DDoubling Construction for CMC Hypersurfaces in Riemannian Manifolds

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1. Introduction

Assume that $n \geq 3$ and that we are given a compact $(n+1)$-dimensional Riemannian manifold $(M, g)$ and a compact $n$-dimensional manifold $\Lambda$. We define $\mathcal{M}(M, g, \Lambda)$ to be the set of immersed hypersurfaces in $M$ which are diffeomorphic to $\Lambda$ and have their mean curvature function which is constant (it is customary to distinguish minimal hypersurfaces whose mean curvature vanishes identically from constant mean curvature hypersurfaces whose mean curvature is constant not equal to 0).

A natural question in differential geometry is to understand $\mathcal{M}(M, g, \Lambda)$. The problem of constructing constant mean curvature hypersurfaces in a given Riemannian manifold is a hard problem [12] which, to our knowledge, has not yet received any satisfactory answer in the case where the ambient manifold is arbitrary (in special geometries (e.g. when the ambient manifold is a homogeneous space) there is by now quite a vast literature even though the full understanding of all constant mean curvature hypersurfaces is far from being complete even in this simple framework).

Adapting the result of White [15], one can prove that, for a generic choice of the metric $g$ on the ambient manifold $M$, the set $\mathcal{M}(M, g, \Lambda)$ is a smooth one dimensional manifold (possibly empty) which might have infinitely many (compact or non compact) connected components. Understanding the possible degeneration of sequences of constant mean curvature surfaces with fixed topology will certainly give some information about $\mathcal{M}(M, g, \Lambda)$ and this will also provide a partial answer to the existence problem. Let us describe some works that have already been done in this direction.

In the case where $\Lambda = S^n$, given $p \in M$, a nondegenerate critical point of the scalar curvature function, Ye has shown in [14] that there exists a local foliation of a neighborhood of $p$ by constant mean curvature embedded $n$-spheres which concentrate at $p$ as their mean curvature tends to infinity. The leaves of this foliation are perturbations of geodesic balls of small radius, which are centered at $p$. As their mean curvature tends to infinity, these hypersurfaces converge to $p$. This result has been extended by Pacard and Xu to the case where the scalar curvature function of the ambient manifold does not necessarily have nondegenerate critical points in which case, the existence of embedded constant mean curvature spheres is proven for any large value of the mean curvature loosing the information about the location
of the hypersurfaces. In the special case where the ambient manifold is a 2-dimensional surface, let us mention the recent work of Schneider [13] which proves that, when \((M, g)\) is a convex 1/4-pinched surface, there exists an embedded constant geodesic curvature curve for any value of the geodesic curvature.

As another example, assume that we are given \(k = 1, \ldots, n - 1\) and a (nondegenerate) compact \(k\)-dimensional minimal submanifold \(K\) immersed in \(M\). We denote by \(SNK\), the spherical normal bundle over \(K\) in \(M\). It is shown in [8] and in [9] that there exist sequences of constant mean curvature embedding of \(SNK\) in \(M\) for large values of the mean curvature. More precisely, these hypersurfaces are small perturbations of geodesic tubes of small radius centered around \(K\) and, again, as their mean curvature tends to infinity, they converge to \(K\).

The above examples, yield the existence of constant mean curvature hypersurfaces with high value of the mean curvature. Moreover, as the mean curvature tend to infinity, these families of hypersurfaces converge to some lower dimensional submanifold of \(M\). In a completely different direction, in the present paper, we show that it is possible to describe another possible degeneration of families of constant mean curvature hypersurfaces in Riemannian manifolds. Under mild assumptions, we prove that a minimal hypersurface \(\Lambda\) immersed in a Riemannian manifold \((M, g)\) is the multiplicity 2 limit of a family of constant mean curvature hypersurfaces whose topology degenerates as their mean curvature tends to 0. The constant mean curvature hypersurfaces we construct have small mean curvature and are obtained by performing the connected sum between two copies of \(\Lambda\) at finitely many carefully chosen points.

This result generalizes some results which have been obtained in the case where the ambient manifold is homogeneous. Recall that, in \(S^{n+1}\) endowed with the standard metric, for \(p, q \geq 1\) such that \(p + q = n\) and any \(a \in (-1, 1)\), the generalized Clifford torus

\[
C_a := S^p \left( \sqrt{\frac{1+a}{2}} \right) \times S^q \left( \sqrt{\frac{1-a}{2}} \right),
\]

is a constant mean curvature hypersurface and is a minimal hypersurface when \(a\) takes the special value \(a_* := \frac{p}{q}\). Butcher and Pacard have shown in [1] and [2] that it is possible to perform the connected sum of two copies of the minimal generalized Clifford torus corresponding to \(a_*\) at finitely many points, to obtain a constant mean curvature hypersurface with mean curvature close to 0. As the mean curvature tends to 0, these constant mean curvature hypersurfaces converge to \(C_{a_*}\), with multiplicity 2. The convergence is smooth away from the points where the connected sum is performed. The choice of the points where the connected sum is performed is a crucial problem and the general idea is that the points have to be symmetrically distributed (balanced) on \(C_{a_*}\). This construction generalizes
a former construction by Ritoré in flat tori of the quotient of $\mathbb{R}^3$ by a 3-dimensional lattice [11]. It is also very closely related to a recent construction by Kapouleas and Yang [6].

2. Statement of the result

Assume that $\Lambda$ is a smooth, compact orientable, minimal hypersurface immersed in a $(n+1)$-dimensional Riemannian manifold $(M, g)$. The Jacobi operator about $\Lambda$ appears in the expression of the second variation of the area functional and is defined by

$$J_\Lambda := \Delta_\Lambda + |A_\Lambda|^2 + \text{Ric}_g(N, N),$$

where $\Delta_\Lambda$ is the Laplace-Beltrami operator on $\Lambda$, $A_\Lambda$ is the second fundamental form, $|A_\Lambda|^2$ is the square of the norm of $A_\Lambda$ (i.e. the sum of the square of the principal curvatures of $\Lambda$). Finally, $\text{Ric}_g$ denotes the Ricci tensor on $(M, g)$ and $N$ denotes a unit normal vector field on $\Lambda$. Recall that :

**Definition 2.1.** A minimal hypersurface $\Lambda$ is said to be nondegenerate if $J_\Lambda : C^{2,\alpha}(\Lambda) \to C^{0,\alpha}(\Lambda)$, is injective.

If $\Lambda$ is nondegenerate, the implicit function theorem guaranties the existence of $\varepsilon_0 > 0$ and a smooth one parameter family of immersed constant mean curvature hypersurfaces $\Lambda_\varepsilon$, for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, whose mean curvature is constant equal to $\varepsilon$. Moreover, $\Lambda_\varepsilon$ is a normal geodesic graph over $\Lambda$ for some function whose $C^{2,\alpha}$ norm is bounded by a constant times $\varepsilon$.

In this paper we are interested in the existence of other families of hypersurfaces which are close to $\Lambda$ (in the sense that they are included in a small tubular neighborhood about $\Lambda$) and which have small constant mean curvature. These constant mean curvature hypersurfaces are obtained by perturbing the connected sum of $\Lambda_\varepsilon$ and $\Lambda_{-\varepsilon}$ at finitely many carefully chosen points, in the spirit of [1], [2], [11] or [6].

Henceforth, we assume that $\Lambda$ is a nondegenerate, compact, orientable minimal hypersurface which is immersed in $M$ and we define $\phi_0$ to be the (unique) solution of

$$J_\Lambda \phi_0 = 1.$$

Our main result reads :

**Theorem 2.1.** Assume that $n \geq 3$ and that $p \in \Lambda$ is a nondegenerate critical point of $\phi_0$. Further assume that $\phi_0(p) \neq 0$. Then, there exist $\varepsilon_0 > 0$ and a one parameter family of compact, connected constant mean curvature hypersurfaces $\hat{\Lambda}_\varepsilon$, for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, which are immersed in $M$ and satisfy the following properties :

(i) The mean curvature $\hat{\Lambda}_\varepsilon$ is constant equal to $\varepsilon$ ;
(ii) Away from any given neighborhood of \( p \) in \( M \), the hypersurface \( \hat{\Lambda}_\varepsilon \) is, for \( \varepsilon \) small enough, a normal geodesic graph over a subset of the disjoint union \( \Lambda_\varepsilon \sqcup \Lambda_{-\varepsilon} \);

(iii) The hypersurface \( \hat{\Lambda}_\varepsilon \) is the connected sum of \( \Lambda_\varepsilon \) and \( \Lambda_{-\varepsilon} \) at points in \( \Lambda_\varepsilon \) and \( \Lambda_{-\varepsilon} \) which are close to \( p \).

Some comments are due. First, this result generalizes to the case where the connected sum of \( \Lambda_\varepsilon \) and \( \Lambda_{-\varepsilon} \) is performed at finitely many points which are nondegenerate critical points of the function \( \phi_0 \) and, applying this result to different choices of points, yields families of constant mean curvature hypersurfaces which are geometrically distinct.

Further informations are available. For example, as \( \varepsilon \) tends to 0
\[
\mathcal{H}^n \downarrow \hat{\Lambda}_\varepsilon \rightarrow 2\mathcal{H}^n \downarrow \Lambda,
\]
and the total curvature density
\[
|A_{\hat{\Lambda}_\varepsilon}|^n \mathcal{H}^n \downarrow \Lambda_\varepsilon \rightarrow 2|A_{\Lambda}|^n \mathcal{H}^n \downarrow \Lambda + c_n \delta_p,
\]
both in the sense of measures, where
\[
c_n := \omega_n^{-1} n^n (n-1)^{n-2} \int_R (\cosh s)^{-n} ds,
\]
is the \( L^n \)-norm of the shape form of a \( n \)-dimensional catenoid and \( \omega_{n-1} \) the volume of the unit sphere in \( \mathbb{R}^n \).

As we will see, the proof of this result is based on a perturbation argument, hence, if \( \Lambda \) is not embedded, \( \hat{\Lambda}_\varepsilon \) will not be embedded either. However, when \( \Lambda \) is embedded, the question of the embeddedness of the hypersurfaces \( \hat{\Lambda}_\varepsilon \) is addressed in the following:

**Corollary 2.1.** Assume that \( \Lambda \) is embedded and further assume that the function \( \phi_0 \) does not change sign. Then, for \( \varepsilon > 0 \) small enough, \( \hat{\Lambda}_\varepsilon \) is embedded. If \( \Lambda \) is embedded and \( \phi_0 \) changes sign, the hypersurfaces \( \hat{\Lambda}_\varepsilon \) are not embedded anymore for any \( \varepsilon > 0 \) small.

As already mentioned, the result of the present paper is very much influenced by the result of Kapouleas and Yang, Butscher and Pacard and by the result of Ritoré where similar doubling constructions are considered when the ambient manifold is either the unit sphere \( S^{n+1} \) or a quotient of \( \mathbb{R}^{n+1} \). However, it is important to realize that the doubling procedure in these papers has to take care of the presence of nontrivial Jacobi fields on the minimal hypersurfaces considered and this, in some sense, forces to work with hypersurfaces which are invariant under the action of a large group of symmetries. As a consequence, in all these papers, the connected sum has to be performed at points which are evenly distributed. In contrast, in the present paper, we implicitly assume that there are no Jacobi fields on the initial hypersurface \( \Lambda \) and this time the critical points of the function \( \phi_0 \) correspond to the points where it is possible to connect the two copies of \( \Lambda \) which have been translated along the normal using the two functions \( \pm \varepsilon \phi_0 \).
To illustrate this we consider the case where $\Lambda = S^n \subset S^{n+1}$, the Jacobi operator reads

$$J_{S^n} = \Delta_{S^n} + n,$$

and is known to be degenerate (because of the action of the group of isometries of $S^{n+1}$). Even though the Jacobi operator $J_{S^n}$ is not injective, the function $\phi_0$ is well defined and negative (since $\phi_0 \equiv -1/n$ as can be easily checked). The function $\phi_0$ being constant, it does not have any nondegenerate critical point. Even though this case does not fall into the framework of the present paper, it is still possible to find a one parameter family of embedded connected constant mean curvature hypersurfaces $\hat{\Lambda}_\varepsilon$ satisfying properties (i)-(iii) of Theorem 2.1. However this time the corresponding set of points where the connected sum can be performed is not related anymore to the critical points of the function $\phi_0$ (which is constant !) but it corresponds to the orbit of a (suitably chosen) discrete group of isometries of $S^{n+1}$. In other words, the main result of this paper applies, working equivariantly with respect to some suitable discrete group of isometries.

### 3. Examples and Comments

In view of Corollary 2.1, it seems important to understand under which condition it is possible to guarantee that the function $\phi_0$ defined in (2.1) does not change sign. Recall that :

**Definition 3.1.** A minimal hypersurface $\Lambda$ is said to be strictly stable if there exists a constant $c > 0$ such that

$$-\int_\Lambda \xi J_\Lambda \xi \, d\text{vol} \geq c \int_\Lambda \xi^2 \, d\text{vol},$$

for all $\xi \in H^1(\Lambda)$.

We have the :

**Lemma 3.1.** A minimal hypersurface $\Lambda$ is strictly stable if and only if $\phi_0$ exists and is negative.

**Proof :** We first assume that $\Lambda$ is strictly stable. Obviously this implies that $J_\Lambda$ is injective and hence $\phi_0$ is well defined. Moreover, the maximum principle implies that $\phi_0 < 0$. Conversely, let us assume that $\phi_0$, solution of $J_\Lambda \phi_0 = 1$ exists and is negative. Let $\lambda$ denote the least eigenvalue of $-J_\Lambda$ and $\psi$ be the eigenfunction associated to the eigenvalue $\lambda$. We assume that $\psi$ is normalized to be positive and have $L^2(\Lambda)$ norm equal to 1. The operator $J_\Lambda$ being self adjoint, we have

$$0 = \int_\Lambda (\psi J_\Lambda \phi_0 - \phi_0 J_\Lambda \psi) \, d\text{vol}. $$

Hence

$$\lambda \int_\Lambda \psi \phi_0 \, d\text{vol} = -\int_\Lambda \psi \, d\text{vol}. $$
This immediately implies that $\lambda > 0$ and hence we conclude that $\Lambda$ is strictly stable.

A straightforward modification of the second half of the proof of the previous Lemma yields:

**Lemma 3.2.** Assume that $\phi_0$ exists and is positive. Then $\Lambda$ is strictly unstable (in the sense that the least eigenvalue of the Jacobi operator $J_\Lambda$ is negative).

We briefly outline the plan of the paper and the construction of $\hat{\Lambda}_\varepsilon$. In section 4, we recall some well known facts about the mean curvature of hypersurfaces in Riemannian manifolds and we derive a general formula for the mean curvature of (small) normal geodesic graphs over a given hypersurface $\Lambda$. In section 5, we derive the expansion of the ambient metric in Fermi coordinates about $\Lambda$.

In section 6, we derive some precise expansions of $\Gamma_p$, Green’s function for the Jacobi operator about $\Lambda$ with pole at some given point $p \in \Lambda$. A description of the $n$-dimensional catenoid together with a description of the Jacobi fields associated to the catenoid are given in section 7.

The construction of a family of approximate solutions is performed in section 8 using the ingredients introduced in the previous sections. The idea is to consider two hypersurfaces which are the normal geodesic graphs over $\Lambda - B_r(p)$ for the functions

$$\pm (\varepsilon \phi_0 - \varepsilon^{n-1} \Gamma_p),$$

where $\varepsilon$ is a small parameter and where $p \in \Lambda$ is fixed (the radius $r_\varepsilon$ is a function of $\varepsilon$ and the parameter $\hat{\varepsilon}$ turns out to be a function of both $\varepsilon$ and the value of $\phi_0$ at the point $p$). We perform the connected sum between these two hypersurfaces using an embedded rescaled copy of the $n$-dimensional catenoid. This yields a hypersurface $\Lambda_{\varepsilon,p}$ whose mean curvature is almost constant equal to $\varepsilon + O(\varepsilon^2)$ away from the region where the catenoid is inserted.

Then, in section 9 we proceed with the analysis of the Jacobi operator about $\Lambda_{\varepsilon,p}$. To this aim, we introduce appropriate weighted spaces on $\Lambda_{\varepsilon,p}$. In section 10, we use a perturbation argument to deform $\Lambda_{\varepsilon,p}$ into a constant mean curvature hypersurface with mean curvature equal to $\varepsilon$. This last step requires a careful choice of the point $p$ where the connected sum is performed and, as we will see, this point has to be chosen to be close to a critical points of the function $\phi_0$. This will complete the proof of Theorem 2.1.

4. **The Mean Curvature**

We gather some basic material concerning the mean curvature of a hypersurface in a Riemannian manifold. All these results are well known but we feel that it makes the reading of the paper easier if we collected them here. Moreover, this will also be the opportunity to introduce some of the
notations we will use throughout the paper. We refer to [3] or [7] for further details.

We assume that $(M, g)$ is a $(n + 1)$-dimensional Riemannian manifold. Let $\Lambda$ be an oriented (possibly immersed) hypersurface in $M$. We do not necessarily assume that $\Lambda$ is a minimal hypersurface. We denote by $N$ a unit normal vector field on $\Lambda$. We denote by $\hat{N}$ a unit normal vector field on $\Lambda$ and $\hat{g}$ will denote the induced metric on $\Lambda$. The mapping

$$F(p, z) := \text{Exp}_p(z \, N(p)),$$

provides a local diffeomorphism from a neighborhood of any $(p, 0) \in \Lambda \times \mathbb{R}$ into a neighborhood of $p$ in $M$. The coordinates associated to $F$ are called Fermi coordinates.

On $\Lambda$, the metric which is induced by the ambient metric $g$, is denoted by $\hat{g}$. Recall that $\hat{h}$, the second fundamental form of $\Lambda$, is defined by

$$\hat{h}(t_1, t_2) = -\hat{g}(\nabla g t_1 N, t_2),$$

for all $t_1, t_2 \in T\Lambda$, where $\nabla g$ denotes the Riemannian connection on $M$. The mean curvature of $\Lambda$ is defined to be

$$H := \text{Tr}\hat{h},$$

and the mean curvature vector is then given by $\vec{H} := H \, N$.

For computational purposes, we recall that the mean curvature appears in the first variation of the area functional. More precisely, given $w$, a sufficiently small smooth function which is defined on $\Lambda$, we consider the surface $\Lambda_w$ which is the normal graph over $\Lambda$ for the function $w$. Namely

$$\Lambda \ni p \mapsto \vec{F}(p, w(p)) \in \Lambda_w.$$ 

We denote by $\text{Vol}(\Lambda_w)$ the volume of the hypersurface $\Lambda_w$. Then

$$D\text{Vol}(\Lambda_w)_{|w=0}(v) = -2 \int_{\Lambda} H \, v \, d\text{vol}_{\hat{g}}.$$ 

In the case where surfaces close to $\Lambda$ are parameterized as graphs over $\Lambda$ using a vector field $\vec{N}$ which is transverse to $\Lambda$ but which is not necessarily a unit normal vector field, the previous formula has to be modified. Let us denote by $\bar{\Lambda}_w$ the surface which is the graph over $\Lambda$, using the vector field $\vec{N}$, for some sufficiently small smooth function $w$. Namely

$$\Lambda \ni p \mapsto \vec{F}(p, w(p)) \in \bar{\Lambda}_w.$$ 

where

$$\vec{F}(p, z) := \text{Exp}_p(z \, \vec{N}(p)).$$

We denote by $\text{Vol}(\bar{\Lambda}_w)$ the volume of this hypersurface. The previous formula has to be changed into

$$(4.3) \quad D\text{Vol}(\bar{\Lambda}_w)_{|w=0}(v) = -2 \int_{\Lambda} (\vec{H} \cdot \vec{N}) \, v \, d\text{vol}_{\hat{g}}.$$
For all $z$ small enough, let us denote by $g_z$ the induced metric on the parallel hypersurface

$$\Lambda_z := \{ \text{Exp}_p(z N(p)) \in M : p \in \Lambda \}.$$ 

In the following result, we give the expression of the mean curvature of $\Lambda_w$ in terms of $w$ and the metric $g_z$. More precisely, we have the:

**Proposition 4.1.** The mean curvature of the surface $\Lambda_w$ is given by the formula

$$H(w) = \text{div}_{g_w} \left( \frac{\nabla^{g_w} w}{\sqrt{1 + |\nabla^{g_w} w|^2_{g_w}}} \right) - \frac{1}{2} \frac{\sqrt{1 + |\nabla^{g_w} w|^2_{g_w}}}{g_w(\nabla^{g_w} w, \nabla^{g_w} w)} \text{Tr}_{g_w} \hat{g}_w\right) - \frac{1}{2} \frac{\sqrt{1 + |\nabla^{g_w} w|^2_{g_w}}}{1 + |\nabla^{g_w} w|^2_{g_w}} \frac{1}{2} \text{Tr}_{g_w} \hat{g}_w,$$

where $\hat{g}_z := \partial_z g_z$.

**Proof:** We keep the notations introduced in the proof of Proposition 5.1. The induced metric $\hat{g}$ on $\Lambda_w$ is given by

$$\hat{g} = g_w + dw \otimes dw,$$

and hence we get

$$\det \hat{g} = (1 + |\nabla^{g_w} w|^2_{g_w}) \det g_w.$$

We can now compute the volume of $\Lambda_w$

$$\text{Vol}(\Lambda_w) = \int_{\Lambda} \sqrt{1 + |\nabla^{g_w} w|^2_{g_w}} \text{dvol}_{g_w},$$

and compute the differential of this functional with respect to $w$. In doing so, one should be careful that the function $w$ appears implicitly each time we compute $g_w$ since this tensor is evaluated at the point $F(p, w(p))$. We find

$$D_w \text{Vol}(\Lambda_w)_{|w}(v) = \int_{\Lambda} \frac{1}{\sqrt{1 + |\nabla^{g_w} w|^2_{g_w}}} g_w(\nabla^{g_w} w, \nabla^{g_w} v) \text{dvol}_{g_w}$$

$$- \frac{1}{2} \int_{\Lambda} \frac{1}{\sqrt{1 + |\nabla^{g_w} w|^2_{g_w}}} \hat{g}_w(\nabla^{g_w} w, \nabla^{g_w} w) v \text{dvol}_{g_w}$$

$$+ \frac{1}{2} \int_{\Lambda} \frac{1}{\sqrt{1 + |\nabla^{g_w} w|^2_{g_w}}} \text{Tr}_{g_w} \hat{g}_w \text{dvol}_{g_w},$$
where \( \dot{g}_z \) is the tensor whose entries is given by \((\partial_z g)_{ij}\). The first term can be integrated by parts to get

\[
D_w \text{Vol}(\Lambda_w)|_w(v) = -\int_\Lambda \text{div}_{g_w} \left( \frac{\nabla^{g_w} w}{\sqrt{1 + |\nabla^{g_w} w|_{g_w}^2}} \right) v \, dv_{g_w} \\
- \frac{1}{2} \int_\Lambda \frac{1}{\sqrt{1 + |\nabla^{g_w} w|_{g_w}^2}} \dot{g}_w(\nabla^{g_w} w, \nabla^{g_w} w) \, v \, dv_{g_w} \\
+ \frac{1}{2} \int_\Lambda \sqrt{1 + |\nabla^{g_w} w|_{g_w}^2} \text{Tr}_{g_w} \dot{g}_w \, v \, dv_{g_w}.
\]

To proceed, observe that, if \( N_w \) denotes the normal vector field about \( \Lambda_w \), we can write

\[
N_w = \frac{1}{\sqrt{1 + |\nabla^{g_w} w|_{g_w}^2}} (X_0 - \nabla^{g_w} w),
\]

and hence we get

\[
dv_{g_w} = g(N_w, X_0) \, dv_{\tilde{g}},
\]

This readily implies that

\[
D_w \text{Vol}(\Lambda_w)|_w(v) = -\int_\Lambda \text{div}_{\tilde{g}} \left( \frac{\nabla^{g_w} w}{\sqrt{1 + |\nabla^{g_w} w|_{g_w}^2}} \right) v \, g(N_w, X_0) \, dv_{\tilde{g}} \\
- \frac{1}{2} \int_\Lambda \frac{1}{\sqrt{1 + |\nabla^{g_w} w|_{g_w}^2}} \dot{g}_w(\nabla^{g_w} w, \nabla^{g_w} w) \, v \, g(N_w, X_0) \, dv_{\tilde{g}} \\
+ \frac{1}{2} \int_\Lambda \sqrt{1 + |\nabla^{g_w} w|_{g_w}^2} \text{Tr}_{g_w} \dot{g}_w \, v \, g(N_w, X_0) \, dv_{\tilde{g}}.
\]

Finally, according to (4.3), the first variation of the volume of \( \Lambda_w \) when deformed using the vector field \( v X_0 \) is given by

\[
D_w \text{Vol}(\Lambda_w)|_w(v) = -\int_\Lambda H(w) \, v \, g(N_w, X_0) \, dv_{\tilde{g}},
\]

where \( H(w) \) denotes the mean curvature of \( \Lambda_w \). The result follows from the comparison between these two formula.

5. Expansion of the metric in Fermi coordinates

Again, we assume that \( \Lambda \) is a compact orientable hypersurface immersed in \( M \) and we keep the notations introduced in the previous section. According to Gauss’ Lemma, close to \( \Lambda \), we can decompose the metric tensor \( g \) into

\[
g = g_z + dz^2.
\]

where we recall that, for \( z \) small enough, \( g_z \) is the induced metric on the parallel hypersurface

\[
\Lambda_z := \{ F(p, z) \in M : p \in \Lambda \}.
\]
It should be clear that
\[ g_{z=0} = \hat{g}, \]
is the induced metric on \( \Lambda \). We now recall the expression of the first terms in the Taylor expansion of \( g_z \) in powers of \( z \). To do this, we need to introduce some notations. Recall that \( \hat{g} \) denotes the induced metric on \( \Lambda \) and \( \hat{h} \) denotes the second fundamental form of \( \Lambda \), namely
\[ \hat{h}(t_1, t_2) := -g(\nabla^g t_1 N, t_2), \]
for all \( t_1, t_2 \in T\Lambda \), where \( \nabla^g \) denotes the Riemannian connection on \( M \). We also define the tensor \( \hat{h} \otimes \hat{h} \) by the formula
\[ (\hat{h} \otimes \hat{h}) ij = \sum_{a,b} \hat{h}_{ia} \hat{g}^{ab} \hat{h}_{bj}, \]
for all \( t_1, t_2 \in T\Lambda \). Observe that, in local coordinates on \( \Lambda \), we have
\[ (\hat{h} \otimes \hat{h}) \]
Finally, we denote by \( R \) the Riemann curvature tensor in \( M \). The next result provides a second order expansion of \( g_z \) in powers of \( z \):

**Proposition 5.1.** The following expansions hold:
\[ g_z = \hat{g} - 2z \hat{h} + z^2 \hat{k} + O(z^3), \]
where the tensor \( \hat{k} \) is defined by
\[ \hat{k} := \hat{h} \otimes \hat{h} + g(R(N, \cdot) N, \cdot). \]

**Proof:** It is easier to work in local coordinates. Given \( p \in \Lambda \), we choose geodesic normal coordinates \( x := (x_1, \ldots, x_n) \) at \( p \) and define
\[ \zeta(x) := \text{Exp}_p \left( \sum_{i=1}^{n} x_i e_i \right), \]
to be the exponential map at \( p \) on \( \Lambda \), where \( e_1, \ldots, e_n \) is an orthonormal basis of \( T\Lambda \) at \( p \). We consider the mapping
\[ \tilde{F}(x, z) := F(\zeta(x), z), \]
which is a local diffeomorphism from a neighborhood of \( 0 \in \mathbb{R}^{n+1} \) into a neighborhood of \( p \) in \( M \). The corresponding coordinate vector fields are denoted by
\[ X_0 := \tilde{F}_*(\partial_z) \quad \text{and} \quad X_j := \tilde{F}_*(\partial_{x_j}), \]
for \( j = 1, \ldots, n \). The curve \( z \mapsto \tilde{F}(x, z) \) is a geodesic and hence we have \( g(X_0, X_0) \equiv 1 \). This also implies that \( \nabla_{X_0} X_0 \equiv 0 \) and therefore, we also get the formula
\[ \partial_z g(X_0, X_j) = g(\nabla_{X_0} X_0, X_j) + g(\nabla_{X_0} X_j, X_0) = g(\nabla_{X_0} X_j, X_0). \]
The vector fields $X_0$ and $X_j$ being coordinate vector fields we have $\nabla_{X_0}X_j = \nabla_{X_j}X_0$ and we conclude that

$$2 \partial_z g(X_0, X_j) = 2 g(\nabla_{X_0}X_j, X_0) = \partial_x_j g(X_0, X_0) = 0.$$ 

Therefore, $g(X_0, X_j)$ does not depend on $z$ and, since on $\Lambda$ this quantity is 0 for $j = 1, \ldots, n$, we conclude that the metric $g$ can be written as

$$g = g_z + dz^2.$$ 

This is nothing but Gauss’ Lemma.

By definition of $\hat{g}$, we have

$$g_z = \hat{g} + O(z).$$

We now derive the next term the expansion of $g_z$ in powers of $z$. To this aim, we compute

$$\partial_z g(X_i, X_j) = g(\nabla_{X_i}X_0, X_j) + g(\nabla_{X_j}X_0, X_i) + 2g(\nabla_{X_i}X_0, \nabla_{X_j}X_0).$$

By definition of the curvature tensor, we can write

$$\nabla_{X_0}X_j = R(X_0, X_j) + \nabla_{X_j}X_0 + \nabla_{[X_0,X_j]},$$

which, using the fact that $X_0$ and $X_j$ are coordinate vector fields, simplifies into

$$\nabla_{X_0}X_j = R(X_0, X_j) + \nabla_{X_j}X_0.$$ 

Since $\nabla_{X_0}X_0 \equiv 0$, we get

$$\nabla_{X_0} \nabla_{X_j}X_0 = R(X_0, X_j)X_0.$$ 

Inserting this into (5.4) yields

$$\partial_z^2 g(X_i, X_j) = 2g(R(X_0, X_i)X_0, X_j) + 2g(\nabla_{X_i}X_0, \nabla_{X_j}X_0).$$

Evaluation at $z = 0$ gives

$$\partial_z^2 g_z |_{x=0} = 2g(R(N, \cdot)N, \cdot) + 2g(\nabla_N, \nabla N).$$ 

The formula then follows at once from Taylor’s expansion. □

Let us now derive some consequence of this result and the result of the previous section. It follows from Proposition 4.1 that the differential of $w \mapsto H(w)$ with respect to $w$ computed at $w = 0$ is given by

$$D_w H |_{w=0} = \Delta_{g_w} + \frac{1}{2} \left( \text{Tr}_{g_w}(\hat{g}_w \otimes \hat{g}_w) - \text{Tr}_{g_w} \hat{g}_w \right),$$
where \( \dot{g}_z := \partial_z g_z \) and \( \ddot{g}_z := \partial^2_z g_z \). According to Proposition 5.1, we have
\[
g_z = \dot{g} - 2 z \dot{h} + z^2 \ddot{k} + \mathcal{O}(z^3).
\]

Therefore,
\[
\frac{1}{2} \left( \text{Tr}_{g_z}(\dot{g}_z \otimes \dot{g}_z) - \text{Tr}_{g_z}(\ddot{g}_z) \right)_{|z=0} = 2 \text{Tr}_g(\dot{h} \otimes \dot{h}) - \sum_{i,j} g^{ij} g(R(N,e_i)N,e_j)
\]
\[
= \text{Tr}_g(\dot{h} \otimes \dot{h}) + \text{Ric}_g(N,N).
\]

We recover the well known fact that the Jacobi operator about \( \Lambda \) is given by
\[
J_{\Lambda} := D_{\omega} H_{|w=0} = \Delta_g + \text{Tr}_g(\dot{h} \otimes \dot{h}) + \text{Ric}_g(N,N),
\]
as expected. The quantity \( \text{Tr}_g(\dot{h} \otimes \dot{h}) \) is usually denoted by \( |A_{\Lambda}|^2 \) and is the square of the norm of the second fundamental form.

6. Green’s function for \( J_{\Lambda} \) with pole at \( p \in \Lambda \)

Assume that \( \Lambda \) is a compact orientable hypersurface immersed in \( M \). Given \( p \in \Lambda \), we define, on the punctured manifold
\[
\Lambda^p := \Lambda - \{ p \},
\]
a smooth function \( d_p \) which is positive and equal to \( \text{dist}(\cdot, p) \) in a fixed neighborhood of \( p \) (say in a geodesic ball of radius \( r_* > 0 \), where \( r_* \) can be fixed independently of the choice of \( p \in \Lambda \)). We further assume that the function \( d_p \) is built in such a way that
\[
d_p \leq \text{dist}_\Lambda(\cdot, p) \leq 2 d_p,
\]
on \( \Lambda^p \), where \( \text{dist}_\Lambda \) denotes the intrinsic distance on \( \Lambda \). We further assume that \( J_{\Lambda} \), the Jacobi operator about \( \Lambda \), is injective.

We define on \( \Lambda^p \) the conformal metric
\[
g^c_p := (d_p)^{-2} \dot{g}.
\]
Let \( \Gamma_p \) be Green’s function associated to \( J_{\Lambda} \) with pole at \( p \). Namely
\[
-J_{\Lambda} \Gamma_p = \omega_n \delta_p,
\]
where the constant \( \omega_n \) is the volume of the unit sphere in \( \mathbb{R}^n \). We have the following result which asserts that, near the point \( p \), the function \( \Gamma_p \) is close to a multiple of the function \( \dot{\Gamma}_p \) defined by
\[
\dot{\Gamma}_p := \frac{1}{n-2} (d_p)^{2-n},
\]
when \( n \geq 3 \). Indeed, in geodesic normal coordinates, the metric on \( \Lambda \) osculates the Euclidean metric up to order 2 and hence, for all \( k \geq 0 \), we find a constant \( C_k > 0 \) independent of \( p \in \Lambda \), such that
\[
|\nabla^k J_{\Lambda}(\Gamma_p - \dot{\Gamma}_p)|_{g^c} \leq C_k (d_p)^{2-n},
\]
where we use the conformal metric $\hat{g}_p$ instead of $\hat{g}$ to estimate the norm of the derivatives of a function. Observe that, using the metric $\hat{g}$, we would have gotten an estimate of the form $|\nabla^k J_\Lambda (\Gamma_p - \hat{\Gamma}_p - \gamma_p)|_{\hat{g}} \leq C_k (d_p)^{2-n-k}$ instead. In any case, this makes it possible to estimate the difference $\Gamma_p - \hat{\Gamma}_p$.

We have the:

**Lemma 6.1.** There exists a smooth function $\gamma_p$ defined on $\Lambda$ and, for all $k \geq 0$, there exists a constant $C_k > 0$, such that the function $\Gamma_p$ satisfies

$$|\nabla^k (\Gamma_p - \hat{\Gamma}_p - \gamma_p)|_{\hat{g}_p} \leq C_k \begin{cases} (d_p)^{4-n}, & \text{when } n \neq 4 \\ (1 + |\log d_p|) & \text{when } n = 4, \end{cases}$$

the estimate being uniform in $p \in \Lambda$. Moreover, when $n \geq 4$, the function $\gamma_p \equiv 0$, while $\gamma_p$ is constant (depending on $p$) when $n = 3$ and, when $n = 2$, the function $\gamma_p$ is, in a neighborhood of $p$, an affine function of the normal geodesic coordinates at $p$. Finally, $p \mapsto \gamma_p$ is smooth and bounded (in $C^\infty$ topology) uniformly in $p \in \Lambda$.

**Proof :** We choose geodesic normal coordinates $x := (x_1, \ldots, x_n)$ at $p$ and define

$$\zeta(x) := \text{Exp}_p \left( \sum_{i=1}^n x_i e_i \right),$$

to be the exponential map at $p$ on $\Lambda$, where $e_1, \ldots, e_n$ is an orthonormal basis of $T\Lambda$ at $p$. In these coordinates, the metric $\hat{g}$ on $\Lambda$ osculates the Euclidean metric up to order 2. This means that we have the expansion

$$\zeta^* \hat{g} = \sum_{i=1}^n dx_i^2 + \sum_{i,j=1}^n O(|x|^2) \, dx_i \, dx_j,$$

and, for all $k \geq 0$, we find that there exists $C_k > 0$ such that

$$|\nabla^k (J_\Lambda (\Gamma - \hat{\Gamma}))|_{\hat{g}} \leq C_k (d_p)^{2-n}.$$

We then define the function $v$ to be the (unique) solution of

$$J_\Lambda v = J_\Lambda (\Gamma - \hat{\Gamma}),$$

so that $\Gamma = \hat{\Gamma} + v$. The existence of $v$ is guaranteed by the fact that $J_\Lambda$ is assumed to be injective and also $J_\Lambda (\Gamma - \hat{\Gamma} - \chi v_0) \in L^q(\Lambda)$, for all $q < \frac{n}{n-2}$.

Now, when $n \geq 5$, we have

$$J_\Lambda (d_p)\nu = \nu (n - 2 + \nu) (d_p)^{\nu-2} + O((d_p)^\nu).$$

and hence the function $(d_p)^{4-n}$ can be used as a supersolution to prove that

$$|\Gamma - \hat{\Gamma} - \chi v_0| \leq C (d_p)^{4-n}.$$

The estimates for the derivatives follow from application of standard Schauder’s estimates [5] on concentric annuli of inner radius $r$ and outer radius $2r$ centered at $p$ and this completes the proof of the result when $n \geq 5$. 

In dimensions $n = 3$ and $4$, the previous analysis has to be modified. We fix $r_* > 0$ smaller than half of the injectivity radius of $\Lambda$ and let $\chi$ be a cutoff function identically equal to 1 in the ball of radius $r_* > 0$ centered at $p$ and identically equal to 0 away from the ball of radius $2r_*$ centered at $p$.

First observe that, expanding the metric in normal geodesic coordinates defined by $\zeta$, we can write the more precise expansion

$$\zeta^* \hat{g} = \sum_{i=1}^{n} dx_i^2 + \sum_{i,j,k,\ell=1}^{n} a_{ijk\ell} x_k dx_i dx_j + \sum_{i,j=1}^{n} O(|x|^3) dx_i dx_j,$$

where $a_{ijk\ell}$ are constants depending on $p$. This implies that, there exist $P \in \mathbb{R}[X_1, \ldots, X_n]$ a homogeneous polynomial of degree 2 and, for all $k \geq 0$, a constant $C_k > 0$ such that

$$|\nabla^k (J_\Lambda (\Gamma - \hat{\Gamma}) \chi_{w_0})|_{\hat{g}_p} \leq C_k (dp)^{3-n}.$$

where $w_0$ is defined by the identity

$$\zeta^* w_0 := |x|^{-n} P(x).$$

We decompose

$$P(x) = a|x|^2 + P_h(x),$$

where $a \in \mathbb{R}$ and $P_h$ is harmonic and homogeneous of degree 2 and we define

$$Q(x) := \begin{cases} \frac{a}{2} |x|^2 - \frac{1}{4} P_h(x), & \text{when } n = 3 \\ \frac{a}{2} \log |x| - \frac{1}{8} P_h(x) & \text{when } n = 4, \end{cases}$$

and hence $Q$ involves log terms when $n = 2$ and $n = 4$. Observe that, by construction, we have

$$\Delta (|x|^{2-n} Q(x)) = |x|^{-n} P(x),$$

where $\Delta$ is the Laplacian in $\mathbb{R}^n$. Then, for all $k \geq 0$, there exists $C_k > 0$ such that

$$|\nabla^k (J_\Lambda (\Gamma - \hat{\Gamma} - \chi v_0))|_{\hat{g}_p} \leq C_k (dp)^{3-n},$$

where the function $v_0$ is defined by the identity

$$\zeta^* v_0 := |x|^{-n} Q(x).$$

We define the function $v$ to be the (unique) solution of

$$J_\Lambda v = J_\Lambda (\Gamma - \hat{\Gamma} - \chi v_0),$$

so that $\Gamma = \hat{\Gamma} + v_0 + v$. Again, the existence of $v$ is guarantied by the fact that $J_\Lambda$ is assumed to be injective and also that $J_\Lambda (\Gamma - \hat{\Gamma} - \chi v_0) \in L^q(\Lambda)$, for all $q < \frac{n}{n-3}$.

When $n = 3$, then $J_\Lambda (\Gamma - \hat{\Gamma} - \chi v_0) \in L^q(\Lambda)$ for all $q > 1$ and hence we conclude that $v \in C^{1,\alpha}(\Lambda)$. The result then follows by taking $\gamma = v(p)$. Finally, when $n = 4$, then $J_\Lambda (\Gamma - \hat{\Gamma} - \chi v_0) \in L^q(\Lambda)$ for all $q \in (1,4)$ and
hence we conclude that $v \in C^{0,\alpha}(\Lambda)$ and the result then follows by taking $\gamma = 0$. 

7. The $n$-dimensional catenoid

The $n$-dimensional catenoid is a minimal hypersurface of revolution which is embedded in $\mathbb{R}^{n+1}$ and which can be parameterized by

$$X : \mathbb{R} \times S^{n-1} \rightarrow \mathbb{R}^{n+1}$$

$$(s, \Theta) \mapsto (\varphi(s) \Theta, \psi(s)),$$

where the functions $\varphi$ and $\psi$ are given explicitly by

$$\varphi(s) := (\cosh((n-1)s))^{\frac{1}{n-1}},$$

and

$$\psi(s) := \int_{0}^{s} \varphi^{2-n} \, dt.$$ 

When $n \geq 3$, it will be convenient to define

$$\psi_\infty := \int_{0}^{+\infty} \varphi^{2-n} \, dt.$$ 

It is easy to check that the hypersurface $C$ parameterized by $X$ is a minimal hypersurface embedded in $\mathbb{R}^{n+1}$ [4]. Indeed, the induced metric on $C$ is given by

$$\bar{g} := \varphi^{2} (ds^{2} + g_{S^{n-1}}),$$

and, if the normal vector field is chosen to be

$$\bar{N} := (-\varphi^{1-n} \Theta, \partial_{s} \log \varphi),$$

one checks that the second fundamental form on $C$ in given by

$$\bar{h} = \varphi^{2-n} \left( (n-1) ds^{2} - g_{S^{n-1}} \right).$$

It is then straightforward to check that the mean curvature of $C$ is given by

$$H = \text{Tr}_{\bar{g}} \bar{h} \equiv 0,$$

and hence this shows that $C$ is a minimal hypersurface. Moreover, we also get the expression of the Jacobi operator about $C$ in these coordinates

$$\bar{J} := \frac{1}{\varphi^{2}} \left( \frac{1}{\varphi^{n-2}} \partial_{s}(\varphi^{n-2} \partial_{s}) + \Delta_{g_{S^{n-1}}} + \frac{n(n-1)}{\varphi^{2n-2}} \right).$$

Of interest to us will be the classification of Jacobi fields for the operator $\bar{J}$ (namely the set of functions $V$ solutions of $\bar{J}V = 0$). We have the:

**Lemma 7.1.** The only Jacobi fields of $\bar{J}$ which tend to 0 at infinity are linear combinations of the functions

$$f_{j}(s, \Theta) := \varphi^{1-n}(s) \Theta \cdot e_{j},$$

for $j = 1, \ldots, n$, where $e_{1}, \ldots, e_{n}$ is an orthonormal basis of $\mathbb{R}^{n}$. 
Proof. Assume that we have a function \( V \in C^2(\mathbb{R} \times S^{n-1}) \) which solves \( J V = 0 \).

We decompose the function \( V \) over the spherical harmonics, namely
\[
V(s, \Theta) = \sum_{j=0}^{+\infty} V_j(s, \Theta),
\]
where
\[
-\Delta_{S^{n-1}} V_j = j (n - 2 + j) V_j.
\]
It is easy to check that each \( V_j \) is a solution of the second order ordinary differential equation
\[
(7.5) \quad \left( \varphi^{2-n} \partial_s (\varphi^{n-2} \partial_s) - j (n - 2 - j) + n (n - 1) \varphi^{2-2n} \right) V_j = 0.
\]

When \( j = 1 \), there exists an explicit solution of this second order ordinary differential equation which is given by \( \varphi^{1-n} \). This can be checked through either an explicit computation or using the fact that the function \( w_j + := \bar{N} e_j = \varphi^{1-n} \Theta \cdot e_j \), which is the scalar product between the normal vector field to \( C \) and the constant vector field \( e_j \), is a Jacobi field associated to the invariance of the mean curvature equation under horizontal translations. The function \( \varphi^{1-n} \) being positive, it can be used as a barrier to prove that \( (7.5) \) has no decaying solution when \( j \geq 2 \). Indeed, since \( \varphi \sim \cosh s \) at \( \pm \infty \), we see that the asymptotic behavior of any solution of \( (7.5) \) at \( +\infty \) is dictated by the asymptotic behavior at \( +\infty \) of the solutions of
\[
(\partial_s^2 + (n - 2) \partial_s - j (n - 2 - j)) w = 0.
\]
Therefore, at \( +\infty \), any solution of \( (7.5) \) either tends to 0 like \( e^{(2-n-j)s} \) or blows up like \( e^{js} \). Similarly, at \( -\infty \), any solution of \( (7.5) \) either decays to 0 like \( e^{(n-2+j)\bar{s}} \) or blows up like \( e^{-js} \). When \( j \geq 2 \), this implies that any bounded solution of \( (7.5) \) is in fact bounded by \( \varphi^{-n} \). The function \( \varphi^{1-n} \) can then be used as a barrier to show that there are no bounded solutions of \( (7.5) \) when \( j \geq 2 \).

When \( j = 1 \), we already know one solution of \( (7.5) \) and this solution decays both at \( \pm \infty \). This implies that the other independent solution of \( (7.5) \) blow up at \( \pm \infty \) like \( \varphi \). In fact, all solutions of \( (7.5) \) are explicitely known and are given by
\[
w^+_i := \varphi^{1-n} \Theta \cdot e_i \quad \text{and} \quad w^\_i := (\partial_s \varphi + \psi \varphi^{1-n}) \Theta \cdot e_i,
\]
for \( i = 1, \ldots, n \), where \( e_1, \ldots, e_n \) is an orthonormal basis of \( \mathbb{R}^n \). As already explained the former arise from the invariance of the mean curvature under horizontal translation while the latter arise from the invariance of the mean curvature under rotations.
When \( j = 0 \), the two independent solutions of (7.5) are also explicitly known and they are given by
\[
\begin{align*}
    w_0^+ &= -\partial_s \log \varphi \\
    w_0^- &= \psi \partial_s \log \varphi - \varphi^{2-n}.
\end{align*}
\]
It is easy to check that no linear combination of \( w_0^\pm \) decays at \( \pm \infty \). This time the former arises from the invariance of the mean curvature equation under vertical translation while the latter arise from the invariance of the mean curvature equal zero equation under dilations. \( \square \)

To complete this section, we provide a crucial expansion of the lower end of the catenoid. Indeed, one can perform the change of variable,
\[
x = \varphi(s) \Theta \in \mathbb{R}^n,
\]
for all \( s > 0 \) and \( \Theta \in S^{n-1} \). Then, the \( n \)-dimensional catenoid can be viewed as a double graph over the horizontal hyperplane in \( \mathbb{R}^{n+1} \). In particular, the lower end of the catenoid is the vertical graph for a function \( u_c \) defined outside the unit ball in \( \mathbb{R}^n \). Moreover, at infinity, the function \( u_c \) can be expanded as
\[
u_c(x) = -\psi_\infty + \frac{1}{n-2} |x|^{2-n} + O(|x|^{4-3n}),
\]
when \( n \geq 3 \). Similar estimates hold for the derivatives of \( u_c \) where a power of \( |x| \) is lost each time we take a derivative of \( u_c \).

8. Construction of the approximate solution

In this section, given \( p \in \Lambda \) such that \( \phi_0(p) \neq 0 \), and given \( \varepsilon > 0 \) small enough, we define a hypersurface \( \Lambda_{\varepsilon,p}^0 \) which is close to the constant mean curvature hypersurface we are looking for. First, we choose
\[
r_\varepsilon := \varepsilon^{1/2}.
\]
To fix the ideas, we assume that \( \phi_0 < 0 \) (this is for example the case when \( \Lambda \) is assumed to be stable). In dimension \( n \geq 3 \), we construct \( \Lambda_{\varepsilon,p}^\pm \) as follows. Away from the geodesic ball of radius \( r_\varepsilon/2 \) centered at \( p \), we consider two normal graphs over \( \Lambda \). One of them will be denoted by \( \Lambda_{\varepsilon,p}^+ \) and is the normal graph over \( \Lambda - B(p, r_\varepsilon/2) \) for the function
\[
w_\varepsilon^+ := -\varepsilon \phi_0 - \varepsilon^{n-1} \Gamma,
\]
while the other one will be denoted by \( \Lambda_{\varepsilon,p}^- \) and is the normal graph over \( \Lambda - B(p, r_\varepsilon/2) \) for the function
\[
w_\varepsilon^- := \varepsilon \phi_0 + \varepsilon^{n-1} \Gamma.
\]
The constant \( \varepsilon \) which appears in these two expressions will be defined soon, and it will be seen to satisfy \( \varepsilon \sim \varepsilon \).
Remark 8.1. When $\phi_0(p) > 0$, the above formula have to be changed into

$$w^\pm_\varepsilon := \pm \varepsilon \phi_0 - \hat{\varepsilon}^{n-1} \Gamma,$$

respectively.

Finally, we consider the $n$-dimensional catenoid which is scaled by a factor $\hat{\varepsilon}$ and which is embedded in $M$ as follows. As in the proof of Lemma 6.1, close to $p$, we consider the Fermi coordinates introduced above and, to parameterize $\Lambda$ near $p$, we consider geodesic normal coordinates $x := (x_1, \ldots, x_n)$ at $p$ and define

$$\zeta(x) = \text{Exp}_p \left( \sum_{i=1}^{n} x_i e_i \right),$$

to be the exponential map at $p$ on $\Lambda$. Here $e_1, \ldots, e_n$ is an orthonormal basis of $T\Lambda$ at $p$.

The embedded rescaled catenoid, which will be denoted by $C_\varepsilon$ is just defined to be the image of

$$\hat{F}_\varepsilon(s, \Theta) := F(\zeta(\hat{\varepsilon} \varphi(s) \Theta), \hat{\varepsilon} \psi(s)),$$

for $\Theta \in S^{n-1}$ and $|s| \leq \log(r_\varepsilon/\varepsilon) + c$, for some constant $c > 0$ fixed large enough independent of $\varepsilon$ so that

$$\hat{\varepsilon} \varphi(\log(r_\varepsilon/\varepsilon) + c) \geq 2 r_\varepsilon.$$

For the sake of simplicity, we assume that $n \geq 5$, minor modifications are needed to handle the case $n = 3$ and 4. Observe that on the annulus of inner radius $r_\varepsilon/2$ and outer radius $2r_\varepsilon$, centered at $p$, $\Lambda^-_{\hat{\varepsilon},p}$ is the normal graph for a function $w^-_\varepsilon$ which can be expanded as

$$w^-_\varepsilon(x) = \varepsilon \phi_0(p) + \varepsilon \nabla \phi_0(p) \cdot x + \hat{\varepsilon}^{n-1} \frac{n}{n-2} |x|^{2-n} + O(\varepsilon^2),$$

while, the lower end of $C_\varepsilon$ is the normal graph for a function $v^-_\varepsilon$ which can be expanded as

$$v^-_\varepsilon(x) = -\hat{\varepsilon} \psi_\infty + \hat{\varepsilon}^{n-1} \frac{n}{n-2} |x|^{2-n} + O(\varepsilon^2).$$

In view of these expansions, it is reasonable to ask that the constants $\varepsilon$ and $\hat{\varepsilon}$ are related by

$$-\varepsilon \phi_0(p) = \hat{\varepsilon} \psi_\infty,$$

so that the constant terms in both expansions agree. Observe that this makes sense since we have assumed that $\phi_0(p) < 0$. Also observe that $\hat{\varepsilon} \sim \varepsilon$ as promised. The hypersurface $\Lambda^\pm_{\hat{\varepsilon},p}$ is then obtained by gluing together the different pieces we have just described using some appropriate cutoff function which passes from 0 to 1 on the annulus of inner radius $r_\varepsilon/2$ and outer radius $2r_\varepsilon$ which is centered at $p$ and, without loss of generality, whose $k$-th partial derivatives are bounded by a constant (depending on $k$) times $\varepsilon^{-k/2}$. 
The Jacobi operator about $\Lambda^\pm_{\epsilon,p}$ is denoted by $J_{\epsilon}$. The next section is devoted to the analysis of this operator acting on appropriately defined weighted spaces.

9. Analysis of the Jacobi operator about $\Lambda^\pm_{\epsilon,p}$

9.1. An a priori estimate. We define a weight function $W_{\epsilon}$ as follows:

Away from the gluing region (namely the region where the embedded rescaled catenoid $C_{\epsilon}$ is glued to $\Lambda^\pm_{\epsilon,p}$), we set

$$W_{\epsilon} := d_{p},$$
onumber

on the part of $\Lambda_{\epsilon}$ which agrees with $\Lambda^\pm_{\epsilon,p}$ and

$$W_{\epsilon} := \epsilon \varphi,$$

on the part of $\Lambda^b_{\epsilon,p}$ which agrees with $C_{\epsilon}$. In the gluing region, we just interpolate smoothly between these two expressions. Again, $W_{\epsilon}$ depends on $p$ even though, to allay notations, we have chosen not to make this apparent.

We can certainly arrange things in such a way that the partial derivatives of $W_{\epsilon}$ are controlled, namely that, for all $k \geq 0$, there exists $C_k > 0$ such that $|\nabla^k W_{\epsilon}| \leq C_k W_{\epsilon}^k$.

Let $g^b_{\epsilon}$ be the induced metric on $\Lambda^b_{\epsilon,p}$. It will be convenient to define the conformal metric

$$g^\#_{\epsilon} := W_{\epsilon}^{-2} g^b_{\epsilon}.$$

In terms of this metric, we define the Hölder weighted norm of a function $w$ defined on $\Lambda^b_{\epsilon,p}$, by

$$\|w\|_{C^{k,\alpha}_{g^\#_{\epsilon}}(\Lambda^b_{\epsilon,p})} := \|W_{\epsilon}^{-\alpha} w\|_{C^{k,\alpha}(\Lambda^b_{\epsilon,p}, g^b_{\epsilon})},$$

namely, on the right hand side, the Hölder norm is computed with respect to the metric $g^\#_{\epsilon}$ and not $g^b_{\epsilon}$.

We will work in the space of functions which satisfy some orthogonality conditions we now describe. We fix a cutoff function $\chi$ which is identically equal to 0 on the pieces of $\Lambda^b_{\epsilon,p}$ which is in the graph of $w_{\epsilon}^\pm$ over the complement of a ball of radius $r_0$ centered at $p$ and $\chi$ is designed to be identically equal to 1 on the piece of of $\Lambda^b_{\epsilon,p}$ which is not in the graph of $w_{\epsilon}^\pm$ over the complement of a ball of radius $r_0/2$ centered at $p$.

We will ask that the functions we are considering satisfy

$$\int_{\Lambda^\#_{\epsilon,p}} w \hat{f}_j \text{dvol}_{g_{\#_{\epsilon}}} = 0,$$

for $j = 1, \ldots, n$, where

$$F^b * \hat{f}_j := \varphi^{-2} f_j F^b * \chi,$$

where $F^b$ is a parameterization of the part of $\Lambda^b_{\epsilon,p}$ which can be written as a normal graph over $C_{\epsilon}$ and where $f_j$ has been defined in Lemma 7.1. For later use, we also define the function $\hat{f}_j$ by

$$F^\# * \hat{f}_j := f_j F^\# * \chi.$$
Using this, we have the:

**Proposition 9.1.** Assume that $n \geq 3$ and $\nu \in (2-n, 0)$ are fixed. Then, there exists $\varepsilon_0 > 0$ and $C > 0$ such that, for all $p \in \Lambda$ and for all $\varepsilon \in (0, \varepsilon_0)$,

$$\|w\|_{C^{2,\alpha}_\nu(L^2_{\varepsilon,p})} \leq C \|J_\varepsilon w\|_{C^{2,\alpha}_\nu(L^2_{\varepsilon,p})}^2,$$

provided $w$ satisfies (9.6).

**Proof.** Observe that, by Schauder's estimates, it is enough to prove that, provided $\varepsilon$ is small enough,

$$\|W_{\varepsilon} - \nu J_\varepsilon w\|_{L^\infty(L^2_{\varepsilon,p})} \leq C \|W_{\varepsilon}^2 - \nu J_\varepsilon w\|_{L^\infty(L^2_{\varepsilon,p})},$$

for some constant $C > 0$ which is independent of $\varepsilon$. The proof of this last estimate is by contradiction. Assume that the result is not true. Then, there would exist a sequence $(\varepsilon_j)_{j \geq 0}$ tending to 0, a sequence of points $(p_j)_{j \geq 0}$, with $p_j \in L^2_{\varepsilon_j,p_j}$, and a sequence of functions $(w_j)_{j \geq 0}$ such that

$$\|W_{\varepsilon_j} - \nu J_{\varepsilon_j} w_j\|_{L^\infty(L^2_{\varepsilon_j,p_j})} = 1, \quad (9.7)$$

and

$$\lim_{j \to \infty} \|W_{\varepsilon_j}^2 - \nu J_{\varepsilon_j} w_j\|_{L^\infty(L^2_{\varepsilon_j,p_j})} = 0.$$

We choose a point $q_j \in L^2_{\varepsilon_j,p_j}$ where (9.7) is achieved and we distinguish three cases according to the behavior of $\text{dist}_M(p_j, q_j)$.

**Case 1.** Assume that, for some subsequence, $\text{dist}_M(q_j, p_j)$ remains bounded away from 0 and $(p_j)_{j \geq 0}$ converges to some point $p \in M$. Then, invoking elliptic estimates together with Ascoli-Arzela’s Theorem, one can extract subsequences, pass to the limit in the equation satisfied by $w_j$ and one ends up with a function $w_\infty$ which is a solution of

$$J_\Lambda w_\infty = 0,$$

in $\Lambda - \{p\}$. Moreover, passing to the limit in (9.7), we get

$$\|(d_{p})^{-\nu} w_\infty\|_{L^\infty(\Lambda)} = 1.$$

Since we have chosen $\nu > 2 - n$, one can see that the singularity at $p$ is removable and hence $w_\infty$ is the in kernel of $J_\Lambda$. Since we have assumed that $\Lambda$ was nondegenerate, this implies that $w_\infty \equiv 0$ which is a contradiction.

**Case 2.** Assume that, for some subsequence, $\text{dist}_M(q_j, p_j)$ tends to 0 and also that $\text{dist}_M(q_j, p_j)/\varepsilon_j$ tends to infinity. Then, one can use the exponential map at $p_j$ to dilate the coordinates by a factor $1/\text{dist}_M(q_j, p_j)$. Invoking elliptic estimates together with Ascoli-Arzela’s Theorem, one can extract subsequences, pass to the limit in the equation satisfied by $w_j$ (in stretched coordinates) and one ends up with a function $w_\infty$ which is defined in $\mathbb{R}^n - \{0\}$ and which is a solution of

$$\Delta_{\mathbb{R}^n} w_\infty = 0,$$
in $\mathbb{R}^n - \{0\}$. Moreover, (9.7) implies that
\[ \| |x|^{-\nu} w_\infty \|_{L^\infty(\mathbb{R}^n)} = 1. \]
Again, since we have chosen $\nu > 2-n$, the singularity at 0 is removable and hence $w_\infty$ is harmonic. Since we have assumed that $\nu < 0$, this implies that $w_\infty$ is bounded and hence $w_\infty \equiv 0$ which is a contradiction.

**Case 3.** Assume that, for some subsequence, $\text{dist}_M(q_j, p_j)/\varepsilon_j$ remains bounded. Then, one can use the exponential map at $p_j$ to dilate coordinates by a factor $1/\text{dist}_M(q_j, p_j)$. Invoking elliptic estimates together with Ascoli-Arzela’s Theorem, one can extract subsequences, pass to the limit in the equation satisfied by $w_j$ (in stretched coordinates) and one ends up with a function $w_\infty$ which is defined on the catenoid $C$ and which is a solution of
\[ \mathcal{J} w_\infty = 0, \]
in $C$. Moreover,
\[ \| \varphi^{-\nu} w_\infty \|_{L^\infty(C)} = 1, \]
and, passing to the limit in (9.6), we also conclude that
\[ \int_C w_\infty \varphi^{-2} f_j \, \text{dvol}_\mathcal{g} = 0, \]
for $j = 1, \ldots, n$, where the function $f_j$ is the one which has been defined in Lemma 7.1. Since we have chosen $\nu < 0$, the result of Lemma 7.1 implies that $w_\infty \equiv 0$ which is a contradiction.

Having ruled out every case, this completes the proof of the result. \qed

Thanks to the previous result, we can prove the:

**Proposition 9.2.** Assume that $n \geq 3$ and that $\nu \in (-1, 0)$ is fixed. There exists $\varepsilon_0 > 0$ and $C > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, for all $p \in \Lambda$ and for all $f \in C^{0, \alpha}(\Lambda^\varepsilon, p)$, there exists $w \in C^{2, \alpha}(\Lambda^{\varepsilon, p})$, solution of
\[ J_\varepsilon w + \sum_{i=1}^n a_i \hat{f}_i = f. \]
In addition, the mapping $f \mapsto w$ is linear and
\[ \| w \|_{C^{2, \alpha}(\Lambda^{\varepsilon, p})} \leq C \| f \|_{C^{0, \alpha}(\Lambda^{\varepsilon, p})}, \]
for some constant $C > 0$ which does not depend on $\varepsilon$ nor on $p \in \Lambda$.

**Proof.** We start with some preliminary estimate and then prove the existence of the solution.

**Step 1.** Assume that the functions $w$, $f$ and the real numbers $a_j$ satisfy
\[ J_\varepsilon w + \sum_{i=1}^n a_i \hat{f}_i = f. \]
We multiply this equation by \( \bar{f}_j \) and integrate the result by parts over \( \Lambda_\varepsilon \) to obtain
\[
\sum_{i=1}^n a_i \int_{\Lambda_\varepsilon, p} \hat{f}_i \bar{f}_j \, d\text{vol}_g^\varepsilon = \int_{\Lambda_\varepsilon, p} f \bar{f}_j \, d\text{vol}_g^\varepsilon - \int_{\Lambda_\varepsilon, p} w J_\varepsilon \bar{f}_j \, d\text{vol}_g^\varepsilon.
\]
Direct estimates, imply that
\[
\int_{\Lambda_\varepsilon, p} \hat{f}_i \bar{f}_j \, d\text{vol}_g^\varepsilon \geq C \varepsilon^n,
\]
and, using the fact that the volume form \( d\text{vol}_g^\varepsilon \) is close to \( \varepsilon^n d\text{vol}_{\bar{g}} \), where \( \bar{g} \) is the induced metric on the catenoid embedded in Euclidean space, we also get
\[
\left| \int_{\Lambda_\varepsilon, p} \hat{f}_i \bar{f}_j \, d\text{vol}_g^\varepsilon \right| \leq C \varepsilon^{n+1},
\]
when \( i \neq j \). Here we have used the fact that this quantity would be equal to 0 if we were working in Euclidean space and integrating over the catenoid, namely
\[
\int_C \varphi^{-2} f_i \bar{f}_j \, d\text{vol}_{\bar{g}} = 0.
\]
Moreover, we have
\[
\left| \int_{\Lambda_\varepsilon, p} f \bar{f}_j \, d\text{vol}_g^\varepsilon \right| \leq C \varepsilon^{n-2+\nu} \| f \|_{C^2,0}^{(n-2)}(\Lambda_\varepsilon),
\]
and, it is easy to evaluate \( J_\varepsilon \bar{f}_j \), to find
\[
\left| \int_{\Lambda_\varepsilon, p} w J_\varepsilon \bar{f}_j \, d\text{vol}_g^\varepsilon \right| \leq C \varepsilon^{n-1+\nu} \| w \|_{C^2,0}^{(n-2)}(\Lambda_\varepsilon,p).
\]
Therefore, we conclude that
\[
|a_j| \leq C \left( \varepsilon^{\nu-1} \| w \|_{C^2,0}^{(n-2)}(\Lambda_\varepsilon, p) + \varepsilon^{\nu-2} \| f \|_{C^2,0}^{(n-2)}(\Lambda_\varepsilon, p) \right),
\]
Finally, using the fact that
\[
\| \hat{f}_j \|_{C^{0,\alpha}_{n-2}(\Lambda_\varepsilon, p)} \leq C \varepsilon^{2-\nu},
\]
(This is in this estimate that the factor \( \varphi^{-2} \) in the definition of \( \hat{f}_j \) and the restriction \( \nu \in (-1, 0) \) are needed when \( n = 2 \)).

These estimates, together with the result of Proposition 9.1, we conclude that
\[
\| w \|_{C^2,0}^{(n-2)}(\Lambda_\varepsilon, p) \leq C \left( \varepsilon \| w \|_{C^2,0}^{(n-2)}(\Lambda_\varepsilon, p) + \| f \|_{C^2,0}^{(n-2)}(\Lambda_\varepsilon, p) \right),
\]
and, hence
\[
\| w \|_{C^2,0}^{(n-2)}(\Lambda_\varepsilon, p) \leq C \| f \|_{C^2,0}^{(n-2)}(\Lambda_\varepsilon, p),
\]
for all \( \varepsilon \) small enough.
Step 2. We define $\Pi$ to be the $L^2$-projection over the orthogonal complement of $\text{Span}\{\bar{f}_j : j = 1, \ldots, n\}$. The operator
$$\tilde{J}_\varepsilon := \Pi \circ J_\varepsilon \circ \Pi,$$
is elliptic and self adjoint. We claim that it is injective for all $\varepsilon$ small enough. Indeed, if $\tilde{J}_\varepsilon v = 0$, then, $w := \Pi v$ satisfies
$$J_\varepsilon w + \sum_{j=1}^{n} a_j \hat{f}_j = 0,$$
for some $a_j \in \mathbb{R}$. Using the previous estimate, we conclude that $v \equiv 0$ and hence $\tilde{J}_\varepsilon$ is injective. The existence of $w$ then follows from Fredholm alternative and this completes the proof of the result. \qed

10. The nonlinear argument

Building on the previous analysis, we would like to perturb $\Lambda^\varepsilon_{x,p}$ to get a constant mean curvature surface with mean curvature equal to $\varepsilon$. We look for a normal geodesic graph over $\Lambda^\varepsilon_{x,p}$ for some small function $w \in C^2_{\nu,\alpha}(\Lambda^\varepsilon_{x,p})$ and expand the mean curvature of this graph as
$$H_\varepsilon(w) := H_\varepsilon + J_\varepsilon w + Q_\varepsilon(w),$$
where $H_\varepsilon$ denotes the mean curvature of $\Lambda^\varepsilon_{x,p}$, $J_\varepsilon$ denotes the Jacobi operator about $\Lambda^\varepsilon_{x,p}$ and $Q_\varepsilon$ collects all nonlinear terms the Taylor’s expansion of $H_\varepsilon(w)$.

The equation we would like to solve reads
$$H_\varepsilon(w) = \varepsilon.$$ 
In view of the result of the previous section, we are going to look for a function $w$ which satisfies (9.6) and which solves

$$J_\varepsilon w + \sum_{i=1}^{n} a_i \bar{f}_i := (\varepsilon - H_\varepsilon) - Q_\varepsilon(w). \tag{10.8}$$

We choose $n \geq 3$ and $\nu \in (-1/3,0)$. We have the :

**Proposition 10.1.** There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0,\varepsilon_0)$, there exists a unique solution $w_\varepsilon$ of (10.8) and (9.6). Moreover,
$$\|w_\varepsilon\|_{C^2_{\nu,\alpha}(\Lambda^\varepsilon_{x,p})} \leq C \varepsilon^{\frac{3-\nu}{2}},$$
for some constant $C > 0$.

**Proof.** Direct estimates using the result of Lemma 11.2 imply that there exists a constant $C > 0$ such that
$$\|H_\varepsilon - \varepsilon\|_{C^0_{\nu,\alpha}(\Lambda^\varepsilon_{x,p})} \leq C(\varepsilon^2 + \varepsilon r_\varepsilon^{1-\nu} + \varepsilon^{3n-3} r_\varepsilon^{4-3n-\nu}) \leq C \varepsilon^{\frac{3-\nu}{2}}.$$
Moreover, using the result of Proposition 4.1, it is easy to check that
$$\|Q_\varepsilon(w_2) - Q_\varepsilon(w_1)\|_{C^0_{\nu,\alpha}(\Lambda^\varepsilon_{x,p})} \leq C \varepsilon^{1+\nu} \|w_2 - w_1\|_{C^0_{\nu,\alpha}(\Lambda^\varepsilon_{x,p})}.$$
provided \( \| w \|_{C^{2,\alpha}(\Lambda^\flat_p)} \leq C \varepsilon^{\frac{3-\nu}{2}} \). The result then follows from the application of a fixed point theorem for contraction mapping.

Observe that, reducing \( \varepsilon_0 \) if this is necessary, we can assume that the solution \( w_\varepsilon \) and the coefficients \( a_1, \ldots, a_n \) depend continuously (and in fact smoothly) on the parameter \( p \in \Lambda \), the proof of this fact offers no difficulty. Summarizing what we have obtained so far, we conclude that the hypersurface \( \Lambda^\flat_{\varepsilon,p} \) which is the normal graph over \( \Lambda_{p,\varepsilon} \) for the function \( w_\varepsilon \) has mean curvature given by

\[
\hat{H}_\varepsilon = \varepsilon + \sum_{j=1}^n a_j \hat{f}_j.
\]

In order to prove Theorem 2.1, it is enough to show that one can choose the point \( p \in \Lambda \) in such a way that \( a_1 = \ldots = a_n = 0 \).

**Proof of Theorem 2.1.** We multiply (10.8) by \( \hat{f}_j \) and integrate the result by parts. The equation \( a_1 = \ldots = a_n = 0 \) reduces to the system

\[
\int_{\Lambda^\flat_{\varepsilon,p}} \hat{f}_j (\varepsilon - H_\varepsilon) \, \text{dvol}_{g_\varepsilon} = \int_{\Lambda_{\varepsilon,p}} w_\varepsilon J_\varepsilon \hat{f}_j \, \text{dvol}_{g_\varepsilon} + \int_{\Lambda^\flat_{\varepsilon,p}} \hat{f}_j Q_\varepsilon(w_\varepsilon) \, \text{dvol}_{g_\varepsilon}.
\]

We estimate each term in turn. First, we have the following easy estimates which follow from the proof of Proposition 9.2

\[
\left| \int_{\Lambda^\flat_{\varepsilon,p}} w_\varepsilon J_\varepsilon \hat{f}_j \, \text{dvol}_{g_\varepsilon} \right| \leq C \varepsilon^{n+1+\frac{\nu}{2}},
\]

and

\[
\left| \int_{\Lambda^\flat_{\varepsilon,p}} \hat{f}_j Q_\varepsilon(w_\varepsilon) \, \text{dvol}_{g_\varepsilon} \right| \leq C \varepsilon^{n+1+3\nu}.\]

Finally, we have the

**Lemma 10.1.** There exists a constant \( c_n > 0 \) such that

\[
\left| \int_{\Lambda^\flat_{\varepsilon,p}} \hat{f}_j (\varepsilon - H_\varepsilon) \, \text{dvol}_{g_\varepsilon} - c_n \varepsilon^n \nabla \phi_0(p) \cdot e_j \right| \leq C \varepsilon^{n+\frac{1}{2}}.
\]

**Proof.** This estimate follows at once from the result of Lemma 11.2.

Collecting the result of the last Lemma together with (10.9) and (10.10) and using the fact that the point \( p \) is a non degenerate critical point of \( \phi_0 \), it is easy to conclude the existence of a point \( p_\varepsilon \) close to \( p \) for which \( a_1 = \ldots = a_n = 0 \). For example, it is enough to apply a degree argument or Browder's fixed point.

This completes the proof of Theorem 2.1.
11. Appendix

We derive some estimates for the mean curvature of the approximate solution $\Lambda^\flat_{\epsilon,p}$.

**Lemma 11.1.** The following estimates hold

\[
\left| \int_{\Lambda^\flat_{\epsilon,p}} \bar{f}_i \bar{f}_j \, d\text{vol}_{g^\flat_{\epsilon}} \right| \leq C \epsilon^{n+1}
\]

and

\[
\left| \int_{\Lambda^\flat_{\epsilon,p}} \bar{f}_i J_\epsilon \, w \, d\text{vol}_{g^\flat_{\epsilon}} \right| \leq C \epsilon^{n+\nu-1} \|w\|_{C^{2,\alpha}(\Lambda^\flat_{\epsilon,p})}
\]

together with similar estimates for the derivatives.

**Proof.** This follows at once from the structure of the metric in Fermi coordinates. □

We denote by $H_\epsilon$ the mean curvature function of $\Lambda^\flat_{\epsilon,p}$. A key point in our analysis is the fact that the leading term in $H_\epsilon$ is odd with respect to $s$. In particular, if we decompose $H_\epsilon = H^e_\epsilon + H^o_\epsilon$, into its even and odd part with respect to $s$, we have the :

**Lemma 11.2.** The following estimates hold for all $\epsilon > 0$ small enough

\[
|H^o_\epsilon| \leq C \begin{cases} 
\epsilon^2 + \epsilon^n W_\epsilon^{-n} + \epsilon^{3n-2} W_\epsilon^{2-3n}, & \text{when } W_\epsilon \geq 2r_\epsilon \\
\epsilon^2 r_\epsilon^{-1} + \epsilon^n W_\epsilon^{-n} + \epsilon^{3n-2} W_\epsilon^{2-3n}, & \text{when } r_\epsilon/2 \leq W_\epsilon \leq 2r_\epsilon \\
\epsilon^2 + \epsilon^n W_\epsilon^{-n}, & \text{when } W_\epsilon \leq r_\epsilon/2
\end{cases}
\]

and

\[
|H^e_\epsilon - \epsilon| \leq C \begin{cases} 
\epsilon^3 + \epsilon^{n+1} W_\epsilon^{-n} + \epsilon^{3n-3} W_\epsilon^{2-3n}, & \text{when } W_\epsilon \geq 2r_\epsilon \\
\epsilon r_\epsilon^{-1} + \epsilon^{3n-3} W_\epsilon^{2-3n}, & \text{when } r_\epsilon/2 \leq W_\epsilon \leq 2r_\epsilon \\
\epsilon, & \text{when } W_\epsilon \leq r_\epsilon/2
\end{cases}
\]

**Proof.** The estimate when $W_\epsilon \leq r_\epsilon/2$, follows at once from the expression of the mean curvature which is given by Proposition 4.1. Indeed, we see that we can write

\[
H(w) = J_\Lambda w + B_1(w, |\nabla w|^2) [w, w] + B_2(w, |\nabla w|^2) [\nabla w, \nabla w] \\
+ B_3(w, |\nabla w|^2) [w, \nabla^2 w] + B_4(w, |\nabla w|^2) [\nabla^2 w, \nabla w, \nabla w]
\]
where $B_i(A,A')$ are smooth multilinear forms depending real analytically on $A$ and $A'$ and the estimates when $W_{\varepsilon} \geq \frac{r_{\varepsilon}}{2}$ follow at once by taking $w = -\varepsilon \phi_0 + \tilde{\varepsilon}^{n-1} \Gamma$ and using the fact that $J_\Lambda \phi_0 = 1$.

In the region where the connected sum is performed, one has to take into account the effect of the cutoff function in the definition of $\Lambda_{\varepsilon,p}^\vee$ which turns out to be the most important term.

In order to estimate the mean curvature when $2\varepsilon \leq W_{\varepsilon} \leq \frac{r_{\varepsilon}}{2}$, we can again use the result of Proposition 4.1 and, this time, we take advantage of the fact that the catenoid is a minimal hypersurface in Euclidean space and hence

$$
(11.11) \quad \text{div}_{g_0} \left( \frac{\nabla g_0 w}{\sqrt{1 + |\nabla g_0 w|^2_{g_0}}} \right) = 0
$$

where $g_0$ stands for the Euclidean metric in $\mathbb{R}^n$.

We decompose $H(w) = H_1(w) + H_2(w)$ where

$$
H_1(w) := \text{div}_{g_0} \left( \frac{\nabla g_0 w}{\sqrt{1 + |\nabla g_0 w|^2_{g_0}}} \right),
$$

and

$$
H_2(w) := \frac{1}{2} \frac{\hat{g}_w(\nabla g_0 w, \nabla w)}{\sqrt{1 + |\nabla g_0 w|^2_{g_0}}} - \sqrt{1 + |\nabla g_0 w|^2_{g_0}} \text{Tr}_{g_0} \hat{g}_w.
$$

Again we decompose each $H_i$ into its odd and even part. Using the arguments as above, it is easy to check that

$$
|H_0^o(v^-_{\varepsilon})| \leq C \varepsilon^2
$$

and also that

$$
|H_2^o(v^-_{\varepsilon}) - \varepsilon| \leq C \varepsilon.
$$

Finally in order to estimate $H_1(v^-_{\varepsilon})$ we use (11.11) together with the expansion of the metric to show that

$$
|H_2^e(v^-_{\varepsilon})| \leq C \varepsilon^n W_{\varepsilon}^{-n}
$$

and also that

$$
|H_2^e(v^-_{\varepsilon}) - \varepsilon| \leq C \varepsilon.
$$

This completes the proof of the result, at least when $W_{\varepsilon} \geq 2\varepsilon$. The estimate in a ball of radius $2\varepsilon$ centered at $p$ is not hard to drive and will be a consequence of the next result.
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