MATHEMATICAL ENTITIES WITHOUT OBJECTS

ON THE REALISM IN MATHEMATICS
AND A POSSIBLE MATHEMATIZATION OF THE (NON)PLATONISM:

DOES PLATONISM DISSOLVE IN MATHEMATICS?

by
Thierry PAUL

Contents

PRELUDE
Repetition, seriality, temporality and reality in mathematics ........................................ 2
1. Realism and (different levels in) mathematics .......... 3
2. Three key examples ........................................ 5
   2.1. Dynamical systems ................................. 5
   2.2. Quantum mathematics .............................. 9
   2.3. Partial differential equations ...................... 14
3. The three examples reunified: what happened? ...... 18
INTERMEZZO
Identity: last call for immediate boarding to temporality! ........................................ 21
4. Platonism and realism revisited .......................... 21
5. Synopsis .................................................. 23
POSTLUDE
Mathematics versus philosophy: the other way ........ 24
References .................................................. 25
PRELUDE
Repetition, seriality, temporality and reality in mathematics

Repetition, Seriality, Temporality are three concepts to be taken carefully when looking at how they operate in the field of mathematics. To be more precise, they acts, at a first glance, more naturally in a field, a space tied to mathematics, surrounding them, mostly inspiring them, but not boiling down to them.

Repetition refers obviously to the confrontation between identity and non-identity: to repeat means that we repeat the same, the identical. Any change breaks repetition. Seriality questions the possibility of decomposition of sequences, i.e. succession of, say, objects in their full diversity, into identical ones: decomposition of diversity into partial identity. Temporality addresses the phenomenon of successive actions parametrized by a "time" belonging to an ordered set. The vocabulary used in this description of the three concepts is totally absent of mathematics. It rather belongs, among others and to restrict the view to academic directions, to natural sciences or philosophy of them: in other words to real situations. Naively there is no intrinsic time - ordered set of parameters indexing mathematical objects - in mathematics, and the question of identity (together with the strongly related concepts of repetition and seriality) is supposed to be defined, settled one for good in the mathematical concept of definition. Naively, the three concepts of Repetition-Seriality-Temporality don’t belong to the core of mathematics. But they do belong, in a fundamental way, to some space upstream and downstream mathematics: downstream is their domain of applications: physics, chemistry, biology, economy, the list can be long. In one word: real situations. The field upstream refers to the famous concept of Platonism in mathematics: very roughly speaking, mathematics would be seating somewhere where the mathematician fishes them in his process of research. One sees that it is in this “somewhere”, upstream to mathematics, if it exists, that Repetition-Seriality-Temporality would be incarnated in mathematics: upstream mathematics would be a kind of a closet where real objects are placed and ordered by identity or chronology.

In conclusion, one can say, less naively than before, that the mathematical pertinence of Repetition-Seriality-Temporality is definitively tied to the general pertinence of realism in mathematics.
1. Realism and (different levels in) mathematics

When thinking the concept of realism in mathematics one faces immediately the opportunity of considering per se the two actors present when mathematics are in action, that is the mathematics themselves and the mathematician at work. Saying that doesn’t mean that mathematics exit by themselves, a priori. That would be a (too) strong Platonist point of view. No, what we are talking about by mentioning the mathematics are the thinking of, in mathematics, the doing of them; and by the mathematician we mean the thinking on, about mathematics, looking at them. The two are very distinct, especially when realism and Platonism are involved. They are even sometimes opposite as we shall see later. There are mathematicians who are Platonist in their look on mathematics, though their own mathematics develop a non Platonist view, in a sense of interior Platonism we propose to discuss in the present text.

Mathematicians and realism: this confrontation seems to look a bit funny, “komisch”, a tentative conciliation, as mathematician have the reputation of being outside any trace of reality\(^1\). We will therefore concentrate on realism inside mathematics. The question we would like to address is to ask if there are entities without objects in mathematics in the sense we shall explain now.

It is usually understood, and maybe wrong in our opinion, that mathematics (once again by this I mean the result of considerations and constructions of the mathematician “at work”) consist of two separate parts: objects (so to speak) and operations, transformations, a certain cooking with thee objects. The important part conceptually being the objects (numbers, manifolds, algebraic structures, etc). More, the dynamical operations leading to a mathematical statement have the tendency to disappear at the moment where the statement is definitively settled. And one must concede that static statement have often a certain aesthetics in mathematics. But what we would like to claim in this article, and prove on three examples taken from contemporary mathematics, is that this distinction, between objects and dynamical operations on them, vanishes very often. More than that: in mathematics, objects merge with operations made on them (for example even numbers are numbers that can be divided by two) and these operations are entities themselves (just

---

\(^1\) If you check on people in the street (at least in France, country per excellence of “pure mathematics”), they would think and express that mathematician are strange people, they have a sixth sense that most of people miss, but they haven’t got any of the five others
because in mathematics everything is precisely well defined, so that operations are written in the same language as objects). It is therefore natural, in mathematics, to try to understand better, to progress, by working directly at the level of these entities that are operations on objects, to perform “operations on these operations” so to speak. But, by doing so, you get sometimes to a point where there is no underlying object any more: new operations without a clear idea on what they acts on. Phrased like that, it seems paradoxical, but we will try to convince the reader that such situations exist in mathematics.

Of course the history of mathematics is long, and there are plenty of situations where the new object corresponding to these new operations was just hidden for a while and eventually appeared in full light (2). But, being not an historian, we would like to concentrate on situations where this happens either intentionally either by a sort of luck of strength, but in the two cases situations where the mathematician at work is aware of what is going on. This “intentionality” will play a key role in the notion of realism we would like to exhibit in mathematics.

And indeed, and this will be a guide for our formulation of a possible “realism without entities”, the mathematics of the twentieth century, far beyond the aforementioned Platonism, took seriously under consideration this systematic exploration (of operations on objects) through the corpus of what is nowadays called “quantum mathematics”. But not only through them. We will give two other examples, in the theory of differential equations.

Our title, “Mathematical entities without objects”, refers to the fact that the realism in mathematics can be seen as a feature, a product of the action of doing mathematics. Could it be something else [2]? Mathematics, by the fact of doing something, are real, they are in some reality. But traditionally this is not what is considered as being the realism, especially in the Platonist point of view: in mathematics should be a “doing” and a “done”. And then after this “distinction”, being debatable according to us, comes the fact of debating about the status of these “done”. Do they exist? Are they real? Or “just” a product of thinking? A weak solution of a partial differential equation [PDE], as we will see very soon in Section 2.3 is not an object given in its task of

---

2. An example is the case of the theory of groups, born out of the study of the transformation of objects, actually not restricted to sciences (think for example to fugues in 17th and 18th theory and the practice of inversion and dilatation of musical themes), before it takes off as a theory of groups per se, that is transformations of ... nothing.
being a solution of a PDE. In fact it solves an equation in the sense that, after the action of averaging against test functions, and a lot of them, and only after this action, the equation is solved. So much work to be done, compared to the simple look at the solution itself, would you say. Well, try to do better, if you can you win one million of Dollars\(^{(3)}\).

The “philosophical, epsitemological” scheme of what we just described and will embody in the next section, is always the same: a “classical”, very well (culturally) defined, mathematical object (an equation, a space, a partition of a space) happens to be isomorphic to a mathematical entity of very different nature (operations, actions of these objects), whose slight generalization (for example the suppression of one of its axioms) not only breaks the given isomorphism, but ruins any attend to make this modified entity isomorphic to any another “classical” object.

Think at a second degree equation posed on real numbers. We all know that sometimes a root can be complex, that means it doesn’t belong to the original space where the equation is settled. We know how to solve the problem: we define the “number” \(i\). Nowadays we can define the complex plane without having in mind the original second degree equation. But this was not the case when the number \(i\), satisfying \(i^2 = -1\), has been invented, as not being part of any set of numbers, just being a notation.

The three examples of the next section will be comparable to the creation of \(\sqrt{-1}\) outside the framework of complex numbers.

2. Three key examples

2.1. Dynamical systems. — Our first example deals with a quite new subject of mathematics: flows with low regularity properties.

The idea that the dynamical movement of rigid objects in our physical space should result from solving differential equation is the great revolutionary discovery of Newton. A way of putting it in a nutshell is the following. Suppose our rigid body is ideally reduced to a point, a point like the one you get by posing the extremity of your pen on a sheet of paper. If now you draw a curve on the paper, you just draw the successive positions of your ideal rigid body. Successive refers to time and positions refers to space: at each moment the body occupies ONE point, and when times evolves it follows a trajectory consisting of the different, successive,
points of the drawn curve. Exactly like when you ask to internet the way of going from one town to another one: the answer is a curve on a map.

At each point of the curve you drew, you can draw a straight line tangent \(^{(4)}\) to the original curve at the point you selected. This set of tangents reflects the dynamical process which produced the curve: when you draw a curve as if you do it in a row, the curve you obtain is very nice, regular, smooth. At the contrary, if you stop your drawing because your phone rings or if somebody touches your hand, the tangent will jump discontinually and the curve will present at this point an angle, a singularity. The same is true for the on-line route finder: if you influence the dynamics of the process by asking, for example, not to take highways, or to avoid centres of towns, or to get the cheapest travel, you will find abrupt changes of directions in your route.

The fundamental problem of the theory of dynamical systems, one of the most productive fields in the mathematics of the last century, consists in going the inverse way: instead of drawing first the curve and then tracing the tangents at each point, let us suppose that the dynamics is given first, that is let us be given a straight line at each point of your sheet of paper. Can we start at any point we wish to pose our pen, and draw a curve, the tangent at any point of it being the straight line which was given initially? And, other important question, when this curve exists, is it unique, or can we draw two curves having the same distributions of tangents? Existence, uniqueness, we entering slowly the vocabulary of mathematics, as we will point out later.

The intuition is that these two (existence, uniqueness) questions should have both a positive answer when the distribution of tangents is continuous: by this one means that, moving a little bit the point on the sheet, the associated tangent should change of direction just a little bit. The reader can have such an intuition by trying her/himself to draw a curve by following a distribution of tangents.

**BUT THIS INTUITION IS WRONG.**

It has been proved at the end of the 19th century that the distribution of tangent must have a stronger property in order to allow the construction of a unique nice curve. Without defining it for the moment, let us name by Lipschitz continuity this extra-property the distribution of tangent must have. 

---

\(^{(4)}\) The tangent to a curve at a chosen point is obtained by taking the straight line passing by two points nearby the point you chose and letting both points going to the chosen point.
A natural question arises immediately: what happens when this Lipschitz continuity condition is NOT satisfied by the distribution of tangents?

It took more than one century to have a hint on the answer to this question ([8, 5, 1]), and we will see that the answer necessitates to ... destroy the underlying space.

In order to understand the philosophical ideas behind this a priori negative phenomenon, we need to rephrase the preceding discussion in an a bit more mathematical language\(^5\).

In mathematics (and in physics), a flow on a given space is defined, once again as the roadmap provided by an on-line route finder, by a curve (route on a map) and a way of assigning to any value \(t\) of the time a point \(X(t)\) in this curve (0 hours (departure: Paris), 2 hours: Tours, 3 hours: Poitiers etc): such an assignment is called in mathematics a function \(t \mapsto X(t)\). While consulting an on-line route finder, you are asked certain constraints you wish your travel to satisfy. For example you might decide not to pay any highway fee. The route finder will then decide to change brutally the direction of your trajectory when you are about to meet a highway toll booth: the tangent to your course will be modified, and the trajectory will follow this new tangent indication. To a certain dynamics (e.g. avoid fees) is assigned a certain trajectory. Such a dynamics is realized in mathematics by the fact of giving on your space a vector field, that is to say a way of associating to each point \(X\) a direction \(f(X)\), a (oriented) straight line, supposed to be tangent to the trajectory. The problem is then to find a curve with the requirement that at each point the curve is tangent to the direction assigned by the vector field at this point. It happens that solving this problem consists mathematically of solving a differential equation written as

\[
\frac{dX}{dt}(t) = f(X(t)), \quad X(0) = X_0.
\]

Here \(X(t)\) is the point on the curve at time \(t\) and \(f(X(t))\) is the direction (vector field) at the point \(X(t)\), as seen before. The new ingredients are \(X_0\), the point of departure which can be chosen in principle anywhere in the space considered, and what is (a bit barbarically) denoted by \(\frac{dX}{dt}(t)\),

\(^5\) The non mathematician reader should not be afraid by the mathematical “vocabulary” used below, but should just concentrate on the (changes of) morphology of the syntax, almost at a graphical level, and try to behave with a low level thinking “à la Teissier” [18].
the “velocity” on the trajectory at time $t$ \(^{(6)}\), a notion anybody knows intuitively. The equality between the “velocity” $\frac{dX}{dt}(t)$ and the direction $f(X(t))$ at each point $X(t)$ is, in particular, the formalization of “the curve is tangent to the vector field”.

As we mentioned earlier, solving this problem \textit{in a unique way and for any initial point} $X_0$ necessitates that the way $f(X)$ depends on $X$ is regular enough, not too much hectic (Lipschitz continuity): otherwise the curve might not exist or the problem might have several solutions \(^{(7)}\).

When this Lipschitz condition is not satisfied and replaced by a weaker hypothesis, “$BV$ regularity” \(^{(8)}\), it happens that (2.1) is still solvable in a unique way, but only \textit{for almost all initial point} $X_0$, not \textit{all initial point} $X_0$. When we release the Lipschitz condition, though the equation (2.1) is still very nice, very smooth so that we naively expect as before the solution to be smooth also, the real solution still exists but only “almost everywhere”, not “everywhere”\(^{1}\). What means “exists almost everywhere”? It means that if you select by chance an initial point, with probability one everything will go nicely, as if the equation was more regular. But nothing prevents you from picking up for initial point, by luck of chance, one of these rare points where, for example, two different trajectories (or even worse, none) can be born at the same time \(^{(9)}\).

The flow can then be defined, but not on the whole space, as the original task was asking for, but on an “almost everywhere” defined space. No trouble with that a priori: we know such spaces, the reals deprived of the rationals is such an example. But in our case the set of remaining points of our space is not known. Or equivalently the bad points are not known (at the contrary of the rationals imbeded in the reals which are perfectly identifiable). More: “almost everywhere space” can change by composition of flows, that is to say if you stop and restart again.

By identifying a “space” by the entity consisting of all the trajectories solutions of a vector field’s equation, instead of defining it by an “object” made of points given (Platonistically?) a priori, you see that when the vector field is Lipschitz, the two definition merge, but when the vector

\begin{itemize}
  \item \footnote{See the link with the definition of the tangent in footnote 4}
  \item \footnote{Think to the situation where exist, near a tollbooth on a given highway, two free highways on each side of it. Though it seems like even.}
  \item \footnote{The definition of $BV$ goes much further the scope of this text. Let us just mention that the two letters B and V refer to “bounded variations”, which indicate that the vector field could be hectic, but not totally crazy.}
  \item \footnote{Think again to the situation of the two free highways near a tollbooth. You will be easily convinced that such situations are very, very rare.}
\end{itemize}
filed is only BV, there is no “object” counterpart to the “entity-like” notion of space.

We have the two following diagrams:

when the vector fields $f$ is Lipschitz,

\[
\begin{align*}
\text{object} & \quad \text{space } M \\
\updownarrow & \\
\text{entity} & \quad \text{equation} \\
\left( \frac{dX}{dt}(t) = f(X(t)), \; X(0) = X_0, \; \text{for all } X_0 \right)
\end{align*}
\]

when the vector fields $f$ is only BV

\[
\begin{align*}
\text{entity} & \quad \text{equation} \\
\left( \frac{dX}{dt}(t) = f(X(t)), \; X(0) = X_0, \; \text{for almost all } X_0 \right)
\end{align*}
\]

\[
\begin{align*}
\updownarrow \\
\text{object} & \quad ?
\end{align*}
\]

NO UNDERLYING OBJECT TO THIS NEW ENTITY.

2.2. Quantum mathematics. — The nowadays quite popular notions of quantum groups and quantum (= noncommutative) geometries have no underlying group or geometry-type objects. Nevertheless they fulfil completely, according to us, the paradigm of realism in the sense
that their structures provide an arsenal of study methods, comparable to the ones available in the “classical” situation.

How are they constructed?

A rigid body, a human body, a landscape, is a geometrical object, a manifold, perfectly understood if one knows enough drawings of it (it can be reconstructed from them). No need to explain more this fact we all experienced. What is a drawing? It is a set of points on a sheet of paper. In what consists the action od drawing? It consists in assigning to each point of the (surface of the) body a point on the sheet: that is to say a function from the body to the sheet. A single drawing doesn’t capture entirely the body but, and Picasso understood this very well, several ones do. This beautiful fact turns into a theorem in mathematics: all the functions on a manifold (e.g. the surface of a rigid body), a manifold, determines it\(^\text{10}\) What we do by stating (and using) this theorem is, when we look at it closer, building a higher level, as in the preceding section. It refers to a dynamical action (“drawing, looking at”) and to a formalism: it belongs definitively to the side of “entities”.

But identity (it again) in mathematics allows the semantic shift

“determines” \(\implies\) “is”:

a manifold “is” the algebra of its functions.

This is a dynamical point of view, an operating one. And this algebra of function inherits form numbers a nice property: one can multiply functions like numbers, and the order by which you do this operation is insignificant\(^\text{11}\): one calls such an algebra a *commutative* algebra (since we can commute the different functions/drawings we multiply without changing the result). And the converse is true: give me any algebra with these property, that is any commutative algebra, then I can construct the underlying manifold. This is the Gelfand Theorem \[^9\]. We can start to build our diagram: at the lower level, ground floor, we put what we would like to call, by convention, the objects like a manifold, considered as a set of points. At the upper level we put an alternative “entity”, the algebra of functions on the corresponding object.

We will now remove one property of the algebra: the commutativity. We leave the commutative algebra, isomorphic to a lower level object, the\(^\text{10}\) To be more precise: the algebra of continuous functions with values in the complex numbers determines a manifold\(^\text{11}\) It seems to us that this *multiplication* is truly incarnate in the cubist portraits. On a single drawing one put/multiply several different views of the body to be drawn. And obviously the order between the different takes is insignificant.
manifold, and get a new one, with a priori nothing under it. Nevertheless the upper level still exists and inherits from the left all the property you need, in principle, to generalize the construction made at the level down.

The question is: can we move by staying on the ground floor and construct an object which would be the classical object to the new upper level? Can we move Left-Right by remaining at the lower level?

Let us draw a picture.

\[
\begin{array}{c}
\text{entities} & \text{commutative algebra} & \rightarrow & \text{noncommutative algebra} \\
\uparrow & & \downarrow \\
\text{objects} & \text{manifold} & \rightarrow & ?
\end{array}
\]

The answer is: no!

\[
\begin{array}{c}
\text{entities} & \text{commutative algebra} & \rightarrow & \text{noncommutative algebra} \\
\uparrow & & \downarrow \\
\text{objects} & \text{manifold} & \rightarrow & \emptyset, \text{nothing}
\end{array}
\]

There is no manifold whose space of functions is noncommutative.

But then, what becomes the arrow at the ground floor? The diagrammatic answer is the following:

\[
\begin{array}{c}
\text{entities} & \text{commutative algebra} & \rightarrow & \text{noncommutative algebra} \\
\uparrow & & \nearrow \\
\text{objects} & \text{manifold} &
\end{array}
\]

The object “manifold” joins the upper level and becomes the entity “noncommutative algebra”, and by another semantic shift one can say:
a non commutative manifold is a non commutative algebra.

This presentation might let think that non-commutative manifold arrives in the field of mathematics just by fantasy of generalization that mathematicians are know to be fond of. This is not true, and the most interesting situation where the lower floor disappears are the ones in which this destruction is performed at (by) the lower floor itself. The simplest case comes from the concept of quotient space, that is to say the set of families of elements of a space, a family being the subset of elements sharing a certain property. For example take a sheet of paper and fill it completely by drawing straight lines on it. The quotient appears as the set of such lines (12). The sheet itself is a sweet set of points, each line is itself a sweet collections of points. What’s about the quotient? Well, it is a fact in mathematics (only in mathematics?), that the set of nice objects included in a nice object too ... might be not nice at all (13).

The reader interested by going a bit further and treating a very simple example, though totally meaningful, can do her/himself the following experience.

Let him/her draw on a sheet of paper a two-dimensional torus in the from of a square whose facing borders are identified, so that each point on the left side is taken as the same of the one on the right side with same altitude. The same construction being done for the up and down sides. If one draws now an oblique straight line passing by the left-down corner and the middle of the right side, the curve drawn by this method will be closed (14): one comes back to the original point after a two “rounds” around the torus. The line will pass two times on the left side: first at its middle, second at the top (which is the same as the bottom after identification of the upper and lower sides).

In you drawn now a parallel line starting now from any point in the lower left half-side, the same argument will apply and the line will cross again the left side only at a point, in the upper half side this time. Therefore one sees that any such straight line passes through one and only one point in the lower left half-side, and to any such point passes one and only one straight line. That is to say that one can label any such straight lines univocally by one point on the lower left half side: the set of such straight lines is identified with the

---

12. A metaphor is the following: when you talk about a packet of spaghettis, you mention in fact two different things. One is the set of the 250 grams of flour its volume contains, the other is the set of 50 spaghettis in contains. The second one is the quotient of the first when you regroup the grams of flour sharing the property of belonging to the same spagetti.

13. When the spaghettis are very well stored in the envelope, you can count the spaghettis and easily determine one from the other, count them etc. But drop the packet of spaghettis on the floor and try to count them without putting them back to the packet!

14. Don’t forget that the right and left sides of the square have been identified, so that when the straight line arrives to the right side, it has to be continued by starting again at the middle of the left side.
lower left half-side. Everything goes well, the quotient we were looking at is just a piece of straight line, a nice geometrical object \(^{(15)}\).

But let us suppose now that one does exactly the same construction but without taking care of the point where you are going to cross the right side. In general, the drawn straight lines won’t be closed any more: you won’t go back to the starting point, but you will, in general, miss it narrowly after many rounds (do the experience!), the line will intersect the left side at an infinite number of points. More: if you continue going around the torus on this line, the set of these intersection points will accumulate everywhere and become dense in the left side. If you now choose another starting point and draw a straight line parallel to the latter, the set of intersection with the left side will look exactly the same as the one for the first straight line, and you won’t be able to distinguish which intersection point belongs to which straight lines, although each point of the left side belongs to one and only one of the two straight lines.

In mathematical language, this means that, in general \(^{(16)}\) the set of straight lines, still ideally well defined as a set, is totally unreachable by any approximation, this last property reflecting the “looks the same” expressed before. Defining the set of drawn straight lines only as “a set” is tautologically possible. But this view is unsatisfactory, as the simple drawing shows, since one cannot differentiate any line from the other by its trace of the left side. In fact if one looks at the (algebra of) continuous functions on this set, one can prove the following theorem:

*Any continuous function on the quotient “space”, set of dense points, is constant everywhere (it asserts the same number to all the points).*

The set of drawings of our set, the algebra of functions on it, is reduced to trivial ones, the ones making no differences between the points, a fully black drawing, with not even a texture “à la Soulages”. Nothing. No classical structure, nothing. But if we lift the whole construction we made on the lower level to the upper one, we find that there is a possibility of describing and “understanding” this space by identifying it with a non commutative algebra \(^{[7]}\). One can calculate, manipulate this set-entity, construct a topology on it, although there is **NO UNDERLYING OBJECT TO THIS NEW ENTITY.**

We insisted a bit heavily of the preceding construction not for torturing the reader, but rather because we believe that it gives a quite faithful image of the mathematician at work: tedious drawings, computations, failures, repetitions, ..., using, pushing to its very extremities a formalism leading to a new paradigm. To put it in a nutshell: the mathematical formalism is a formalism in action \(^{[3, 4]}\).

---

15. The attentive reader might have noticed that if we start the straight line from the very bottom of the lower left half side, this line will cross twice the segment. But this means that the bottom and the middle of the lower left half side are on the same straight line. They have therefore to be identified. And the quotient is thereby not a segment but a segment with the two extremities identified (like during the construction of the torus on the sheet of paper), that is a circle, topologically.

16. Namely, for almost all values of the slope of the straight line.
In order to conclude this section, we would like to point out\(^{(17)}\) a clear difference between the object-entity dialectic present in the present article and the dynamics of “generalization-extension” so familiar in mathematics, where, though it is widely generalized, the underlying object never really disappears. An example of this is group theory already evoked in footnote 2: abstract groups are transformations of ... nothing, but it happens that a very efficient way of studying groups is to let them “act” on different types of objects in the framework of representations theory. Let us give another example: Analysis situs by Poincaré \cite{17} views objects as new entities, but without removing the lower level. Maybe the upper level becomes more important, the concepts of fundamental group and simplicial homology becoming even necessary to understand the underlying level, but the underlying object never disappears.

2.3. Partial differential equations. — Our last example will be a bit more technical, and might be skipped by the uninterested reader, without damage for the comprehension of the core of this article.

Solving the Navier-Stokes equations, fundamental equations of hydrodynamics, is limited, up to nowadays, to defining “weak” solutions. We would like in this section to show how this concept of weak solution, that we are going to explain later on, enter perfectly the notion of entity as considered before: it is an entity considered as the solution (because the important fact here is that there is only one), as a substitute to a “true” solution, an object which is, for Navier-Stoke problem, still unknown and might never exist.

Let us be a bit precise, without too much technicality\(^{(18)}\). A partial differential equation (PDE) consists in a function \(u\) (the unknown), for example a function \(x \mapsto u(x)\) as defined in Section 2.1 which sends (real) numbers \(x\) to (real) numbers \(u(x)\), an operator \(P : u \mapsto P(u)\), a “fonction” with sends the function \(u : x \mapsto u(x)\) to another function \(P(u) : x \mapsto P(u)(x)\), and an “equality to zero”:

\[
P(u) = 0.
\]

What is means by this is that \(P(u)\) is a function, and one look at a function \(u\) such that \(P(u)\) is the null function, the function identically equal to zero:

we want to fine THE \(u\) such that \(P(u)(x) = 0\) FOR ALL number \(x\).

We see here arising a question concerning identity: “identically equal to 0” means what we just wrote. But checking something “for all number” is a long

\(^{17}\) This paragraph follows a demand of clarification by Jochen Br"uning, Jorgen Jost and Bernard Teissier.

\(^{18}\) Once again the reader shouldn’t be afraid by the coming equations, though they are a bit more “serious” in this section than in the two preceding ones. She/he can comfort her/himself by thinking that reading without (fully) understanding is also an integral part of the mathematician’s activity.
task, very long task. And doing that task patiently is boring and the risk of missing “some xt” is high. An alternative way consists in taking averages of $p(x)$ with a given probability distribution. What does this mean? Just that we will sum the numbers $u(x)$ defined out of “almost” all numbers $x$ by weighting by the importance we want to assert to $x$. Just like insurance or political survey companies do. The reader might argue that we still miss some values of $x$ by the “almost” intrinsic to the concept of average. The answer is that, if we take all the averages with all probabilities distributions, then we determine the value of $P(u)(x)$ for all $x$ (19).

Taking the average of a function $f$ with a probability distribution $\varphi$ is written in mathematics the following way (20)

$$\int_{\mathbb{R}} \varphi(x)f(x)ds.$$  

And what we just wrote can be formalized as

(2.3) \[ P(u)(x) = 0 \iff \left[ \int_{\mathbb{R}} \varphi(x)P(u)(x)ds = 0 \text{ for ALL functions } \varphi \right] \]

It is a matter of fact (a very unpleasant fact) that in many cases the mathematician finds out that a prototype model of solution $u$ she/he’s working at is such that, for some few points $y$, $P(u)(y)$ is infinite and therefore cannot be defined properly (21). In this case $u$, of course, cannot pretend to be a solution of (2.2) as we just defined, but these “bad” points $y$ are sometimes so few that we just would like to be able to ignore them. In order to do that we first remark that, usually in these situations, these points are so few, so isolated, that they disappear when taking an average of the left hand side of (2.2). But one doesn’t want to take averages with all probabilities, since, because of (2.3) this would be equivalent to (2.2). What to do?

It happens that solving the right hand side of (2.3) by restricting it to some special functions called meaningfully “test functions” is easier than solving the left hand side, namely (2.2).

19. The reason is simple, take a probability asserting the maximal value to a number $x_0$ and the value zero to all others. Obviously the corresponding average will be equal to $P(u)(x_0)$. What else?

20. The reader not familiar with infinitesimal calculus might remember that, when averages are taken over integers numbers one write $\sum_i \varphi(i)f(i)$. The following notation (it is just a notation) is obtained my formally “translating” $\sum_i \ldots$ to $\int_{\mathbb{R}} \ldots dx$.

21. In many cases, such as the Navier-Stokes equations, the operator $P$ when acting on $u$ conveys the speed of variation of the number $u(x)$ when $x$ varies just a little bit. This quantity becomes obviously infinite at a point $y$ where, e.g., $u$ is not continuous, that is presents jumps.
We will say that $u$ is a weak solution of (2.2) if

\[(2.4) \quad \int_{\mathbb{R}} \varphi(x) P(u)(x) ds = 0 \quad \text{for all “TEST functions” } \varphi\]

Why do we say that such a function $u$ satisfying (2.4) is a weak solution of (2.2)? Well because if $u$ was a nice solution of (2.2), then (2.4) would be true \textit{for every function}, \textit{not only a test function} and therefore (2.4) would be equivalent to (2.2), as every function whose integration against every function is equal to 0 is itself equal to 0.

We propose to define as “objects” the contents of (2.2) and the bracket in the right place in (2.3), and as “entities” the brackets in the left place in (2.3) and the contents of (2.4). We have

\[
\begin{align*}
\text{object} & \quad \iff \quad \text{entity} \\
\left\{ P(u) = 0 \right\} & \iff \left\{ \int_{\mathbb{R}} \varphi(x) P(u)(x) ds = 0 \quad \text{for all functions } \varphi \right\}
\end{align*}
\]

and, since test functions are functions, after all,

\[
\begin{align*}
\text{object} & \quad \implies \quad \text{entity} \\
\left\{ P(u) = 0 \right\} & \implies \left\{ \int_{\mathbb{R}} \varphi(x) P(u)(x) ds = 0 \quad \text{for all “TEST functions” } \varphi \right\}
\end{align*}
\]

But as we mentioned before

\[
\begin{align*}
\text{entity} & \quad \implies \quad \text{object} \\
\left\{ \int_{\mathbb{R}} \varphi(x) P(u)(x) ds = 0 \quad \text{for all “TEST functions” } \varphi \right\} & \not\implies \left\{ P(u) = 0 \right\}
\end{align*}
\]

Under this last entity there is no clear existence of a true solution, nothing which plays the role of the complex plane for the second degree equation. (2.4) is definitively different from (2.2):

\textbf{NO UNDERLYING OBJECT TO THIS NEW ENTITY (22)}

---

22. Though we won’t develop it here, let us make a link with sections 2.1 and 2.2 by mentioning that we believe that we are facing here an epistemological shift: the geometrical space becomes a functional space i.e. a space of functions. One can consult [16] for other studies of mutations of the notion of spaces.
We get the two following diagrams.

**entities** \[ \int_\mathbb{R} \psi(x)P(u)(x)dx \rightarrow \int_\mathbb{R} \psi(x)P(u)(x)dx \]
for all functions \( \psi \)

\[ \downarrow \]

**objects** \( P(u) = 0, \text{PDE} \)
\[ \rightarrow \emptyset, \text{nothing "like a PDE"} \]

**entities** \[ \int_\mathbb{R} \psi(x)P(u)(x)dx \rightarrow \int_\mathbb{R} \psi(x)P(u)(x)dx \]
for all functions \( \psi \)

\[ \uparrow \]

**objects** \( P(u) = 0, \text{PDE} \)
3. The three examples reunified: what happened?

Let us resume the three diagrams corresponding to the three situations we investigated earlier:

the three unanswered questions (or equivalently wrongly answered by a forced essentialist-type answer: “∅-nothing”)

<table>
<thead>
<tr>
<th>entities</th>
<th>$\frac{dX}{dt} = f(X(t)), X(0) = X_0$</th>
<th>$\frac{dX}{dt} = f(X(t)), X(0) = X_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>for all initial point $X_0$</td>
<td>for almost initial point $X_0$</td>
</tr>
<tr>
<td>↓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>objects</td>
<td>space $\mathcal{M}$</td>
<td>$\emptyset$, nothing “like a space”</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>entities</th>
<th>commutative algebra $\rightarrow$ noncommutative algebra of functions on $\mathcal{M}$ of operators</th>
</tr>
</thead>
<tbody>
<tr>
<td>↓</td>
<td></td>
</tr>
<tr>
<td>objects</td>
<td>manifold $\rightarrow$ $\emptyset$, nothing “like a manifold”</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>entities</th>
<th>$\int_{\mathbb{R}} \psi(x)P(u)(x)dx$</th>
<th>$\int_{\mathbb{R}} \psi(x)P(u)(x)dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>for all functions $\psi$</td>
<td>for every test functions $\varphi$</td>
</tr>
<tr>
<td>↓</td>
<td></td>
<td></td>
</tr>
<tr>
<td>objects</td>
<td>$P(u) = 0$, PDE $\rightarrow$ $\emptyset$, nothing “like a PDE”</td>
<td></td>
</tr>
</tbody>
</table>
were finally answered in an existentialist way by

$$\frac{dX}{dt} = f(X(t)), X(0) = X_0$$

for all initial point $X_0$ for almost initial point $X_0$

$$\downarrow \quad \uparrow$$

**objects**

space $\mathcal{M}$

**entities**

commutative algebra of functions on $\mathcal{M}$ → noncommutative algebra of operators

$$\downarrow \quad \uparrow$$

**objects**

manifold

**entities**

$$\int_\mathbb{R} \psi(x) P(u(x) dx$$

for all functions $\psi$ for every test functions $\varphi$

$$\downarrow \quad \uparrow$$

**objects**

$P(u) = 0$, PDE

Objects disappear at the benefit of entities.

**What happened?**

We claim that one attends here to a sort of mathematization of the non-Platonism, in the sense that not only the mathematical entities are not somewhere, waiting to be discovered (as being upstairs a non existing ground floor), there are not even expressible by the standard essentialist language of mathematics available at the time of their creation. More, they influence the language into its paradoxical extremities: a non commutative manifold is stricto
senso a non sense as there is nothing in the definition of a manifold to be multiplied, commutatively or not (check on wikipedia!).

But how could the mathematics be non-Platonist, even anti-Platonist sometimes, in their own being without being Platonic when viewed by the mathematicians? More precisely, if the mathematics would do themselves the job, the job of inventing a new realism over the traditional ones, it could just mean that they are, somewhere, themselves, that they happen to be outside a thinking of mathematicians inside of which they could be only a construction.

The answer of this “paradox” lies, according to us, in the finality of these entities without objects, which is to compensate the luck of a classical, standard definition of underlying objects or to compensate their luck of ability in solving an equation. It is when they face a dead end that mathematics take off to the upper level: this realism is not a new one, it is just the right one.

The realism of an equation lies in its solutions. In the case of a nice gentle partial differential equation (PDE) the solutions are nice functions on a nice space, and it is to this space, this nice one, that they, the solutions, provide the status of realism. But for the Navier-Stokes equations, the solutions exist only in a weak sense \(23\), i.e. without a clear spatial counterpart. But forget this luck of “clear spacial counterpart”, the situation is the same, one continues to do mathematics, to compute, to estimate: realism = solutions.

In fact, what is space? It is the repository of a movement.

Do you see space ? NO.
Do you see movement? YES.

The realism belongs to the movement, identified with a nice space in the good situations, to an “almost everywhere”, a non commutative one in the bad ones: realism=movement \(24\). In the three situations examined in Section 2, realism lies in on the upper floor, isomorphic to the “natural” object for the good situations, isomorphic to itself, and only itself in the “bad” ones. This is the meaning of the north-east arrows in the diagrams.

---

23. Up to nowadays.

24. In the good cases, a solution of a PDE is “pushed” by a flow: that’s what you see on a flowing river. One identifies a stream of moving particles with a “push forward”, one identifies a flow with the solution of a PDE. The meaning of this is that one can solve some PDEs as in Section 2.3, by solving some flow equations, that is by the theory of dynamical systems as in Section 2.1. When the motion, the underlying flow is long, fast, chaotic, the movement is unbearable, impossible to see. What is left to be apprehended are some geometric features (invariants), for example eddies. Therefore, in the chaotic situations, the real space for dynamical systems consists in a set of geometrical objects quite similar to the construction on the torus of Section 2.2: a quantum (noncommutative) one. And precisely the solution of the PDE driving the quantum evolution (Schrödinger equation) conveys such a noncommutative space of invariants when looking at large values of the size of the system and time of evolution \(11\): realism=space of invariants. This last identification leads to a unified probabilistic view of quantum and classical mechanics where quantum indeterminism and classical unpredictability merge \(14, 15\).
The diagrams in page 18 do not commute, the ones in page 19 do.

**INTERMEZZO**

**Identity: last call for immediate boarding to temporality!**

Let us come back to the theme evoked in the prelude: Identity-Repetition-Seriality, or more generally and synthetically identity versus temporality. Indeed, discussing identity together with repetition-seriality seems to us to realize an attempt to look at identity versus temporality: repetition-seriality refers to identity in a temporality of repetition.

Naively, identity belongs to the ground floor and temporality (namely action, process, dynamics) seats on the upper level. We tried to show in Section 3 that temporarily possibly creates a kind of non-existing objects that we named entities. Weak solutions of PDEs, almost everywhere defined flows, noncommutative manifold are such examples of entities without clear underlying identified objects. Identified: the word refers clearly to the concept of identity, a concept that disappeared from the lower level in such situations.

But one of the goals of mathematics, under the angle we chose to look at in this article, is precisely to give an identity to these entities. Weak solutions, which appeared first as worst, lesser evil, are perfectly, nowadays, identifiable: they got by themselves their own identity.

Identity jumped form lower to upper level.

The temporality of ... nothing happens to be only a transient phenomenon, which, after taking off, creates its own identity.

4. Platonism and realism revisited

Would we say, for this would put somehow Platonism and Realism in duality:

- the object belongs to the mathematician,
- the entity to the mathematics?

What’s remarkable in the three example studied in Section 2 is that there is no a priori willing behind the disappearance of the object. Nevertheless the replacing, the “standing for” entity is very often (supposed to be) just a tool, something the ontology of which to be fixed later. And after all, it often happens that the new entity is as comfortable as the original object and the choice of the floor to seat in insignificant. More than that, it is insignificant that there is or not a lower level. The essentialist status of the ground floor
doesn’t matter (as the existence of god for existentialists in Saint Germain des Prés). A few questions arise.

Would we put therefore the realism at the entities level defined earlier? Yes, definitively, because entities are dynamical.

Isn’t there no need of talking of an (even nonexisting) underlying object? Isn’t the entities we are talking about just new objects? Is it the case in a pure abstract way? Yes, strictly at a technical level.

But nevertheless, one continues to talk about a “space”, though it disappeared: the entity replaced the object, but not quite for our mind since one continues to remember the object. We talk about a noncommutative space (non sense stricto senso as we saw before), a space defined almost everywhere, without any reference to an immutable set and a weak solution, though the equation is not stricto senso solved anymore.

Here appears again the Platonist paradox introduced in Section 3 that we can revisit now: how is it possible to set a question of Platonism inside the mathematics themselves, to criticize it, to let the mathematics decide, without being Platonist ourselves the mathematicians? After all, the mathematics could own this very Platonist property of being given first, and looking at themselves in a non-, an anti-Platonist way.

Definitively, an answer consists to overcome this difficulty by being non Platonist as mathematician, only way, for us, to let going on the creation of these non Platonic mathematics. Otherwise the realism, in the traditional sense of the word, would be everywhere present and, by then, paradoxes would start to proliferate.

More interestingly, can this conception of a form of realism strictly inside mathematics, with nothing from the traditional one to be eager to, be exported outside of mathematics? The noncommutative space is a space by the fact that topology, among other, can be defined on it. That is to say, it is its own properties, the look one has on it, the way of indirectly handling it, which transform an object into an entity, on which the realism is real.

Exporting this outside mathematics, in “real life” so to speak, would constitute a fantastic issue (25), to be added to the already long list of services rendered by mathematics to human community.

Sonate que me veux-tu ? Elle veut être écoutée.
A. Boucourechliev [6]

25. The reader can consult [10, 12, 13] for a similar attempt concerning mathematics versus music and mathematics versus quantum mechanics.
5. Synopsis

Five Key ideas (kinematics of the article)

1. Realism inside mathematics leads to the question: Platonism inside mathematics?
2. Necessity in mathematics to dynamically reinterpret objects: probably one of the lessons of the 20th century mathematics.
3. Importance in mathematics of the formalism, formalism in action: temporality.
4. Structure: extension without non extended counterpart.
5. Without object: reference to a culture, but “tradition=trahison”: the diagrams with essentialist bottoms should never commute.

Five Key (e)motions (dynamics of the article)

REALISM SEATS AT THE (UPPER) FLOOR OF OPERATIONS, NOT AT THE (LOWER) FLOOR OF OBJECTS.

⇓

SOMETIMES YOU CAN GO DOWN, SOMETIMES NOT.

⇓

THE MATHEMATICS MATHEMATIZE THIS IDEA, BY MAKING THE UPPER FLOOR PRECISE.

⇓

THE UPPER FLOOR IS THE ONE OF ENTITIES, THE GROUND FLOOR THE ONE OF OBJECTS, SOMETIMES NONEXISTING.

⇓

ENTITIES ARE IN A PROCESS OF THINKING, OF OPERATING, AND DECIDE OF THEIR OWN PLATONISM (EXISTENCE OR NOT OF UNDERLYING OBJECTS).
POSTLUDE
Mathematics versus philosophy: the other way

We believe that this discussion offers the opportunity of considering an example of interaction between mathematics and philosophy which goes “the other way”. Indeed the traditional discussion concerning Platonism involves mathematics as outside philosophy, watched by it, and looks at what they do when creativity is in action: do they invent or do they discover?

It seems to us that in the situations described in this note, and which belong to recent mathematics, everything works the other way. It is inside mathematics that is considered the question of discovering (something already existing [i.e. at the lower level]) or creating (something new [i.e. which belongs only to the upper level]): a non commutative manifold does not exist somewhere else than in the framework of “its” non commutative algebra of functions (functions ..., on the (non existing) manifold itself).

By putting mathematical Platonism in its strict interior, mathematics might have closed the debate concerning its own reality.
References


June 6, 2017

Thierry PAUL, CMLS, Ecole polytechnique, CNRS, Université Paris-Saclay, 91128 Palaiseau, Cedex, France • E-mail : thierry.paul@polytechnique.edu, http://www.cmls.polytechnique.fr/perso/paul/