ANSWER TO A QUESTION BY BURR AND ERDŐS ON RESTRICTED ADDITION, AND RELATED RESULTS

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Abstract. We study the gaps in the sequence of sums of \( h \) pairwise distinct elements of a given sequence \( \mathcal{A} \) in relation with the gaps in the sequence of sums of \( h \) not necessarily distinct elements of \( \mathcal{A} \). We present several results on this topic. One of them gives a negative answer to a question by Burr and Erdős.

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1. Introduction

In [1], Erdős writes:

Here is a really recent problem of Burr and myself: An infinite sequence of integers \( a_1 < a_2 < \cdots \) is called an asymptotic basis of order \( k \), if every large integer is the sum of \( k \) or fewer of the \( a \)'s. Let now \( b_1 < b_2 < \cdots \) be the sequence of integers which is (sic) the sum of \( k \) or fewer distinct \( a \)'s. Is it true that

\[
\limsup (b_{i+1} - b_i) < \infty.
\]

In other words the gaps between the \( b \)'s are bounded. The bound may of course depend on \( k \) and on the sequence \( a_1 < a_2 < \cdots \).

For \( h \geq 1 \), we will use the following notation for addition and restricted addition: \( h\mathcal{A} \) will denote the set of sums of \( h \) not necessarily distinct elements of \( \mathcal{A} \), and \( h \times \mathcal{A} \), the set of sums of \( h \) pairwise distinct elements of \( \mathcal{A} \).

If \( \mathcal{A} \) is an increasing sequence of integers \( a_1 < a_2 < \cdots \), the largest asymptotic gap in \( \mathcal{A} \), that is

\[
\limsup_{i \to +\infty} (a_{i+1} - a_i),
\]

is denoted by \( \Delta(\mathcal{A}) \).

We shall write \( \mathcal{A} \sim \mathbb{N} \) to denote that a set of integers \( \mathcal{A} \) contains all but finitely many positive integers. According to the Erdős-Burr definition, a set of integers \( \mathcal{A} \) is an asymptotic basis of order \( h \) if \( h \) is the smallest integer such that \( \bigcup_{j=1}^{h} j\mathcal{A} \sim \mathbb{N} \), or equivalently such that

\[
h(\mathcal{A} \cup \{0\}) \sim \mathbb{N}.
\]

The lower asymptotic density of a set of integers \( \mathcal{A} \) is defined by

\[
d(\mathcal{A}) = \liminf_{x \to +\infty} \frac{|\{a \in \mathcal{A} \text{ such that } 1 \leq a \leq x\}|}{x},
\]

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where the notation $|F|$ denotes the cardinality of a finite set $F$.

The question of Burr and Erdős takes the shorter form: is it true that if $h(\{0\} \cup \mathcal{A}) \sim \mathbb{N}$, then

$$\Delta(\mathcal{A} \cup 2 \times \mathcal{A} \cup \cdots \cup h \times \mathcal{A}) < +\infty?$$

We may also ask the following even more natural question: is it true that $\mathcal{A} \cup 2 \times \mathcal{A} \cup \cdots \cup h \times \mathcal{A}$ has a positive lower asymptotic density, as it was conjectured in [2].

We will show that the answer to both questions is no, except if $h = 2$:

**Theorem 1.** (i) If $(\mathcal{A} \cup 2 \mathcal{A}) \sim \mathbb{N}$ then $$\Delta(\mathcal{A} \cup 2 \times \mathcal{A}) \leq 2.$$ If $2\mathcal{A} \sim \mathbb{N}$ then $\Delta(2 \times \mathcal{A}) \leq 2$.

(ii) Let $h \geq 3$. There exists a set $\mathcal{A}$ such that $h(\{0\} \cup \mathcal{A}) \sim \mathbb{N}$ and $$\Delta(\mathcal{A} \cup 2 \times \mathcal{A} \cup \cdots \cup h \times \mathcal{A}) = +\infty.$$ There exists a set $\mathcal{A}$ such that $h\mathcal{A} \sim \mathbb{N}$ and $\Delta(h \times \mathcal{A}) = +\infty$.

The restricted order of an asymptotic basis $\mathcal{A}$, if it exists, is defined as the smallest integer $h$ such that any large enough integer is the sum of $h$ or fewer pairwise distinct elements of $\mathcal{A}$. We denote it by $\text{ord}_r(\mathcal{A})$. In general, asymptotic bases do not have to possess a (finite) restricted order. However, in the special case of asymptotic bases of order 2, the situation is more simple and can be precisely described (see [7] and [6]): indeed, being given an arbitrary asymptotic basis $\mathcal{A}$ of order 2, its restricted order is known to exist and to satisfy $2 \leq \text{ord}_r(\mathcal{A}) \leq 4$; moreover any integral value in this range can be achieved with asymptotic bases $\mathcal{A}$ such that $2\mathcal{A} = \mathbb{N}$. In particular, there exist asymptotic bases $\mathcal{A}$ containing 0 verifying $\text{ord}_r(\mathcal{A}) > 2$ and for which we consequently have $\Delta(2 \times \mathcal{A}) = \Delta(\mathcal{A} \cup 2 \times \mathcal{A}) \geq 2$. This shows that assertion (i) in Theorem 1 is optimal.

Having Theorem 1 at hand, the next natural question is then: assume that $h\mathcal{A} \sim \mathbb{N}$, that is $h\mathcal{A}$ contains all but finitely many positive integers, is it true that there exists an integer $k$ such that $\Delta(k \times \mathcal{A}) < +\infty$? If so, $k$ could depend on $\mathcal{A}$. But, suppose that such a $k$ exists for all $\mathcal{A}$ satisfying $h\mathcal{A} \sim \mathbb{N}$: is this value of $k$ uniformly (with respect to $\mathcal{A}$) bounded from above (in term of $h$)? If so, write $k(h)$ for the maximal possible value:

$$k(h) = \max_{h\mathcal{A} \sim \mathbb{N}} \min\{k \in \mathbb{N} \text{ such that } \Delta(k \times \mathcal{A}) \text{ is finite}\}.$$ 

Theorem 1 implies that $k(2)$ does exist and is equal to 2. No other value of $k(h)$ is known but we believe that the following conjecture is true.

**Conjecture 2.** The function $k(h)$ is well-defined in the sense that for any integer $h \geq 1$, $k(h)$ is finite.

If this conjecture is true, what is the asymptotic behaviour of $k(h)$? Our proof of Theorem 1 will be based on an explicit counterexample to the Erdős-Burr conjecture. This construction will lead in fact to a lower bound of $k(h)$, which obviously implies Theorem 1 for $h \geq 3$. 

Theorem 3. Let \( h \geq 2 \). We have
\[
k(h) \geq 2^{h-2} + h - 1.
\]

This study is closely related to the following problem: If \( \mathcal{A} \) is an asymptotic basis of order \( h \) which admits a (finite) restricted order \( \text{ord}_r(\mathcal{A}) \), is it true that \( \text{ord}_r(\mathcal{A}) \) is bounded in terms of \( h \)? If so, let us define \( f(h) \) to be the maximal possible value taken by \( \text{ord}_r(\mathcal{A}) \), when \( \mathcal{A} \) runs over the bases of order \( h \) having a finite restricted order. For \( h = 2 \), the question has been completely solved in [6] where it is shown that \( f(2) = 4 \). For \( h \geq 3 \), if we reuse the example leading to the bound of Theorem 3, we obtain an explicit lower bound for \( f(h) \).

Theorem 4. Let \( h \geq 3 \). One has
\[
f(h) \geq 2^{h-2} + h - 1.
\]

In another direction, we can study, for a given set of positive integers \( \mathcal{A} \), the asymptotic behaviour of the sequence \( (\Delta(h \times \mathcal{A}))_{h \geq h_0} \). The first observation is that this sequence is well-defined for some \( h_0 \) as soon as \( \Delta(h_0 \times \mathcal{A}) \) is finite. Indeed we have the following proposition.

Proposition 5. Let \( \mathcal{A} \) be a set of positive integers. Assume that \( \Delta(h_0 \times \mathcal{A}) \) is finite for some integer \( h_0 \), then for any \( h \geq h_0 \), \( \Delta(h \times \mathcal{A}) \) is finite.

This result implies that
\[
k(h) = 1 + \max_{h_0 \leq h} \max \{ k \in \mathbb{N} \text{ such that } \Delta(k \times \mathcal{A}) = +\infty \}.
\]

According to what obviously happens in the case of usual addition, it would be of some interest to establish, for any given set of integers \( \mathcal{A} \), the monotonicity of the sequence \( (\Delta(h \times \mathcal{A}))_{h \geq 1} \):

Conjecture 6. Let \( \mathcal{A} \) be a set of positive integers, then the sequence \( (\Delta(h \times \mathcal{A}))_{h \geq 1} \) is non-increasing.

We will observe firstly the following:

Proposition 7. Let \( \mathcal{A} \) be a set of positive integers, then
\[
\Delta(3 \times \mathcal{A}) \leq \Delta(2 \times \mathcal{A}).
\]

More interestingly, we will show the following partial result in the direction of Conjecture 6:

Theorem 8. Let \( \mathcal{A} \) be a set of positive integers. Then there exists an increasing sequence of integers \( (h_j)_{j \geq 1} \) such that \( (\Delta(h_j \times \mathcal{A}))_{j \geq 1} \) is non-increasing.

This theorem clearly implies that for a given set of positive integers \( \mathcal{A} \), the inequality \( \Delta((h+1) \times \mathcal{A}) \leq \Delta(h \times \mathcal{A}) \) holds for infinitely many positive integers \( h \). Theorem 8 is a direct consequence of the following more precise result.

Theorem 9. Let \( \mathcal{A} \) be a set of positive integers and \( h \) be the smallest positive integer such that \( \Delta(h \times \mathcal{A}) \) is finite. Then there exists an increasing sequence of integers \( (h_j)_{j \geq 1} \) with \( h_0 = h \) such that for any \( j \geq 1 \), one has \( h_j + 2 \leq h_{j+1} \leq h_j + h + 1 \) and \( \Delta(h_{j+1} \times \mathcal{A}) \leq \Delta(h_j \times \mathcal{A}) \).
This shows that for a given set of positive integers $\mathcal{A}$, the inequality $\Delta((h + 1) \times \mathcal{A}) \leq \Delta(h \times \mathcal{A})$ holds for any $h$ belonging to some set of positive integers having a positive lower asymptotic density bounded from below by $1/(h + 1)$.

Let $\mathcal{A}$ be a set of integers satisfying the weaker condition $\delta h \mathcal{A} > 0$ (instead of $h \mathcal{A} \sim \mathbb{N}$). We will establish in Theorem 10 that the validity of Conjecture 2 would imply that $\Delta(h \times \mathcal{A})$ is finite for some integer $k$ under this weaker condition. Clearly this result, if true, could not be uniform in $\mathcal{A}$. Henceforth, we introduce, for $\beta > 0$, the quantity

$$k_1(\beta, h) = \max \min \{k \in \mathbb{N} \text{ such that } \Delta(k \times \mathcal{A}) \text{ is finite}\}.$$ 

Our final result shows that $k_1$ is as well-defined as $k$, in some sense.

**Theorem 10.** Assume that Conjecture 2 holds. Then for any real number $\beta$ such that $0 < \beta \leq 1$ and any positive integer $h$, we have

$$k_1(\beta, h) \leq k \left\lceil \left(1 + \frac{1}{h} \right) \frac{1}{\beta} h \right\rceil,$$

where $\lceil u \rceil$ is the ceiling of $u$.

2. The proofs

For any real numbers $x$ and $y$, $[x, y]$ and $[x, y)$ will denote the sets of all integers $n$ (called intervals of integers) such that $x \leq n \leq y$ and $x \leq n < y$ respectively.

**Proof of Theorems 1, 3 and 4.** Let us first consider the case $h = 2$. Clearly the odd elements in $2\mathcal{A}$ do belong to $2 \times \mathcal{A}$. This implies that if $2\mathcal{A} \sim \mathbb{N}$, then $\Delta(2 \times \mathcal{A}) \leq 2$. This also implies that the odd elements in $\mathcal{A} \cup 2\mathcal{A}$ are in $\mathcal{A} \cup (2 \times \mathcal{A})$. It follows that $\mathcal{A} \cup 2\mathcal{A} \sim \mathbb{N}$ implies $\Delta(\mathcal{A} \cup (2 \times \mathcal{A})) \leq 2$.

In the case $h \geq 3$, it is enough to construct an explicit example. We first introduce the sequence defined by $x_0 = h$ and $x_{n+1} = (3 \cdot 2^{h-2} - 1)x_n^2 + hx_n$ for $n \geq 0$, and let

$$\mathcal{A}_n = [0, x_n^2) \cup \{2^j x_n^2 : j = 0, 1, 2, \ldots, h - 2\}.$$ 

Finally we define

$$\mathcal{A} = \{0\} \cup \bigcup_{n \geq 0} (x_n + \mathcal{A}_n).$$

Since any positive integer less than or equal to $2^{h-1} - 2$ can be written as a sum of at most $h - 2$ (distinct) powers of 2 taken from $\{2^j : j = 0, 1, \ldots, h - 2\}$, any integer in $[0, (2^{h-1} - 1)x_n^2)$ can be written as a sum of $h - 1$ elements of $\mathcal{A}_n$. Thus it follows

$$[0, (3 \cdot 2^{h-2} - 1)x_n^2) \subset [0, 2^{h-2}x_n^2) + [0, (2^{h-1} - 1)x_n^2) \subset [0, 2^{h-2}x_n^2) + (h - 1)\mathcal{A}_n \subset h\mathcal{A}_n.$$ 

We therefore infer that $hx_n, x_{n+1}) \subset h(x_n + \mathcal{A}_n)$. Moreover, since $hx_n \leq x_n^2$, we have $[x_n, hx_n] \subset [x_n, x_n^2] \subset x_n + \mathcal{A}_n$. It follows that, for any $n \geq 0$, we have

$$[x_n, x_{n+1}) \subset h((x_n + \mathcal{A}_n) \cup \{0\}) \subset h\mathcal{A}.$$ 

Consequently $h\mathcal{A} \sim \mathbb{N}$.

On the other hand, $(h - 1)\mathcal{A} \not\sim \mathbb{N}$. Indeed, this assertion follows from the more precise fact that, for any $n \geq 0$, no integer in the range $[(2^{h-1} - 1)x_n^2 + (h - 1)x_n + 1, 2^{h-1}x_n^2 - 1]$ (an
interval of integers with a length tending to infinity with $n$) can be written as a sum of $h - 1$ elements of $A$. Let us prove this fact by contradiction and assume the existence of an integer

$$u \in [(2^{h-1} - 1)x_n^2 + (h - 1)x_n + 1, 2^{h-1}x_n^2 - 1] \cap (h - 1)A.$$ 

Since we have (using $h \geq 3$)

$$u \leq 2^{h-1}x_n^2 - 1 < x_{n+1},$$

we deduce that

$$u \in (h - 1)\left(\{0\} \cup \bigcup_{i=0}^{n} (x_i + A_i)\right) \subset (h - 1)((0, x_n + x_n^2] \cup \{2^jx_n^2 + x_n : j = 1, 2, \ldots, h - 2\}).$$

In other words, we can express $u$ as a sum of the form

$$u = \alpha_{h-2}(2^{h-2}x_n^2 + x_n) + \cdots + \alpha_1(2x_n^2 + x_n) + \rho(x_n + x_n^2)$$

$$= (2^{h-2}\alpha_{h-2} + \cdots + 2\alpha_1 + \rho)x_n^2 + (\alpha_{h-2} + \cdots + \alpha_1 + \rho)x_n,$$

with $\alpha_1, \ldots, \alpha_{h-2} \in \mathbb{N}$, $\rho$ a positive real number and

$$\alpha_{h-2} + \cdots + \alpha_1 + \rho \leq h - 1.$$ 

If we denote by $[\rho]$ the integral part of $\rho$, this implies that

$$(2^{h-2}\alpha_{h-2} + \cdots + 2\alpha_1 + [\rho])x_n^2 \leq u \leq (2^{h-2}\alpha_{h-2} + \cdots + 2\alpha_1 + \rho)x_n^2 + (h - 1)x_n$$

and in view of $u \in [(2^{h-1} - 1)x_n^2 + (h - 1)x_n + 1, 2^{h-1}x_n^2 - 1]$, we deduce that

$$2^{h-2}\alpha_{h-2} + \cdots + 2\alpha_1 + [\rho] \leq 2^{h-1} - 1$$

and

$$2^{h-2}\alpha_{h-2} + \cdots + 2\alpha_1 + \rho \geq 2^{h-1} - 1.$$ 

We therefore obtain $2^{h-2}\alpha_{h-2} + \cdots + 2\alpha_1 + [\rho] = 2^{h-1} - 1$. We conclude by the facts that $\alpha_{h-2} + \cdots + \alpha_1 + [\rho] \leq h - 1$ and that the only decomposition of $2^{h-1} - 1$ as a sum of at most $h - 1$ powers of 2 is $2^{h-1} - 1 = 1 + 2 + 2^2 + \cdots + 2^{h-2}$ that $\alpha_1 = \cdots = \alpha_{h-2} = [\rho] = 1$. From this, we deduce that $\rho \leq h - 1 - \alpha_1 - \cdots - \alpha_{h-2} = 1$ and finally $\rho = 1$ which gives

$$u = (2^{h-1} - 1)x_n^2 + (h - 1)x_n,$$

a contradiction. Since $hA \sim \mathbb{N}$, we deduce that $A$ is an asymptotic basis of order $h$.

Concerning restricted addition, we see that for $l \geq h - 2$, we have

$$\max(l \times A_n) \leq (2^{h-1} - 2)x_n^2 + (l - h + 2)x_n^2 = (2^{h-1} + l - h)x_n^2.$$ 

Hence

$$x_{n+1} - \max(l \times (x_n + A_n)) \geq (2^{h-2} - l + h - 1)x_n^2 + (h - l)x_n.$$ 

If $l \leq 2^{h-2} + h - 2$, then $x_{n+1} - \max(l \times (x_n + A_n)) \geq x_n^2 - (2^{h-2} - 2)x_n$ which tends to infinity as $n$ tends to infinity. It follows that $k(h) \geq 2^{h-2} + h - 1$, as asserted in Theorem 3.

We now complete the proof of Theorem 4. It is clear from the preceding computations that if the basis $A$ defined above has a (finite) restricted order $\text{ord}_r(A)$ then it must satisfy $\text{ord}_r(A) \geq 2^{h-2} + h - 1$. Our goal is to prove that $\text{ord}_r(A)$ exists. We will show more precisely
that \( \text{ord}_r(A) = 2^{h-2} + h - 1 \). For this purpose, it is enough to prove that any sufficiently large integer is a sum of at most \( 2^{h-2} + h - 1 \) distinct elements of \( A \).

It is readily seen that if \( n \) is large enough, any integer in \([x_n, 2^{h-2}x_n^2 + x_n]\) is a sum of at most \( 2^{h-2} \) integers of \([x_n, x_n^2 + x_n] \subset x_n + A \). Moreover for any integer \( m \) in \([0, 2^{h-1} - 1]\), there exists some integer \( t(m) \) verifying \( 0 \leq t(m) \leq h - 1 \) such that

\[
z_m = mx_n^2 + t(m)x_n
\]
can be written as a sum of at most \( h - 1 \) distinct elements of \([x_n + 2jx_n^2 : j = 0, 1, 2, \ldots, h - 2]\) \( \subset x_n + A \). In particular, we observe that \( t(0) = 0 \) and \( t(2^{h-1} - 1) = h - 1 \). If we assume that \( n \) is large enough, then for any arbitrary integer \( m \) the difference \( z_{m+1} - z_m \) which satisfies \( 0 \leq z_{m+1} - z_m \leq x_n^2 + (h - 1)x_n \) is less than the length of the interval \([x_n, 2^{h-2}x_n^2 + x_n]\) by our assumption \( h \geq 3 \). Thus we infer that any integer in the sumset

\[
[x_n, 2^{h-2}x_n^2 + x_n] + \{z_m : 0 \leq m \leq 2^{h-1} - 1\} = [x_n, 2^{h-2}x_n^2 + z_{2^{h-1} - 1} + x_n)
\]
is a sum of at most \( 2^{h-2} + h - 1 \) distinct elements of \( x_n + A \). Since \( z_{2^{h-1} - 1} = (2^{h-1} - 1)x_n^2 + (h - 1)x_n \), we deduce that any integer in \([x_n, x_n + 1]\) is a sum of at most \( 2^{h-2} + h - 1 \) distinct elements of \( x_n + A \). This being true for any large enough integer \( n \), it follows that the basis \( A \), which is of order \( h \), has a restricted order equal to \( 2^{h-2} + h - 1 \).

This ends the proof of Theorem 4. \( \square \)

**Proof of Proposition 5.** We denote by \( a_1 < a_2 < \cdots \) the (increasing sequence of) elements of \( A \) and by \( b_1 < b_2 < \cdots \) the elements of \( h \times A \). We assume that \( \Delta(h \times A) = \lim \sup_{i \to +\infty} (b_{i+1} - b_i) \) is finite.

We define \( i_0 \) to be the smallest integer such that \( b_{i_0} > a_1 + a_2 + \cdots + a_h \). Hence, for any \( i \geq i_0 \), there exists an element of \( A \), \( \alpha(i) \in \{a_1, a_2, \ldots, a_h\} \) such that \( b_i \in h \times (A \setminus \{\alpha(i)\}) \); in particular this gives \( c_i = \alpha(i) + b_i \in (h+1) \times A \) for \( i \geq i_0 \).

If \( i \geq i_0 \) is large enough, then \( (b_{i+1} - b_i) \leq \Delta(h \times A) \). Let \( j \) be the smallest integer greater than \( i \) such that \( c_j > c_i \). We have

\[
0 < c_j - c_i \leq c_j - c_{j-1} = (b_j - b_{j-1}) + (\alpha(j) - \alpha(j - 1)) \leq \Delta(h \times A) + (a_h - a_1).
\]

This shows that for any large enough \( c_i \in (h+1) \times A \), there exists \( c_j \in (h+1) \times A \) such that \( 1 \leq c_j - c_i \leq \Delta(h \times A) + (a_h - a_1) \). From this, it clearly follows that

\[
\Delta((h+1) \times A) \leq \Delta(h \times A) + (a_h - a_1),
\]

thus in particular \( \Delta((h+1) \times A) \) is finite.

Proposition 5 follows by an easy induction. \( \square \)

**Proof of Proposition 7.** Let \( X = \{x_1 < x_2 < \cdots < x_i < \cdots \} \) be a set of positive integers. We denote \( D(X) = \max_{i \geq 1}(x_{i+1} - x_i) \) and recall that \( \Delta(X) = \lim \sup_{i \to +\infty}(x_{i+1} - x_i) \).

Let \( d > 0 \). We shall say that \( X \) \( d \)-covers an interval of integers \( I \) if the union of the balls centered on the elements of \( X \) with radius \( d/2 \) contains \( I \). In other words:

for all \( r \in I \), there exists \( x \in X \) such that \( |x - r| \leq d/2 \).
Let \( \mathcal{A} = \{a_1 < a_2 < \cdots < a_i < \cdots \} \). Assume \( d = \Delta(2 \times \mathcal{A}) < +\infty \). There exists an \( x_0 \) such that \([x_0, +\infty)\) is \( d \)-covered by \( 2 \times \mathcal{A} \). We shall see that for any \( a_i \in \mathcal{A} \) large enough, the interval \([a_i + x_0, a_{i+1} + x_0]\) is \( d \)-covered by \( 3 \times \mathcal{A} \). This will imply \( \Delta(3 \times \mathcal{A}) \leq d = \Delta(2 \times \mathcal{A}) \).

First case: if \( a_{i+1} \leq 2a_i - x_0 - d/2 \), then \( a_i + ((2 \times \mathcal{A}) \cap [0, a_i]) \) is contained in \( 3 \times \mathcal{A} \) and \( d \)-covers \([a_i + x_0, 2a_i - d/2]\) which contains \([a_i + x_0, a_{i+1} + x_0]\) by assumption.

Second case: if \( a_{i+1} > 2a_i - x_0 - d/2 \), then

\[
(2 \times \mathcal{A}) \cap \left[ \frac{3a_i}{2}, a_{i+1} \right) \subset 2 \times \left( \mathcal{A} \cap \left[ \frac{a_i}{2}, a_i \right) \right).
\]

Indeed, if \( a \) and \( b \) are two distinct elements of \( \mathcal{A} \) such that \( 3a_i/2 \leq a + b < a_{i+1} \), then \( a \leq a_i \) and \( b \leq a_i \); consequently we must have \( a \geq a_i/2 \) and \( b \geq a_i/2 \).

Let \( a \in \mathcal{A} \) such that \( d/2 + x_0 < a < a_i/2 - d \) (we may always find such an \( a \) if \( a_i \) is large enough). Then

\[
a + \left( (2 \times \mathcal{A}) \cap \left[ \frac{3a_i}{2}, a_{i+1} \right) \right) \subset 3 \times \mathcal{A}.
\]

Since \( [3a_i/2, a_{i+1}] \) is \( d \)-covered by \( 2 \times \mathcal{A} \), the interval \( [3a_i/2 + d/2 + a, a + a_{i+1} - d/2] \) is \( d \)-covered by \( 3 \times \mathcal{A} \). Since, in view of the choice made for \( a \), \( 3a_i/2 + d/2 + a \leq 2a_i - d/2 \) and \( a + a_{i+1} - d/2 \geq a_{i+1} + x_0 \), we infer that \([2a_i - d/2, a_{i+1} + x_0]\) is \( d \)-covered by \( 3 \times \mathcal{A} \). Moreover, the interval of integers \([a_i + x_0, 2a_i - d/2]\) is \( d \)-covered by \( a_i + ((2 \times \mathcal{A}) \cap [0, a_i]) \).

Therefore we conclude that \([a_i + x_0, a_{i+1} + x_0]\) is \( d \)-covered by \( 3 \times \mathcal{A} \). \( \square \)

Proof of Theorem 9. Let \( \mathcal{A} \) be such that \( d = \Delta(h \times \mathcal{A}) < +\infty \). This implies that for any sufficiently large \( x \),

\[
A(x) = |\mathcal{A} \cap [1, x]| \geq Cx^{1/h},
\]

for some positive constant \( C \) depending only on \( d \). Now, the number of subsets of \( \mathcal{A} \cap [1, x] \) with cardinality \( h + 1 \) is equal to the binomial coefficient \( \binom{A(x)}{h+1} \gg x^{1+1/h} \) where the implied constant depends on both \( \mathcal{A} \) and \( h \). Choose an \( x \) such that \( \binom{A(x)}{h+1} \geq (h + 2)!h^{h+2}x \). It thus exists an integer \( n \) less than \((h + 1)x \) such that

\[
n = a_1^{(i)} + \cdots + a_h^{(i)}, \quad \text{for } i = 1, \ldots, (h + 1)!h^{h+2},
\]

where the \((h + 1)!h^{h+2}\) sets \( E_i = \{a_1^{(i)}, \ldots, a_h^{(i)}\} \) of \( h + 1 \) pairwise distinct elements of \( \mathcal{A} \) are distinct. We now make use of the following intersection theorem for systems of sets due to Erdős and Rado (cf. Theorem III of [3]):

**Lemma** (Erdős-Rado). Let \( m, q, r \) be positive integers and \( E_i, 1 \leq i \leq m, \) be sets of cardinality at most \( r \). If \( m \geq r!q^{r+1} \), then there exist an increasing sequence \( i_1 < i_2 < \cdots < i_{q+1} \) and a set \( F \) such that \( E_{i_j} \cap E_{i_k} = F \) as soon as \( 1 \leq j < k \leq q + 1 \).

By applying this result with \( q = h \) and \( r = h + 1 \), we obtain that there are \( h + 1 \) sets \( E_{ij}, j = 1, \ldots, h + 1 \), and a set \( F \), with \( 0 \leq |F| \leq h - 1 \), such that \( E_{ij} \cap E_{ik} = F \) if \( 1 \leq j \neq k \leq h + 1 \). Observe that we must have \( 0 \leq |F| \leq h - 1 \) since the \( E_{ij} \)'s are distinct and the sum of all elements of \( E_i \) is equal to \( n \) for any \( i \). We obtain that the integer

\[
n' = n - \sum_{a \in F} a
\]
can be written as a sum of \( h + 1 - |F| \) pairwise distinct elements of \( A \) in at least \( h + 1 \) ways, such that all summands occurring in any of these representations of \( n' \) in \((h+1 - |F|) \times A\) are pairwise distinct (equivalently, this means that the set \( \bigcup_{j=1}^{h+1} E_j \setminus F \) has exactly \((h + 1)(h + 1 - |F|)\) distinct elements). This shows that

\[
n' + (h \times A) \subset (2h + 1 - |F|) \times A,
\]

and finally \( \Delta(h \times A) = \Delta(n' + (h \times A)) \geq \Delta(h_1 \times A) \), where \( h_1 = 2h + 1 - |F| \).

Iterating this process, we get an increasing sequence \((h_j)_{j \geq 0}\), with \( h_0 = h \), such that

\[
\Delta(h_j \times A) = \Delta(n' + (h_j \times A)) \geq \Delta(h_{j+1} \times A),
\]

where \( h_{j+1} \) is of the form \( h_j + h + 1 - |F_j| \) for some set \( F_j \) satisfying \( 0 \leq |F_j| \leq h - 1 \). We conclude that \( h_j + 2 \leq h_{j+1} \leq h_j + h + 1 \), as stated. \( \square \)

**Proof of Theorem 10.** Let \( h \) be a positive integer and \( A \) be a sequence of integers. We put \( B = hA \) and assume that \( d_B \geq \beta > 0 \). Define

\[
j = \left\lceil \frac{1}{h} \right\rceil.
\]

We thus have

\[
jd_B \geq 1 + \frac{1}{h} \geq 1 \geq d_j B.
\]

By Kneser’s theorem on addition of sequences of integers (cf. [9, 10], [4] or [12]), we obtain that there exist an integer \( g \geq 1 \) and a sequence \( B_1 \) of integers such that

\[
B \subset B_1, \quad g + B_1 \subset B_1, \quad jB_1 \setminus jB \text{ is finite},
\]

and

\[
d_j B_1 \geq jd_B - \frac{j - 1}{g}.
\]

We may assume that \( g \) is the smallest integer satisfying these conditions.

Since \( d_B \geq d_B = \beta \), we deduce from the previous inequality that

\[
g \leq \frac{j - 1}{j\beta - 1}.
\]

Hence \( g \leq (j - 1)h \leq jh \).

We denote by \( \overline{A} \subset \mathbb{Z}/g\mathbb{Z} \) the image of \( A \) by the canonical homomorphism of \( \mathbb{Z} \) onto \( \mathbb{Z}/g\mathbb{Z} \), the group of residue classes modulo \( g \). Let \( H \) be the period of \( g\overline{A} \), that is the subgroup of \( \mathbb{Z}/g\mathbb{Z} \) formed by the elements \( c \) such that \( c + g\overline{A} = g\overline{A} \). Since \( g \leq jh \), the sumset \( jh\overline{A} = j\overline{B} = j\overline{B_1} \) satisfies

\[
j\overline{B_1} + H = j\overline{B_1}.
\]

It therefore follows from the minimality of \( g \) that \( H = \{0\} \). Thus, from a repeated application of Kneser’s theorem on addition of sets in an abelian group (see [9, 10], [8] or [11]), we deduce

\[
g \geq |g\overline{A}| \geq g(|\overline{A}| - 1) + 1,
\]

which implies \( |\overline{A}| = 1 \). Therefore there exists an integer \( a_0 \) such that any element of \( A \) can be written in the form \( a_0 + gx \) for some integer \( x \). We define \( A_1 = \{(a - a_0)/g : a \in A\} \subset \mathbb{N} \).
Since $jhA = jB \sim jB_1$, we get $jhA_1 \sim \mathbb{N}$. Assuming the validity of Conjecture 2, we obtain that $\Delta(k(jh) \times A_1)$ is finite, and accordingly $\Delta(k(jh) \times A) < +\infty$. □

References