimétriques et problèmes additifs, prépublication EC 95/04 de l'Institut Blaise Pascal (1995).

[49] Y.O. Hamidoune A. Lladó, O. Serra, Minimum order of loop networks of given degree and girth, Graphs Combin. 11 (1995), 131–138.

[50] Y.O. Hamidoune, On weighted sequence sums, Combin. Probab. Comput. 4 (1995), 363–367.

[51] Y.o. Hamidoune, On inverse additive problems, Rapport de Recherche EC 95/01, Institut Blaise Pascal, Paris, 1995.

[52] Y.O. Hamidoune, An isoperimetric method in additive theory, J. Algebra 179 (1996), 622–630.

[53] Y.O. Hamidoune, Additive group theory applied to network topology, Combinatorial network theory, 1–39, Appl. Optim., 1, Kluwer Acad. Publ., Dordrecht, 1996.

[54] Y.O. Hamidoune, O. Serra, On small cuts separating an abelian Cayley graph into two equal parts, Math. Systems Theory 29 (1996), 407–409.

[55] Y.O. Hamidoune, A. Lladó, O. Serra, Small cutsets in quasiminimal Cayley graphs, Discrete Math. 159 (1996), 131–142.

[56] Y.O. Hamidoune, O. Rødseth, On bases for σ-finite groups, Math. Scand. 78 (1996), 246–254.

[57] Y.O. Hamidoune, G. Zémor, On zero-free subset sums, Acta Arith. 78 (1996), 143–152.

[58] Y.O. Hamidoune, On weighted sums in abelian groups, Discrete Math. 162 (1996), 127–132.

[59] Y.O. Hamidoune, Subsets with small sums in abelian groups, I: the Vosper property, European J. Combin. 18 (1997), 541–556.

[60] Y.O. Hamidoune, M. Las Vergnas, Directed switching games, II. The arborescence game, Graphs and combinatorics (Marseille, 1995), Discrete Math. 165/166 (1997), 395–402.

[61] Y.O. Hamidoune, Erdős-Heilbronn Problem, Encyclopaedia of Mathematics Supplement I, pp 242, Kluwer Acad. Publishers, 1997.

[62] Y.O. Hamidoune, Adding distinct congruence classes, Combin. Probab. Comput. 7 (1998), 81–87

[63] Y.O. Hamidoune, On the Diophantine Frobenius problem, Portugal. Math. 55 (1998), 425–449.

[64] W. Gao, Y.O. Hamidoune, Zero sums in abelian groups, Combin. Probab. Comput. 7 (1998), 261–263.

[65] Y.O. Hamidoune, O. Ordaz, A. Ortuño, On a combinatorial theorem of Erdös, Ginzburg and Ziv, Combin. Probab. Comput. 7 (1998), 403–412.

[66] Y.O. Hamidoune, A. Lladó, O. Serra, On subsets with small product in torsion-free groups, Combinatorica 18 (1998), 529–540.

[67] W. Gao, Y.O. Hamidoune, On additive bases, Acta Arith. 88 (1999), 233–237.

[68] Y.O. Hamidoune, On small subset product in a group, Structure Theory of set addition, Astérique 258 (1999), 281–308.

[69] Y.O. Hamidoune, A. Lladó, O. Serra, An isoperimetric problem in Cayley graphs, Theory Comput. Syst. 32 (1999), 507–516.

[70] Y.O. Hamidoune, A. Lladó, O. Serra, On sets with a small subset sum, Combin. Probab. Comput. 8 (1999), 461–466.

[71] Y.O. Hamidoune, A. Lladó, O. Serra, Sets with a small subset sum and a conjecture by Diderrich. Paul Erdős and his mathematics (Budapest, 1999), 99–101, János Bolyai Math. Soc., Budapest, 1999.

[72] Y.O. Hamidoune, A. Lladó, O. Serra, R. Tindell, On isoperimetric connectivity in vertex-transitive graphs, SIAM J. Discrete Math. 13 (2000), 139–144.

[73] Y.O. Hamidoune, Subsets with a small sum II: the critical pair problem, European J. Combin. 21 (2000), 231–239.

[74] Y.O. Hamidoune, Ø. Rødseth, An inverse theorem modulo p, Acta Arith. 92 (2000), 251-262.

[75] Y.O. Hamidoune, A. Lladó and O. Serra, On restricted sums, Combin. Probab. Comput. 9 (2000), 513-518.

[76] Y.O. Hamidoune, Some results in additive number theory I: The critical pair theory, Acta Arith. 96 (2000), 97-119.

[77] Y.O. Hamidoune, A. Geroldinger, Zero-sumfree sequences in cyclic groups and some arithmetical application, J. Théor. Nombres Bordeaux 14 (2002), 221–239.

[78] Y.O. Hamidoune, A. Plagne, A generalization of Freiman's $3k-3$ theorem, Acta Arith. 103 (2002), 147–156.

[79] W. Gao, Y.O. Hamidoune, A. Lladó, O. Serra, Covering a finite abelian group by subset sums, Combinatorica 23 (2003), 599–611.

[80] Y.O. Hamidoune, Subsequence sums, Combin. Probab. Comput. 12 (2003), 413–425.

[81] Y.O. Hamidoune, A. Plagne, A new critical pair theorem applied to sum-free sets in abelian groups, Comment. Math. Helv. 79 (2004), 183–207.

[82] Y.O. Hamidoune, D. Quiroz, On subsequence weighted products, Combin. Probab. Comput. 14 (2005), 485–489.

[83] Y.O. Hamidoune, A. Plagne, A multiple set version of the $3k-3$ theorem. Rev. Mat. Iberoam. 21 (2005), 133–161.

[84] Y.O. Hamidoune, O. Serra, G. Zémor, On the critical pair theory in $\mathbb{Z}/p\mathbb{Z}$. Acta Arith. 121 (2006), 99–115.

[85] Y.O. Hamidoune, On iterated image size for point-symmetric relations, Combin. Probab. Comput. 17 (2008), 61–66.

[86] Y.O. Hamidoune, O. Serra, G. Zémor, On the critical pair theory in abelian groups: beyond Chowla's theorem, Combinatorica 28 (2008), 441–467.

[87] Y.O. Hamidoune, A. Montejano, O. Serra, Rainbow-free three colorings in abelian groups, Sixth Conference on Discrete Mathematics and Computer Science 389–393, Univ. Lleida, Lleida, 2008.

[88] Y.O. Hamidoune, A. Lladó, O.Serra, On complete subsets of the cyclic group, J. Combin. Theory Ser. A 115 (2008), 1279–1285.

[89] Y.O. Hamidoune, O. Serra, G. Zémor, On some subgroup chains related to Kneser's theorem, J. Théor. Nombres Bordeaux 20 (2008), 125–130.

[90] Y.O. Hamidoune, Some additive applications of the isoperimetric approach, Ann. Inst. Fourier (Grenoble) 58 (2008), 2007–2036.

[91] Y.O. Hamidoune, The global isoperimetric methodology applied to Kneser's Theorem, unpublished (2007), `arXiv:0708.2191`.

[92] Y.O. Hamidoune, A weighted generalization of Gao's $n+D-1$ theorem, Combin. Probab. Comput. 17 (2008), 793–798.

[93] Y.O. Hamidoune, O. Serra. A note on Pollard's Theorem, unpublished (2008), `arXiv:0804.2593`.

[94] J. Cilleruelo, Y. O. Hamidoune, O. Serra, On sums of dilates, Combin. Probab. Comput. 18 (2009), 871–880.

[95] Y.O. Hamidoune, The isoperimetric method. Combinatorial number theory and additive group theory, 241–252, Adv. Courses Math. CRM Barcelona, Birkhäuser Verlag, Basel, 2009.

[96] W. Gao, Y.O. Hamidoune, G. Wang, Distinct length modular zero-sum subsequences: a proof of Graham's conjecture, J. Number Theory 130 (2010), 1425–1431.

[97] Y.O. Hamidoune, Extensions of the Scherk-Kemperman theorem. J. Combin. Theory Ser. A 117 (2010)974–980.

[98] J. Cilleruelo, Y.O. Hamidoune, O. Serra, Addition theorems in acyclic semigroups. Additive number theory, 99–104, Springer, New York, 2010.

[99] Y.O. Hamidoune, I. P. da Silva, Distinct matroid base weights and additive theory. Additive number theory, 145–151, Springer, New York, 2010.

[100] Y.O. Hamidoune, On Minkowski product size: The Vosper's property, unpublished (2010), `arXiv:1004.3010`.

[101] Y.O. Hamidoune, J.Rué, A lower bound for the size of a Minkowski sum of dilates, Combin. Probab. Comput. 20 (2011), 249–256.

[102] Y.O. Hamidoune, A structure theory for small sum subsets, Acta Arith. 147 (2011), 303–327.

[103] Y.O. Hamidoune, A.Lladó, S. C. López, Vertex-transitive graphs that remain connected after failure of a vertex and its neighbors. J. Graph Theory 67 (2011), 124–138.

[104] Y.O. Hamidoune, Counting certain pairings in arbitrary groups, Combin. Probab. Comput. 20 (2011), 855–865.

[105] Y.O. Hamidoune, Hyper-atoms applied to the critical pair theory, submitted to Combinatorica, (see `arXiv:1102.2099`).

[106] Y.O. Hamidoune, Topology of Cayley graphs applied to inverse additive problems, Proc. 3rd International Workshop on Optimal Networks Topologies IWONT 2010, Barcelona, 2011 (see `arXiv:1011.1797`).

[107] B. Girard, S. Griffiths, Y.O. Hamidoune, k -sums in abelian groups, Combin. Probab. Comput. 21 (2012), 582–596.

[108] Y.O. Hamidoune, A. Lladó, S. C. López, On vosperian and superconnected vertex-transitive digraphs, Graphs Combin. 29 (2013), 241–251.

[109] É. Balandraud, B. Girard, S. Griffiths, Y.O. Hamidoune, Subset sums in abelian groups, this volume.

[110] Y.O. Hamidoune, S. López, A. Plagne, Large restricted sumsets in general abelian groups, this volume.

[111] Y.O. Hamidoune, Two Inverse results, Combinatorica, to appear, `arXiv:1006.5074`.

[A] A. O. Bah, Les dames du désert, Jeux et Stratégies 27 (1984), 46–48.

[B] T. Diagana, Remembering Yahya Ould Hamidoune, Notices of AMS 59 (2012), 827.

[C] A. Plagne, *Yahya ould Hamidoune*, grand Mauritanien, homme singulier, mathématicien d'exception, Gaz. Math. 129 (2011), 123–129.

[D] A. Plagne, *Yahya Ould Hamidoune*: *the* Mauritanian mathematician: 1948 – 11 March 2011, Combin. Probab. Comput. 20 (2011), 641–645.

CMLS, École polytechnique, 91128 Palaiseau Cedex, France

Universitat Politècnica de Catalunya, Barcelona-Tech, España

IMB, Université Bordeaux 1, 351, cours de la Libération, 33405 Talence Cedex, France

E-mail address: `plagne@math.polytechnique.fr`
E-mail address: `oserra@ma4.upc.edu`
E-mail address: `zemor@math.u-bordeaux1.fr`