EXTREMAL METRICS ON BLOW-UPS

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1. Introduction

Let \((M, J)\) be a compact complex manifold of complex dimension \(m\). Suppose that \(c\) is a Kähler class on \(M\). If \(\omega \in c\) is a Kähler form, denote by \(s(\omega)\) the scalar curvature of (the riemannian metric associated to) the Kähler form \(\omega\). Calabi proposed the study of the functional

\[
c^+ \ni \omega \mapsto \int_M s(\omega)^2 \frac{\omega^m}{m!}
\]

where \(c^+\) is the set of positive Kähler forms in \(c\). Its critical points are now called extremal Kähler metrics; one verifies that \(\omega\) is extremal if and only if the hamiltonian vector field \(X_{s(\omega)}\) is real holomorphic, that is

\[
L_{X_{s(\omega)}} J = 0,
\]

where \(L\) denotes the Lie derivative.

If \(s(\omega)\) is a constant, then (1.1) is trivially satisfied, and so the study of extremal Kähler metrics is a natural generalization of the study of Kähler metrics of constant scalar curvature (and in particular Kähler–Einstein metrics). If \((M, J)\) admits no non-trivial holomorphic vector fields, then every extremal Kähler metric must have constant scalar curvature, but in the opposite case it is really essential to allow extremal Kähler metrics of non-constant scalar curvature to have a good theory: for example, if a Kähler class \(c_0\) contains a metric of constant scalar curvature, then every nearby Kähler class will contain an extremal metric [21], but in the presence of holomorphic vector fields these will not in general have constant scalar curvature. As another illustration of this phenomenon, we may point to Theorem 2.5 below.

An important feature of extremal Kähler metrics is that they must have the maximal possible symmetry allowed by the complex manifold \(M\). More precisely, Calabi proved that the identity-component of the isometry group of any extremal metric must be a maximal compact subgroup of the identity-component of the group of biholomorphic maps of \(M\) to itself. This fact soon led to the construction of a compact complex manifold which cannot admit extremal metrics in any Kähler class [23].
At the time of writing, the main problem in the subject is to obtain necessary and sufficient conditions on \((M, J, c)\) for the existence of an extremal Kähler metric in \(c\). The question of uniqueness was settled by Chen and Tian in [12]: if \(\omega_1\) and \(\omega_2\) are extremal metrics in \(c\) then there is a holomorphic symmetry \(g\) of \(M\) such that \(g^*(\omega_2) = \omega_1\). Following the work of various authors (e.g. [37, 26, 35, 13]) it is now clear that the existence problem is related to a suitable stability condition (in the sense of GIT [29]) for \((M, J, c)\).

In view of the difficulty of the general existence problem, we shall study here the following simpler problem: let \(M\) be a compact complex manifold with an extremal Kähler metric \(\omega\). Under what conditions does the blow-up \(\tilde{M}\) of \(M\) at a finite set of points admit an extremal Kähler metric? This is a natural extension of the work of the first two authors [1, 2] which answers the corresponding question for Kähler metrics of constant scalar curvature.

2. Statement of the results

Let \(M\) be a compact complex manifold of complex dimension \(m\), equipped with a Kähler form \(\omega\). Denote by \(K\) the group of biholomorphic self-maps of \(M\) which are also exact symplectomorphisms of \((M, \omega)\). Then \(K\) is a compact Lie group and there is a normalized moment map

\[ \xi : M \to \mathfrak{k}, \]

where \(\mathfrak{k}\) is the Lie algebra of \(K\). Thus \(\xi\) satisfies the following conditions:

- For each \(X \in \mathfrak{k}\), \(\langle \xi, X \rangle\) is a hamiltonian generating function for \(X\); (2.2)
- \(\xi\) is equivariant: \(\xi(kx) = \text{Ad}(k)\xi(x)\) for all \(x \in M\) and \(k \in K\); (2.3)
- For each \(X \in \mathfrak{k}\), \(\int_M \langle \xi, X \rangle d\text{vol}_\omega = 0\). (2.4)

Let \(p_1, \ldots, p_n\) be points of \(M\), denote by \(\tilde{M}\) or \(\text{Bl}_{p_1, \ldots, p_n}(M)\) the blow-up of \(M\) at these points and denote by \(\pi : \tilde{M} \to M\) the natural map. To place our main result about extremal Kähler metrics in context, recall the main theorem of [2] regarding Kähler metrics of constant scalar curvature (CSCK metrics):

**Theorem 2.1.** Suppose \((M, \omega)\) has constant scalar curvature. Suppose that

1. there exist \(a_j > 0\) such that \(\sum_{j=1}^{n} a_j^{m-1} \xi(p_j) = 0\) (balancing condition);
2. \(\mathbb{R}\xi(p_1) + \cdots + \mathbb{R}\xi(p_n) = \mathfrak{k}\) (genericity condition).
Then there exist $\varepsilon_0 > 0$, $c > 0$ and $\theta > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ there is a CSCK metric $\omega_\varepsilon$ on $\widetilde{M}$ in the Kähler class

$$\pi^*[\omega] - \varepsilon^2 \sum_{j=1}^n \tilde{a}_j PD(E_j),$$

where $\tilde{a}_j$ depends upon $\varepsilon$ and $|\tilde{a}_j - a_j| \leq c\varepsilon^\theta$ as $\varepsilon \to 0$.

Furthermore, if

(iii) there is no non-zero element of $k$ which vanishes at $p_1, \ldots, p_n$

then we can assume that $\tilde{a}_j = a_j$.

Suppose now that $(M, \omega)$ is an extremal Kähler manifold. We would like an analogue of Theorem 2.1, asserting that under certain conditions, $\widetilde{M}$ carries a family of extremal Kähler metrics $\omega_\varepsilon$. If we have such a family, then we will in particular have a family of extremal vector fields, $X_\varepsilon$ say, on $\widetilde{M}$. Now our constructions are essentially perturbative away from the exceptional divisors, and it follows from this that $X_\varepsilon \to X_s$ away from these divisors, and this suggests that we must assume that $X_s$ must lift to $\widetilde{M}$. This is the case if and only if $X_s(p_j) = 0$ for $j = 1, \ldots, n$, and this motivates one of the main assumptions of Theorem 2.2 below.

The other assumptions arise from the aforementioned theorem of Calabi on the structure of the automorphism group of $M$. As observed by LeBrun and Simanca [22] this means, in effect, that one can fix the isometry group in advance. (Since one knows that the isometry group will be a maximal compact subgroup, and any two such subgroups are conjugate, different choices of maximal compact subgroup correspond to pull-back of the metric by holomorphic symmetries of $M$.)

We shall exploit a variant of this idea, where we fix in advance a torus $T \subset K$ that will act isometrically on the extremal Kähler metrics we construct on $\widetilde{M}$. Now $k \in K$ lifts to act on $\widetilde{M}$ if and only if $k$ fixes the $p_j$. Turning this around, if we fix $T$, then the points $p_j$ at which we can blow up must all be contained in $\text{Fix}(T)$; combining with the previous requirement that $X_s$ should lift to $\widetilde{M}$, we see that $T$ should be chosen so that its lie algebra $t$ contains $X_s$.

Having established this basic geometric set-up, we can state a theorem precisely parallel to Theorem 2.1, where the role of the moment map $\xi$ is taken by a `relative moment map' $\xi''$. More precisely, define $H$ to be the centralizer of $T$ in $K$, denote by $h$ the Lie algebra of $H$, and put $H'' = H/T$ with Lie algebra $h'' = h/t$. If $X \in h$, denote by $X''$ the projection of $X$ to $h''$. By the equivariance of $\xi$, if $p \in \text{Fix}(T)$, then $\xi(p) \in h$, so the relative moment map $\xi''(p)$, the projection of $\xi(p)$ to $h''$, is well defined. (The reader is referred to §9 for more details of this.)

Our main theorem can now be stated as follows:
Theorem 2.2. Let $(M, \omega)$ be an extremal Kähler manifold with extremal vector field $X_s$. Let $T \subset K$ be a torus with $X_s \in \mathfrak{t}$. Suppose that $p_j \in \text{Fix}(T)$ and

(i) there exist $a_j > 0$ such that $\sum_{j=1}^{n} a_j^{m-1} \xi''(p_j) = 0$ (balancing condition);
(ii) $\Re \xi''(p_1) + \cdots + \Re \xi''(p_n) = h''$ (genericity condition).

Then there exist $\varepsilon_0 > 0, c > 0$ and $\theta > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ there is an extremal Kähler metric $\omega_{\varepsilon}$ on $\tilde{M}$ in the Kähler class

$$\pi^* [\omega] - \varepsilon^2 \sum_{j=1}^{n} \tilde{a}_j PD[E_j],$$

where $\tilde{a}_j$ depends upon $\varepsilon$ and $|\tilde{a}_j - a_j| \leq c\varepsilon^\theta$ as $\varepsilon \to 0$.

Furthermore, if

(iii) there is no non-zero element of $h''$ which vanishes at $p_1, \ldots, p_n$ then we can assume that $\tilde{a}_j = a_j$. 

As explained in §9 the group $H''$ acts on the symplectic manifold $\text{Fix}(T)$ with moment map $\xi''$ (Lemma 9.1). Thus conditions (i)–(iii) of Theorem 2.2 are exact analogues of the corresponding conditions of Theorem 2.1 except that $M$ has been replaced by $\text{Fix}(T)$ and $K$ has been replaced by $H'' = H/T$.

It should also be pointed out that if conditions (i) and (ii) hold but condition (iii) fails, then one can always replace $T$ by a larger torus $T_1$ so that (i)–(iii) are all satisfied: see Proposition 9.3.

2.1. Extremal versus CSC metrics. If $\omega$ is an extremal metric with non-constant scalar curvature, then Theorem 2.2 will construct an extremal metric on $\tilde{M}$ which also has non-constant scalar curvature. If, however, $\omega$ is a CSC Kähler form, then we can ask whether or not the extremal Kähler metrics obtained on $\tilde{M}$ have constant or non-constant scalar curvature. The following result gives a simple answer to this question:

Proposition 2.3. Under the assumptions of Theorem 2.2, suppose that $\omega$ has constant scalar curvature. If

$$\sum_{j=1}^{n} a_j^{m-1} \xi(p_j) \neq 0,$$

then the Kähler metric $\omega_{\varepsilon}$ is extremal with non-constant scalar curvature.

2.2. Toric varieties. Although the balancing condition of Theorem 2.2 is subtle, it becomes trivial if we take $T$ to be a maximal torus in $K$. In this case $H = T$, $h'' = 0$ and so conditions (i)–(iii) of Theorem 2.2 become vacuous. Thus this Theorem applies at once if $X_s \in \mathfrak{t}$. However, $X_s$ lies in the centre of $\mathfrak{t}$ (see §3.3) and so $X_s$ is contained in the Lie algebra of any maximal torus.
Hence:

**Theorem 2.4.** Let \((M, \omega)\) be a compact extremal Kähler manifold and let \(T\) be a maximal torus of \(K\). Then for any subset \(\{p_1, \ldots, p_n\} \subset \text{Fix}(T)\) there exist extremal Kähler metrics \(\omega_\varepsilon\) on \(\text{Bl}_{p_1, \ldots, p_n}(M)\) with the properties described in Theorem 2.2.

This simple statement provides many important examples. First of all, we have

**Theorem 2.5.** Suppose that \(p_1, \ldots, p_n\) are \(n \leq m + 1\) points in general position in complex projective space \(\mathbb{C}P^m\). Suppose that \(a_j > 0\) are arbitrary. Then there exists an extremal Kähler metric \(\omega_\varepsilon\) on \(\text{Bl}_{p_1, \ldots, p_n}(\mathbb{C}P^m)\), with

\[
[\omega_\varepsilon] = \pi^*[\omega_{FS}] - \varepsilon^2 \sum a_j PD[E_j].
\]

Moreover \(\omega_\varepsilon\) is not CSC unless \(n = m + 1\) and all the \(a_j\) are equal.

**Proof.** Since the \(p_j\) are in general position, we may choose homogeneous coordinates \([Z_0 : \cdots : Z_m]\) so that

\[
p_1 = [1 : 0 : 0 : \cdots : 0], p_2 = [0 : 1 : 0 : \cdots : 0], \ldots
\]

Denote by \(\omega_{FS}\) the standard Fubini–Study metric determined by these coordinates, normalized so that \([\omega_{FS}]\) is Poincaré-dual to a hyperplane \(\mathbb{C}P^{m-1} \subset \mathbb{C}P^m\). The group \(K\) is then isomorphic to \(PSU(m+1)\) and the moment map is given by

\[
\xi(z) = i \frac{z \otimes z^* - |z|^2/(m+1)}{|z|^2},
\]

where \(\mathfrak{g}\) has been identified with the algebra of trace-free skew-hermitian matrices (see, for example, [19] for this). The diagonal matrices give a maximal torus \(T^{m+1} \subset U_{m+1}\) and the image \(T\) of \(T^{m+1}\) in \(K\) is a maximal torus with \(\text{Fix}(T) = \{p_1, p_2, \ldots, p_{n+1}\}\). Since \(\omega_{FS}\) is a Kähler-Einstein metric, it is certainly extremal, and so Theorem 2.4 applies to give the first part of the theorem.

To prove the second part, note that

\[
\sum A_j \xi(p_j) = i \text{diag}(A_1 - \overline{A}, A_2 - \overline{A}, \ldots, A_n - \overline{A}),
\]

where

\[
\overline{A} = \frac{A_1 + \cdots + A_n}{m+1}.
\]

It now follows from Proposition 2.3 that \(\omega_\varepsilon\) will not be CSC unless \(n = m + 1\) and all the \(A_j\) are equal. \(\square\)

**Remark 2.6.** It was proved in [2] that in the case when \(n = m + 1\) and all \(A_j\) are equal, the Kähler class

\[
[\omega_\varepsilon] = \pi^*[\omega_{FS}] - \varepsilon^2 \sum PD[E_j]
\]

does indeed contain a CSCK metric.
The reader is referred to §8 for a more detailed discussion of extremal Kähler metrics on blow-ups of $\mathbb{C}P^2$. In particular it is proved there that there exist non-toric extremal Kähler metrics on $\text{Bl}_{p_1,\ldots,p_n}(\mathbb{C}P^2)$ for every $n \geq 3$. These examples are of particular interest also thanks to Della Vedova’s result [14] showing optimality of our assumptions in some of these cases.

Since the appearance of the first version of this paper, Theorem 2.5 has been used by Chen, LeBrun and Weber [11] and He [18] to prove that all Kähler classes on $M = \text{Bl}_{p_1,p_2}(\mathbb{C}P^2)$ of the form $\pi^*\omega_{\text{FS}} - \varepsilon^2 a(\text{PD}[E_1] + \text{PD}[E_2])$ have an extremal representative. (This result implies the existence of non-Kähler, Einstein metrics of positive scalar curvature on $M$.) We should also mention the recent work of Chen and He [10] where the existence of extremal metrics in certain Kähler classes on $\text{Bl}_{p_1,p_2,p_3}(\mathbb{C}P^2)$ is proved via the Calabi flow.

On the other hand Donaldson [13] has produced examples of Kähler classes on certain iterated toric blow-ups of $\mathbb{C}P^2$ which do not admit extremal Kähler metrics. Thus we cannot expect to be able to make the constant $\varepsilon_0$ of Theorem 2.2 arbitrarily large in general.

Theorem 2.4 also gives a simple general statement for blow-ups of toric extremal Kähler manifolds:

**Theorem 2.7.** Let $M$ be a compact complex manifold of dimension $m$ acted on by a complex torus $T^c$ so that a dense open subset of $M$ is biholomorphic to $T^c$. Suppose that $\omega$ is a toric extremal Kähler metric on $M$, so that there exists a compact $m$-dimensional torus $T \subset T^c$ acting isometrically on $(M, \omega)$, with moment map $\xi$. Then for any subset $\{p_1, \ldots, p_n\} \subset \text{Fix}(T)$, $\text{Bl}_{p_1,\ldots,p_n}(M)$ admits extremal Kähler metrics.

**Proof.** Let $K$ be the symmetry group of $(M, \omega)$. By hypothesis, $T \subset K$. But it is not possible for a torus of dimension $> m$ to act effectively and by exact symplectomorphisms on a symplectic manifold of dimension $2m$, so $T$ must be a maximal torus of $K$. The result follows at once from Theorem 2.4. \hfill \Box

**Remark 2.8.** Donaldson has recently (December 2007) announced a complete account of the existence problem for toric CSCK metrics on compact complex surfaces.

We end this section with a general result:

**Theorem 2.9.** Let $(M, \omega)$ be a compact extremal Kähler manifold. Then there exist at least two points $p_1, p_2$ of $M$ such that $\text{Bl}_{p_1,p_2}(M)$ admits extremal Kähler metrics.

**Proof.** By Theorem 2.4, it is enough to show that there are at least two points in $\text{Fix}(T)$, where $T$ is a maximal torus of $K$. This is a well-known result from symplectic geometry: a sketch proof runs as follows. Take a smooth function $f$ on $M$ such that the flow of $X_f$ is dense in $T$. Then the critical points of $f$ correspond to the fixed points of $T$. Since $f$
must have a maximum and a minimum, we see that \( \operatorname{Fix}(T) \) must contain at least two points. \( \square \)

2.3. **Outline of the paper.** The rest of this paper is organized as follows. §3 sets up the notation and gives essential geometrical background. §4 gives a careful gluing construction of the blow-up and gives an outline of the ‘domain-splitting method’ which is the analytical tool used to prove Theorem 2.2. In §5 we give the necessary analysis of the linearization of the extremal Kähler condition and then in section §6 we use this to prove nonlinear perturbation results for extremal Kähler metrics. The proof of our main result is completed in section §7; the last two sections are devoted to a discussion of examples and the geometry of the balancing condition appearing in the main theorem.

The analysis in §§5–7 is very close to that of [1, 2], and the reader is referred to these papers for a more detailed treatment.

3. **Notation and Background**

3.1. **Conventions.** Let \((M, g, J, \omega)\) be a Kähler manifold of complex dimension \(m\). Thus \(g\) is a riemannian metric, \(J\) is an integrable, \(g\)-orthogonal almost-complex structure, and \(\omega\) is the corresponding Kähler form, so that

\[
\omega(X, Y) = g(JX, Y), \quad g(X, Y) = \omega(X, JY)
\]  

for any vector fields \(X\) and \(Y\) on \(M\). The operator \(J\) is viewed initially as an automorphism of \(TM\) satisfying \(J^2 = -1\). The corresponding action on \(T^*M\) is defined by

\[
J\alpha(X) = -\alpha(JX)
\]  

so that the action of \(J\) commutes with the musical isomorphisms (raising and lowering indices with \(g\)).

Then \(T^{1,0}\) corresponds to the \(+i\)-eigenspace of \(J\) while \(\Lambda^{1,0}\) corresponds to the \(-i\)-eigenspace of \(J\). In particular, we have formulae such as

\[
\bar{\partial}f = \frac{i}{2}(df - iJdf), \quad \text{and} \quad Jdf = i(\bar{\partial}f - \partial f)
\]  

By abuse of notation, we often refer to \(\omega\) (rather than \(g\)) as a Kähler metric, speak of isometries of \(\omega\) or the scalar curvature of \(\omega\) and so on.

3.2. **Symmetry groups and algebras.** The group \(\operatorname{Aut}(M, J)\) is by definition the set of all biholomorphic self-maps of \(M\). Its Lie algebra \(\mathfrak{a}(M, J)\) consists of the real-holomorphic vector fields:

\[
\mathfrak{a}(M, J) = \{ X \in \operatorname{Vect}(M) : L_X J = 0 \}.
\]  

Here \(L_X\) denotes Lie derivative with respect to \(X\), so

\[
(L_X J)Y = [X, JY] - J[X, Y]
\]
for any vector field $Y$. From this formula, it is easy to show that $X$ is real-holomorphic iff the corresponding section $X - iJX$ of $T^{1,0}$ is holomorphic.

Now we consider the symmetries of $(M, \omega)$ as a symplectic manifold. Recall that if $f \in C^\infty(M)$, the corresponding Hamiltonian vector field $X_f$ is defined by the equation

$$\omega(X_f, -) = -df, \quad X_f = J\nabla f,$$

(3.6)

these two being equivalent by (3.1).

Any Hamiltonian vector field is an infinitesimal symmetry of the symplectic form: $L_{X_f} \omega = 0$. The group $\text{Exact}(M, \omega)$ of exact symplectomorphisms is therefore defined as the set of diffeomorphisms which arise as finite-time flows generated by Hamiltonian vector fields. By construction, the Lie algebra of $\text{Exact}(M, \omega)$ is isomorphic to $C^\infty(M)/\mathbb{R}$, for the vector field $X_f$ does not change if a constant is added to $f$.

Put $K = \text{Exact}(M, \omega) \cap \text{Aut}(M, J)$, and denote its Lie algebra by $\mathfrak{k}$. It will sometimes be convenient to distinguish between $\mathfrak{k}$ as an abstract Lie algebra and as a set of vector fields on $M$. In particular, we can define an invariant inner product on $\mathfrak{k}$ by integration over $M$:

$$X \cdot Y = \int_M g(\sigma(X), \sigma(Y)) \, d\text{vol}_g$$

Let $\xi_\omega = \xi : M \to \mathfrak{k}$ be the normalized moment map for the action of $K$ on $M$ introduced in §2.

The structure of $K$ is clarified by the introduction of the operator $P : C^\infty(M) \to C^\infty(M, T^* \otimes T)$ defined by

$$Pf = \frac{1}{2}L_{X_f}J$$

(3.7)

This is a second-order differential operator and it is clear that $\ker(P)/\mathbb{R}$ is isomorphic to $\mathfrak{k}$ via the dual of the moment map. More precisely, the dual of $\xi$ maps $\mathfrak{k}$ into $C^\infty(M)$ and the image is the orthogonal complement of $\mathbb{R}$ in $\ker(P)$.

A straightforward computation gives

$$P^* Pf = \frac{1}{2} \Delta^2 f + r(\nabla^2 f) + \frac{1}{2}(\nabla s)f$$

where $r$ is the Ricci curvature and $s$ is the scalar curvature of $\omega$.

3.2.1. Variation of moment map. Let $T \subset K$ be a torus, and let $\varphi$ be a $T$-invariant function such that $\tilde{\omega} := \omega + i\partial\bar{\partial}\varphi$ is another Kähler form. Then $T$ preserves $\tilde{\omega}$ and we can ask for a momentum map $\tilde{\xi}$ for the action of $T$ on $(M, \tilde{\omega})$.

**Proposition 3.1.** The map

$$\tilde{\xi} : M \to \mathfrak{t}, \quad \langle \tilde{\xi}, X \rangle = \langle \xi, X \rangle - \frac{1}{2}JX \varphi$$

(3.8)

is a moment map for the action of $T$ on $(M, \tilde{\omega})$. 
Proof. Unravelling the definitions, we see that (3.8) is a hamiltonian for $X$ with respect to $\tilde{\omega}$ if and only if
\[ i\partial\bar{\partial}\varphi(X,Y) = \frac{1}{2} Y(JX \varphi) \] (3.9)
holds for every vector field $Y$ on $M$. Now we have
\[
2i\partial\bar{\partial}\varphi(X,Y) = d(Jd\varphi(X,Y)) = X(Jd\varphi,Y) - Y(Jd\varphi,X) - (Jd\varphi,[X,Y]) \] (3.10)
But $X\varphi = 0$ as $\varphi$ is $T$-invariant, so this can also be written as
\[
2i\partial\bar{\partial}\varphi(X,Y) = Y(JX \varphi) - (JY \varphi) - (L_X J Y \varphi) \] (3.11)
since $X$ is real-holomorphic. This proves (3.9). It is clear that $\tilde{\xi}$ is $T$-equivariant, so it is a moment map for the action of $T$ on $(M,\tilde{\omega})$, as required. □

Remark 3.2. Even if $\xi$ is normalized in the sense that $\langle\xi,X\rangle$ has zero mean on $M$, for all $X \in \mathfrak{k}$, the moment map $\tilde{\xi}$ will not generally be normalized in this sense.

3.3. Extremal Kähler metrics. We recall the definition of an extremal Kähler metric:

Definition 3.3. Let $(M,\omega,J)$ be Kähler manifold. The metric $\omega$ is said to be extremal if the hamiltonian vector field generated by the scalar curvature $s(\omega)$ of $\omega$ is real-holomorphic,
\[ L_{X_{\omega}} J = 0. \]
We call $X_{\omega}$ the extremal vector field corresponding to $\omega$.

Remark 3.4. Let $M$ be compact and suppose that $\omega$ is an extremal Kähler metric on $M$. Denote by $\hat{K}$ the identity-component of the isometry group of $(M,\omega)$ and let $\text{Aut}_0(M,J)$ be as above. Then Calabi [9] proved that $\hat{K}$ is a maximal compact subgroup of $\text{Aut}_0(M,J)$ and that $K \subset \hat{K}$ is a normal subgroup, with abelian quotient.

Note that $X_{\omega}$ must lie in the centre of $\mathfrak{k}$. Indeed, if $k \in K$, then $k^* \omega = \omega$, $k^*(s) = s$ and it follows that $X_{s}$ is $K$-invariant, as claimed.

Let $\omega'$ be another extremal Kähler metric on $M$. Then, as observed by LeBrun and Simanca [22], the corresponding identity component $\hat{K}'$ of the isometry group of $(M,\omega')$ must be conjugate to $\hat{K}$ inside $\text{Aut}_0(M,J)$. Thus there is a metric equivalent to $\omega'$ up to the action of $\text{Aut}_0(M,J)$ and with isometry group equal to $\hat{K}$. This is the sense in which we may fix the isometry group in advance when studying the perturbation theory of extremal Kähler metrics on $M$. We need a slight variant of this idea, because the
group of holomorphic automorphisms itself will change when we make a blow-up of $M$. Indeed,

$$\text{Aut}_0(\tilde{M}, \tilde{J}) = \{ \gamma \in \text{Aut}_0(M, J) : \gamma(p_j) = p_j \text{ for } j = 1, \ldots, n \}. \quad (3.12)$$

Rather than work with a maximal compact subgroup of $\text{Aut}_0(\tilde{M}, \tilde{J})$ we shall fix a compact torus $T \subset K$, with $X_s \in t$. In our later application, the points $p_j$ will be chosen so as to lie in $\text{Fix}(T)$ and the isometry group of the extremal Kähler metric constructed on $\tilde{M}$ will contain $T$.

The main result of this section is as follows:

**Proposition 3.5.** Let $(M, J)$ be a compact complex manifold, with extremal Kähler metric $\omega$. Let $T \subset K$ be torus with $X_s \in t$, and let $\xi$ be the normalized moment map. Set

$$U = \{ f \in C^\infty(M)^T : \omega + i\partial\bar{\partial}f > 0 \}, \quad (3.13)$$

where the superscript $T$ denotes the $T$-invariant part. Let

$$\Phi : U \times t \times \mathbb{R} \to C^\infty(M)^T$$

$$\Phi(f, X, c) = s(\omega + i\partial\bar{\partial}f) - \langle \xi, X_s + X \rangle + \frac{1}{2} J(X_s + X)f - \overline{s(\omega)} - c, \quad (3.14)$$

where $\overline{s(\omega)}$ denotes the mean value of $s(\omega)$ over $M$. Then $\tilde{\omega} := \omega + i\partial\bar{\partial}f$ is extremal, with extremal vector field $X_s + X$ if and only if there exists $c \in \mathbb{R}$ such that $\Phi(f, X, c) = 0$; moreover $\Phi$ has an expansion of the form

$$\Phi(f, X, c) = - P^* Pf - \langle \xi, X \rangle - c$$

$$- \frac{1}{2} J X f + Q(f) \quad (3.15)$$

where $Q$ is the nonlinear part of the operator $f \mapsto s(\omega + i\partial\bar{\partial}f)$. In particular, $Q$ has the form

$$Q(f) = A(\nabla^2 f, \nabla^2 f \otimes \nabla^4 f) + B(\nabla^2 f, \nabla^3 f \otimes \nabla^3 f) + C(\nabla^2 f, \nabla^2 f \otimes \nabla^2 f) \quad (3.16)$$

where $A(U, V), B(U, V), C(U, V)$ are real-analytic in $U$, linear in $V$ and all coefficients are smooth functions on $M$ (depending upon $\omega$ and its derivatives).

**Proof.** From Proposition 3.1,

$$\langle \xi, X_s + X \rangle - \frac{1}{2} J(X_s + X)f$$

is a Hamiltonian for $X_s + X$. It follows at once that if $\Phi(f, X, c) = 0$, then $\tilde{\omega}$ is extremal, with extremal vector field $X_s + X$. Notice that $\Phi(0, 0, 0) = s(\omega) - \langle \xi, X_s \rangle - \overline{s(\omega)}$ vanishes because we have assumed $\xi$ to be normalized.

From the formulae

$$\rho(\tilde{\omega}) - \rho(\omega) = - i\partial\bar{\partial}\log(\tilde{\omega}^m/\omega^m) \quad (3.17)$$

$$s(\tilde{\omega}) = \text{tr}_{\tilde{\omega}}(\rho(\tilde{\omega})) \quad (3.18)$$
one obtains the well-known expansion
\[ s(\tilde{\omega}) = s(\omega) - \frac{1}{2} \Delta^2 f - r(\nabla^2 f) + Q(f) \] (3.19)
where \( Q(f) \) is the nonlinear part of the scalar curvature operator and has a decomposition of the form (3.16). Using the formula for \( P^* P \), we see that
\[ s(\omega + i\partial\bar{\partial}f) = s(\omega) - P^* Pf + \frac{1}{2}(\nabla s)f + Q(f). \]

Hence
\[ \Phi(f, X, c) = -P^* Pf + \frac{1}{2} \nabla s f - \xi_\omega \cdot X + \frac{1}{2} JX_s f - c + \frac{1}{2} JX f + Q(f). \]

Since \( JX_s = -\nabla s \), the two first-order terms cancel, completing the proof of (3.15). \( \square \)

Let us end this section with the following easy regularity statement for extremal Kähler metrics, noted by LeBrun and Simanca:

**Corollary 3.6.** Let the notation be as above, but assume that \( f \in C^{3,\alpha}(M) \), where \( 0 < \alpha < 1 \). Then if \( \omega + i\partial\bar{\partial}f \) is extremal (in the distributional sense), it follows that \( f \) is smooth.

**Proof.** See Proposition 4 of [22]. \( \square \)

### 4. The Domain-splitting method

In this section we shall describe a construction of families of Kähler metrics on \( \tilde{M} = \text{Bl}_{p_1,\ldots,p_n}(M) \) which will motivate and underpin the analytical work of the next three sections. We will also establish some notation that will be used in the remainder of the paper.

#### 4.1. \( \tilde{M} \) as a union of manifolds with boundary

Let \( M \) be a compact complex manifold of complex dimension \( m \), let \( p \) be a point of \( M \) and let \( \tilde{M} = \text{Bl}_p(M) \). We wish to construct \( \text{Bl}_p(M) \) by identifying the boundaries of two manifolds with boundary.

Let \( (z_1, \ldots, z_m) \) be a system of local holomorphic coordinates centred at \( p \) in \( M \) \((z_j(p) = 0 \text{ for } j = 1, \ldots, n)\). Suppose, without loss of generality, that these coordinates are defined for \( |z| < 1 \). Let \( \mathbb{C}^m \) have standard linear coordinates \((u_1, \ldots, u_m)\) and use these coordinates away from the exceptional divisor also on \( N = \text{Bl}_0(\mathbb{C}^m) \). Pick \( r \in (0, \frac{1}{2}) \), \( \varepsilon > 0 \) and put \( R = r/\varepsilon \). Then the map
\[ t_\varepsilon : \left\{ \frac{1}{2}r < |z| < 2r \right\} \rightarrow \left\{ \frac{1}{2}R < |u| < 2R \right\} \quad (4.1) \]
\[ t_\varepsilon : z \mapsto z/\varepsilon, \quad (4.2) \]
is biholomorphic between annular regions contained in \( M \) and \( N \). Using \( t_\varepsilon \) to identify these regions, we get an explicit model of \( M \).
We repackage this in terms of the manifolds with boundary
\[ M_r := M \setminus \{|z| < r\} \text{ and } N_R := N \setminus \{|u| > R\}. \] (4.3)
That is, we think of \( \tilde{M} \) as the union of \( M_r \) and \( N_R \), glued by \( \iota_\varepsilon \) along their boundaries:
\[ \tilde{M} = M_r \sqcup_{\iota_\varepsilon} N_R. \] (4.4)

4.2. **Group actions on \( \tilde{M} \).** Let \( \Gamma \) be a subgroup of \( \text{Aut}(M,J) \) with \( p \in \text{Fix}(\Gamma) \). By a classical result [6, 2], one can then assume that the coordinates \((z_1, \ldots, z_m)\) near \( p \) are \( \Gamma \)-linear: that is, each element of \( \Gamma \) acts as a complex-linear map relative to these coordinates. Thus we may regard \( \Gamma \) as a subgroup of \( \text{GL}_m(\mathbb{C}) \).

Now the linear action of \( \text{GL}_m(\mathbb{C}) \) on \( \mathbb{C}^m \) lifts uniquely to a holomorphic action on \( \mathbb{C}^m \), which agrees with the linear action outside the exceptional divisor. In particular, as a subgroup of \( \text{GL}_m(\mathbb{C}) \), \( \Gamma \) acts naturally on \( \mathbb{C}^m \) through the coordinates \((u_1, \ldots, u_m)\) and clearly the actions of \( \Gamma \) on \( M \) and on \( \mathbb{C}^m \) are matched up by the identification \( \iota_\varepsilon \).

Hence we see that the action of \( \Gamma \) on \( M \) lifts to \( \tilde{M} \) and that we may assume that this action is the standard action of a subgroup of \( \text{GL}_m(\mathbb{C}) \) in a neighbourhood of the exceptional divisor.

4.3. **Kähler metrics with given isometry group on \( \tilde{M} \).** Now let \( \omega \) be a given Kähler metric on \( M \). Then it is natural to choose the holomorphic coordinates \( z_j \) near \( p \) to be normal with respect to \( \omega \). It was shown in [2] that in the presence of an isometry group \( \Gamma \), acting holomorphically on \( M \), with \( p \in \text{Fix}(\Gamma) \), the \( z_j \) can be chosen so as to be simultaneously normal and \( \Gamma \)-linear. Thus we may assume that
\[ \omega|\{|z| < 1\} = i\partial \overline{\partial} \left( \frac{1}{2} |z|^2 + \varphi(z) \right), \] (4.5)
that \( \Gamma \subset U(m) \), that \( \varphi(z) \) is \( \Gamma \)-invariant, and that \( \varphi(z) = O(|z|^4) \) for small \( |z| \).

Recall that on \( \mathbb{C}^m \) there is a standard metric \( \eta \) due to Burns and Simanca [33], with the following properties:

(i) \( s(\eta) = 0 \);

(ii) \( \eta = i\partial \overline{\partial} \left( \frac{1}{2} |u|^2 + \psi_m(u) \right) \) for \( |u| \geq 1 \), where \( \psi_2(u) = 2 \log |u| \) and \( \psi_m(u) = -|u|^{4-2m} + O_\infty(|u|^{2-2m}) \) if \( m \geq 2 \);

(iii) the standard action of \( U(m) \) on \( N \) preserves \( \eta \) (so \( \psi_m \) is \( U(m) \)-invariant).

(Here the notation \( O_\infty(|u|^j) \) denotes a function \( F \) with the property that \( |\nabla^j F| = O(|u|^{\delta-j}) \) for all \( j \geq 0 \).)

From these ingredients we can construct \( \Gamma \)-invariant Kähler metrics on \( \tilde{M} \) as follows. Suppose \( f_1 \in C^\infty(M_r)^\Gamma \) such that \( \omega + i\partial \overline{\partial} f_1 > 0 \) on \( M_r \). Suppose that \( f_2 \in C^\infty(N_R)^\Gamma \) is such that \( \omega_2 = \eta + i\partial \overline{\partial} f_2 > 0 \) on \( N_R \). To compare these metrics under the identification \( \iota_\varepsilon \), it is convenient to transfer the data to a standard annulus. So let
\[ A^o = \{ x \in \mathbb{C}^m : 1 \leq |x| \leq 2 \}, \quad A^i = \{ x \in \mathbb{C}^m : \frac{1}{2} \leq |x| \leq 1 \}, \quad A = A^i \cup A^o, \] (4.6)
and write

\[ S = A^0 \cap A^i = \{|x| = 1\} \]  \hspace{1cm} (4.7)

for the unit sphere in \( \mathbb{C}^m \). Define \( j_a(x) = ax \), and by abuse of notation, regard \( j_r \) as an identification of \( A^0 \) with the subset \( \{r \leq |z| \leq 2r\} \) of \( M_r \) and \( j_R \) as an identification of \( A^i \) with the subset \( \{\frac{1}{2}R \leq |u| \leq R\} \) of \( N_R \).

Then

\[ \omega^o = j^*_r(\omega + i\partial \bar{\partial} f_1) = i\partial \bar{\partial}[\frac{1}{2}|x|^2/r^2 + \psi^o(x)] \]  \hspace{1cm} (4.8)

and

\[ \omega^i = \varepsilon^2 j^*_R(\eta + i\partial \bar{\partial} f_2) = i\partial \bar{\partial}[\frac{1}{2}|x|^2/r^2 + \psi^i(x)] \]  \hspace{1cm} (4.9)

Thus if \( \psi^i \) and \( \psi^o \) patch together smoothly across \( S \), we can regard \( \omega^1 \) and \( \varepsilon^2 \omega^2 \) as patching together to give a smooth metric on \( \tilde{M} \).

4.4. Cauchy-data matching for extremal metrics. Now suppose that \( \omega^1 \) and \( \omega^2 \) are extremal on \( M_r \) and \( N_R \) respectively, with extremal vector fields \( X^1 \) and \( X^2 \). The next result gives useful conditions for \( \omega^1 \) and \( \varepsilon^2 \omega^2 \) to patch together to form a smooth extremal Kähler metric on \( \tilde{M} \).

**Proposition 4.1.** Let \( M_r \), \( N_R \) and \( \tilde{M} \) be as above. Suppose \( T \) fixes \( p \) and so acts in standard fashion on \( M_r \) and \( N_R \). Suppose further that the action is hamiltonian with respect to \( \omega^1 \) on \( M_r \) and \( \omega^2 \) on \( N_R \), with moment maps \( \xi^1 \) and \( \xi^2 \) respectively. Suppose that

\[ s(\omega^1) = \langle \xi^1, X^1 \rangle + c_1, \]  \hspace{1cm} (4.10)

\[ s(\omega^2) = \varepsilon^4 \langle \xi^2, X^2 \rangle + \varepsilon^2 c_2, \]  \hspace{1cm} (4.11)

\[ \varepsilon^2 \int_{|z| = r} s(\omega^1) = \int_{|z| = r} t^*_\varepsilon s(\omega^2) \]  \hspace{1cm} (4.12)

and that \( X^1 \) and \( X^2 \) agree along \( |z| = r \).

Then if the functions \( \psi^o \) and \( \psi^i \) of (4.8) and (4.9) agree to third order at \( S \), then \( \omega^1 \) and \( \varepsilon^2 \omega^2 \) patch together to give a smooth eK metric on \( \tilde{M} \).

**Proof.** We transfer all the data to the standard annulus as above. Denote by \( \omega \) the form on \( A \) given by \( \omega^o \) in \( A^0 \) and by \( \omega^i \) in \( A^i \). Similarly denote by \( \xi : A \rightarrow t \) the moment map given by \( \xi^o \) in \( A^0 \) and by \( \xi^i \) in \( A^i \) and by \( \psi \) the function equal to \( \psi^i \) in \( A^i \) and \( \psi^o \) in \( A^0 \).

Observe first that the third-order agreement of \( \psi^i \) and \( \psi^o \) across \( S \) implies that \( \psi \) is \( C^{3,1} \). Moreover, since \( j^*_a X_1 \) and \( j^*_p X_2 \) are holomorphic and agree across \( S \), they must patch together to give a holomorphic vector field \( X \), say, on \( A \). It follows that \( \omega \) is at
least $C^{1,1}$ across $S$ so that the 1-form $\omega(X, -)$ is also $C^{1,1}$. Now the conditions
\begin{equation}
\mathbf{s}(\omega^o) = \langle \xi^o, X \rangle + c^o \tag{4.13}
\end{equation}
\begin{equation}
\mathbf{s}(\omega^i) = \langle \xi^i, X \rangle + c^i \tag{4.14}
\end{equation}
show that $\mathbf{s}(\omega^o)$ and $\mathbf{s}(\omega^i)$ differ by a constant at $S$, and (4.12) is designed precisely to guarantee that this constant must be $0$. Hence $f$ defined by $\langle \xi^o, X \rangle + c^o$ in $A^o$ and $\langle \xi^i, X \rangle + c^i$ in $A^i$ is in $C^{2,1}$. By Corollary 3.6, $\omega$ is smooth on $A$. \qed

In the next two sections we shall construct infinite-dimensional families of extremal Kähler metrics on $M_r$ and $N_R$. We will then show that it is possible to find a pair of metrics, one from each of these families, which satisfy the conditions of this Proposition. The details of this latter argument are given in §7.

5. Linear analysis

This section is devoted to the study of the linearization of the extremal metric equation identified in Proposition 3.5, namely
\begin{equation}
L : C^\infty(M)^T \times t \times \mathbb{R} \rightarrow C^\infty(M)^T \tag{5.1}
L : (f, X, \mu) \mapsto -P^* P f - \langle \xi, X \rangle - \mu
\end{equation}
More precisely, we shall study this operator on the punctured manifold $(M \setminus \{p_j\}, \omega)$ and on $(N, \eta)$ and find settings in which it is surjective in each case. The last part of this section reviews interior and exterior biharmonic extensions from data on the unit sphere in $\mathbb{C}^m$ as these will also be needed later.

5.1. Preliminaries and set-up. We begin with some notation. Set
\begin{equation}
B(a) = \{ x \in \mathbb{C}^m : |x| < a \}
\end{equation}
and write $B = B(1)$. Further, set
\begin{equation}
\overline{B}^* = \{ x \in \mathbb{C}^m : 0 < |x| \leq 1 \}, \; \overline{C} = \{ x \in \mathbb{C}^m : |x| \geq 1 \}.
\end{equation}
Recall also the definitions of the annuli $A^i$ and $A^o$ from (4.6) and denote by $j_r$ the scaling transformation $j_r(x) = rx$.

We now introduce weighted Hölder spaces for the domains $\overline{B}^*$ and $\overline{C}$.

**Definition 5.1.** Let $\ell \in \mathbb{N}$, $\alpha \in (0, 1)$ and $\delta \in \mathbb{R}$. If $f \in C^\ell_{loc}(\overline{B}^*)$, we put
\begin{equation}
\| f \|_{C^\ell_{\alpha}(\overline{B}^*)} := \sup_{0 < a \leq 1} a^{-\delta} \| j_r f \|_{C^\ell_{\alpha}(A^i)}. \tag{5.2}
\end{equation}
If \( f \in C^\ell_{\text{loc}}(\overline{C}) \), we put
\[
\|f\|_{C^\ell_{\delta,\alpha}(\overline{C})} := \sup_{a \geq 1} a^{-\delta} \|j_a^* f\|_{C^\ell_{\alpha}(A^\vee)}.
\] (5.3)

The space \( C^\ell_{\delta,\alpha}(\overline{B}^r) \) is defined as the completion of \( C^\infty_0(\overline{B}^r) \) in the norm (5.2) and the space \( C^\ell_{\delta,\alpha}(\overline{C}) \) is defined as the completion of \( C^\infty_0(\overline{C}) \) in the norm (5.3).

It is to be understood that on the RHS of (5.2) we are taking the \((\ell,\alpha)\) norm of the restriction of \( j_a^* f \) to \( A^\vee \). This convention will be used without comment in the rest of this paper. Note that the weight \( \delta \) gives the allowed growth rate of functions at 0 or \( \infty \). Thus \( f_\alpha(x) = |x|^\alpha \) lies in \( C^\ell_{\delta,\alpha}(\overline{B}^r) \) if and only if \( \alpha \geq \delta \) and it lies in \( C^\ell_{\delta,\alpha}(\overline{C}) \) if and only if \( \alpha \leq \delta \).

Now suppose that \( M \) is a compact manifold of complex dimension \( m \), and that \( p \) is a point of \( M \). Let \( \zeta : B(2) \to M \) be a given diffeomorphism with \( \zeta(0) = p \). Then we define
\[
\|f\|_{C^\ell_{\delta,\alpha}(M \setminus p)} = \|\zeta^*(f)\|_{C^\ell_{\delta,\alpha}(\overline{B}^r)} + \|f\|_{C^\ell_{\delta,\alpha}(M \setminus \zeta(B))}
\] (5.4)
where \( B(a) \) is the open ball of radius \( a \) in \( \mathbb{C}^m \) and \( B = B(1) \). Similarly if \( p_1, \ldots, p_n \) is a finite set of points of \( M \) and \( M^* = M \setminus \{p_1, \ldots, p_n\} \) we define the space \( C^\ell_{\delta,\alpha}(M^*) \) by choosing a diffeomorphism of \( B(2) \) with a neighbourhood of each of the points \( p_j \) and extending the definition (5.4) in the obvious way.

In the rest of this paper we suppose given a compact complex manifold \( M \), of complex dimension \( m \geq 2 \), with an extremal Kähler metric \( \omega \) and isometry group \( K \) with normalized moment map \( \xi : M \to \mathfrak{k} \) as in §3. We also fix a torus \( T \) whose lie algebra contains the extremal vector field \( X_\alpha \) and a finite set of points \( p_1, \ldots, p_n \) in \( \text{Fix}(T) \). We also suppose that for each \( j \) we have a given system of \( T \)-linear normal coordinates \( z_j = (z_j^1, \ldots, z_j^m) \) centred at \( p_j \) and defined, without loss of generality, for \( |z_j| < 4 \). Write
\[
B_j(r) = \{ p \in M : |z_j(p)| < r \},
\]
\[
\overline{B}_j^*(r) = \{ p \in M : 0 < |z_j(p)| \leq r \},
\]
and so on. Assume that the balls \( B_j(4) \) are pairwise disjoint and write
\[
M_r = M(4) = M \setminus \cup_j B_j(r).
\]

We also denote by \( N \) the blow-up of \( \mathbb{C}^m \) in the origin, and by \( \eta \) the Burns–Simanca metric. We use standard coordinates \( u = (u^1, \ldots, u^m) \) defined in particular for \( |u| \geq 1 \) in \( N \), and write \( N_R = N(R) = N \setminus \{ u : |u| > R \} \), for \( R \geq 1 \).

Analogously to (5.4), we introduce weighted Hölder spaces on \( N \) through the definition
\[
\|f\|_{C^\ell_{\delta,\alpha}(N)} = \|f\|_{C^\ell_{\alpha}(\overline{N})} + \|f\|_{C^\ell_{\delta,\alpha}(N(1))}
\] (5.5)
and define $C_δ^ℓ,α(N)$ as the completion of $C_δ^∞(N)$ in this norm. The reader is reminded of the convention whereby the first term on the right-hand side is interpreted as the norm of the restriction of $f$ to $C$ and the second term is interpreted as the norm of the restriction of $f$ to $N(1)$.

5.2. Analysis of $L$ on $M^*$. In $B_j^*$, define the function $G_j$ by

$$G_j(z_j) = \begin{cases} -\log |z_j| & \text{if } m = 2; \\ |z_j|^{4-2m} & \text{if } m \geq 3. \end{cases}$$

Then $\Delta^2 G_j = 0$ in $B_j^*$, where $\Delta$ is the laplacian of the flat metric in the coordinates $z_j$. Using this, and the fact that $\omega$ osculates the standard euclidean metric to order 2, it can be shown that

$$P_ω^* P_ω G_j \in C_2^{0,α}(B_j^*)$$

and hence that there exist $T$-invariant functions $\tilde{G}_j$ with the properties

$$P_ω^* P_ω \tilde{G}_j = 0 \text{ in } B_j^*, \quad \tilde{G}_j - G_j \in C_6^{4,α}(B_j^*)$$

for any $δ > 0$. (In fact, we can take $δ = 0$ if $m \geq 4$.) This is explained in greater detail in [2].

With the functions $\tilde{G}_j$ at hand, we define the deficiency spaces

$$D_0 := \text{Span}\{\chi_1, \ldots, \chi_n\}, \quad D_1 := \text{Span}\{\chi_1 \tilde{G}_1, \ldots, \chi_n \tilde{G}_n\},$$

where $\chi_j$ is a $T$-invariant cut-off function identically equal to 1 in $B_j^*(\frac{1}{2})$ and identically equal to 0 in $M \setminus B_j$. Now set

$$D = \begin{cases} D_0 \oplus D_1 & \text{if } m = 2; \\ D_0 & \text{if } m \geq 3. \end{cases}$$

(5.7)

The main result of this section is the following.

**Proposition 5.2.** Let the data be as in §5.1 and suppose that condition (ii) of Theorem 2.2 is satisfied, namely

$$\Re \xi''(p_1) + \cdots + \Re \xi''(p_n) = b''.$$  

(5.8)

Let $δ \in (0, 1)$. Then the map

$$\mathcal{L}_δ : (C_4^{4,α}(M^*)^T \oplus D) \times t \times \mathbb{R} \longrightarrow C_2^{0,α}(M^*)^T,$$

$$\mathcal{L}_δ : (f, X, μ) \longmapsto -P^* Pf - \langle ξ, X \rangle - μ$$

(5.9), (5.10)

given by (5.1) (with $f \in C_4^{4,α}(M^*)^T \oplus D$) is surjective and the dimension of the null-space is $n + 1 + \dim t$. 

Proof. The proof of this result can be deduced from the general theory of \('b\) elliptic operators (also known as asymptotically cylindrical operators) [24], [28] and [27]. We shall give an almost self-contained proof.

We start by noting that 
\[
C_{-2m+\delta}^{0,\alpha}(M^*) \subset L^p(M) \quad \text{for all} \quad p < 1 + \frac{\delta}{2m}.
\]
Thus the equation
\[
P^*_\omega P_\omega u + \langle \xi, X \rangle + \mu + \sum_{j=1}^n b_j \delta_{p_j} = f, \quad (X \in t, \mu \text{ a constant}) \tag{5.11}
\]
can be viewed as a distributional equation on \(M\) if \(f \in C_{-2m+\delta}^{0,\alpha}(M^*)\) and \(\delta > 0\), and will be solvable for \(u\) if and only if \(f - \sum b_j \delta_{p_j} - \langle \xi, X \rangle - \mu\) is annihilated by the \(T\)-invariant part of the null-space of \(P^*P\).

From our discussion of \(P^*P\), given \(f \in L^p(M)^T\), we have a unique decomposition
\[
f = f_0 + f' + f'' + f^\perp \tag{5.12}
\]
where \(f_0\) is a constant, \(X_f' \in t, X_f'' \in h''\) and all 4 terms on the right-hand side are pairwise \(L^2\)-orthogonal. Hence (5.11) is solvable if and only if
\[
\sum b_j + \mu \text{Vol}(M) = f_0 \text{Vol}(M); \tag{5.13}
\]
\[
\sum b_j \xi'(p_j) + X = X_f'; \tag{5.14}
\]
\[
\sum b_j \xi''(p_j) = X_f''. \tag{5.15}
\]
where we have written \(\xi'\) for the projection of \(\xi\) into \(t\). Now by (5.8) we can always solve for the \(b_j\) in (5.15). Having done so, (5.13) and (5.14) then fix \(\mu\) and \(X\).

To complete the proof, we invoke the general regularity theory for elliptic \('b'\)-differential operators, which gives that \(u \in (C^1_{\delta}(M^*) \oplus D)^T\). The computation of the dimension of the kernel will not be used in this paper and is left to the reader. \(\Box\)

In the proof of the Proposition we used the genericity condition of Theorem 2.2. Our next result brings the balancing condition into play.

**Proposition 5.3.** Let the data be as above, and suppose that condition (i) of Theorem 2.2 holds, that is, there exist positive constants \(a_j \in \mathbb{R}\) such that
\[
\sum_j a_j^{m-1} \xi''(p_j) = 0. \tag{5.16}
\]
Then there exists \((\Gamma_a, X_a, \mu_a) \in C^\infty(M^*)^T \times t \times \mathbb{R}\) such that
\[
\mathcal{L}(\Gamma_a, X_a, \mu_a) = 0. \tag{5.17}
\]
Moreover, $\Gamma_a$ has the following asymptotic behaviour near $p_j$:

$$\Gamma(a(z_j)) + a_j^{-m+1}G_j(z_j) = \begin{cases} b_j + c_j(z_j) + O_\infty(|z_j|^2 \log |z_j|) & \text{when } m = 2; \\ b_j + d_j \log |z_j| + O_\infty(|z_j|) & \text{if } m \geq 3. \end{cases}$$

Here $b_j$ and $d_j$ are constants, $c_j$ is a linear functional on $C^m$ and $O_\infty(|z|^a \log |z|^b)$ denotes a smooth function $f$ defined away from 0, for which

$$|V_1 \ldots V_k f| \leq C_k |z|^a \log |z|^b$$

for any smooth vector fields $V_1, \ldots, V_k$ all of which vanish at 0.

**Proof.** Consider the distributional equation

$$P^* P \Gamma(a) + \langle \xi, X_a \rangle + c_a = -c_m \sum_{j} a_j^{-m-1} \delta_{p_j}$$

on $M$, where

$$c_m = \begin{cases} 2|S^3| & \text{if } m = 2; \\ 4(m - 1)(m - 2)|S^{2m-1}| & \text{if } m \geq 3. \end{cases}$$

Condition (5.16) is used to prove the existence of $(\Gamma(a), X_a, c_a)$ exactly as in the previous Proposition. The reader is referred to [2] for a discussion of the asymptotics of $\Gamma_a$ near each of the $p_j$.

5.3. **Analysis of $L$ on $(N, \eta)$.** The following result is proved in [2]:

**Proposition 5.4.** Let $\delta \in (0, 1)$. Then the operator

$$\tilde{L}_\delta : C^{4,\alpha}(N)_T \to C^{0,\alpha}(N)_T,$$

$$\tilde{L}_\delta : f \mapsto -P_\eta^* P_\eta f$$

is surjective. Its kernel is one-dimensional and consists of the constant functions.

5.4. **Biharmonic extensions.** The following results are concerned with biharmonic extensions either on the complement of the unit ball or on the unit ball of $C^m$ of boundary data defined on the unit sphere. Recall the notation

$$\overline{B^*} = \{ x \in \mathbb{C}^m : 0 < |x| \leq 1 \}, \quad \overline{C} = \{ x \in \mathbb{C}^m : |x| \geq 1 \}, \quad S = \{ x \in \mathbb{C}^m : |x| = 1 \}.$$

**Proposition 5.5.** Assume $h \in C^{4,\alpha}(S)_T$ and $k \in C^{2,\alpha}(S)_T$ are given.

(i) If

$$\int_S (4m h - k) \, dvol_{\text{eucl}} = 0,$$

there exists a function $W^i = W^i_{h,k} \in C^{4,\alpha}(\overline{B^*})_T$ such that

$$\Delta^2 W^i = 0 \text{ in } B, \quad W^i|S = h, \Delta W^i|S = k.$$
Moreover, there is a constant $c$ such that
\[ \|W^i\|_{C^{4,\alpha}_1(\tilde{B}^*)} \leq c (\|h\|_{C^{4,\alpha}(S)} + \|k\|_{C^{2,\alpha}(S)}). \]

(ii) Instead of (5.21), suppose that
\[ \int_{\partial B_1} k \, d\text{vol}_{\text{eucl}} = 0. \] (5.22)

Then there exists a function $W^o = W^o_{h,k} \in C^{4,\alpha}_{3-2m}(\bar{C})$ such that
\[ \Delta^2 W^o = 0, \quad \text{in } \bar{C}, \quad W^o|S = h, \Delta W^o|S = k. \]

Moreover,
\[ \|W^o\|_{C^{4,\alpha}_{3-2m}(C)} \leq c (\|h\|_{C^{4,\alpha}(S)} + \|k\|_{C^{2,\alpha}(S)}). \]

Let us briefly comment on the conditions (5.21) and (5.22). To this end, let us concentrate on the case where both $h$ and $k$ are constant functions in which case their bi-harmonic extensions $W^i$ and $W^o$ are given explicitly by
\[ W^i(z) = a + b |z|^2, \]
and
\[ W^o(z) = c |z|^{4-2m} + d |z|^{2-2m}, \]
when $m \geq 3$ and
\[ W^o(z) = c \log |z| + d |z|^{-2}, \]
when $m = 2$. It is easy to see that $W^i \in C^{4,\alpha}_1(\tilde{B}^*)$ if and only if $a = 0$ and that $W^o \in C^{2,\alpha}_{3-2m}(C_1)$ if and only if $c = 0$. This computation underlies the constraints on $(h,k)$ in the statement of Proposition 5.5.

6. Nonlinear perturbation results

In this section we shift attention from $M^*$ and $N$ to the manifolds with boundary $M_r$ and $N_R$ defined in §5.1. We shall build on the linear analysis of the previous section to construct infinite-dimensional families of extremal Kähler perturbations of $\omega$ on $M_r$ and of $\eta$ on $N_R$.

The role of the analysis of the previous section is to provide uniform control on the right-inverse of $L$ on $M_r$ and $\tilde{L}$ on $N_R$ as $r \to 0$ and $R \to \infty$ respectively. To make this precise, denote by $C^{\ell,\alpha}_\delta(M_r)$ the set of restrictions of elements of $C^{\ell,\alpha}_\delta(M^*)$ to $M_r$, with the induced norm. Similarly, denote by $C^{\ell,\alpha}_\delta(N_R)$ the set of restrictions of elements of $C^{\ell,\alpha}_\delta(N)$ to $N_R$ with the induced norm. For every $r > 0$, the $C^{\ell,\alpha}_\delta(M_r)$-norm is equivalent to the $C^{\ell,\alpha}_\delta(M_r)$-norm, but the equivalence is not uniform as $r$ tends to 0. A similar remark applies to the $C^{\ell,\alpha}_\delta(N_R)$-norm.
The restriction map $C^\ell,\alpha(M^*) \to C^\ell,\alpha(M_r)$ is uniformly bounded in $r$, essentially by definition of the norms. Conversely, we can define a uniformly bounded extension linear operator $\mathcal{E}_{r,\delta} : C^{0,\alpha}_\delta(M_r)^T \to C^{0,\alpha}_\delta(M^*)^T$. This is defined by pulling back by the radial projection map $\overline{B}_j(r) \to S_j(r) (z \mapsto rz/|z|)$ and then multiplying by a $T$-invariant cut-off function which vanishes in $B_j(r/2)$, say. (An explicit formula can be found in [2].)

Combining these ideas, we have

**Proposition 6.1.** Let $\delta \in (0,1)$. The operator

$$\mathcal{L} : C^{4,\alpha}_{4-2m+\delta}(M_r)^T \oplus D \times t \times \mathbb{R} \to C^{0,\alpha}_{-2m+\delta}(M_r)$$

has a right inverse $\mathcal{G} = \mathcal{G}_{\delta,r}$ if $0 < \delta < 1$, with operator norm bounded independent of $r$.

**Proof.** Just compose the right-inverse of Proposition 5.2 with the extension operator $\mathcal{E}_{r,\delta}$. □

The corresponding result for $N_R$ is

**Proposition 6.2.** Let $\delta \in (0,1)$. The operator

$$\tilde{\mathcal{L}} : C^{4,\alpha}_{\delta}(N_R) \to C^{0,\alpha}_{\delta-4}(N_R)$$

has a right inverse $\tilde{\mathcal{G}}_{\delta,R}$ with operator norm bounded independent of $R$.

6.1. **Perturbation of $\omega$.** Recall the notation

$$M(r) = M \setminus \cup_j B_j(r).$$

In this section we construct a family of extremal Kähler perturbations of $(M_r,\omega)$ (for sufficiently small $r$) parameterized, roughly speaking, by boundary data for the biharmonic equation on $M_r$. The perturbation takes place in three steps. The first involves the solution $(\Gamma,X,\mu) := (\Gamma_a,X_a,\mu_a)$ of the linearized equation from Proposition 5.3. The second incorporates the boundary data, and uses the biharmonic extensions of Proposition 5.5. After these two steps we have a family of ‘approximately extremal Kähler metrics’ on $M_r$; in the final step the contraction mapping theorem is applied to perturb these approximate solutions to give genuine solutions of the problem.

Set

$$r_\varepsilon := \varepsilon^{\frac{2m-1}{2m+1}}, \quad \text{so that} \quad \varepsilon = r_\varepsilon r^2_{\varepsilon}, \quad \text{for} \quad j = 1, \ldots, n.$$  \hspace{1cm} (6.1)

Denote by $A_j(r_\varepsilon)$ the closed annulus $\overline{B}_j(2r_\varepsilon) \setminus B_j(r_\varepsilon)$ and by $j_\varepsilon : A_j \to A_j(r_\varepsilon)$ the map $x \mapsto r_\varepsilon x$. (This abuse of notation should cause no confusion in what follows.)

To specify the boundary data, assume given

$$h_j \in C^{4,\alpha}(S)^T \quad \text{and} \quad k_j \in C^{2,\alpha}(S)^T,$$

for $j = 1, \ldots, n$, satisfying

$$\|h_j\|_{C^{4,\alpha}(S)} + \|k_j\|_{C^{2,\alpha}(S)} \leq \kappa r_\varepsilon^4,$$  \hspace{1cm} (6.2)
where \( \kappa > 0 \) will be fixed later on. As usual, the superscript \( T \) here indicates that the functions \( h_j \) and \( k_j \) are supposed to be \( T \)-invariant. We further assume that
\[
\int_S k_j \, dv_{geucl} = 0,
\]
so that part (ii) of Proposition 5.5 applies. To keep notation short we set
\[
h := (h_1, \ldots, h_n) \quad \text{and} \quad k := (k_1, \ldots, k_n).
\]

Then we can define a \( T \)-invariant function \( W_{\varepsilon, h, k} \) which is identically equal to 0 in \( M(2) \) and, for \( j = 1, \ldots, n \) is given by
\[
W_{\varepsilon, h, k}(z_j) := \chi_j(z_j) W_{h_j, k_j}(z/r_{\varepsilon}),
\]
in \( B_j(2) \). Here \( \chi_j \) is a \( T \)-invariant cutoff function identically equal to 1 in \( B_j \) and identically equal to 0 in \( M \setminus B_j(2) \). Observe that the function \( W_{\varepsilon, h, k} \) depends (linearly) on \( h \) and \( k \) and is \( T \)-invariant.

This being understood, the idea is to look for an extremal Kähler metric defined in \( M(r_{\varepsilon}) \) as a perturbation of
\[
\omega + i \partial \bar{\partial} (\varepsilon^{2m-2} \Gamma + W_{\varepsilon, h, k}).
\]
The purpose of the next Proposition is to show that this perturbation is possible provided \( \varepsilon \) is small enough.

**Proposition 6.3.** Let conditions (i) and (ii) of Theorem 2.2 hold. Let \( (h, k) \) be boundary data as above, with components satisfying (6.2) and (6.3) for some given \( \kappa > 0 \). Then there exists \( \varepsilon_{\kappa} > 0 \) such that for all \( \varepsilon \in (0, \varepsilon_{\kappa}) \) one can find a function \( \varphi_{\varepsilon, h, k} \in C^{4,\alpha}(M_{\varepsilon}) \), a vector field \( Y_{\varepsilon, h, k} \in \mathfrak{t} \) and a constant \( \lambda_{\varepsilon, h, k} \in \mathbb{R} \) such that
\[
\omega_{\varepsilon, h, k} = \omega + i \partial \bar{\partial} (\varepsilon^{2m-2} \Gamma + W_{\varepsilon, h, k} + \varphi_{\varepsilon, h, k}),
\]
is an extremal Kähler metric on \( M(r_{\varepsilon}) \), with extremal vector field
\[
X_{\varepsilon, h, k} = X_s + \varepsilon^{2m-2} X + Y_{\varepsilon, h, k}.
\]
Let
\[
s_{\varepsilon, h, k} = s(\omega_{\varepsilon, h, k}),
\]
and, for \( j = 1, \ldots, n \), let
\[
\varphi^j_{\varepsilon, h, k} = \varphi_{\varepsilon, h, k}|A_j(r_{\varepsilon})
\]
Then there exist constants \( c > 0 \) and \( \theta > 0 \) independent of \( \kappa \) such that
\[
\|Y_{\varepsilon, h, k}\| + \sup_{M(r_{\varepsilon})} |s_{\varepsilon, h, k} - s(\omega)| \leq c \varepsilon^{\theta}
\]
and, for each \( j \),
\[
\|s^j_{\varepsilon} \varphi^j_{\varepsilon, h, k}\|_{C^{4,\alpha}(A^0)} \leq c \varepsilon^4.
\]
Suppose further that \((h', k')\) are another set of boundary data satisfying (6.2) and (6.3). Then we have
\[
\|Y_{\varepsilon, h, k} - Y_{\varepsilon, h', k'}\| + \sup_{M(r_\varepsilon)} |s_{\varepsilon, h, k} - s_{\varepsilon, h', k'}| \leq c\varepsilon^\theta \|(h - h', k - k')\|
\]
and
\[
\|J^2 \varphi_{\varepsilon, h, k} - J^2 \varphi_{\varepsilon, h', k'}\|_{C^{4,\alpha}(A^o)} \leq c\varepsilon^\theta \|(h - h', k - k')\|. \tag{6.5}
\]
On the right-hand side of these estimates, the norm is the product norm on \((C^{4,\alpha}(S) \times C^{2,\alpha}(S))^n\).

**Remark 6.4.** The constant \(\theta\) can be found explicitly but its value is not relevant for the forthcoming analysis. A key point in the statement of Proposition 6.3 is that \(c\) and \(\theta\) can be assumed to be independent of \(\kappa\) by taking \(\varepsilon, \kappa\) sufficiently small. This is crucial in §7, where we will have to allow for the possibility of \(\kappa\) being some large constant.

**Proof.** Suppressing the notational dependence on the parameters \(\varepsilon, h, k\), for the moment, we shall seek \((f, Y, \lambda) \in C^{4,\alpha}(M(r_\varepsilon))^T \times t \times \mathbb{R}\) such that
\[
\Phi(\varepsilon^{2m-2} \Gamma + W + \varphi, \varepsilon^{2m-2} X + Y, \varepsilon^{2m-2} \mu + \lambda) = 0 \tag{6.6}
\]
on \(M(r_\varepsilon)\), where \((\Gamma, X, \mu)\) comes from Proposition 5.3. Note that the \(C^2\) norm of \(\varepsilon^{2m-2} \Gamma + W\) over \(M(r_\varepsilon)\) tends to zero as \(\varepsilon \to 0\), so that \(\omega + i\partial\bar{\partial}(\varepsilon^{2m-2} \Gamma + W)\) is a genuine Kähler metric once \(\varepsilon\) is small enough. Similarly, it will follow from the construction that \(\varphi_{\varepsilon, h, k}\) is small in \(C^2\), so that (6.6) is well-defined.

Applying the result of Proposition 3.5 and using the fact that \(\mathcal{L}(\Gamma, X, \mu) = 0\), (6.6) is equivalent to
\[
\mathcal{L}(f, Y, \mu) = P^* PW + \frac{1}{2} J(\varepsilon^{2m-2} X + Y)(\varepsilon^{2m-2} \Gamma + W + \varphi) + Q(\varepsilon^{2m-2} \Gamma + W + \varphi). \tag{6.7}
\]
Recall from Proposition 6.1 the uniform right-inverse \(\mathcal{G} = \mathcal{G}_{r, \delta}\) of \(\mathcal{L}\). Using this we see that a solution of the fixed-point problem
\[
(\varphi, Y, \mu) = \mathcal{N}(\varphi, Y, \mu), \tag{6.8}
\]
where the nonlinear operator \(\mathcal{N}\) is defined by
\[
\mathcal{N}(\varphi, Y, \mu) = \mathcal{G} P^* PW
\]
\[
+ \mathcal{G} \left( \frac{1}{2} J(\varepsilon^{2m-2} X + Y)(\varepsilon^{2m-2} \Gamma + W + \varphi) + Q(\varepsilon^{2m-2} \Gamma + W + \varphi) \right). \tag{6.9}
\]
gives a solution of (6.7).

To solve (6.8), fix \(\delta \in (0, 1)\) when \(m \geq 3\) or \(\delta \in (0, \frac{2}{3})\) when \(m = 2\), and set
\[
\mathcal{F} := (C^{4,\alpha}_{4-2m+\delta}(M^*)^T \oplus D) \times t \times \mathbb{R},
\]
endowed with the product norm.

The existence of a solution to (6.8) will follow from the contraction mapping theorem once we have the following estimates:
Lemma 6.5. With the conditions of the Proposition, there exist constants $c > 0$, $c_\kappa > 0$ and $\varepsilon_\kappa > 0$ such that, for all $\varepsilon \in (0, \varepsilon_\kappa)$

$$\|N(h, k, 0, 0)\|_F \leq c_\kappa (r_\varepsilon^{2m+1} + \varepsilon^{4m-4} r_\varepsilon^{2m-2})$$

(6.10)

and

$$\|N(h, k; \varphi, Y) - N(h, k; \varphi', Y')\|_F \leq c_\kappa \varepsilon^{2m-2} r_\varepsilon^{2m-2} \|(\varphi - \varphi', Y - Y', 0)\|_F.$$

(6.11)

Finally,

$$\|N(h, k; \varphi, Y) - N(h', k'; \varphi, Y)\|_F \leq c_\kappa (r_\varepsilon^{2m-3} + \varepsilon^{2m-2} r_\varepsilon^{2m-2}\delta) \|(h - h', k - k')\|_{(C^4, \alpha)^n(C^2, \alpha)^n_F}$$

provided $(\varphi, Y, 0)$ and $(\varphi', Y', 0)$ belong to the ball centered at the origin and of radius $2c_\kappa (r_\varepsilon^{2m+1} + \varepsilon^{4m-4} r_\varepsilon^{2m-2})$ in $F$, and provided the components of $h, h'$ and $k, k'$ satisfy (6.2) and (6.3).

Proof. The proofs of these estimates are very close to computations already done in [1] and [2]. We shall therefore describe the proof of (6.10) and leave the rest of the proof to the reader. Suppressing again the notational dependence on the parameters $\varepsilon, h, k$, we start by using Proposition 5.5 to estimate

$$\|W\|_{C^4_3 (M(r_\varepsilon))} \leq c_\kappa r_\varepsilon^{2m+1}.$$

(6.13)

Now by construction, $\Delta^2 W = 0$ in $B_j(\frac{1}{2}) \setminus B_j(r_\varepsilon)$ (here $\Delta$ is the Euclidean Laplacian), hence

$$P_\omega^* P_\omega W = (P_\omega^* P_\omega - \Delta^2) W \text{ in } B_j(\frac{1}{2}) \setminus B_j(r_\varepsilon).$$

Making use of the fact that the coordinates near $p_j$ are chosen to be normal, we get the existence of a constant $c_\kappa > 0$ such that

$$\|P_\omega^* P_\omega W\|_{C^0_{-2m+4}(M(r_\varepsilon))} \leq c_\kappa r_\varepsilon^{2m+1} (1 + r_\varepsilon^{1-\delta}) \leq c_\kappa r_\varepsilon^{2m+1},$$

using $1 - \delta > 0$ to simplify the last estimate.

Next, using the fact that $X$ vanishes at all $p_j$, we estimate $\varepsilon^{2m-2} J X (\varepsilon^{2m-2} \Gamma + W)$ by

$$\|\varepsilon^{2m-2} J X (\varepsilon^{2m-2} \Gamma + W)\|_{C^0_{-2m+4}(M(r_\varepsilon))} \leq c\varepsilon^{4m-4},$$

where the constant $c > 0$ does not depend on $\kappa$ provided $\varepsilon$ is chosen small enough, say $\varepsilon \in (0, \varepsilon_\kappa)$.

Finally, we use the structure of the nonlinear operator $Q$ along with the estimate (6.13) to get

$$\|Q_\omega (\nabla^2 (\varepsilon^{2m-2} \Gamma a + W_\varepsilon h, k))\|_{C^0_{-2m+3}(M(r_\varepsilon))} \leq c\varepsilon^{4m-4} (1 + r_\varepsilon^{2-2m-\delta}) \leq c\varepsilon^{4m-4} r_\varepsilon^{2-2m-\delta},$$

(6.14)
for some constant $c > 0$ which does not depend on $\kappa$ provided $\varepsilon$ stays small enough. This completes the proof of (6.10). □

Reducing $\varepsilon_\kappa > 0$ if necessary, we can assume that,

$$c_\kappa \varepsilon^{2m-2} r_\varepsilon^{2-2m-\delta} \leq \frac{1}{2} \tag{6.14}$$

for all $\varepsilon \in (0, \varepsilon_\kappa)$. Then (6.10) and (6.11) show that

$$(\varphi, Y, \lambda) \mapsto N(h, k; \varphi, Y),$$

is a contraction from

$$\left\{ (\varphi, Y, \lambda) \in \mathcal{F} : \| (\varphi, Y, \lambda) \| \leq 2 c_\kappa (r_\varepsilon^{2m+1} + \varepsilon^{4m-4} r_\varepsilon^{4-2m-\delta}) \right\},$$

into itself and hence has a unique fixed point $(\varphi_\varepsilon, h, k, Y_\varepsilon, \lambda_\varepsilon)$ in this set. This fixed point yields a solution of (6.6) in $M(r_\varepsilon)$ and hence provides an extremal Kähler form on $M(r_\varepsilon)$. The estimates in Proposition 6.3 follow from the corresponding estimates in Lemma 6.5. Since the $\kappa$-dependent constants in this Lemma are all multiplied by a positive power of $\varepsilon$, it follows that by decreasing $\varepsilon_\kappa$ if necessary we can assume that the constant $c$ in Proposition 6.3 is independent of $\kappa$. The condition $\delta \in (0, \frac{2}{3})$ is required to have the relevant estimates in dimension $m = 2$. □

6.2. Perturbation of $\eta$. The goal of this section is to find a family of extremal perturbations of the Burns–Simanca metric $\eta$ on $N(R)$, with prescribed extremal vector field. This family will be parameterized essentially by boundary data for the biharmonic equation on $N_R$.

Fix

$$R_\varepsilon := \frac{r_\varepsilon}{\varepsilon} = \varepsilon^{- \frac{2}{2m+1}}. \tag{6.15}$$

Denote by

$$\tilde{\eta}_\varepsilon(x) = R_\varepsilon x/a$$

and regard this as an identification of $A^i$ with the annulus $\{ R_\varepsilon / 2a \leq |u| \leq R_\varepsilon / e/a \}$ in $N(R_\varepsilon / a)$.

Assume given $h \in C^{4,\alpha}(S)^T$ and $k \in C^{2,\alpha}(S)^T$ satisfying

$$\|h\|_{C^{4,\alpha}(S)} + \|k\|_{C^{2,\alpha}(S)} \leq \kappa R_\varepsilon^{3-2m}, \tag{6.16}$$

and

$$\int_S (4mh - k) \, dvol_{eucl} = 0, \tag{6.17}$$

Here the constant $\kappa > 0$ will be fixed later on. We define in $N(R_\varepsilon / a)$ the function

$$W_{\varepsilon,a,h,k}(u) := \tilde{\chi}(u) W_{h,k}(au/R_\varepsilon) \tag{6.18}$$

for some constant $c > 0$ which does not depend on $\kappa$ provided $\varepsilon$ stays small enough. This completes the proof of (6.10). □

Reducing $\varepsilon_\kappa > 0$ if necessary, we can assume that,

$$c_\kappa \varepsilon^{2m-2} r_\varepsilon^{2-2m-\delta} \leq \frac{1}{2} \tag{6.14}$$

for all $\varepsilon \in (0, \varepsilon_\kappa)$. Then (6.10) and (6.11) show that

$$(\varphi, Y, \lambda) \mapsto N(h, k; \varphi, Y),$$

is a contraction from

$$\left\{ (\varphi, Y, \lambda) \in \mathcal{F} : \| (\varphi, Y, \lambda) \| \leq 2 c_\kappa (r_\varepsilon^{2m+1} + \varepsilon^{4m-4} r_\varepsilon^{4-2m-\delta}) \right\},$$

into itself and hence has a unique fixed point $(\varphi_\varepsilon, h, k, Y_\varepsilon, \lambda_\varepsilon)$ in this set. This fixed point yields a solution of (6.6) in $M(r_\varepsilon)$ and hence provides an extremal Kähler form on $M(r_\varepsilon)$. The estimates in Proposition 6.3 follow from the corresponding estimates in Lemma 6.5. Since the $\kappa$-dependent constants in this Lemma are all multiplied by a positive power of $\varepsilon$, it follows that by decreasing $\varepsilon_\kappa$ if necessary we can assume that the constant $c$ in Proposition 6.3 is independent of $\kappa$. The condition $\delta \in (0, \frac{2}{3})$ is required to have the relevant estimates in dimension $m = 2$. □

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Fix

$$R_\varepsilon := \frac{r_\varepsilon}{\varepsilon} = \varepsilon^{- \frac{2}{2m+1}}. \tag{6.15}$$

Denote by

$$\tilde{\eta}_\varepsilon(x) = R_\varepsilon x/a$$

and regard this as an identification of $A^i$ with the annulus $\{ R_\varepsilon / 2a \leq |u| \leq R_\varepsilon / e/a \}$ in $N(R_\varepsilon / a)$.

Assume given $h \in C^{4,\alpha}(S)^T$ and $k \in C^{2,\alpha}(S)^T$ satisfying

$$\|h\|_{C^{4,\alpha}(S)} + \|k\|_{C^{2,\alpha}(S)} \leq \kappa R_\varepsilon^{3-2m}, \tag{6.16}$$

and

$$\int_S (4mh - k) \, dvol_{eucl} = 0, \tag{6.17}$$

Here the constant $\kappa > 0$ will be fixed later on. We define in $N(R_\varepsilon / a)$ the function

$$W_{\varepsilon,a,h,k}(u) := \tilde{\chi}(u) W_{h,k}(au/R_\varepsilon) \tag{6.18}$$
where \( \tilde{\chi} \) is a \( T \)-invariant cutoff function which is identically equal to 1 in \( C_2 \) and identically equal to 0 in \( N_1 \) and \( W_{h,k}^j \) has been defined in Proposition 5.5. Then \( W_{\varepsilon,a,h,k} \) is \( T \)-invariant and depends linearly on \((h,k)\).

The counterpart of Proposition 6.3 reads as follows:

**Proposition 6.6.** Suppose \((X, \nu) \in t \times \mathbb{R} \) with
\[
|\nu| + \|X\| \leq c_0. 
\]
Let \( a_{\text{min}} \) and \( a_{\text{max}} \) be two positive constants and suppose that
\[
a_{\text{min}} \leq a \leq a_{\text{max}}. 
\]
Let \((h,k)\) be boundary data satisfying (6.16) and (6.17) for some given \( \kappa \). Then there exists \( \varepsilon_\kappa > 0 \) such that for any \( \varepsilon \in (0, \varepsilon_\kappa) \) one can find \( \phi = \varphi_{\varepsilon,a,\nu,X,h,k} \in C^4_\alpha(N(R_\varepsilon/a)) \) such that
\[
\tilde{\eta} := \eta + i\partial\bar{\partial}(W_{h,k} + \varphi)
\]
is an extremal Kähler form on \( N(R_\varepsilon/a) \), with extremal vector field \( \varepsilon^4 X \), and such that
\[
\frac{1}{|S|} \int_S j_\varepsilon^* \tilde{s} \, d\text{vol}_{g_{eucl}} = \varepsilon^2 \nu,
\]
where \( \tilde{s} = s(\tilde{\eta}) \).

Let \((a', h', k', \nu', X')\) satisfy the same conditions as \((a, h, k, \nu, X)\) and put \( \varphi' = \varphi_{\varepsilon,a',\nu',X',h',k'} \). Also write \( j_{\varepsilon,a}(x) = R_\varepsilon x/a, j_{\varepsilon,a'}(x) = R_\varepsilon x/a' \). There exists a constant \( c > 0 \) independent of \( \kappa \), such that
\[
\|j_{\varepsilon,a}^* \varphi \|_{C^4_\alpha(A^1)} \leq c R_\varepsilon^{3-2m}.
\]
and, for any \( \delta \in (0,1) \),
\[
\|j_{\varepsilon,a}^* \varphi - j_{\varepsilon,a'}^* \varphi' \|_{C^4_\alpha(A^1)} \leq c (R_\varepsilon^{3-1} \|(h - h', k - k')\|_{C^4_\alpha(S) \times C^2_\alpha(S)} + R_\varepsilon^{3-2m} (|\nu - \nu'| + |a - a'|) + R_\varepsilon^{1-4m} \|X - X'|)). \tag{6.21}
\]

**Remark 6.7.** As in Proposition 6.3, the crucial point here is that the constant \( c \) can be taken to be independent of \( \kappa \), although it will depend upon \( c_0, a_{\text{min}} \) and \( a_{\text{max}} \).

**Proof.** Let \( \zeta \) be a hamiltonian generating function for \( X \) with respect to \( \eta \),
\[
-d\zeta = \eta(X, -). \tag{6.22}
\]
Writing \( W = W_{\varepsilon,h,k} \), we shall solve the equation
\[
s(a^2 \eta + i\partial\bar{\partial}(W + \varphi) = \varepsilon^4 (a^2 \zeta - \frac{1}{2}JX(W + \varphi) - \mu) \tag{6.23}
\]
where the constant \( \mu \) is chosen so that
\[
\frac{1}{|S|} \int_S j_\varepsilon^* (a^2 \zeta - \frac{1}{2}J(W + f) - \mu) = \varepsilon^{-2} \nu. \tag{6.24}
\]
Now
\[ s(a^2 \eta + i \partial \bar{\partial} f) = s(a^2 (\eta + ia^{-2} \partial \bar{\partial} f)) = a^{-2} s(\eta + ia^{-2} \partial \bar{\partial} f), \]
so we can rewrite (6.23) as
\[ -a^{-4} P_{\eta}^* P_{\eta} (W + \varphi) + a^{-2} Q_{\eta}(a^{-2}(W + \varphi)) = \varepsilon^4 (a^2 \zeta - \frac{1}{2} JX(W + \varphi) - \mu), \]
(6.25)
since \( s(\eta) = 0. \)

Using the uniform right inverse \( \tilde{G}_{R,\delta} \) of Proposition 6.2, we reformulate this as a fixed-point problem
\[ \varphi = \tilde{N}(a, \nu, X, h, k; \varphi), \]
(6.26)
where
\[ \tilde{N}(a, \nu, X, h, k; \varphi) = \tilde{G}(P_{\eta}^* P_{\eta} \varphi + \varepsilon^4 a^2 (a^2 \zeta - \frac{1}{2} JX(W + \varphi) - \mu) - a^2 Q_{\eta}(W + \varphi)) \]
(6.27)
where \( \mu \) and \( \nu \) are related by (6.24). The existence of our fixed point will follow from the:

**Lemma 6.8.** Let the hypotheses and notation be as in Proposition 6.6 There exists \( c > 0 \) (independent of \( \kappa \)), \( c_\kappa > 0 \) and there exists \( \varepsilon_\kappa > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_\kappa) \)
\[ \|\tilde{N}(\varepsilon, a, \nu, X, h, k; 0)\|_{\tilde{F}} \leq c R_{\varepsilon}^{3-2m-\delta}, \]
(6.28)
Moreover, we have
\[ \|\tilde{N}(\varepsilon, a, \nu, X, h, k; \varphi) - \tilde{N}(\varepsilon, a, \nu, X, h, k; \varphi')\|_{\tilde{F}} \leq c_\kappa (R_{\varepsilon}^{3-2m-\delta} + \varepsilon^4 R_{\varepsilon}^4) \|\varphi - \varphi'\|_{\tilde{F}} \]
(6.29)
and
\[ \|\tilde{N}(\varepsilon, a, \nu, X, h, k; \varphi) - \tilde{N}(\varepsilon, a', \nu', X', h', k'; \varphi)\|_{\tilde{F}} \leq c_\kappa (R_{\varepsilon}^{-1} \|(h - h', k - k')\|_{C^4,\alpha} + R_{\varepsilon}^{3-2m-\delta} (|\nu - \nu'| + |a - a'|) + R_{\varepsilon}^{4-4m-\delta} \|X - X'\|), \]
(6.30)
provided \( \varphi, \varphi' \in \tilde{F} \), satisfy
\[ \|\varphi\|_{\tilde{F}} + \|\varphi'\|_{\tilde{F}} \leq 2 c R_{\varepsilon}^{3-2m-\delta}, \]
\( h, h' \) and \( k, k' \) satisfy (6.16), \( \nu, \nu' \) and \( X, X' \) satisfy (6.19) and \( a, a' \) satisfy (6.20).

**Proof.** The proof is identical to that of the corresponding result in [2]; we omit it. \( \square \)

Reducing \( \varepsilon_\kappa > 0 \) if necessary, we can assume that,
\[ c_\kappa (R_{\varepsilon}^{3-2m-\delta} + \varepsilon^4 R_{\varepsilon}^4) \leq \frac{1}{2} \]
(6.31)
for all \( \varepsilon \in (0, \varepsilon_\kappa) \). Then (6.28) and (6.29) show that
\[ \varphi \mapsto \tilde{N}(\varepsilon, a, \nu, X, h, k; \varphi) \]
is a contraction from
\[ \{ \varphi \in \tilde{F} : \|\varphi\|_{\tilde{F}} \leq 2 c R_{\varepsilon}^{3-2m-\delta} \}, \]
into itself and hence \( N \) has a unique fixed point \( \varphi_{\varepsilon,a,\nu,X,h,k} \) in this set. This fixed point gives the required extremal \( \mathcal{K} \)ahler perturbation \( \tilde{\eta} \) of \( \eta \) in \( N(R_{\varepsilon}/a) \). The estimates in the second part of Proposition 6.6 follow at once from the corresponding estimates in Lemma 6.8, increasing the value of \( c_\kappa \) and reducing \( \varepsilon_\kappa \) if this is necessary. \( \square \)

7. Matching Cauchy data for extremal metrics

Building on the analysis of the previous sections we complete the proof of Theorem 2.2, following the idea of Cauchy-data matching described in §4.4. As far as technicalities are concerned the proof is extremely close to the one in [2], so we shall keep the discussion brief. The argument is made slightly more complicated because in §6.1 and §6.2 we were not, in fact, able to find perturbations with arbitrarily assigned boundary data: recall that in each case one degree of freedom was lost because of the constraints (6.3) and (6.17). On the other hand, in §6.2 we built in the additional freedom to rescale the metric \( \eta \) by an arbitrary positive factor. We shall use this to recover, by hand, the degrees of freedom lost in (6.3) and (6.17).

Before we proceed, a word about notation. In this section \( O_{C^\ell,\alpha}(F) \) will refer to a function whose \( C^\ell,\alpha \)-norm is bounded by \( F \) times a constant independent of \( \varepsilon \) and also independent of \( \kappa \) provided \( \varepsilon \) is chosen small enough (but which might depend on \( m, \omega \), the points \( p_j \) and the coefficients \( a_j \)). In general this function will be a nonlinear operator of the data.

7.1. Extremal metrics on \( M(r_{\varepsilon}) \). Recall the notation from §5.1 and §6.1. In particular, we have boundary data

\[
\mathbf{h} = (h_1, \ldots, h_n), \quad \mathbf{k} = (k_1, \ldots, k_n)
\]

satisfying the bound (6.2) and the constraint

\[
\int_S k_j = 0 \tag{7.1}
\]

for each \( j \). We continue to use the identification \( j_{\varepsilon} : A^0 \to A_j(r_{\varepsilon}) \) to transfer data from \( M(r_{\varepsilon}) \) to the standard annulus \( A^0 = \{1 \leq |x| \leq 2\} \subset \mathbb{C}^m \).

Then we have:

**Lemma 7.1.** With the notation of Proposition 6.3, we have

\[
\omega^{o,j} := j_{\varepsilon}^* (\omega_{\varepsilon,h,k}|A_j(r_{\varepsilon})) = i\partial \overline{\partial} \left( \frac{1}{2} |x|^2/r_{\varepsilon}^2 + \psi^{o,j} \right).
\]

where

\[
\psi^{o,j} = -a_j^{m-1} \varepsilon^{2m-2} r_{\varepsilon}^{4-2m} G_m + W_{h_j,k_j}^0 + O_{C^4,o}(r_{\varepsilon}^4) \text{ in } A^0. \tag{7.2}
\]

Here \( G_2(x) = -\log |x| \) and \( G_m(x) = |x|^{4-2m} \) if \( m \geq 3 \).
Proof. This follows immediately from Proposition 6.3 and Proposition 5.3. When \( m = 2 \), we use the freedom to change the Kähler potential by a constant to achieve this form, cf. [2]. □

For the purposes of Cauchy-data matching, we want to rewrite this in terms of data \((h', k')\) which no longer satisfies the constraint (7.1). If we change the boundary data functions \( h_j \) and \( k_j \) into \( h'_j \) and \( k'_j \) defined by

\[
\begin{align*}
  h'_j &= (\tilde{a}_j^{m-1} - a_j^{m-1}) r_\epsilon^{4-2m} \varepsilon^{2m-2} + h_j \\
  k'_j &= 4 (m-2) (a_j^{m-1} - \tilde{a}_j^{m-1}) \varepsilon^{2m-2} r_\epsilon^{4-2m} + k_j
\end{align*}
\]

when \( m \geq 3 \) and

\[
\begin{align*}
  h'_j &= h_j \\
  k'_j &= 4 (a_j - \tilde{a}_j) \varepsilon^2 + k_j
\end{align*}
\]

when \( m = 2 \). Now if \( k \) is a constant, we extend the definition of \( W_{0,h,k}^{\alpha} \) by setting

\[
W_{0,h,k}^{\alpha}(x) := \frac{k}{4(m-2)} (|x|^{2-2m} - |x|^{4-2m}),
\]

when \( m \geq 3 \) and

\[
W_{0,h,k}^{\alpha}(x) = \frac{k}{4} \log |x|,
\]

when \( m = 2 \). Then we can trivially rewrite (7.2) in the form

\[
\psi^{\alpha,j} = -\tilde{a}_j^{m-1} r_\epsilon^{4-2m} \varepsilon^{2m-2} G_j + W_{h',k'}^{\alpha} + O_{C^4,\alpha}(r_\epsilon^4)
\]

and \((h'_j, k'_j) \in C^{4,\alpha}(S) \times C^{2,\alpha}(S)\) are no longer required to satisfy (7.1).

7.2. Extremal metrics on \( N_j \). Let the notation be as in the previous section. In particular, (7.3) or (7.4) determine a value \( \tilde{a}_j > 0 \) for each \( j \). Let \( \hat{a}_j = \sqrt{\tilde{a}_j} \). Decreasing \( \varepsilon_\kappa \) if necessary, we may assume that the \( \hat{a}_j \) lies in some given interval \([a_{\min}, a_{\max}]\), where \( a_{\min} > 0 \), for each \( j \). Let \( N_j = N(R_\epsilon/\hat{a}_j) \).

By the discussion in §4, for each \( j \) the restriction of \( X_{\varepsilon,h,k} \), the extremal vector field of \( \omega_{\varepsilon,h,k} \), corresponds to an element \( X_j \) of \( T \), relative to a choice of \( T \)-linear coordinates defined near \( p_j \). Denote by the same symbol the lift of this vector field to \( N_j \). Similarly, set

\[
\nu_j = \frac{1}{|S|} \int_S \frac{1}{r_\epsilon^4} s_{\varepsilon,h,k}.
\]

Further, let

\[
\tilde{h} = (\tilde{h}_1, \ldots, \tilde{h}_n), \quad \tilde{k} = (\tilde{k}_1, \ldots, \tilde{k}_n)
\]

be boundary data as in §6.2 so that in particular

\[
\int_S (4m\tilde{h}_j - \tilde{k}_j) dvol_{g_{\text{eucl}}} = 0.
\]
Applying Proposition 6.6 we get a metric $\tilde{\eta}_j = \tilde{\eta}_{\varepsilon, \hat{a}_j, \nu_j, X_j, \tilde{h}_j, \tilde{k}_j}$ with extremal vector field $\varepsilon^4 X_j$ and satisfying
\[ \frac{1}{|S|} \int_S \tilde{\eta}_{\varepsilon, \hat{a}_j} S(\varepsilon^2 \tilde{\eta}_j) = \nu_j. \] (7.7)

**Lemma 7.2.** Denote by $j_{\varepsilon, \hat{a}_j}$ the identification
\[ A^i \rightarrow \{ \frac{1}{2} R_\varepsilon / \hat{a}_j \leq |u| \leq R_\varepsilon / \hat{a}_j \}, \quad j_{\varepsilon, \hat{a}_j}(x) = R_\varepsilon x / \hat{a}_j. \]

Then we have
\[ \omega^{i,j} := \varepsilon^2 j_{\varepsilon, \hat{a}_j} \tilde{\eta}_j = i \partial \bar{\partial} \left( \frac{1}{2} |x|^2 / r_\varepsilon^2 + \psi^{i,j} \right) \] (7.8)
where
\[ \psi^{i,j} = -\tilde{a}_j^{-m-1} \varepsilon^{2m-2} r_\varepsilon^{4-2m} G_m + \varepsilon^2 W^{i}_{\tilde{h}_j, \tilde{k}_j} + O_{C^4, \alpha}(r_\varepsilon^4) \text{ in } A^i. \] (7.9)

Again, we want a formulation of this result in which the boundary data is not constrained by (7.6). This time we note that if $h$ is a constant function on $S$, then the (bounded) interior biharmonic extension is just $W^i_{h_j, k_j}$. So if we drop the constraint (7.6) the above expression for $\psi^{i,j}$ only changes by a constant, and this does not affect the Kähler form. For later convenience, set $\tilde{h}_j' = \varepsilon^2 \tilde{h}_j$, $\tilde{k}_j' = \varepsilon^2 \tilde{k}_j$, so that (7.9) becomes
\[ \psi^{i,j} = -\tilde{a}_j^{-m-1} \varepsilon^{2m-2} r_\varepsilon^{4-2m} G_m + W^{i}_{\tilde{h}_j', \tilde{k}_j'} + O_{C^4, \alpha}(r_\varepsilon^4) \text{ in } A^i. \] (7.10)
and we no longer assume that $(\tilde{h}_j', \tilde{k}_j')$ satisfy (7.6).

We are now almost ready to apply Proposition 4.1. Indeed, for each $j$, we have a Kähler form $\omega^{i,j}$ on $A^i$ and a Kähler form $\omega^{o,j}$ on $A^o$. These forms are extremal, with extremal vector fields that agree along the interface $S$, and the and the mean values of $s(\omega^{i,j})$ and $s(\omega^{o,j})$ are also equal. We have also matched, by hand, the coefficients of $G_m$ in the two expansions.

Thus it remains to show that the parameters $(h', k')$, $(\tilde{h}', \tilde{k}')$ can be chosen so as to match $\psi^{i,j}$ and $\psi^{o,j}$ to third order at $S$. This is the purpose of the next section.

### 7.3. The final step

First, we note that it is enough to solve the system of equations
\[ \psi^{o,j} = \psi^{i,j}, \quad \partial_r \psi^{o,j} = \partial_r \psi^{i,j}, \quad \Delta \psi^{o,j} = \Delta \psi^{i,j}, \quad \partial_r \Delta \psi^{o,j} = \partial_r \Delta \psi^{i,j} \text{ on } S. \] (7.11)
where $\partial_r$ is the radial vector field on $\mathbb{C}^m$. Indeed, it is easy to see that these conditions are satisfied if and only if $\partial_r^k \psi^{i,j} = \partial_r^k \psi^{o,j}$ on $S$ for $k = 0, \ldots, 3$. 
Inserting the formulae for $\psi^i,j$ and $\psi^{o,j}$ into (7.11), we obtain

$$\begin{align*}
W^o_{h,j,k} &= W^i_{h,j,k} + O_{C^4,\alpha}(r_\varepsilon^4) \\
\partial_r W^o_{h,j,k} &= \partial_r W^i_{h,j,k} + O_{C^3,\alpha}(r_\varepsilon^4) \\
\Delta W^o_{h,j,k} &= \Delta W^i_{h,j,k} + O_{C^2,\alpha}(r_\varepsilon^4) \\
\partial_r \Delta W^o_{h,j,k} &= \partial_r \Delta W^i_{h,j,k} + O_{C^1,\alpha}(r_\varepsilon^4),
\end{align*}$$

(7.12)
on $S$. By definition of $W^o_{h,k}$ and $W^i_{h,k}$, the first and third equations reduce to

$$\begin{align*}
h^\prime_j &= \tilde{h}^\prime_j + O_{C^4,\alpha}(r_\varepsilon^4) \\
k^\prime_j &= \tilde{k}^\prime_j + O_{C^2,\alpha}(r_\varepsilon^4),
\end{align*}$$

(7.13)
Inserting these into the second and fourth sets of equations and using the linearity of the mappings $(h,k)\mapsto W^o_{h,k}$ and $(h,k)\mapsto W^i_{h,k}$, the second and fourth equations become

$$\begin{align*}
\partial_r W^o_{h,j,k} &= \partial_r W^i_{h,j,k} + O_{C^3,\alpha}(r_\varepsilon^4) \\
\partial_r \Delta W^o_{h,j,k} &= \partial_r \Delta W^i_{h,j,k} + O_{C^1,\alpha}(r_\varepsilon^4),
\end{align*}$$

(7.14)
for all $j = 1, \ldots, n$. We now make use of the following result whose proof can be found in [1]:

**Lemma 7.3.** The mapping

$$\mathcal{P} : \ C^{4,\alpha}(S) \times C^{2,\alpha}(S) \longrightarrow C^{3,\alpha}(S) \times C^{1,\alpha}(S)$$

$$(h,k) \longmapsto (\partial_r (W^i_{h,k} - W^o_{h,k}), \partial_r \Delta (W^i_{h,k} - W^o_{h,k})),$$

is an isomorphism.

Using Lemma 7.3, we see that (7.14) reduces to

$$\begin{align*}
h^\prime_j &= O_{C^4,\alpha}(r_\varepsilon^4) \\
k^\prime_j &= O_{C^2,\alpha}(r_\varepsilon^4),
\end{align*}$$

(7.15)
for all $j = 1, \ldots, n$. This, together with (7.13), yields a fixed point problem which can be written as

$$(h', \tilde{h}', k', \tilde{k}') = S_\varepsilon(h', \tilde{h}', k', \tilde{k}),$$

and we know from (7.13) and (7.15) that the nonlinear operator $S_\varepsilon$ satisfies

$$\|S_\varepsilon(h', \tilde{h}', k', \tilde{k}')\|_{(C^{4,\alpha})^{2n} \times (C^{2,\alpha})^{2n}} \leq c_0 r_\varepsilon^4,$$
for some constant $c_0 > 0$ which does not depend on $\kappa$, provided $\varepsilon \in (0, \varepsilon_\kappa)$. We finally choose

$$\kappa = 2c_0,$$

and $\varepsilon \in (0, \varepsilon_\kappa)$. We have therefore proved that $S_\varepsilon$ is a map from

$$A_\varepsilon := \left\{ (h', \tilde{h}', k', \tilde{k}') \in (C^{4,\alpha})^{2n} \times (C^{2,\alpha})^{2n} : \| (h', \tilde{h}', k', \tilde{k}') \|_{(C^{4,\alpha})^{2n} \times (C^{2,\alpha})^{2n}} \leq \kappa r^4 \varepsilon \right\},$$

into itself. It follows from Propositions 6.3 and 6.6 that, reducing $\varepsilon_\kappa$ if necessary, $S_\varepsilon$ is a contraction mapping from $A_\varepsilon$ into itself for all $\varepsilon \in (0, \varepsilon_\kappa)$. Therefore, $S_\varepsilon$ has a fixed point in this set. This completes the proof of the existence of a solution of (7.11) and achieves the required third-order matching of $\psi^{i,j}$ and $\psi^{o,j}$ across $S$.

This shows the existence of an extremal Kähler metric $\omega_\varepsilon$, say, on $\tilde{M}$ as claimed in our main result, Theorem 2.2. We note that the Kähler class is equal to

$$\pi^*[\omega] - \varepsilon^2 \sum_{j=1}^n \tilde{a}_j PD[E_j],$$

and that we have the control

$$|\tilde{a}_j - a_j| \leq c \varepsilon^\theta,$$

as required.

Since the construction of the Kähler form $\omega_\varepsilon$ is performed using fixed point theorems for contraction mappings, it is clear that $\omega_\varepsilon$ depends continuously on the parameters of the construction (such as the Kähler form $\omega$, the points $p_j$ and the coefficients $a_j$). In particular, if $h'' = 0$, conditions (i)-(iii) in Theorem 2.2 are vacuous and the parameters can be freely chosen. We claim, moreover, that in this case the Kähler class can be exactly prescribed (last statement of the Theorem). To see this, let

$$\Sigma = \{ a_j \in \mathbb{R}^n : a_j > 0, a_1 + \cdots + a_n = 1 \}.$$ 

If $B \subset \Sigma$ is any closed ball, then our construction gives a mapping $B \to \Sigma$

$$(a_1, \ldots, a_n) \mapsto (\tilde{a}_1, \ldots, \tilde{a}_n),$$

provided that $\varepsilon$ is less than some constant depending upon $B$. Moreover, this map is a small perturbation of the identity, and so degree theory shows that any given point $(b_1, \ldots, b_n) \in B$ is in the image of (7.16), that is, there exists $a_j$ so that $\tilde{a}_j = b_j$ as required.

If $h'' \neq 0$ a more sophisticated argument is needed. This is contained in the proof of Theorem 9.7 below. The proof of Proposition 2.3 also follows directly from the construction. Indeed, when $\omega$ is a constant scalar curvature Kähler form, then $X_\omega = 0$. However,
in the expansion of Proposition 6.3 one directly sees that the scalar curvature of \( \omega_{\varepsilon} \) will not be constant if the vector field \( X_\alpha \) is not zero. Now

\[
X_\alpha = 0 \quad \text{if and only if} \quad \sum_{j=1}^{n} a_j^{m-1} \xi_\omega(p_j) = 0.
\]

This completes the proof of Proposition 2.3

8. AN EXAMPLE: EXTREMAL METRICS ON THE BLOW UP OF \( \mathbb{C}P^2 \)

We explain how the results of this paper can be used to obtain extremal metrics on \( \text{Bl}_{p_1,\ldots,p_3}(\mathbb{C}P^2) \) for various configurations of points and Kähler classes.

First of all, if \( p_1, p_2, p_3 \) are in general position, Theorem 2.5 gives extremal Kähler metrics on \( \text{Bl}_{p_1, p_2, p_3}(\mathbb{C}P^2) \), in Kähler classes of the form

\[
\pi^* [\omega_{FS}] - \varepsilon^2 (a_1 \text{PD}[E_1] + a_2 \text{PD}[E_2] + a_3 \text{PD}[E_3]),
\]

provided \( a_1, a_2, a_3 > 0 \) are fixed and \( \varepsilon \in (0, \varepsilon_0) \), where \( \varepsilon_0 > 0 \) is small enough.

Next recall that Calabi constructed an explicit extremal metric \( \omega_C \) on \( \text{Bl}_{p_1}(\mathbb{C}P^2) \). If \( p_1, p_2, p_3 \) are in general position, then we can pick a maximal torus \( T \) in the isometry group of \( \omega_C \) to fix \( p_2 \) and \( p_3 \). Applying Theorem 2.4 now gives extremal Kähler metrics in Kähler classes of the form

\[
\pi^* [\omega_{FS}] - (a_1 \text{PD}[E_1] + \varepsilon^2 a_2 \text{PD}[E_2] + \varepsilon^2 a_3 \text{PD}[E_3]),
\]

provided \( a_1 \in (0, 1), a_2, a_3 \) are fixed and \( \varepsilon \in (0, \varepsilon_0) \), where \( \varepsilon_0 > 0 \) is small enough.

We can obtain extremal representatives in other Kähler classes using the so-called Cremona transformation (see e.g. [15] pages 397-399). We wish to thank M. Abreu for bringing this to our attention.

Let \( p_1, p_2, p_3 \) be three points in \( \mathbb{C}P^2 \) not lying on a line, and denote by \( \ell_{ij} \) the projective line joining \( p_i \) to \( p_j \). Let \( X = \text{Bl}_{p_1, p_2, p_3}(\mathbb{C}P^2) \); inside \( X \) we have the exceptional divisors \( E_j = \pi^{-1}(p_j) \) and the proper transforms \( F_1 \) of \( \ell_{23} \), \( F_2 \) of \( \ell_{13} \) and \( F_3 \) of \( \ell_{12} \). Now the \( F_j \) satisfy exactly the same incidence relations as the \( E_j \) (\( F_j \cdot F_k = -\delta_{jk} \)) and so there is a blow-down map \( \rho : X \to \mathbb{C}P^2 \) contracting the \( F_j \) to points \( f_j \), say. The process is symmetrical: \( \rho(E_1) \) is the line joining \( f_2 \) to \( f_3 \), \( \rho(E_2) \) is the line joining \( f_1 \) to \( f_3 \) and \( \rho(E_3) \) is the line joining \( f_1 \) to \( f_2 \). Thus we have a birational map from \( C \mathbb{P}^2 \) to itself given in coordinates by \( (z_0 : z_1 : z_2) \mapsto (z_0^{-1} : z_1^{-1} : z_2^{-1}) \).

In \( X \), we have the following relations:

\[
\begin{align*}
\pi^* [\omega_{FS}] &= 2\rho^* [\omega_{FS}] - \text{PD}[F_1] - \text{PD}[F_2] - \text{PD}[F_3], \\
\text{PD}[E_1] &= \rho^* [\omega_{FS}] - \text{PD}[F_2] - \text{PD}[F_3], \\
\text{PD}[E_2] &= \rho^* [\omega_{FS}] - \text{PD}[F_3] - \text{PD}[F_1], \\
\text{PD}[E_3] &= \rho^* [\omega_{FS}] - \text{PD}[F_1] - \text{PD}[F_2].
\end{align*}
\]
Hence

\[
\pi^* [\omega_{FS}] - \alpha_1 PD[E_1] - \alpha_2 PD[E_2] - \alpha_3 PD[E_3] = (2 - \alpha_1 - \alpha_2 - \alpha_3) \rho^*[\omega_{FS}]
- (1 - \alpha_2 - \alpha_3) PD[F_1] - (1 - \alpha_1 - \alpha_3) PD[F_2] - (1 - \alpha_1 - \alpha_2) PD[F_3]
\]

An application of Theorem 2.2 implies that, on \( Bl_{p_1, p_2, p_3} \mathbb{CP}^2 \), the Kähler classes

\[
\pi^* [\omega_{FS}] - (1 - \alpha_2 - \alpha_3) PD[F_1] - (1 - \alpha_1 - \alpha_3) PD[F_2] - (1 - \alpha_1 - \alpha_2) PD[F_3]
\]

have extremal representatives provided \( a_1, a_2, a_3 > 0 \) are fixed and \( \varepsilon \) is small enough.

We should recall also that when we blow up three points in general position, the Kähler classes

\[
\pi^* [\omega_{FS}] - (a_1 PD[E_1] + a_2 PD[E_2] + a_3 PD[E_3]),
\]

where all the \( a_j \) are sufficiently close to \( \frac{1}{3} \), also have extremal representatives. This follows by combining the existence of a Kähler–Einstein metric on the three-point blow-up of \( \mathbb{CP}^2 \) due to Siu–Tian–Yau [34, 38] with the deformation theory of LeBrun–Simanca [21].

Finally, we consider the existence of extremal Kähler metrics on \( Bl_{p_1, \ldots, p_n} (\mathbb{CP}^2) \) for \( n \geq 3 \), where the points are now no longer in general position, so leaving the world of toric geometry.

Consider the action of \( S^1 \) on \( \mathbb{CP}^2 \) given by

\[
\theta(z_0 : z_1 : z_2) = [\theta^{-2}z_0 : \theta z_1 : \theta z_2]
\]

The fixed-set \( F \) is the union \( \{p_0\} \cup L \), where \( p_0 = [1 : 0 : 0] \) and \( L \) is given by \( z_0 = 0 \).

Using the formula (2.6) for the moment map \( \xi_3 \) of the action of \( PSU(3) \) on \( \mathbb{CP}^2 \), we compute

\[
\xi_3(p_0) = i \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}
\]

and

\[
\xi_3(0 : z_1 : z_2) = \begin{pmatrix} 0 & 0 \\ 0 & \xi_2(z_1 : z_2) \end{pmatrix} + i \begin{pmatrix} -\frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{6} & 0 \\ 0 & 0 & \frac{1}{6} \end{pmatrix}
\]

where the moment map \( \xi_2 \) for the action of \( PSU(2) \) on \( L \) appears in the second formula. Identifying \( h'' \) with the orthogonal complement of \( i \text{diag}(2, -1, -1) \) inside \( h \), a copy of \( \text{su}(2) \), we get

\[
\xi''_3(p_0) = 0
\]

and

\[
\xi''_3(0 : z_1 : z_2) = \xi_2(z_1 : z_2)
\]
Hence if \( p_1, \ldots, p_n \) lie on \( L \), then
\[
\sum a_j \xi_3''(p_j) = 0 \text{ if and only if } \sum a_j \xi_2(p_j) = 0.
\]
Now \( \xi_2 \) is just the standard embedding of \( \mathbb{C}P^1 \) in \( \mathbb{R}^3 = \mathfrak{su}(2) \). So the latter condition is satisfied if and only if the centre of mass of the configuration of particles of mass \( a_j \) at \( p_j \in S^2 \) is at the origin in \( \mathbb{R}^3 \).

These considerations lead to

**Proposition 8.1.** Let \( L \) and \( p_0 = [1 : 0 : 0] \) as above. If \( n \geq 3 \), then one can choose the \( p_1, \ldots, p_n \in L \) and \( a_j > 0 \) so that \( \text{Bl}_{p_1, \ldots, p_n}(\mathbb{C}P^2) \) admits extremal metrics in Kähler classes of the form
\[
\pi^*[\omega_{FS}] - \varepsilon^2 \sum_{j=1}^{n} a_j PD[E_j].
\]
Furthermore, if the \( a_j \) and \( p_j \) are so chosen, then for any \( a_0 > 0 \), there exist extremal metrics in Kähler classes of the form
\[
\pi^*[\omega_{FS}] - \varepsilon^2 \left( a_0 PD[E_0] + \sum_{j=1}^{n} a_j PD[E_j] \right).
\]
Moreover, these metrics have constant scalar curvature if and only if
\[
a_0 = \frac{1}{2} \sum_{j=1}^{n} a_j. \tag{8.1}
\]

**Proof.** From the interpretation of \( \xi_3'' \) in terms of the centre of mass of the configuration of point-masses described above, it is clear that if \( n \geq 2 \) one can always find configurations with \( \sum_{j=1}^{n} a_j \xi_2(p_j) = 0 \).

If \( n = 2 \), we get nothing new; the result is contained in Theorem 2.5.

If \( n = 3 \), then it is clear that \( p_1, p_2, p_3 \) must lie on a great circle on \( S^2 \). In particular,
\[
\mathbb{R}\xi''(p_1) + \mathbb{R}\xi''(p_2) + \mathbb{R}\xi''(p_3) \neq \mathfrak{su}(2).
\]
However, we can exploit a symmetry trick here to overcome this problem. Let \( \iota(z_0 : z_1 : z_2) = (z_0 : z_2 : z_1) \), and choose \( p_1, p_2, p_3 \) so that \( \iota(p_1) = p_1, \iota(p_2) = p_3 \), and assume \( a_2 = a_3 \). In terms of the sphere, \( \iota \) corresponds to a half-turn, with \( p_1 \) on the axis, so it is clear that we can choose the \( a_j \) and \( p_j \) so that the centre of mass is still at the origin.

Now we apply our construction equivariantly with respect to the action of \( \iota \). Now the \( \xi''(p_j) \) do span the \( \iota \)-invariant part of \( \mathfrak{su}(2) \), the conditions of Theorem 2.2 are satisfied, and we conclude that there exist extremal Kähler metrics in such Kähler classes.

If \( n \geq 4 \), and the \( p_j \) do not all lie on a great circle of \( S^2 \), then conditions (i)–(iii) of Theorem 2.2 are satisfied, and so we get extremal metrics as claimed. (If the points lie
on a great circle but the configuration is preserved by a half-turn, then we can argue as above to get extremal Kähler metrics in this case also.)

Finally since $\xi''(p_0) = 0$, it is clear that we can also blow up at this point and assign an arbitrary positive value to the weight $a_0$. This completes the proof apart from the very last statement. Given that our configuration satisfies $\sum a_j \xi''(p_j) = 0$, we see that

$$a_0 \xi_3(p_0) + \sum_{j=1}^{n} a_j \xi_3(p_j) = i(a_0 - \frac{1}{2} \sum_{j=1}^{n} a_j) \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & -\frac{1}{3} & 0 \\ 0 & 0 & -\frac{1}{3} \end{pmatrix}.$$  

Thus if (8.1) does not hold $a_0 \xi_3(p_0) + \sum a_j \xi_3(p_j) \neq 0$ and by Proposition 2.3 we have an extremal Kähler metric with non-constant scalar curvature.

Conversely, suppose that (8.1) does hold. Then we must blow up $p_0$ as well as $n$ points on $L$. Since $\xi_3(p_0)$ spans $t$ and the projections to $h''$ of the $\xi''(p_j)$ span $h''$, we see that the elements $\xi_3(p_0), \ldots, \xi_3(p_n)$ span $h$. Now we can apply [2] (working equivariantly with respect to the given $S^1$-action) to produce Kähler metrics of constant scalar curvature in these Kähler classes.

Notice that the $a_j$ cannot be arbitrarily assigned here. A moment’s thought reveals that it is not possible for a weighted configuration of $n$ distinct points on the sphere to have centre-of-mass at the origin if the mass of any one of the points is greater than or equal to half the total. In fact, it has been shown by Della Vedova [14] that in this case the corresponding polarized manifold is not relatively K-stable, forbidding the existence of an extremal metric in the corresponding Kähler class thanks to the work of Székelyhidi. [35].

9. The geometry of the balancing condition and the Kempf–Ness principle

For the contents of this section, we acknowledge useful discussions with Frances Kirwan, Gabor Székelyhidi and Richard Thomas.

Let $Z = M \times \cdots \times M$ ($n$ factors in the product). Let $\Omega = \sum A_j \text{pr}^*_j(\omega)$, where $\text{pr}_j$ denotes the projection on the $j$-th factor. Then

$$\mu(p_1, \ldots, p_n) = \sum A_j \xi(p_j)$$

is a moment map for the diagonal action of $K$ on $(Z, \Omega)$. Thus to understand condition (i) of Theorem 2.2, it is helpful to review general properties of the moment map in symplectic geometry.
9.1. Relative moment maps. Let $T \subset K$ be a torus and let $H$ denote the centralizer of $T$ in $K$. We shall use $\sigma : \mathfrak{k} \to \text{Vect}(M)$ to denote the action of $\mathfrak{k}$ on $M$. Let $H'' = H/T$ and denote by $\mu''$ the composition of $\mu$ with the projection to $\mathfrak{h}/\mathfrak{t}$. The following lemma reinterprets $\mu''$ as a moment map.

**Lemma 9.1.** The group $H''$ acts holomorphically and symplectically on $\text{Fix}(T)$ and $\mu''$ may be viewed as a moment map for this action.

**Proof.** If $x \in \text{Fix}(T)$ and $t \in T$, then by the equivariance of $\mu$,

$$\mu(x) = \mu(tx) = \text{Ad}(t)\mu(x)$$

so $\mu(x) \in \mathfrak{h}$. For $X \in \mathfrak{h}$, it is clear that $\langle \mu, X \rangle$ is a hamiltonian generating function for $\sigma(X)$; moreover this function clearly vanishes if $X \in \mathfrak{t}$. Thus $\mu''$ inherits the necessary properties from $\mu$. \qed

**Remark 9.2.** We saw an example of this in the previous section. There we had a one-dimensional torus acting on $\mathbb{C}P^2$, fixing a point and a line $L$. By direct computation, we found that the projection to $\mathfrak{h}$ of the restriction of $\xi_3$ to $L$ could be naturally identified with $\xi_2$, the moment map for the action of $H'' \simeq \text{PSU}(2)$ on $L \simeq \mathbb{C}P^1$.

In Theorem 2.2 condition (iii) plays an important role. Let us now show that if we allow ourselves to change the choice of torus, we can always reduce to the situation where this condition is satisfied. Thus suppose, in the situation of the previous Lemma, we have $p \in \text{Fix}(T)$, $\mu''(p) = 0$, but $\text{Stab}_{H''}(p)$ is of positive dimension. Let $T_1 \supset T$ be a maximal torus in $\text{Stab}_{K}(p)$, $H_1$ the centralizer of $T_1$ in $K$ and $\mu''_1$ the corresponding relative moment map. At the level of Lie algebras, we clearly have

$$\mathfrak{t} \subset \mathfrak{t}_1 \subset \mathfrak{h}_1 \subset \mathfrak{h}.$$  

By the Lemma, since $p \in \text{Fix}(T_1)$, $\mu(p) \in \mathfrak{h}_1$, and so $\mu''(p) = 0$ implies $\mu''_1(p) = 0$. By choice of $T_1$, $\text{Stab}_{H''_1}(p)$ is finite.

Applying this in the case of interest to us:

**Proposition 9.3.** Let the notation be as in Theorem 2.2. Suppose that conditions (i) and (ii) of that theorem hold. Then by replacing $T$ by a larger torus $T_1$, we can assume that conditions (i), (ii) and (iii) all hold, with $\mathfrak{h}$ and $\xi''$ replaced by $\mathfrak{h}_1''$ and $\xi''_1$.

**Proof.** This follows from what has just been said, replacing $T$ by a maximal torus $T_1$ containing $T$ in $\text{Stab}_{K}(p_1, \ldots, p_n)$. It is clear that if the generic condition (ii) holds for $\mathfrak{h}$ and $\xi''$, then it continues to hold for $\mathfrak{h}_1''$ and $\xi''_1$. \qed

9.2. Action of the complex group $G$. The balancing condition (ii) in Theorem 2.2 is clearly a closed condition, which appears to impose strong conditions on the choice of weights $a_j$ and points $p_j$ that can be used in our construction. However, this overlooks
the possibility of using the complexification $G$ of $K$ to give us further degrees of freedom. We shall see in this section, that, in a suitable sense, once we use this additional freedom, our construction ‘is open in the points and the weights’.

In this discussion, the following definition will be very handy:

**Definition 9.4.** Let $M$ be a compact Kähler manifold with extremal Kähler metric $\omega$. By a *weighted configuration* on $M$ we mean a pair of ordered sets $(a_1, \ldots, a_n), (p_1, \ldots, p_n)$ where the $a_j$ are positive real numbers and the $p_j$ are points of $M$. A weighted configuration $(a, p)$ will be called *admissible* if, for any $\varepsilon \in (0, \varepsilon_0)$, there exists an extremal Kähler metric on $\text{Bl}_{p_1, \ldots, p_n}(M)$ in the Kähler class

$$[\omega_\varepsilon] = \pi^*[\omega] - \varepsilon^2 \sum_{j=1}^{n} a_j P D[E_j].$$

If $(a, p)$ is an admissible configuration and $g \in G$ is such that $g(p_j) = p_j$, then $g$ lifts to define a holomorphic map

$$g : \text{Bl}_{q_1, \ldots, q_n}(M) \to \text{Bl}_{p_1, \ldots, p_n}(M).$$

Then $g^*$ of an extremal Kähler metric on $\text{Bl}_{p_1, \ldots, p_n}(M)$ will be an extremal metric on $\text{Bl}_{q_1, \ldots, q_n}(M)$, so we see that $(a, q)$ is also an admissible configuration.

The key to proving stronger results using this idea is the following:

**Lemma 9.5.** Let

$$\Phi : \mathfrak{k} \times M \to \mathfrak{k}, \quad \Phi(X, p) = \mu(\exp(iX)p).$$

Then the partial derivative $D_1 \Phi$ with respect to $X$,

$$D_1 \Phi(0, p) : \mathfrak{k} \to \mathfrak{k}$$

can be identified with the bilinear form

$$(X, Y) \mapsto -g_p(\sigma_p(X), \sigma_p(Y)).$$

In particular, $D_1 \Phi(0, p)$ is an isomorphism if and only if $\text{Stab}_K(p)$ is finite.

**Proof.** We have

$$D_1 \Phi(0, p)(X) = (J\sigma(X)\mu)_p = (J\sigma_p(X), d\mu_p).$$

Pairing with $Y \in \mathfrak{k}$, we get (with minor abuse of notation)

$$D_1 \Phi(0, p)(X, Y) = -g_p(\sigma_p(X), \sigma_p(Y))$$

from the definition of the moment map. The result follows. \qed

The first consequence of this result is the following, using the implicit function theorem.
Proposition 9.6. Suppose that $\text{Stab}_p(K)$ is finite for every point $p$ of $\mu^{-1}(0)$. Then there exists $\delta > 0$ and a tubular neighbourhood $N$ of $\mu^{-1}(0)$ in $M$, such that the map
\[(X,p) \mapsto \exp(iX)p\] (9.1)
give a fibre-preserving diffeomorphism
\[
\{X \in \mathfrak{k} : |X| < \delta\} \times \mu^{-1}(0) \to N.
\]
Similarly if $\text{Stab}_K(p_0)$ is finite for some $p_0 \in \mu^{-1}(0)$, there exists $\delta > 0$ and a neighbourhood $W$ of $p_0$ in $\mu^{-1}(0)$ such that the map (9.1) gives a fibre-preserving diffeomorphism
\[
\{X \in \mathfrak{k} : |X| < \delta\} \times W \to U,
\]
where $U$ is a tubular neighbourhood of $W$ in $M$.

The second result relates to the situation where the symplectic form $\Omega$ depends smoothly on parameters $a \in A$, say, with $\Omega_0 = \Omega$. Assume that for each $a \in A$, the $K$-action preserves $\Omega_a$ and there is a smoothly varying choice of moment map $\mu_a$, with $\mu_0 = \mu$. If we replace $\Phi$ by the map
\[
\mathfrak{k} \times M \times A \to \mathfrak{k}, \quad (X, p, a) \mapsto \mu_a(\exp(iX)p)
\]
then, under the same assumption that $\text{Stab}_K(p)$ is finite, the partial derivative with respect to $X$ is still an isomorphism and we conclude that, given $q$ sufficiently close to $p$, and $a$ sufficiently close to 0 we can find $X$ so that $\mu_a(\exp(iX)q) = 0$ if $\mu(p) = 0$.

We can apply these ideas to prove the following:

Theorem 9.7. Let the data be as in Theorem 2.2, and suppose that conditions (i) and (ii) of that Theorem hold. Let $T_1 \supset T$ be a maximal torus in $\text{Stab}_K(p_1, \ldots, p_n)$. Then all weighted configurations $(b, q)$ with $b$ sufficiently close to $a$ and $q$ sufficiently close to $p$ in $\text{Fix}(T_1)$, are admissible.

Proof. By what we have proved in §7.3, given these data, there is a map $a \mapsto \bar{a}$, depending upon $\varepsilon$, such that $(\bar{a}, p)$ is admissible. With $T_1$ chosen as in the statement, the stabilizer of $(p_1, \ldots, p_n)$ in $H_1''$ is finite, so that we can apply the above results to the action of $H_1''$ on $F_1 \times \cdots F_n$, where $F_j$ is the connected component of $T_1$ containing $p_j$. In particular, if we consider the family of metrics $\Gamma = \{g^* \omega : g = \exp(iX), X \in \mathfrak{k}, |X| \leq \delta\}$, then we can apply the construction uniformly to any metric in $\Gamma$. It follows from Proposition 9.6 that given a nearby configuration $(a', p')$ with $p' \in F_1 \times \cdots \times F_n$, we can find a metric $\omega' \in \Gamma$ to which the above construction is applicable, showing that $(\bar{a}', p')$ is admissible. We can now use the fact that the map $a \mapsto \bar{a}$ tends to the identity on compact subsets of the parameter space $A$ as at the end of §7.3 to conclude that any given configuration $(a', p')$ is admissible, provided that $a'$ and $p'$ are sufficiently close to $a$ and $p$. \hfill $\Box$

Remark 9.8. This is the appropriate version of our construction being ‘open’ in the configurations.
Remark 9.9. Although we shall not make any serious use of this fact, it should be pointed out that there is an intimate relationship between the relative moment maps and the critical points of the norm-square $f(x) = |\mu(x)|^2$ of the moment map [19]. It is not hard to show that if $x$ is a critical point of $f$, then either $\mu(x) = 0$ (absolute minimum) or there is a non-trivial torus $T$ such that $x \in \text{Fix}(T)$. Taking $T$ to be a maximal torus in $\text{Stab}_K(x)$, we then have that among all points $y$ of $\text{Fix}(T)$, $y$ is a critical point of $f$ if and only if $\mu''(y) = 0$. Thus the somewhat intricate geometric set-up of this paper emerges naturally from the Morse theory of the norm-square of the moment map.

9.3. Kempf–Ness principle and Morse theory of $|\mu|^2$. Return to our general set-up of a compact Kähler manifold $(Z, \Omega)$ acted on by a compact group $K$ with moment map $\mu$ and complexification $G$. We saw in the previous section that if $\text{Stab}_K(p)$ is finite for all $p \in \mu^{-1}(0)$, then $G\mu^{-1}(0)$ contains an open neighbourhood of $\mu^{-1}(0)$. In our application, a consequence of this was the admissibility of configurations $p'$ in an open neighbourhood in $F_1 \times \cdots \times F_n$ of a given admissible configuration $p$. According to the Kempf–Ness principle, we actually obtain admissibility for a dense open set of configurations $p \in F_1 \times \cdots \times F_n$.

If $(Z, \Omega)$ is a projective variety (so $\Omega$ can be represented as the curvature of an ample line-bundle) then geometric invariant theory [29] shows that $G\mu^{-1}(0)$ is dense in $M$ if $\text{Stab}_K(p)$ is finite for every $p \in \mu^{-1}(0)$.

An analogous result holds also if $(Z, \Omega)$ is only Kähler; this was explained to us by Kirwan and follows from [19].

To explain why such a statement might be true, one tries to treat the norm-square $f(x) = |\mu(x)|^2$ of the moment map as a Morse function on $Z$. For $p \in M$, consider the (downward) gradient flow of $f$ emanating from $p$, namely the curve $\gamma_p : [0, \infty) \to M$ satisfying

$$\frac{d}{dt} \gamma_p(t) = -\nabla f(\gamma_p(t)), \quad \gamma_p(0) = p. \quad (9.2)$$

It is easy to show that $\gamma_p(t) \in Gp$ for all $t$ and that $t \mapsto f(\gamma_p(t))$ is non-increasing.

In the spirit of Morse theory, denote by $S$ the set of points $p \in M$ such that $\gamma_p(t)$ is arbitrarily close to a critical set of positive index for $t \gg 0$. It was shown by Kirwan that $S$ is a stratified set of codimension $\geq 2$, even though $f$ is not in general non-degenerate in the sense of Morse or Morse–Bott. In particular, the complement $U = M \setminus S$ is an open dense set of $M$ and if $p \in U$ then $\gamma_p(t)$ is arbitrarily close to a critical set of index 0 for $t \gg 0$. It follows that for such $p$, $\gamma_p(t)$ is arbitrarily close to $\mu^{-1}(0)$ for $t \gg 0$.

Combining these facts we have a version of the Kempf–Ness principle:

**Theorem 9.10.** (Kirwan) Suppose that $Y = \mu^{-1}(0) \neq \emptyset$ and that $\text{Stab}_K(p)$ is finite for all $p \in Y$. Then $GY = U$ is a dense open subset of $Z$.

**Proof.** (Sketch.) Given the finite-stabilizer hypothesis, there is a tubular neighbourhood $N$ of $\mu^{-1}(0)$ as in Proposition 9.6 such that $G\mu^{-1}(0)$ contains $N$. On the other hand,
if \( p \in U \), there exists \( \tau > 0 \) such that \( \gamma_p(\tau) \in N \). Moreover, \( \gamma_p(\tau) = g_0(p) \) for some \( g_0 \in G \). But then there exists \( q \in \mu^{-1}(0) \) and \( g_1 \in G \) such that \( g_1(q) = \gamma_p(\tau) \). Hence \( p = g^{-1}_0 g_1(q) \) as required. \( \square \)

We can now strengthen Theorem 9.7 as follows.

**Theorem 9.11.** Let \( M \) be a compact Kähler manifold with extremal Kähler metric \( \omega \). Suppose that \( T \subset K \) is a torus with \( X_s \in \mathfrak{t} \). Fix a connected component \( Z \) of \( \text{Fix}(T) \) inside \( M \times \cdots \times M \). Denote by \( \mu \) the moment map

\[
\mu(p_1, \ldots, p_n) = \sum a^{m-1}_j \xi''(p_j)
\]

Suppose that (i) \( \mu^{-1}(0) \) is non-empty; (ii) for some point \( (p_1, \ldots, p_n) \) of \( \mu^{-1}(0) \), the values \( \xi''(p_j) \) span \( \mathfrak{h}'' \); (iii) for every point \( (p_1, \ldots, p_n) \) of \( \mu^{-1}(0) \), \( \text{Stab}_{\mathfrak{h}''}(p_1, \ldots, p_n) \) is finite. Then there is a dense open subset \( U_0 \subset Z \) such that if \( q \in U_0 \), \( (a, q) \) is an admissible configuration.

**Proof.** We know that there is a dense open set \( U \subset Z \) such that if \( q \in U \), then there exists \( g \in G'' \) (the complexification of \( H'' \)) with \( g(q) \in \mu^{-1}(0) \). As discussed in [2], the set of points \( p \in \mu^{-1}(0) \) for which (ii) does not hold is some closed subset \( F \) of \( \mu^{-1}(0) \). Moreover, \( F \) is nowhere dense in \( \mu^{-1}(0) \) if, as we are assuming, it is not the whole of \( F \).

Suppose, for a contradiction, that there exists a ball \( B \) whose closure \( \overline{B} \) is contained in \( U \), but such that \( G(B) \cap \mu^{-1}(0) \subset F \). Then by a simple covering argument, there exists an element \( g \in G'' \) such that \( g(B) \subset N \) our good tubular neighbourhood of \( \mu^{-1}(0) \). Of course \( g(B) \) is open in \( N \), and it follows from this that \( G(B) \cap \mu^{-1}(0) \) must be open in \( \mu^{-1}(0) \). But \( F \) is nowhere dense, so we have a contradiction. It follows that the subset \( U_0 \) of \( U \) of configurations of points \( q \) for which \( Gq \cap \mu^{-1}(0) \) lies in \( \mu^{-1}(0) \setminus F \) is still open and dense in \( Z \). \( \square \)

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