THE TODA SYSTEM AND MULTIPLE-END SOLUTIONS OF AUTONOMOUS PLANAR ELLIPTIC PROBLEMS

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ABSTRACT. We prove the existence of a new class of entire, positive solutions for the classical elliptic problem \( \Delta u - u + u^p = 0 \) in \( \mathbb{R}^2 \), when \( p > 2 \). The solutions we construct are obtained by perturbing the function
\[
\sum_{j=1}^{k} w(\text{dist}(\cdot, \gamma_j)),
\]
where \( k \geq 1 \), \( w \) is the unique even, positive, non constant solution of \( w'' - w + w^p = 0 \) in \( \mathbb{R} \) and where the curves \( \gamma_j \) are the graphs of the functions \( f_1, \ldots, f_k \) which are solutions of the Toda system
\[
e^2 f'''_j = e^{f_{j-1}} - f_j - e^{f_{j+1}} - f_{j+1}
\]
with \( f_0 \equiv -\infty \) and \( f_{k+1} \equiv +\infty \). This result provides a surprising link between the solutions of the Toda system and entire solutions of the above semilinear elliptic equation.

1. Introduction

This paper is concerned with the existence of entire, positive solutions of the classical semilinear elliptic problem
\[(1.1) \Delta u - u + u^p = 0,\]
in \( \mathbb{R}^N \), where \( p > 2 \). Equation (1.1) arises for instance in the study of standing-wave for the standard nonlinear Schrödinger equation
\[i \psi_t = \Delta \psi + |\psi|^{p-1} \psi,\]
where typically \( p = 3 \) and where one looks for solutions of the form \( \psi(t, x) = e^{-it} u(x) \). This problem also arises as a nonlinear model in Turing’s biological theory of pattern formation [45] such as the Gray-Scott or Gierer-Meinhardt systems [19, 18].

The solutions of (1.1) which are positive and decay to zero at infinity are well understood. In particular, it is well known that, provided
\[1 < p < \begin{cases} k \frac{N+2}{N-2} & \text{if } N \geq 3, \\ +\infty & \text{if } N = 1, 2, \end{cases}\]
problem (1.1) has a positive, radially symmetric solution which tends to 0 at infinity, which is usually called the ground state. We refer to [44, 4] for a proof. This solution is unique [25] and actually any positive solution of (1.1) which vanishes at infinity must be radially symmetric about some point [17].

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In the last two decades, problem (1.1) and its variations have been broadly treated in the PDE literature. The variations are mostly of two types: Either (1.1) is changed into a non-autonomous problem with a potential depending on the space variable or (1.1) is considered in a bounded domain under suitable boundary conditions. Typically, in both versions a small parameter is introduced so that (1.1) can be understood as a singular perturbation problem. We refer the reader to [2, 3, 7, 11, 12, 13, 16, 20, 21, 27, 29, 30, 39, 37, 38] and references therein. Many constructions in the literature refer to multi-bump solutions which are built out of a perturbation of the superposition of finitely many scaled copies of the ground state. Usually the location of the points where the copies of the ground state are centered is determined by the critical points of some function involving the geometry of the underlying domain.

Much less is known about solutions to this equation which are defined in the entire space and which do not tend to 0 at infinity. Entire solutions of (1.1) are known to be bounded thanks to [42]. Observe that the radially symmetric solution of (1.1) in $\mathbb{R}^N$ can be trivially extended as a solution of (1.1) which is defined in $\mathbb{R}^{N+1}$ and which only depends on $N$ variables. Starting from this solution, a new class of entire, positive solutions has been discovered by N. Dancer [9], these solutions will be described in the next section. In [28], A. Malchiodi has constructed entire, positive solutions of (1.1) by perturbing infinitely many ground states periodically arranged along a finite number of half lines meeting at a point. In contrast, the solutions we construct in the present paper are obtained by perturbing finitely many copies of the ground state in dimension $N$, which are trivially extended in dimension $N + 1$ to be independent of the last variable. The solutions of A. Malchiodi and our solutions are qualitatively different but they belong to the same general class of entire solutions of (1.1). We shall comment this later on.

1.1. Dancer’s solutions of the nonlinear Schrödinger equation. For the sake of simplicity, we now restrict our attention to the two dimensional case ($N = 2$) and we consider the existence of entire, positive solutions of (1.1). To begin with, we recall the existence of the one dimensional bump, which we will denote in the sequel by $w_0$, namely the unique positive solution of

\[(1.2)\quad w'' - w + w^p = 0,\]

which is defined on $\mathbb{R}$, tends to 0 at $\pm \infty$ and which is normalized so that $w'(0) = 0$.

Using the function $w$, we can define a positive, entire solution of (1.1) by extending trivially $w$ in one space variable. With slight abuse of notation we will still denote by $w$ this solution. If we agree that $x = (x, z)$ denotes a point in $\mathbb{R}^2$, then

\[w(x) = w(x).\]

More generally, we can also define a family of positive, entire solutions of equation (1.1) by

\[x \mapsto w(a \cdot x - c),\]

where $a \in \mathbb{R}^2$ has norm equal to 1 and where $c \in \mathbb{R}$. By analogy with the above terminology, we will christen these solutions single bump-lines. A natural question is the classification of the entire solutions of (1.1). Unfortunately (or fortunately) the above described 2 dimensional family of functions does not exhaust the set of entire solutions of (1.1) in $\mathbb{R}^2$. Even though these solutions were found to be isolated in a uniform topology by J. Busca and P. Felmer in [6], a new class of solutions
was discovered by N. Dancer in [9] using a bifurcation argument. This new class of solutions forms a 4-parameter family of entire, positive solutions of (1.1) which are singly periodic. Let us briefly review their construction since they play a central role in our analysis: We consider solutions of (1.1) which are $T$-periodic in the $z$-variable, namely

$$(1.3) \quad u(x, z + T) = u(x, z),$$

for all $(x, z) \in \mathbb{R}^2$ and we regard $T > 0$ as a bifurcation parameter. Obviously, the function $w$, which only depends on $z$, is $T$-periodic for any value of $T > 0$. The nonlinear equation (1.1) linearized about $w_0$ is given by

$$L := \partial_x^2 + \partial_z^2 - 1 + p w^{p-1}(x).$$

The spectrum of the operator

$$(1.4) \quad L_0 := \partial_x^2 - 1 + p w^{p-1}(x),$$

is well understood and will be described more carefully in the next sections. It is well known that $-L_0$ has a unique negative eigenvalue $\lambda_1$, which corresponds to the bottom of the spectrum, with associated (positive) eigenfunction which will be denoted by $Z$ (and which is normalize to have $L^2$-norm equal to 1). Also, 0 is always an eigenvalue of $-L_0$ with associated eigenfunction given by $w'_0$ (this reflects the fact that equation (1.2) is autonomous) and all other eigenvalues are positive. Now, observe that, as long as

$$0 < T < T_1 := \frac{2\pi}{\sqrt{\lambda_1}},$$

the operator $L$ has a one dimensional $L^\infty$-kernel spanned by the function $w'$. When $T = T_1$ the $L^\infty$-kernel also includes the linear combinations of

$$(x, z) \mapsto Z(x) \cos(\sqrt{\lambda_1} z), \quad \text{and} \quad (x, z) \mapsto Z(x) \sin(\sqrt{\lambda_1} z).$$

Working in the class of functions which are even with respect to both $x$ and $z$ and which are $T$ periodic in $z$, Crandall-Rabinowitz bifurcation Theorem can be successfully applied to prove the existence of a smooth branch of solutions of (1.1) bifurcating at $T = T_0$. The solutions belonging to the bifurcated branch will be denoted by $u(\cdot; \varepsilon)$, where $\varepsilon \in \mathbb{R}$ is close to 0 (when $\varepsilon = 0$, $u(\cdot; \varepsilon)$ coincides with $w$). They depend smoothly on $\varepsilon$. Each $u(\cdot; \varepsilon)$ is an entire, bounded solution of (1.1) which is even with respect to $x$ and $y$ periodic in $x$, with fundamental period $T_\varepsilon$ smoothly depending on $\varepsilon$. Moreover, the period $T_\varepsilon$ can be expanded as

$$T_\varepsilon = \frac{2\pi}{\sqrt{\lambda_1}} + O(\varepsilon),$$

as $\varepsilon$ tends to 0.

The solutions $u(\cdot; \varepsilon)$ belonging to the branch of bifurcated solutions are uniformly close to $w$ and, as $\varepsilon$ tends to 0, their asymptotic form can be expanded as

$$u(x, z; \varepsilon) = w(x) + \varepsilon Z(x) \cos\left(\frac{2\pi}{T_\varepsilon} z\right) + e^{-|x|} O_{L^\infty(\mathbb{R}^2)}(\varepsilon^2).$$

The notation $O_{L^\infty(\mathbb{R}^2)}(\varepsilon^2)$ above refers to the fact that this function is bounded by a constant times $\varepsilon^2$ in $L^\infty$-norm. Obviously, the group of isometries acts on the
set of solutions of (1.1) and, since \( w_\varepsilon \) is not invariant anymore under translations along the \( x \) axis we find that

\[
(x, z) \mapsto u(x, z + \varphi; \varepsilon),
\]

is also a solution of (1.1), for all \( \varphi \in \mathbb{R} \). Roughly speaking, the parameter \( \varepsilon \) represents the amplitude and the parameter \( \varphi \) the phase shift of the oscillations superposed over \( w_0 \). It is time to introduce the parameters

\[
\delta := \varepsilon \cos \left( \frac{2\pi}{T} \varphi \right), \quad \text{and} \quad \tau := \varepsilon \sin \left( \frac{2\pi}{T} \varphi \right),
\]

which amounts to consider that \( \varepsilon \) and \( \varphi \) are (in some sense) polar coordinates in the plane of parameters \( (\delta, \tau) \) in a neighborhood of \( 0 \) in \( \mathbb{R}^2 \). Finally, we define the function \( w_{\delta, \tau} \) by

\[
w_{\delta, \tau}(x, z) := u(x, z + \varphi; \varepsilon).
\]

In what follows, we refer to the functions \( w_{\delta, \tau} \) as Dancer’s solution of parameter \( \delta \) and \( \tau \).

1.2. The statement of the main result. As already mentioned, the purpose of this paper is to construct a new type of positive, entire solutions of (1.1) in \( \mathbb{R}^2 \) which have multiple ends asymptotic to properly translated and rotated copies of Dancer’s solutions. To start with, we give a precise definition of what we mean by a multi ended solution of (1.1).

**Definition 1.1.** For all \( k \geq 1 \), we say that an entire solution \( u \) of (1.1) has \( 2k \) ends if there exist a compact \( K \subset \mathbb{R}^2 \), constants \( C, c > 0 \) and, for all \( j = 1, \ldots, 2k \), an oriented half line

\[
\Lambda_j := \{ x \in \mathbb{R}^2 | a_j^\perp \cdot x + b_j = 0 , \ a_j \cdot x > 0 \},
\]

with \( a_j \in S^1 \subset \mathbb{R}^2 \), a constant \( b_j \in \mathbb{R} \) and parameters \( \delta_j, \tau_j \in \mathbb{R} \), such that

\[
\left\| e^{c|x|} \left| u - \sum_{j=1}^{2k} u_j \right| \right\|_{L^\infty(\mathbb{R}^2 \setminus K)} \leq C,
\]

where

\[
u_j(x) := w_{\delta_j, \tau_j}(a_j^\perp \cdot x + b_j, a_j \cdot x),
\]

and where \( \perp \) denotes the rotation by \( \pi/2 \) in \( \mathbb{R}^2 \). The lines \( \Lambda_j \) are called the ends of the solution \( u \).

In order to construct \( 2k \)-ended solutions, the idea is to look for solutions of (1.1) which are close to the function

\[
\sum_{j=1}^{k} w_0(\text{dist}(\cdot, \gamma_j)),
\]

where \( \gamma_j \) are embedded curves which are asymptotic to oriented half lines at infinity. We assume that, for \( j = 1, \ldots, k \), the curve \( \Gamma_j \) is the graph of the function \( f_j \) over the \( x \)-axis

\[
\gamma_j := \{ (x, z) \in \mathbb{R}^2 | x = f_j(z) \}.
\]

It turns out that, in order for the construction to be successful, the functions \( f_j \) which define the curves \( \gamma_j \) have to be chosen very carefully and in fact they are
related to a nonlinear second order system of differential equations (a Toda system) which is given by
\[ c_p^2 f_j'' = e^{f_{j-1} - f_j} - e^{f_j - f_{j+1}}, \]
for \( j = 1, \ldots, k \), where we agree that \( f_0 \equiv -\infty \) and \( f_{k+1} \equiv +\infty \) and where \( c_p > 0 \) is an explicit positive constant which will be specified later on (see (4.44) for a precise definition of \( c_p \)). This Toda system is a classical model describing the scattering of \( k \) particles distributed on a straight line, which interact only with their closest neighbors with forces depending exponentially on their mutual distances. A complete analysis of the Toda system can be found for instance in [23, 36] and, in section 2, we will recall the main results needed for our analysis.

Observe that, if \( f := (f_1, \ldots, f_k) \) is a solution of this system, then for all \( \alpha > 0 \),
\[ f_\alpha := (f_\alpha, 1, \ldots, f_\alpha, k) \]
defined by
\[ f_\alpha, j(z) := f_j(\alpha z) - 2 \left( j - \frac{k + 1}{2} \right) \log \alpha, \]
is also a solution of (1.9). In our construction, we will exploit this scaling property of the Toda system. The corresponding graphs will be denoted by \( \gamma_{\alpha,j} \).

As we will see later, the functions \( f_j \), solutions of the Toda system, are asymptotically linear at infinity. In fact, for each solution, there exists \( a_j^\pm, b_j^\pm \in \mathbb{R} \) and \( \vartheta > 0 \) such that
\[ f_j(z) = a_j^\pm z + b_j^\pm + O((\cosh z)^{-\vartheta}), \]
at \( \pm\infty \) (with upper index + when \( z > 0 \) and upper index − when \( z < 0 \)). Keeping the above notations, our main result reads:

**Theorem 1.1.** Assume that \( N = 2, p > 2 \) and \( k \geq 2 \) are fixed. Further assume that we are given \( f := (f_1, \ldots, f_k) \) a solution of the Toda system (1.9). Then, there exists \( \vartheta > 0 \) and, for all sufficiently small \( \alpha > 0 \), there exists a solution of (1.1) with \( 2k \) ends which are asymptotic to the \( 2k \) half lines \( \Lambda^+_j(\alpha) \) (resp. \( \Lambda^-_j(\alpha) \)), for \( j = 1, \ldots, k \), which depend on \( \alpha \) and are the graphs of the functions
\[ z \mapsto \alpha (a_j^\pm + O(\alpha^{\vartheta})) z + b_j^\pm - 2 \left( j - \frac{k + 1}{2} \right) \log \alpha + O(\alpha^{\vartheta}), \]
for \( z > 0 \) and upper index + (resp. \( z < 0 \) and upper index −). Moreover, along the end \( \Lambda^+_j(\alpha) \) the solution is asymptotic to some Dancer’s solution whose parameters \( (\delta_j^\pm(\alpha), \tau_j^\pm(\alpha)) \) are close to 0, depend on \( \alpha \) and vary from end to end.

The description of the asymptotics of the solutions we construct depend on \( 8k \) parameters. Indeed, the description of an end, which is an oriented half line, requires two parameters and the description of Dancer’s solution to which our solution is asymptotic along an end, requires the knowledge of two parameters which are Dancer’s parameter and the translation along the end. The proof of the above theorem starts with building an approximate solution
\[ \sum_{j=1}^k w(\text{dist}(\cdot, \gamma_{\alpha,j})), \]
where \( \gamma_{\alpha,j} \) are the graphs of the functions \( f_{\alpha,j} \). Next, we allow some more flexibility in our approximate solution by introducing \( 8k \) parameters which account for small modifications of the approximate solutions away from a (large) compact set.
particular, we allow ourselves to translate and rotate slightly the end of the curves \( \gamma_{\alpha,j} \) and we allow ourselves to change \( w_0 \) into any Dancer’s solution with small parameters. As we will see in the proof, we will have to fix half of these parameters leaving the other half free. This implies that the solutions we construct belong to some \( 4k \) dimensional smooth family of solutions. This dimension count is in agreement with the result in [24] which computes the dimension of the space of \( 2k \)-ended, positive, entire solutions of (1.1).

As already mentioned, our result complements the recent result of Malchiodi [28] where solutions of (1.1) which finite number of ends are also constructed. Malchiodi’s solutions and our solutions are qualitatively completely different however they belong to the same class of entire solutions. Indeed, away from a compact set, all solutions constructed are asymptotic to a finite number of simply periodic solutions of (1.1) which are arranged along oriented half lines. In fact the solutions of Malchiodi and our solutions should be understood as two extremes since they correspond to two different parts of the compactification of the moduli space of solutions of (1.1). We shall further comment on this fact in the next section.

To be complete, let us mention that a similar construction has been obtained by the authors of the present paper for the Allen-Cahn equation

\[
\Delta u + u - u^3 = 0,
\]

in \( \mathbb{R}^2 \) [15]. In this case, Toda’s system also plays a central role in the construction but, in opposition to the analysis of the present paper, there is uniqueness of the profile which is used to construct the approximate solutions (namely there is no bifurcation phenomena which would lead to simply periodic solution as in the case for (1.1)) and this simplifies considerably the technical analysis.

1.3. **Geometric counterpart of our result.** One of the striking features of the existence result in Theorem 1.1, which is a purely PDE result, is that its counterparts can be found in geometric framework. To illustrate this, we will concentrate on what is perhaps the most appealing one: the analogy between the theory of complete constant mean curvature surfaces in Euclidean 3-space and the theory of entire solutions of (1.1). For simplicity we will restrict ourselves to constant mean curvature surfaces in \( \mathbb{R}^3 \) which have embedded coplanar ends. In the following we will draw parallels between these geometric objects and solutions of (1.1). This will also shed light on the relation between the different solutions of (1.1).

Embedded constant mean curvature surfaces of revolution were found by Delaunay in the mid 19th century [10]. They constitute a smooth one-parameter family of singly periodic surfaces \( D_t \), for \( t \in (0, 1] \), which interpolate between the cylinder \( D_1 = S^1(1) \times \mathbb{R} \) and the singular surface \( D_0 := \lim_{t \to 0} D_t \), which is the union of infinitely many spheres of radius \( 1/2 \) centered at each of the points \( (0,0,n) \), \( n \in \mathbb{Z} \). The Delaunay surface \( D_t \) can be parametrized by

\[
X_t(x,z) = (\varphi(x) \cos z, \varphi(x) \sin z, \psi(x)) \in D_t \subset \mathbb{R}^3,
\]

for \( (x,z) \in \mathbb{R} \times \mathbb{R}/2\pi \mathbb{Z} \). Here, the function \( \varphi \) is the smooth solution of

\[
(\varphi')^2 + \left( \frac{\varphi^2 + t^2}{2} \right)^2 = \varphi^2,
\]
and the function $\psi$ is defined (up to a constant) by

$$\psi' = \frac{\varphi^2 + t}{2}.$$

As already mentioned, when $t = 1$, the Delaunay surface is nothing but a right circular cylinder $D_1 = S^1(1) \times \mathbb{R}$, with the unit circle as its cross section. This cylinder is clearly invariant under the continuous group of vertical translations, in the same way that the single bump-line solution of (1.1) is invariant under a one parameter group of translations. It is then natural to agree on the correspondence between

$$\begin{array}{c|c}
\text{The cylinder} & \text{The single bump-line} \\
D_1 = S^1 \times \mathbb{R} & (x, z) \mapsto w_0(z)
\end{array}$$

Let us denote by $w_{gr}$ the unique radially symmetric, decaying solution of (1.1) (i.e. the ground state). Inspection of the other end of the Delaunay family, namely when the parameter $t$ tends to 0, suggests the correspondence between

$$\begin{array}{c|c}
\text{The sphere} & \text{The radially symmetric solution} \\
S^1(1/2) & (x, z) \mapsto w_{gr}(\sqrt{x^2 + z^2})
\end{array}$$

It is tempting to extend this correspondence for the whole range of the Delaunay parameter by associating the intermediate Delaunay surfaces with Dancer’s solutions. To do this, first of all, we need to find a curve in the function space that would represent these solutions. However, since we do not have any explicit formula for Dancer’s solution it is not immediately obvious how this curve should be defined. A natural possibility is to built a one parameter family solution of (1.1) by using the variational structure of the problem as follows: let $S_T = (0, T) \times \mathbb{R}$ and consider a least energy (mountain pass) solution in $H^1(S_T)$ for the energy

$$\frac{1}{2} \int_{S_T} |\nabla u|^2 + \frac{1}{2} \int_{S_T} u^2 - \frac{1}{p+1} \int_{S_T} u^{p+1},$$

for $T > 0$. We denote the least energy solution by $u_T$. Let us summarize what has been proven about it as $T$ varies between $T = 0$ and $T = \infty$ in [5]. In general the curve $T \mapsto u_T$ is analytic except for possibly finitely many $T$ (see also [1] for related results). After translating and reflecting with respect to line $x = T/2$, it can be shown that for all $T > 0$, $u_T > 0$ must be even in $x$ and with respect to the line $x = T/2$, it has a maximum located at $(T/2, 0)$ and it is non-increasing in $x$ and $y$ away from it. Moreover when $T < T_0$ the least energy solution is precisely the homoclinic while for $T > T_0$ it must depend on 2 variables in a non-trivial way, and as long as $T - T_0$ is small it is the bifurcating solution described above. For $T$ sufficiently large the least energy solution is unique and as $T \to \infty$ it converges, uniformly on compacts, to $w_{gr}$.

To give further credit to this correspondence, let us recall that the Jacobi operator about the cylinder $D_1$ corresponds to the linearized mean curvature operator when nearby surfaces are considered as normal graphs over $D_1$. In the above parameterization, the Jacobi operator reads $J_1 = (\partial_x^2 + \partial_z^2 + 1)$. In this geometric context, it plays the role of the linear operator $L$ which is the linearization of (1.1) about the single bump-line solution $w_0$. Hence we have the correspondence

$$\begin{array}{c|c}
\text{The Jacobi operator} & \text{The linearized operator} \\
J_1 = (\partial_x^2 + \partial_z^2 + 1) & L = \partial_x^2 + \partial_z^2 - 1 + p w_0^{p-1}.
\end{array}$$
Notice that the emergence of the family of Delaunay surfaces due to the loss of
stability of a cylinder when its height varies is the analogue to the emergence of the
Dancer solutions through a bifurcation from the homoclinic branch at $T = T_0$.

In our construction bounded and linearly growing elements of the kernel of the
linearized operator $L$ play a crucial role. They correspond to the natural invariances
of the problem: translations and rotations and also the derivative of the solution
with respect to Dancer's parameter. From this point of view, they turn out to have
the same geometric interpretation as the bounded and linearly growing elements
in the kernel of the Jacobi operator $J_1$, which again correspond to translations,
rotations and the derivative with respect to Delaunay's parameter.

With these analogies in mind, we can now translate our main result into the
constant mean curvature surface framework. The result of Theorem 1.1 corresponds
to the connected sum of finitely many copies of the cylinder $S^1(1) \times \mathbb{R}$ which have
a common plane of symmetry. The connected sum construction is performed by
inserting small catenoidal necks between two consecutive cylinders and this can be
done in such a way that the ends of the resulting surface are coplanar. Such a result,
in the context of constant mean curvature surfaces, follows at once from [33]. It is
observed that, once the connected sum is performed, the ends of the cylinder have
to be slightly bent and moreover, the ends cannot be kept asymptotic to the ends of
right cylinders but have to be asymptotic to Delaunay ends, in agreement with the
result of Theorem 1.1. In [33] a 4$k$ parameter family of constant mean curvature
surfaces whose ends are coplanar and asymptotic to Delaunay is constructed in the
same way our solutions belong to a 4$k$ dimensional family.

There is yet another difference between the two cases which indeed is much more
substantial. The Toda system which governs the location of the multiple bump lines
does not have a counterpart in the connected sum construction of the constant mean
curvature surfaces. This difference is due to the strong interactions between the
bump line in the context of semilinear elliptic equations.

Another (older) construction of complete noncompact constant mean curvature
surfaces was performed by N. Kapouleas [22] (see also [32]) starting with finitely
many halves of Delaunay surfaces with parameter $t$ close to 0 which are connected
to a central sphere. The corresponding solutions of (1.1) are the one which have
been constructed by A. Malchiodi in [28].

It is well known that the story of complete constant mean curvature surfaces
in $\mathbb{R}^3$ parallels that of complete locally conformally flat metrics with constant,
positive scalar curvature. Therefore, it is not surprising that there should be a cor-
respondence between these objects in conformal geometry and solutions of (1.1).
For example, Delaunay surfaces and Dancer solutions should now be replaced by
Fowler solutions which correspond to constant scalar curvature metrics on the cylin-
der $\mathbb{R} \times S^{n-1}$ which are conformal to the product metric $dz^2 + g_{S^{n-1}}$, when $n \geq 3$.
These are given by
$$v^{n-2}(dz^2 + g_{S^{n-1}}),$$
where $z \mapsto v(z)$ is a smooth positive solution of
$$(v')^2 - v^2 + \frac{n-2}{n} v^{\frac{2n}{n-2}} = -\frac{2}{n} \tau^2.$$
When $\tau = 1$ and $v \equiv 1$ the solution is a straight cylinder while as $\tau$ tends to 0
the metrics converge on compacts to the round metric on the unit sphere. The
connected sum construction for such Fowler type metrics was carried out by R.
Mazzeo, D. Pollack and K. Uhlenberk [35] (where it is called the dipole construction). N. Kapouleas’ construction mentioned above is due to Schoen [43] (see also R. Mazzeo and F. Pacard [32]).

2. The Toda System and its Linearization

In this paper, the Toda system (1.9) plays a crucial role and thus we collect here some important material concerning its analysis. The results are borrowed from [23, 36] and we refer the reader to these works for more details. The analysis as presented here is identical to the one which can be found in [15], hence we shall only present the key features of the results which are needed.

2.1. The Toda System. We are interested in the understanding of the solutions of the Hamiltonian system

\[ c_p^2 f''_j = e^{f_{j-1} - f_j} - e^{f_{j} - f_{j+1}}, \]

where we agree that \( f_0 \equiv -\infty \) and \( f_{k+1} \equiv +\infty \). We introduce the functions

\[ r_j := f_{j+1} - f_j + 2 \log c_p, \]

for \( j = 1, \ldots, k - 1 \). It is easy to check that, if \( f \) is a solution of (2.1), then \( r := (r_1, \ldots, r_{k-1}) \) is a solution of the following nonlinear system

\[ r'' - Me^{-r} = 0 \]

where the \((k-1) \times (k-1)\) matrix \( M \) is given by

\[
M := \begin{pmatrix}
2 & -1 & 0 & \ldots & 0 \\
-1 & \ddots & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \ddots & \ddots & -1 \\
0 & \ldots & 0 & -1 & 2
\end{pmatrix},
\]

and where \( e^{-r} \) is the vector whose entries are given by

\[ e^{-r} := (e^{-r_1}, \ldots, e^{-r_{k-1}}). \]

Conversely, given a solution \( r \) of (2.3) and \( \bar{p}, \bar{q} \in \mathbb{R} \), the functions

\[ f_j = \frac{1}{k} \left( \sum_{i=0}^{j-1} i r_i - \sum_{i=0}^{k-j} i r_{k-i} \right) + \bar{p} i + \bar{q} + 2 \left( \frac{k-1}{2} - j \right) \log c_p, \]

for \( j = 1, \ldots, k \) (we agree that \( r_0 = r_k \equiv 0 \)), are solutions of (2.1).

The explicit expression of the solutions of (2.3) can be found in section 7.7 of [23]. To describe it, we need to be given \( w := (w_1, \ldots, w_k) \in \mathbb{R}^k \) such that

\[ \sum_{j=1}^{k} w_j = 0, \quad \text{and} \quad w_{j+1} > w_j, \quad j = 1, \ldots, k - 1 \]

and \( g := (g_1, \ldots, g_k) \in \mathbb{R}^k \) such that

\[ \prod_{j=1}^{k} g_j = 1, \quad \text{and} \quad g_j > 0, \quad \text{for} \quad j = 1, \ldots, k. \]
Finally, for \( j = 2, \ldots, k - 1 \), we define the function

\[
\Phi_j(\mathbf{g}, \mathbf{w}; t) := \sum_{1 \leq i_1 < \cdots < i_j \leq k} R_{i_1 \cdots i_j} (\mathbf{w}) g_{i_1} \cdots g_{i_j} e^{-t(w_{i_1} + \cdots + w_{i_j})},
\]

(see formula 7.7.10 in [23]) where \( R_{i_1 \cdots i_j} \) are rational functions of the entries of the vector \( \mathbf{w} \) whose precise form can be found in section 7.5 of [23]. We also agree that

\[
\Phi_0 = \Phi_k \equiv 1.
\]

It is proven in [23] that all solutions of (2.1) are of the form

\[
(2.7) \quad r_j(t) = -\log \Phi_{j-1}(\mathbf{g}, \mathbf{w}; t) + 2 \log \Phi_j(\mathbf{g}, \mathbf{w}; t) - \log \Phi_{j+1}(\mathbf{g}, \mathbf{w}; t)
\]

for some choice of \( \mathbf{g} \) and \( \mathbf{w} \). Observe that we have a \( 2k \) family of solutions of (2.3) since \( \mathbf{g} \) and \( \mathbf{w} \) provide \( 2(k - 1) \) independent parameters to which we have to add the parameters \( \bar{\rho} \) and \( \bar{q} \).

The next result is also borrowed from [23], [36]. It describes the asymptotics of the solutions of (2.3) (see Theorem 7.7.2 of [23]):

**Lemma 2.1.** Let \( \tau_0 > 0 \) be defined by

\[
(2.8) \quad \tau := \min_{j=1,\ldots,k-1} (w_{j+1} - w_j).
\]

Then, for \( j = 1, \ldots, k - 1 \), the following expansion holds

\[
(2.9) \quad r_j(t) = c_j t - d_j + \epsilon_j^+ (\mathbf{c}) + \mathcal{O}(e^{-\tau}),
\]

as \( t \) tends to \( +\infty \) and

\[
(2.10) \quad r_j(t) = -c_{k-j} t + d_{k-j} + \epsilon_j^- (\mathbf{c}) + \mathcal{O}(e^{-\tau}),
\]

as \( t \) tends to \( -\infty \), where, for \( j = 1, \ldots, k - 1 \),

\[
(2.11) \quad c_j := w_{j+1} - w_j, \quad d_j := \log g_{j+1} - \log g_j,
\]

and where \( \epsilon_j^\pm \) are smooth functions of \( \mathbf{c} := (c_1, \ldots, c_{k-1}) \).

**2.2. The linearized Toda system.** We assume that \( \mathbf{f} = (f_1, \ldots, f_k) \) is a solution of (2.1) described in the previous section. The linearized system associated to linearization of (2.1) about the solution \( \mathbf{q} \), reads as

\[
(2.12) \quad c_{\rho}^2 \mathbf{v}''' + \mathbf{N} \mathbf{v} = \mathbf{z},
\]

where the \( k \times k \) matrix \( \mathbf{N} \) has coefficients which are exponentially decaying at \( \pm \infty \). We analyze the solvability of the above linear problem in the space \( C_{\rho}^{\ell, \mu}(\mathbb{R}; \mathbb{R}^k) \) of \( C^{\ell, \mu} \) vector valued functions \( \mathbf{v} \) which satisfy

\[
(2.13) \quad \| \mathbf{v} \|_{C_{\rho}^{\ell, \mu}(\mathbb{R}; \mathbb{R}^k)} := \| (\cosh t)^{\theta} \mathbf{v} \|_{C^{\ell, \mu}(\mathbb{R})} < \infty.
\]

We will write

\[
(2.14) \quad v_j^\ell := \partial_{c_j} \mathbf{f} \quad \text{and} \quad v_j^q := \partial_{d_j} \mathbf{f},
\]

for \( j = 1, \ldots, k - 1 \), and

\[
(2.15) \quad v_j^\rho := \partial_{\rho} \mathbf{f} \quad \text{and} \quad v_j^q := \partial_{\bar{q}} \mathbf{f}.
\]

Since the solution \( \mathbf{f} \), as described in (2.4), depends smoothly on the parameters \( c_1, \ldots, c_{k-1} \) and \( d_1, \ldots, d_{k-1} \) as well as the parameters \( \bar{q} \) and \( \bar{\rho} \), there yields \( 2k \) linearly independent solutions of the homogeneous problem \( c_{\rho}^2 \mathbf{v}''' + \mathbf{N} \mathbf{v} = 0 \). It
follows from the result of Lemma 2.1 that, as \( t \) tends to \( \pm \infty \), the vector valued functions \( v^\sharp_j \) and \( v^\flat_j \) can be decomposed as

\[
v^\sharp_j = a^\sharp_j,\pm t + b^\sharp_j,\pm + O_C((\cosh t)^{-\theta}),
\]
and

\[
v^\flat_j = b^\flat_j,\pm + O_C((\cosh t)^{-\theta}),
\]
where \( a^\sharp_j,\pm \) and \( b^\sharp_j,\pm \), \( b^\flat_j,\pm \) are fixed vectors in \( \mathbb{R}^k \). Moreover, \( \{a^\sharp_j,\iota : j = 1, \ldots, k\} \) and \( \{b^\flat_j,\iota : j = 1, \ldots, k\} \) are basis of \( \mathbb{R}^k \), for \( \iota = \pm \).

We now define the deficiency space

\[
D := \text{Span}\ \left\{ \chi^\pm v^\sharp_j, \chi^\pm v^\flat_j : j = 1, \ldots, k \right\},
\]
where we recall that \( \chi^+ \) (resp. \( \chi^- \)) is a cutoff function identically equal to 1 for \( t > 1 \) (resp. for \( t < -1 \)) and identically equal to 0 for \( t < -1 \) (resp. for \( t > 1 \)) and \( \chi^+ + \chi^- \equiv 1 \). Observe that \( D \) is \( 4k \) dimensional and contains

\[
K := \text{Span}\ \left\{ v^\sharp_j, v^\flat_j : j = 1, \ldots, k \right\},
\]
which is the \( 2k \) dimensional space of homogeneous solutions of \( c_\theta^2 v'' + Nv = 0 \).

Therefore, we can certainly decompose

\[
(2.12) \quad D = K \oplus E.
\]
where \( E \) is a complement of \( K \) in \( D \). With this decomposition at hand, we have the following result which follows from standard arguments in ordinary differential equations [15].

**Lemma 2.2.** Assume that \( \theta > 0 \). Then the mapping

\[
T : C^{2,\mu}_\theta(\mathbb{R}; \mathbb{R}^k) \oplus \mathcal{E} \rightarrow C^{0,\mu}_\theta(\mathbb{R}; \mathbb{R}^k)
\]

\[
v \mapsto c_\theta^2 v'' + Nv
\]
is an isomorphism.

Observe that there the definition of \( T \) depends on the choice of \( \mathcal{E} \).

### 2.3. Another important ODE

We end up with the discussion of a simple linear problem which, even though it is not directly related to the Toda system considered above, also plays an important role in the sequel. The problem we have in mind reads

\[
e'' + \kappa^2 e = g,
\]
where \( \kappa > 0 \) is a fixed constant and where \( g \in C^{2,\mu}_\theta(\mathbb{R}) \), for some \( \theta < 0 \). We are interested in solutions to this problem which decay exponentially at infinity. It is clear that

\[
e(t) = -\frac{1}{\kappa} \cos(\kappa t) \int_{-\infty}^t \sin(\kappa \zeta) g(\zeta) d\zeta + \frac{1}{\kappa} \sin(\kappa t) \int_{-\infty}^t \cos(\kappa \zeta) g(\zeta) d\zeta,
\]
is a solution of (2.14) and one can check that this function decays exponentially at \( -\infty \). If we assume that in addition

\[
\int_{-\infty}^\infty g(\zeta) \sin(\kappa \zeta) d\zeta = 0, \quad \int_{-\infty}^\infty g(\zeta) \cos(\kappa \zeta) d\zeta = 0,
\]

then

\[
e(\infty) = 0.
\]
then $\epsilon \in C^2_\varrho (\mathbb{R})$ and we have
\begin{equation}
\|\epsilon\|_{C^2_\varrho (\mathbb{R})} \leq C \|g\|_{C^0_\varrho (\mathbb{R})},
\end{equation}
for some constant $C > 0$. The necessity of imposing (2.16) to get solutions which decay at infinity has its direct consequence for our construction of solutions of (1.1). As we will see, it is precisely because of (2.16) that we can only fix arbitrarily the amplitudes and phase shifts of only half of the ends. $2k$ ends (say all lower ends if we chose so) of the bump lines and we need to adjust suitably the amplitudes and the phase shifts of the remaining $2k$ ends and thus we have only $2k$ (and not $4k$ as one might expect) free parameters corresponding to the amplitudes and the phase shifts.

3. The approximate solution

3.1. Local coordinates near model bump lines. We will fix from now on a solution to the Toda system $f$ with the properties described in the previous section. We will also choose $v \in \mathcal{E}$. We will assume that
\begin{equation}
\|v\|_{\mathcal{E}} \leq \alpha^{\kappa_1},
\end{equation}
where $\kappa_1 > 0$ is a small number to be chosen later on. With these two functions at hand we define for each $j = 1, \ldots, k$ the model for a bump line to be the curve:
\[ \tilde{\gamma}_{\alpha,j} = \{ x = (x, z) \in \mathbb{R}^2 \mid x = f_{\alpha,j}(z) + v_j(\alpha z) \}, \]
where $f_\alpha = (f_{\alpha,1}, \ldots, f_{\alpha,k})$ is the rescaled solution to the Toda system, see (1.10).

We will introduce local coordinates associated with each $\tilde{\gamma}_{\alpha,j}$. For convenience we will denote $\tilde{f}_\alpha (z) = f_\alpha (z) + v(\alpha z)$. We will fix the orientation of $\tilde{\gamma}_{\alpha,j}$ in such a way that the pair of vectors $(T_{\alpha,j}, N_{\alpha,j})$, where the unit tangent $T_{\alpha,j} = \frac{1}{\sqrt{1 + \alpha(f'_{\alpha,j})^2}}(\alpha f'_{\alpha,j}, 1)$ and the unit normal $N_{\alpha,j} = \frac{1}{\sqrt{1 + (\alpha f'_{\alpha,j})^2}}(1, -\alpha f'_{\alpha,j})$ are negatively oriented (and the functions $f'_{\alpha,j}$ are evaluated at $\alpha z$). Let $z_j$ be the arc length parameter on $\tilde{\gamma}_{\alpha,j}$ i.e.
\begin{equation}
 z_j = \int_0^z \sqrt{1 + \alpha^2(f'_{\alpha,j})^2(\alpha \zeta)} \, d\zeta,
\end{equation}
and let $q_{\alpha,j} = q_{\alpha,j}(z_j)$ be the corresponding arc length parametrization.

As it turns out the true asymptotic behavior of the bump line is not exactly linear but it has an extra exponentially small correction. This correction is an unknown to be determined, and in fact this is one of the most important steps in this paper which involves the linearized Toda system discussed in the previous section. To describe this perturbation we let $h = (h_1, \ldots, h_k)$ to be a fixed function such that
\begin{equation}
\|h\|_{C^2_\varrho (\mathbb{R}; \mathbb{R}^k)} \leq \alpha^{\kappa_2},
\end{equation}
with some small parameter $\kappa_2$. In the sequel we will use the function $h$ of the stretched argument $\alpha z$, namely we will write $h(\alpha z)$. To measure the size of this function it is more suitable to use the weights of the form $(\cosh z)^{\theta \alpha}$ rather than $(\cosh z)^\theta$. Thus we will see norms like $\|\cdot\|_{C^1_\varrho (\mathbb{R}; \mathbb{R}^k)}$. In general we have the following relations:
\begin{equation}
\|h\|_{C^1_\varrho (\mathbb{R}; \mathbb{R}^k)} \leq \|h\|_{C^0_\varrho (\mathbb{R}; \mathbb{R}^k)}, \quad \|h\|_{C^2_\varrho (\mathbb{R}; \mathbb{R}^k)} \leq \alpha^{-\ell - \mu} \|h\|_{C^0_\varrho (\mathbb{R}; \mathbb{R}^k)}.
\end{equation}
These relations will be used for the function $h$ as well as for several similar type functions appearing below without specially mentioning them. Thus for instance from (3.3) and (3.4) it follows:

$$\|h\|_{C^{\infty}_x(R;\mathbb{R}^k)} \leq \varepsilon \alpha^{k_2}.$$  

A neighborhood of the curve $\bar{\gamma}_{\alpha,i}$ can be parametrized in the following way:

$$x = X_{\alpha,i}(x_j, z_j) = q_{\alpha,i}(z_j) + (x_j + h_j(\alpha z_j)) N_{\alpha,i}(z_j).$$

Notice that $t_j = x_j + h_j(\alpha z_j)$ is simply the signed distance to $\bar{\gamma}_{\alpha,i}$. For this reason our local coordinates can be seen as shifted with respect to the Fermi coordinates of the curve $\bar{\gamma}_{\alpha,i}$.

The distance function is not a smooth function in the whole $\mathbb{R}^2$ however we observe that given $f$, $\nu$ there exists a maximal subset of $\mathbb{R}^2$ in which $t_j$ is a smooth function for $j = 1, \ldots, k$. Using the asymptotic (linear) behavior of $f, \nu(\alpha z)$ and estimate (3.1) it is not hard to prove that this set contains the set:

$$V_\zeta = \{x = (x, z) \mid |x| \leq \frac{\zeta}{\alpha} \sqrt{1 + z^2}\},$$

with certain small constant $\zeta$. Indeed, the Fermi coordinates are defined as long as the map $(t_j, z_j) \mapsto x$ is one-to-one. Using the fact that the curvature of each $\bar{\gamma}_{\alpha,i}$,

$$k_{\alpha,i}(z_j) \sim \alpha^2 (\cosh z_j)^{-\delta \alpha},$$

and also the asymptotic behavior as $|z_j| \to \infty$:

$$\bar{\gamma}_{\alpha,i}(z_j) \sim (O(\alpha)|z_j| + O(\log \frac{1}{\alpha}), z_j(1 + O(\alpha^2))),$$

one can show that for each small $\zeta_j$ and each sufficiently small $\alpha$ the Fermi coordinates are well defined around $\bar{\gamma}_{\alpha,i}(z_j)$ as long as:

$$|t_j| \leq \frac{\zeta_j}{\alpha} \sqrt{1 + z_j^2}.$$

Noting that the distance between $\bar{\gamma}_{\alpha,i}$ and any other curve, say $\bar{\gamma}_{\alpha,i}$, behaves like

$$\text{dist} \left(\bar{\gamma}_{\alpha,i}(z_j), \bar{\gamma}_{\alpha,i}\right) \sim O(\alpha)|z_j| + O(\log \frac{1}{\alpha}),$$

we conclude that the constant $\zeta$ in the definition of the set $V_\zeta$ can be taken as small as we wish and also, using (3.3), that it can be chosen in such a way that:

$$x \in V_\zeta \implies |x_j| = |t_j - h_j(\alpha z_j)| \leq \frac{\zeta_j}{\alpha} \sqrt{1 + z_j^2}, \quad x = X_{\alpha,i}(x_j, z_j).$$

To accomplish this it suffices to take $V_\zeta$ to be the intersection of all the sets where (3.6) is satisfied.

In the sequel we will use convenient notation: for a given function $f : V_\zeta \to \mathbb{R}$ we set:

$$(3.8) \quad X_{\alpha,i}^* f(x_j, z_j) = (f \circ X_{\alpha,i})(x_j, z_j).$$

We will also need a simple relation between the coordinates $x_j$ and $x_i$, which follows from the definition of the curves $\bar{\gamma}_{\alpha,i}$ together with elementary geometry. By definition of the coordinates (3.5) we get:

$$(3.9) \quad x_i = [q_{\alpha,i}(z_j) - q_{\alpha,i}(z_i)] (1 + O(\alpha^2)) + O(\alpha^{k_2}) + x_j (1 + O(\alpha^2)),
\quad z_i = z_j (1 + O(\alpha^2)) + O(\alpha)(x_j - x_i) + O(\alpha^{1+k_2}).$$
Since
\[ q_{\alpha,i}(z_i) - q_{\alpha,i}(z_j) = \mathcal{O}(\alpha)(z_j - z_i), \]
in \( V_{\varepsilon} \) we have
\[ x_j - x_i = 2(i - j) \log \frac{1}{\alpha} + \mathcal{O}(\alpha^2)x_j + \mathcal{O}(\alpha)z_j + \mathcal{O}(\alpha^{\varepsilon/2}). \]
\[ z_i - z_j = \mathcal{O}(\alpha^2)z_j + \mathcal{O}(\alpha \log \frac{1}{\alpha}) + \mathcal{O}(\alpha^3)x_j. \]
as \( \alpha \) tends to 0.

### 3.2. Laplacian in the local coordinates

It will be useful to have the expression of the Laplacian in the coordinates defined in (3.5). Let \( k_{\alpha,j} \) be the curvature of the curve \( \tilde{\gamma}_{\alpha,j} \), which in its natural parametrization is given by:
\[ k_{\alpha,j} = \frac{\alpha^2 \tilde{f}_{\alpha,j}'(\alpha z)}{(1 + \alpha^2(\tilde{f}_{\alpha,j}(\alpha z))^2)^{3/2}}, \]
\[ z_j = \int_{0}^{z} \sqrt{1 + \alpha^2(\tilde{f}_{\alpha,j}(\alpha \zeta))^2} \, d\zeta. \]
We define the function \( A_j \) by
\[ A_j := 1 - (x_j + h_j)k_{\alpha,j}. \]
With this notation the following expression for the Laplacian is easy to derive:
\[ \Delta = \frac{1}{A_j} \left\{ \partial_{x_j} \left( \frac{A_j^2 + \alpha^2(h_j')^2}{A_j} \partial_{x_j} \right) - \partial_{z_j} \left( \frac{\alpha h_j'}{A_j} \partial_{z_j} \right) - \partial_{x_j} \left( \frac{\alpha^2 h_j'}{A_j} \partial_{x_j} \right) + \partial_{z_j} \left( \frac{1}{A_j} \partial_{z_j} \right) \right\}. \]
This formula can be written in the form:
\[ \Delta = \partial_{x_j}^2 + \partial_{z_j}^2 + a_{11,j} \partial_{x_j}^2 + a_{12,j} \partial_{z_j} \partial_{x_j} + a_{22,j} \partial_{z_j}^2 + b_{1,j} \partial_{x_j} + b_{2,j} \partial_{z_j}, \]
where:
\[ a_{11,j} = \frac{\alpha^2(h_j')^2}{A_j^2}, \quad a_{12,j} = -\frac{2\alpha h_j'}{A_j}, \quad a_{22,j} = \frac{1 - A_j^2}{A_j^2}, \]
\[ b_{1,j} = \frac{1}{A_j}(-k_{\alpha,j} A_j^2 - \alpha^2 h_j'' A_j + \alpha^2(h_j')^2 k_{\alpha,j} - \alpha(x_j + h_j)h_j' k_{\alpha,j}'), \]
\[ b_{2,j} = \frac{1}{A_j^2}(h_j + x_j) k_{\alpha,j}'. \]
The reader should keep in mind that functions \( h_j, k_{\alpha,j} \) are taken as functions of \( \alpha z_j \). Additionally we recall that
\[ k_{\alpha,j} = \mathcal{O}_{C^2_{\theta\alpha}(\mathbb{R})}(\alpha^2), \quad k_{\alpha,j}' = \mathcal{O}_{C^2_{\theta\alpha}(\mathbb{R})}(\alpha^3), \]
and consequently we have in \( V_{\varepsilon} \), taking into account (3.7):
\[ a_{11,j} = \mathcal{O}_{C^{0,2}(\mathbb{R})}(\alpha^2), \quad a_{12,j} = \mathcal{O}_{C^{0,1}(\mathbb{R})}(\alpha), \quad a_{22,j} = \mathcal{O}_{C^{0,2}(\mathbb{R})}(\alpha^2(1 + |x_j|)), \]
\[ b_{1,j} = \mathcal{O}_{C^{0,2}(\mathbb{R})}(\alpha^2(1 + |x_j|)), \quad b_{2,j} = \mathcal{O}_{C^{0,2}(\mathbb{R})}(\alpha^3(1 + |x_j|)). \]
3.3. Asymptotic formulas for the homoclinic and Dancer’s solution. In this section we will list some well known or standard properties of the functions we will use in the sequel. We will use them without special making any special reference since there are rather ubiquitous. First we recall that for the homoclinic solution defined in (1.2), we have:

\[ w(x) = e^{-|x|} + O(e^{-2|x|}), \quad \text{as } |x| \to \infty. \]

Second, let us recall that the linearized operator

\[
L = \partial_x^2 - 1 + p w^p, \quad (3.16)
\]

has a unique principal eigenvalue \( \lambda_1 > 0 \) with corresponding eigenfunction \( Z(x) > 0 \). In fact we have

\[
\lambda_1 = \frac{1}{4} (p-1)(p+3), \quad Z = \frac{w^{(p+1)/2}}{\sqrt{\int_R w^{p+1}}},
\]

and in particular

\[
Z(x) = e^{-\frac{p+1}{2}|x|} + O(e^{-(p+1)|x|}), \quad \text{as } |x| \to \infty.
\]

It is also known that \( \lambda_2 = 0 \) and the corresponding eigenfunction is \( w' \) while the rest of the spectrum is strictly negative.

Finally, using the result of [9] and standard facts about bifurcating solutions, we find that, for sufficiently small \( \delta \) and \( \tau \), Dancer’s solutions can be expanded as

\[
w_{\delta,\tau}(x,z) = w(x) + Z(x) \left( \delta \cos(\sqrt{\lambda_1} y) + \tau \sin(\sqrt{\lambda_1} y) \right) + e^{-|x|} O_{L^\infty(R^2)}(|\delta|^2 + |\tau|^2),
\]

in the strip \( R \times [0, T] \), for \( \varepsilon = \sqrt{\delta^2 + \tau^2} \).

3.4. Definition of the approximate solution. Before giving a precise definition of the approximate solution let us explain the ingredients from which it is built. Considering just one of the bump lines we require that its lower and upper ends be asymptotic to two (possibly distinct) Dancer solutions. These two functions are “glued” together by some cutoff function. Let us observe that this way the amplitudes and the phase shifts of the ends do not change along the end of the bump line but instead are fixed. This is possible because the ends, whose shape is determined through the Toda system, are asymptotically linear. However, in the middle the bump line is curved and there the amplitude and the phase shift must be allowed to vary. This is quite analogous to bending of a corrugated, plastic pipe which "wrinkles" are stretched on the outside but piled up on the inside. To achieve this extra degree of freedom a function, whose local form is given by \( e_j(\alpha x_j) Z(x_j) \) is added to our approximation. Comparing with the asymptotic formula for the Dancer solution we see that this form of the extra correction to the approximate solution is natural.

Let us be more precise now. We will consider vector functions \( e \in C_0^{2,\mu}(\mathbb{R}, \mathbb{R}^k) \) with the property:

\[
\|e\|_{C_0^{2,\mu}(\mathbb{R}, \mathbb{R}^k)} \leq \alpha^{2+\kappa_2}
\]

(3.18)
where $\kappa_3$ is a small number to be chosen later on. In addition we will use $4k$ real parameters $\delta_\pm = (\delta_{\pm,1},\ldots,\delta_{\pm,k})$ and $\tau_\pm = (\tau_{\pm,1},\ldots,\tau_{\pm,k})$, such that with some small $\kappa_4$:

$$\|\delta_\pm\| + \|\tau_\pm\| \leq \alpha^{1+\kappa_4}. \tag{3.19}$$

Denoting by $w$ the homoclinic solution, by $w_{\delta,\tau}$ the Dancer solution to (1.1) and by $Z$ the principal eigenvector of the operator $L$ defined in (3.16) we define (using the notation (3.8)) functions:

$$X_{\alpha,j}^* w_{\pm,j}(x_j, z_j) = w_{\delta_{\pm,j},\tau_{\pm,j}}(x_j, z_j),$$

$$X_{\alpha,j}^* w_0(x_j) = w(x_j),$$

$$X_{\alpha,j}^* Z_j(x_j, z_j) = Z(x_j). \tag{3.20}$$

Now, let $\Xi_\pm \geq 0$, $\Xi_0 \geq 0$ be cutoff functions such that

$$\Xi_+(t) + \Xi_0(t) + \Xi_-(t) = 1, \quad \forall t \in \mathbb{R},$$

$$\text{supp} \, \Xi_+ = (1, \infty), \quad \text{supp} \, \Xi_0 = (-2, 2), \quad \text{supp} \, \Xi_- = (-\infty, -1),$$

and let

$$X_{\alpha,j}^* \Xi_{\pm,j}(x_j, z_j) := \Xi_{\pm}(\alpha z_j), \quad X_{\alpha,j}^* \Xi_0(x_j, z_j) := \Xi_0(\alpha z_j).$$

We will introduce the following convenient notation:

$$\tilde{w}_j = \Xi_{+,j} w_{+,j} + \Xi_{0,j} w_0,j + \Xi_{-,j} w_{-,j}. \tag{3.21}$$

Given these notations we will define the approximate solution of (1.1) in $\mathcal{V}_\varsigma$ by:

$$\tilde{w}(x) = \sum_{j=1}^k \tilde{w}_j + e_j(\alpha z_j)Z_j. \tag{3.22}$$

Notice that $\tilde{w}$ depends on the parameters $f_\alpha, \nu, h, e, \delta_\pm, \tau_\pm$. We will not emphasize this dependence unless necessary. Taking now a smooth cutoff function $\eta_\varsigma$ supported in $\mathcal{V}_\varsigma$ and such that $\eta_\varsigma \equiv 1$ in $\mathcal{V}_{2}^\varsigma$ we define the global approximate solution of (1.1) by:

$$w := \eta_\varsigma \left(\sum_{j=1}^k \tilde{w}_j + e_j(\alpha z_j)Z_j\right) = \eta_\varsigma \tilde{w}. \tag{3.23}$$

4. Proof of Theorem 1.1

4.1. Reduction to the nonlinear projected problem. For the proof of the theorem it is convenient to modify (1.1) slightly. As customary we will consider initially

$$\Delta u - u + u_+^p = 0, \tag{4.1}$$

where $u_+$ is the positive part of $u$. The modification of the nonlinearity has no effect on the preceding considerations. Also, once the existence of a solution to (4.1) is established, as an immediate consequence of the maximum principle we will obtain the existence for (1.1) as well.

Let $\rho$ be a cutoff function such that

$$\rho(s) = \begin{cases} 1, & |s| \leq \frac{3}{2}, \\ 0, & |s| > \frac{3}{2}. \end{cases} \tag{4.2}$$
We define:
\[ X_{\alpha,j}^* \rho_j = \rho \left( \frac{x_j}{\log \frac{1}{\alpha}} \right). \]
(4.3)

Finally, we define the function \( w'_{0,j} \) by:
\[ X_{\alpha,j}^* w'_{0,j} = w'(x_j), \]
where \( w \) is the homoclinic solution of (1.1).

We look for a solution to (4.1) in the form \( u = w + \varphi \) where \( \varphi \) is a function to be determined. Denoting by \( S(u) \) the nonlinear Schrödinger operator in (4.1) we expand:
\[ S(w + \varphi) = L(\varphi) + S(w) + N(\varphi), \]
where \( S(w) \) is defined in (4.15) and
\[ L(\varphi) = \Delta \varphi - \varphi + pw^{p-1} \varphi, \]
\[ N(\varphi) = (w + \varphi)^p - w^p - pw^{p-1} \varphi. \]

This way our problem can be written in the form:
\[ L(\varphi) + S(w) + N(\varphi) = 0, \]
and in principle it should be possible to reduce to a fixed point problem for the nonlinear function
\[ \varphi + L^{-1}(S(w) + N(\varphi)) = 0, \]
provided that the operator \( L^{-1} \) is, in a suitable sense, bounded. But this is of course what we do not expect in general since in some sense \( L \) is a small perturbation, at least near a fixed bump line, of the operator
\[ L = (\partial_x^2 + \partial_z^2 - 1 + pw^{p-1}), \]
which has bounded kernel spanned by the functions \( w'(x) \), and \( Z(x) \cos(\sqrt{\lambda_1}z) \), \( Z(x) \sin(\sqrt{\lambda_1}z) \).

To deal with this (indeed fundamental) difficulty we will reduce the problem to the following projected nonlinear problem:
\[ L(\varphi) + S(w) + N(\varphi) + \sum_{j=1}^{k} c_j w'_{0,j} \rho_j + \sum_{j=1}^{k} d_j Z_j \rho_j = 0. \]
(4.4)

In the following sections we will describe:

1. how to solve (4.4) for the unknowns \( \varphi \) and \( c = (c_1, \ldots, c_k) \), \( d = (d_1, \ldots, d_k) \) with given fixed parameters \( v, h, e, \delta_\pm, \tau_\pm \), and

2. we will show how to adjust these parameters to achieve \( c \equiv 0, d \equiv 0. \)

This clearly will yield a solution to (4.1) (and (1.1)) as described in Theorem 1.1.

4.2. The decomposition procedure. In this section we will explain how to decompose the projected nonlinear problem into \( k + 1 \) coupled equations. The advantage of this procedure is that we can deal separately with \( k \) problems, each of which is associated with a single bump line, and an extra \((k + 1)\)th problem that accounts for a cumulative, far field behavior of the bump lines.
To begin with we need to introduce cutoff functions $\chi$, $\chi_j$, $j = 1, \ldots, k$ as follows:

\[
\chi(s) = \begin{cases} 
1, & |s| \leq \frac{7}{15}, \\
0, & |s| > \frac{15}{16}.
\end{cases}
\]

We define:

\[
X^*_\alpha,j \chi_j = \chi \left( \frac{X_j}{\log \frac{1}{\alpha}} \right).
\]

Comparing this with the definition of the cutoff functions $\rho_j$, $\rho$ in (4.2)–(4.3) we see that

\[
\chi_j \rho_j = \rho_j, \quad \chi_j \chi_i = 0, \quad j \neq i.
\]

This last statement follows from the fact that the distance between any two model bump lines is at least like $2 \log \frac{1}{\alpha} + O(1)$ and the definition of $\chi_j$.

We look for a solution of (4.4) in the form

\[
\varphi = \sum_{j=1}^k \phi_j \rho_j + \psi.
\]

It is straightforward to check that this function is the solution if we require that functions $\phi_j$, $j = 1, \ldots, k$ and $\psi$ satisfy the following system of equations:

\[
\begin{align*}
\chi_j L(\phi_j) + c_j w_{0,j} \chi_j + d_j Z_j \chi_j &= \chi_j (S(w) + N) - \chi_j (L - \Delta + 1) \psi, \\
(\Delta - 1) \psi &= (1 - \sum_{i=1}^k \rho_i) (S(w) + N) - \sum_{i=1}^k (L(\phi_i \rho_i) - \rho_i L(\phi_i)) \\
&- (1 - \sum_{i=1}^k \rho_i) (L - \Delta + 1) \psi,
\end{align*}
\]

where $N = N(\sum_{j=1}^k \phi_j \rho_j + \psi)$. Indeed, multiplying (4.9) by $\rho_j$, using (4.7) and adding all the equations we get (4.4). This is a coupled system however the coupling terms are of higher order (in $\alpha$). Additionally the linear operator on the right hand side of (4.9) expressed in the local coordinates is a small perturbation of the basic linearized operator $L$ already seen above. We will take advantage of these facts in what follows.

We further recast (4.9)–(4.10). Clearly $\phi_j$ is a solution of (4.9) if

\[
[\partial_{x_j}^2 + \partial_{z_j}^2 + g'(w_{0,j})] X^*_\alpha,j \phi_j = X^*_\alpha,j k_j - X^*_\alpha,j (c_j w'_{0,j} \chi_j) - X^*_\alpha,j (d_j Z_j \chi_j),
\]

where

\[
X^*_\alpha,j k_j = X^*_\alpha,j [\chi_j \{ S(w) + N \}] - X^*_\alpha,j (\chi_j ( L - \Delta + 1) \psi) - X^*_\alpha,j (\chi_j L(\phi_j)) + (X^*_\alpha,j \chi_j) [\partial_{x_j}^2 + \partial_{z_j}^2 + g'(w_{0,j})] X^*_\alpha,j \phi_j.
\]

Again this is evident because of (4.7). We observe that (4.11) can be seen as an equation in $(x_j, z_j) \in \mathbb{R}^2$. In particular functions $X^*_\alpha,j \phi_j$, as solutions of (4.11) are defined for all $(x_j, z_j) \in \mathbb{R}^2$, although in reality these variables correspond to the local coordinates of $\gamma_{\alpha,j}$ in a subset of $\mathbb{R}^2$ only. It is important to remember that this subset contains $\text{supp} \chi_j$. 

Let us now consider equation (4.10). Denoting \( \phi = (\phi_1, \ldots, \phi_k) \) and the right hand side of (4.10) by \( Q = Q(\phi, \psi) \) we can write:
\[
(\Delta - 1)\psi = Q(\phi, \psi).
\]
This way (4.9)–(4.10) is reduced to the system of equations given by (4.11) and (4.13). This is a nonlinear system for the unknowns \( \phi_j, j = 1, \ldots, k \) and \( \psi \) with functions \( c_j \) and \( d_j \) to be determined as well. Because (4.11) carries all long range interactions between the bump lines we will refer to it and its modifications as the interaction system. Equation (4.13) will be called the background equation.

4.3. The error of the initial approximation. Let us analyze the right hand sides of the (4.11), (4.13). We introduce the following weighted Hölder norms:
\[
\|\phi\|_{C^{\tau,\nu}(\mathbb{R}^2)} = \sup_{x \in \mathbb{R}^2} \left( (\cosh x)^\sigma (\cosh z)^\tau \|\phi\|_{C^{\nu}(B_1(x))} \right).
\]

The error of the global approximation \( w \) is defined by:
\[
S(w) = \Delta w - w + w^\ell.
\]
This function depends in particular on the parameters \( v, h, e, \delta, \tau, \tau_\pm \), and although this dependence is usually not emphasized sometimes it will be necessary to denote:
\[
S(w) = S(w; v, h, e, \delta, \tau, \tau_\pm).
\]
We always assume that these parameters satisfy estimates (3.1), (3.3), (3.18) and (3.19) with some fixed \( \kappa_i > 0, i = 1, \ldots, 4 \). In particular we notice that the most involved is the dependence of the error on \( h \) through the local variables \( (x_j, z_j) \).

We will go back to this issue in more detail later.

We state the main result of this section.

Proposition 4.1. The function \( S(w; v, h, e, \delta, \tau, \tau_\pm) \) is a continuous function of its parameters and for each sufficiently small \( \alpha \) the following estimate holds:
\[
\|X_{\alpha,\ell}(\chi_j S(w))\|_{C^{\nu,\mu}(\mathbb{R}^2)} \leq C\alpha^2,
\]
where \( 0 < \sigma < \min\{p - 2, 1\} \), \( \theta \in (0, \vartheta) \) and \( \vartheta \) is the constant defined in (2.8). Moreover \( S(w) \) is a Lipschitz function of its parameters \( h, e \), and denoting \( S^{(\ell)} = S(w; v, h^{(\ell)}, e^{(\ell)}, \delta, \tau, \tau_\pm) \), \( \ell = 1, 2 \), we have:
\[
\|X_{\alpha,\ell}^{(1)}(\chi_j S^{(1)}) - X_{\alpha,\ell}^{(2)}(\chi_j S^{(2)})\|_{C^{\nu,\mu}(\mathbb{R}^2)} \leq C(\alpha^2\|h^{(1)} - h^{(2)}\|_{C^{2,\nu}(\mathbb{R}^2)} + \|e^{(1)} - e^{(2)}\|_{C^{2,\nu}(\mathbb{R}^2)}).
\]

Observe that we regard functions \( X_{\alpha,\ell}(\chi_j S(w)) \) as defined on the whole plane \( \mathbb{R}^2 \). This is correct since these functions are supported in the region where the local coordinates are well defined.

The proof of this lemma is fairly technical but at the same standard (see [14, 15] for similar results) and is postponed to section 5. We should make a comment regarding the Lipschitz property (4.17). We observe that expressing the error \( S^{(\ell)} \) in local variables \( (x_j, z_j) \) we have to use relations (3.9) to express variables \( (x_j, z_j) \) in terms of \( (x, z) \). These relations involve the components of the function \( h^{(\ell)} \) as higher order terms. Using the Implicit Function Theorem one can prove that in fact local coordinates with respect to different bump lines are \( C^{2,\nu} \) functions of the local coordinates of one fixed line.
So far we have estimated the error near the bump lines. Another proposition is needed to estimate the norm in the complement of the sets $\text{supp} \rho_j$. Recall that we have $S(\omega) \equiv 0$ in $\mathbb{R}^2 \setminus V_\xi$. We will denote

$$V_\xi^\alpha = V_\xi \setminus \bigcup_{j=1}^k \text{supp} \rho_j.$$ 

**Proposition 4.2.** Under the hypothesis of the previous Proposition we have for each $j = 1, \ldots, k$:

\begin{equation}
\|S(\omega)(\cosh z_j)^{\theta_\alpha}\|_{C^0,\mu(V_\xi)} \leq C\alpha^{2+\frac{3}{2}\sigma}.
\end{equation}

Similarly to (4.17) we have

\begin{equation}
\|S^{(2)} - S^{(1)}\|_{C^0,\mu(V_\xi)} \leq C\alpha^{2\sigma}\|h^{(1)} - h^{(2)}\|_{C^{\sigma,\mu}(\mathbb{R}^4;\mathbb{R})} + \|e^{(1)} - e^{(2)}\|_{C^{\sigma,\mu}(\mathbb{R}^4;\mathbb{R})}.
\end{equation}

We prove this result in section 5. Here we comment only that in Proposition 4.2 we consider the error as a function of the variable $x \in \mathbb{R}^2$ and the weight function depends in particular on $z$, since $z_j = z_j(z)$ by its definition as the arc length parameter of $\gamma_{\alpha,j}$.

**4.4. Existence of the background function.** In order to solve the system (4.11)–(4.13) we will use the Banach fixed point theorem. A convenient way to implement it is to solve first (4.13) with given $\phi$. To accomplish this we need to make some assumptions regarding the initial size of the functions $\phi_j$. We will assume from now on that functions $\phi_j$ are such that with $\sigma$ and $\theta$ as in the hypothesis of Proposition 4.1 we have

\begin{equation}
\|X_{\alpha,j}\phi_j\|_{C^{\sigma,\mu}(\mathbb{R}^2)} < \infty, \quad j = 1, \ldots, k.
\end{equation}

We assume above that $X_{\alpha,j}\phi_j$ is a function defined in the whole plane and the weight functions are taken with respect to the variables $(x_j, z_j)$. We have the following Lemma:

**Lemma 4.1.** Assuming that (4.20) holds there exists a unique solution of (4.13) such that for all $j = 1, \ldots, k$ we have:

\begin{equation}
\|\psi(\cosh z_j)^{\theta_\alpha}\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{3+\sigma}(\alpha^2 + \sum_{j=1}^k \|X_{\alpha,j}\phi_j\|_{C^{2,\mu}(\mathbb{R}^2)}).
\end{equation}

In addition $\psi$ is a continuous function of the parameters $v, h, e, \delta, \tau, \tau_\pm$ and a Lipschitz function of $\phi$ and also of the parameters $h, e$ and the following estimates hold:

\begin{equation}
\|(\phi^{(1)}) - (\phi^{(2)})\)(\cosh z_j)^{\theta_\alpha}\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{3+\sigma}(\alpha^2 + \sum_{j=1}^k \|X_{\alpha,j}(\phi^{(1)} - \phi^{(2)})\|_{C^{2,\mu}(\mathbb{R}^2)}),
\end{equation}

and

\begin{equation}
\|(\psi^{(1)}, e^{(1)}) - (\psi^{(2)}, e^{(2)})\)(\cosh z_j)^{\theta_\alpha}\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{3+\sigma}(\alpha^2\|h_1 - h_2\|_{C^{2,\mu}(\mathbb{R}^4;\mathbb{R})} + \|e_1 - e_2\|_{C^{2,\mu}(\mathbb{R}^4;\mathbb{R})}).
\end{equation}

The proof of this Lemma is postponed to section 6.
4.5. Invertibility of the basic linearized operator. We will develop now the main functional analytic tool needed to solve the system of equations (4.11). Let us recall the definition of the basic linearized operator $L$ in (3.16):

$$ L = (\partial_x^2 + \partial_z^2 - 1 + p w^{p-1}). $$

We will consider the problem of existence of the unique solution of

(4.24) \[ L(\phi) = h, \quad \text{in} \ \mathbb{R}^2, \]

which additionally satisfies:

(4.25) \[ \int_{\mathbb{R}} w'(x)\phi(x, z) \, dx = \int_{\mathbb{R}} Z(x)\phi(x, z) \, dx. \]

We will assume below that

(4.26) \[ \int_{\mathbb{R}} w'(x)h(x, z) \, dx = \int_{\mathbb{R}} Z(x)h(x, z) \, dx, \]

and

(4.27) \[ \|((\cosh x)^a(\cosh z)^a h)\|_{C^0,\mu(\mathbb{R}^2)} < \infty. \]

**Proposition 4.3.** There exists an $a_0 > 0$ such that given $h$ satisfying (4.26)–(4.27) with $\sigma \in (0,1)$, $a \in [0, a_0)$, there exists a unique bounded solution $\phi = T(h)$ to problem (4.24) which defines a bounded linear operator of $h$ in the sense that

\[ \|((\cosh x)^a(\cosh z)^a \phi)\|_{C^2,\mu(\mathbb{R}^2)} \leq C \|((\cosh x)^a(\cosh z)^a h)\|_{C^0,\mu(\mathbb{R}^2)}, \]

and $\phi$ satisfies additionally the orthogonality conditions (4.25).

The proof of this Proposition is postponed to section 7.

We will adopt the above theory to deal with the system of nonlinear and nonlocal equations (4.11).

4.6. Existence of solutions to the interaction system. Given what we said above we will describe the procedure that will give the solution of (4.11). By what we said in previous section we are reduced to considering the following fixed point problem

(4.28) \[ X_{\alpha, j}^* \phi_j = T(X_{\alpha, j}^* k_j - X_{\alpha, j}^* (c_j w_{0,j}^* \chi_j) - X_{\alpha, j}^* (d_j Z_j \chi_j)), \]

where $c_j$ and $d_j$ must be chosen in such a way that the orthogonality conditions in (4.26) are satisfied. These conditions read in this case:

(4.29) \[
\begin{align*}
    c_j \int_{\mathbb{R}} X_{\alpha, j}^*(w_{0,j}^* )^2 \chi_j ) \, dx_j &= \int_{\mathbb{R}} X_{\alpha, j}^*(k_j w_{0,j}^* ) \, dx_j, \\
    d_j \int_{\mathbb{R}} X_{\alpha, j}^*(Z_j \chi_j) \, dx_j &= \int_{\mathbb{R}} X_{\alpha, j}^*(k_j Z) \, dx_j.
\end{align*}
\]

Let us make a comment about the structure of the system (4.28). Of course it can be written, alternatively as a system of PDEs:

(4.30) \[ [\partial^2_{x_j} + \partial^2_{z_j} + g_j(w(x_j))] X_{\alpha, j}^* \phi_j = X_{\alpha, j}^* k_j - X_{\alpha, j}^* (c_j w_{0,j}^* \chi_j) - X_{\alpha, j}^* (d_j Z_j \chi_j). \]

This system is coupled only through the background function $\psi$ (hidden in $X_{\alpha, j}^* k_j$) considered in each equation restricted to the set $\text{supp} \chi_j$. As given in Lemma 4.1 this function is a function of $x = (x, z) \in \mathbb{R}^2$. Since we can express these variables in terms of the local coordinates in $\text{supp} \chi_j \subset V_\zeta$ we are justified in writing something like $X_{\alpha, j}^*(\chi_j \psi)$. Similar observation applies to other functions appearing on the
right hand side of (4.30). The key point is that functions $X_{α,j}^* k_j$ are supported in $V_ε$, where the local coordinates of all curves $\bar{γ}_{α,j}$ are well defined.

We will examine the size of the functions $X_{α,j}^* k_j$ in the weighted Hölder norms.

**Lemma 4.2.** We assume that
\begin{equation}
\|X_{α,j}^* φ_j\|_{C^{2,σ}_{α,ν,ε}(R^2)} ≤ α^2 σ.
\end{equation}

With the notations of Proposition 4.1 the following estimate holds for $j = 1, \ldots, k$:
\begin{equation}
\|X_{α,j}^* k_j\|_{C^{0,ν}_{α,ν,ε}(R^2)} ≤ Cα^2 + Cα^2 σ \sum_{i=1}^k \|X_{α,i}^* φ_i\|_{C^{2,σ}_{α,ν,ε}(R^2)}.
\end{equation}

Moreover, functions $X_{α,j}^* k_j$ are Lipschitz as functions of $φ$ and we have
\begin{equation}
\|X_{α,j}^* k_j(φ^{(1)}) - X_{α,j}^* k_j(φ^{(2)})\|_{C^{0,ν}_{α,ν,ε}(R^2)} ≤ Cα^2 σ \sum_{j=1}^k \|X_{α,j}^* φ_j^{(1)} - X_{α,j}^* φ_j^{(2)}\|_{C^{2,σ}_{α,ν,ε}(R^2)}.
\end{equation}

We prove this Lemma in section 8.

We will now turn our attention to functions $c_j, d_j$ given by (4.29). It is easy to see that we have in fact:
\begin{equation}
\|c_j\|_{C^{0,ν}_{α,ν,ε}(R^2)} + \|d_j\|_{C^{0,ν}_{α,ν,ε}(R^2)} ≤ C\|X_{α,j}^* k_j\|_{C^{0,ν}_{α,ν,ε}(R^2)},
\end{equation}

and consequently,
\begin{align*}
\|X_{α,j}^* (c_j w_{α,j} x_j)\|_{C^{0,ν}_{α,ν,ε}(R^2)} + \|X_{α,j}^* (d_j Z x_j)\|_{C^{0,ν}_{α,ν,ε}(R^2)} ≤ C\|X_{α,j}^* k_j\|_{C^{0,ν}_{α,ν,ε}(R^2)}
\end{align*}

\begin{align*}
≤ Cα^2 + Cα^2 σ \sum_{i=1}^k \|X_{α,i}^* φ_i\|_{C^{2,σ}_{α,ν,ε}(R^2)},
\end{align*}

by (4.32). The Lipschitz property of the functions $c_j, d_j$ in terms of the unknowns $φ_j$ is also clear. Using these facts, the results of Lemma 4.2 and (4.28) we can apply Banach contraction mapping theorem to conclude:

**Proposition 4.4.** The interaction system (4.28)-(4.29) has a unique solution $φ = (φ_1, \ldots, φ_k)$ such that
\begin{equation}
\sum_{j=1}^k \|X_{α,j}^* φ_j\|_{C^{2,σ}_{α,ν,ε}(R^2)} ≤ Cα^2.
\end{equation}

The proof of this proposition is rather straightforward. We need to set up the fixed point scheme for the operator defined in (4.28) in the space of functions $φ: (R^2)^k \rightarrow R^k$ with the weighted norm defined, component by component, as in the statement of the proposition. We do this in the set of functions satisfying in addition (4.31). Observe that while $X_{α,j}^* k_j$ depends on the component functions of $φ$ the coupling between the equation is only through the operator $ψ$, which is nonlocal but easy to handle thanks to Lemma 4.1. We leave the details of the proof to the reader.

In the sequel we will need one more property of the solution of the interaction system. We observe that $X_{α,j}^* φ_j$ is a function of the parameters $v, h, e, δ, τ, r$. As for the nature of the dependence of $X_{α,j}^* φ_j$ on these parameters we have:
Lemma 4.3. The solution of the system \((4.28)-(4.29)\) is a continuous function of the parameters \(v, h, e, \delta_{\pm}, \tau_{\pm}\) and a Lipschitz function of \(h, e\). Moreover we have:

\[
\|X^{(1)}_{\alpha,j} \phi_j(h^{(1)}, e^{(1)}) - X^{(2)}_{\alpha,j} \phi_j(h^{(2)}, e^{(2)})\|_{c^2_{\nu,\sigma}(R^2)} \leq C\alpha^2\|h^{(1)} - h^{(2)}\|_{c^2_{\nu,\sigma}(R^2)} + C\|e^{(1)} - e^{(2)}\|_{c^2_{\nu,\sigma}(R^2)}.
\]

(4.36)

To prove Lemma 4.3 we observe that the operator defined in \((4.28)\) is a uniform contraction in the set of functions satisfying \((4.31)\) as long as \((3.1), (3.3), (3.18)\) and \((3.19)\) are satisfied. In addition for each fixed \(\phi\) the right hand side of \((4.28)\) is a continuous function of the parameters \(v, h, e, \delta_{\pm}, \tau_{\pm}\) and Lipschitz function of \(h, e\). This follows from Proposition 4.1, Lemma 4.1. From the Banach contraction mapping theorem we conclude \((4.36)\).

We will finish this section with the discussion of the rate of decay of the solution \(\varphi\) to \((4.4)\) which is given by \((4.8)\), namely

\[
\varphi = \sum_{j=1}^k \phi_j \rho_j + \psi,
\]

in terms of the original variables \(x = (x, z)\) rather than the local variables. We observe that whenever

\[
\|X^*_{\alpha,j}(\rho_j \phi_j)\|_{c^2_{\nu,\sigma,\theta}(R^2)} \leq C\alpha^2,
\]

then

\[
\|\rho_j \phi_j\|_{c^2_{\nu,\sigma,\theta}(R^2)} \leq C\alpha^{2-\sigma_j(k+1)}
\]

since, because of \((3.5)\), we have:

\[
x = x_j(1 + O(\alpha^2)) + z_j O(\alpha) + 2(j - (k - 1) + 1) \log \frac{1}{\alpha}.
\]

(4.38)

Of course in \((4.37)\) we must take \(\sigma_j < \sigma\) and \(\theta_j < \theta\). Estimate of a similar type can be shown for the background function \(\psi\) as well by a slight modification of the proof of Lemma 4.1 (see also Remark 6.1 in section 6). Thus taking \(\sigma_j\) sufficiently small we get:

\[
\|\varphi\|_{c^2_{\nu,\sigma,\theta, \alpha}(R^2)} \leq C\alpha,
\]

which is the estimate we claimed in the statement of Theorem 1.1 (see \((1.8)\) in Definition 1.1).

4.7. Derivation of the reduced equations. In order to finish the proof of the Theorem 1.1 we need to adjust the (so far undetermined) parameters \(v, h, e, \delta_{\pm}, \tau_{\pm}\) in such a way that \(c_j = 0, d_j = 0\). In other words, by \((4.29)\), we need:

\[
\int_{\mathbb{R}} X^*_{\alpha,j}(k_j w^0_{0,j}) \, dx_j = 0,
\]

(4.39)

\[
\int_{\mathbb{R}} X^*_{\alpha,j}(k_j Z) \, dx_j = 0.
\]

(4.40)

We will refer to \((4.39)\) as the reduced system. We will first show that it is equivalent to a nonlinear and nonlocal system of second order in variables \(h = (h_1, \ldots, h_k)\) and \(e = (e_1, \ldots, e_k)\). This is a system of \(2k\) equations with \(2k\) unknowns. The first \(k\) equation which determine \(h\) have the form, to main order, of the linearized Toda.
system discussed already in section 2. In particular a solution which decays exponentially exists only if we can chose suitably the unknown function \( v = (v_1, \ldots, v_k) \).

On the other hand the system for \( e \) consists of decoupled (to main order) linear equations of the form considered in section 2.3. As we have indicated each of the \( k \) equations requires 2 extra solvability conditions if we seek solutions in the exponentially decaying class. These requirements lead to \( 2k \) constraints on \( 4k \) parameters \( \delta_\pm, \tau_\pm \). Considering \((4.39)\) we have the following

**Proposition 4.5.** Equations \((4.39)\) are equivalent to the following system of equations:

\[
(4.41) \quad c_p(h + v)'' + N(h + v) = P, \quad h = (h_1, \ldots, h_k), \quad v = (v_1, \ldots, v_k),
\]

and \( N = (N_1, \ldots, N_k)^T \), where

\[
(4.42) \quad N_j = -e^{f_j-1-f_j}e_{j-1} + [e^{f_j-1-f_j} + e^{f_j-2+f_{j+1}}]e_j - e^{f_j-2+f_{j+1}}e_{j+1},
\]

and \( e_j \) are the vectors of the canonical basis in \( \mathbb{R}^k \). The function \( P \) satisfies:

\[
(4.43) \quad \|P\|_{\mathcal{C}^{0,\nu}(\mathbb{R};\mathbb{R}^k)} \leq C\alpha^\nu,
\]

where we choose

\[
\nu_1 = \min\{1 - \mu, 2\kappa_1 - \mu, 2\kappa_2 - \mu, 1 + \kappa_3 - \mu, \kappa_4 + \kappa_4 - \mu, \frac{3}{4}\sigma - \mu\},
\]

provided that \((3.1), (3.3), (3.18)\) and \((3.19)\) are satisfied. The constant \( c_p \) is defined by

\[
(4.44) \quad c_p = \frac{\int_R (w')^2 \, dx}{\pi \int_R w^{p-1} e^x \, dx} > 0.
\]

In addition \( P \) is a continuos function of \( v, h, e, \delta_\pm, \tau_\pm \) and a Lipschitz function of \( h, e \) and we have:

\[
(4.45) \quad \|P(h^{(1)}, e^{(1)}, \cdot) - P(h^{(2)}, e^{(2)}, \cdot)\|_{\mathcal{C}^{0,\nu}(\mathbb{R};\mathbb{R}^k)} \leq C\alpha^{\nu_1 - \mu} (\|h^{(1)} - h^{(2)}\|_{\mathcal{C}^{0,\nu}(\mathbb{R};\mathbb{R}^k)} + \|e^{(1)} - e^{(2)}\|_{\mathcal{C}^{0,\nu}(\mathbb{R};\mathbb{R}^k)})
\]

**Proof.** It is not hard to show that the main order terms in the projection of the function \( X_{\alpha,j}^* k_j \) onto \( w_{0,j}' \) come from the projection of \( X_{\alpha,j}^* (\chi_j S(w)) \). Accepting this fact for now (in section 9 we will provide some more details to justify this claim) we will focus on computing the asymptotic form of this term. In order to make the calculations more accessible we will assume that \( k = 2 \). This way we are able to emphasize the important points without obscuring them with complicated notations. We will compute first the projection of \( X_{\alpha,1}^* (\chi_1 S(w)) \) onto \( w_{0,1}' \). Expressing \( \Delta \) in local coordinates, using the notation \((3.13)-(3.14)\), and neglecting the higher order terms (in \( \alpha \)) we get

\[
(4.46) \quad \int_R X_{\alpha,1}^* (\chi_1 S(w)w_{0,1}') \, dx_1 \sim \int_R b_{1,1}(\partial_{x_1} w_{0,1})^2 \, dx_1 + p \int_R w_{0,1}^{p-1} w_{0,2} \partial_{x_1} w_{0,1} \, dx_1.
\]
In section 9 we will show that the difference between the left and the right member in (4.46) is negligible. Now to compute the integrals we use (3.14) to get:

\[ \int_R b_{1,1}(\partial x, w_{0,1})^2 \, dx_1 = \int_R (\partial x, w_{0,1})^2 [-\kappa_{1,1}A_1^{-1} - \alpha^2 h''_1 A_1^{-2} + \alpha^2 (h'_1)^2 k_{\alpha,1} - \alpha(x_1 + h_1)h'_1 k'_{\alpha,1}] \, dx_1 \]

\[ = -\alpha^2 (f''_1 + h''_1) \int_R (w')^2 \, dx + O_{C^0_{\alpha,\mu}(\mathbb{R})}(\omega^2) (\|h_1\|_{C^2_{\alpha,\mu}(\mathbb{R})}^2 + \|f_1\|_{C^2_{\alpha,\mu}(\mathbb{R})}^2) \]

where we use (3.12) to replace \( k_{\alpha,1} \) by \( f''_1 \). Notice that the exponential weights we take are like \((\cosh z)^\theta\). In other words, in estimating \( P \) we take \( C^0_{\alpha,\mu}(\mathbb{R}) \) norm instead of \( C^0_{\alpha,\mu}(\mathbb{R}) \), which is the norm in which we have actually measured the errors. This entails loss of a power of \( \alpha \) hence the remainder is a factor of \( \omega^3 - \mu \). This small detail, which we have already mentioned in (3.4) will be present in all subsequent calculations. Finally we remind the reader that above all functions of the arc length \( z_1 \) are taken of the scaled argument \( \alpha z_1 \).

To compute the second term in (4.46) we will use a refinement of (3.9) which reads:

\[ x_2 = [q_{\alpha,1}(z_1) + h_1(\alpha z_1) - q_{\alpha,2}(z_1) - h_2(\alpha z_1)] (1 + O(\alpha^2 + \mu)) \]

\[ + O(\omega^2 + \mu) |z_1| + x_1 (1 + O(\omega^2 + \mu)). \]

Using this we can write:

\[ \int_R w_{0,1}^{-1} \partial x, w_{0,1} \, dx_1 = \int_R w_{0,1}^{-1}(x_1)w'(x_1)w(x_1 + q_{\alpha,1}(z_1) - q_{\alpha,2}(z_1)) \, dx_1 \]

\[ + \int_R w_{0,1}^{-1}(x_1)w'(x_1)[w(x_2) - w(x_1 + q_{\alpha,1}(z_1) - q_{\alpha,2}(z_1))] \, dx_1, \]

where \( q_{\alpha,j}(z_1) = q_{\alpha,j}(z_1) + h_j(\alpha z_1) \). To evaluate the first integral above we observe that its leading order behavior comes from integration over the set where \( |x_1| \leq \frac{3}{2} \log \frac{1}{\omega} \), which means \( x_2 < -\frac{1}{2} \log \frac{1}{\omega} + O(1) \). Using the asymptotic formula

\[ w(x) = e^{-|x|} + O((\cosh x)^{-\mu}), \]

and denoting:

\[ c_1 = (p \int_R w_{0,1}(x)w'(x)e^{-x} \, dx), \]

we get:

\[ \int_R w_{0,1}^{-1}(x_1)w'(x_1)w(x_1 + q_{\alpha,1}(z_1) - q_{\alpha,2}(z_1) - h_2(\alpha z_1)) \, dx_1 \]

\[ = c_1 \exp \left( q_{\alpha,1}(z_1) - q_{\alpha,2}(z_1) \right) (1 + O(\omega^2 + \mu)) \]

\[ = c_1 e^{-f_1(\alpha z_1) - f_2(\alpha z_1)} \]

\[ + c_1 \alpha^2 e^{-f_1(\alpha z_1) - f_2(\alpha z_1)}(h_1(\alpha z_1) + v_1(\alpha z_1) - h_2(\alpha z_1)) \]

\[ + O(\omega^2 + \mu)(\omega^2 + \mu). \]
The second term in (4.49) is estimated in a similar way. Notice that since \( w'(x) < 0, x > 0 \) therefore the factor \( c_1 = p \int_0^x w^{p-1}(x)w'(x)e^x \, dx < 0. \)

In combining (4.47), (4.49) and (4.50) we use the fact that \( f \) is a solution of the Toda system (1.9). In this manner we get:

\[
\begin{align*}
c_p(h_1 + v_1)'' + e f_1 - f_2(h_1 + v_1 - h_2 - v_2) &= O_{C_0^\infty(R)}(\alpha^{\nu_1}).
\end{align*}
\]

Analogous calculations can be done of course for the projection onto \( w_2, w_0 \). This gives the assertion of the Proposition, (4.43), except for the detailed calculations which we will discuss in section 9.

Finally, for the continuity and the Lipschitz property (4.45) of \( P \) we observe that the former follows from the corresponding statements for \( S(w), \psi \) and \( \phi \), see Proposition 4.1, Lemma 4.1, Lemma 4.3 respectively. The details are somewhat tedious but at the same time standard. \( \square \)

We will now turn our attention to the second projected equation (4.40). We have:

**Proposition 4.6.** Formula (4.40) is equivalent to the following system of equations:

\[
(4.51) \quad e'' + \frac{\lambda_1}{\alpha^2}e = Q,
\]

where

\[
(4.52) \quad \|Q\|_{C_0^\infty(R; R^k)} \leq C \alpha^{\nu_1}.
\]

In addition statement (4.45) holds, with obvious modifications, for \( Q \) in place of \( P \).

**Proof.** We will again present simply the main point in the proof and postpone some details to section 9. We consider the leading order term in (4.40)

\[
(4.53) \quad \int_{R} X_{\alpha, j}^*(\chi_j S(w))X_{\alpha, j}^* Z_j \, dx_j \sim \int_{R} [\partial_{x_j}^2 + \partial_{z_j}^2 + g_p'(w(x_j))](e_j(\alpha z_j)Z_j)\chi_j Z_j \, dx_j.
\]

Terms that we have neglected above are of smaller order, and in fact they satisfy an estimate similar to (4.52) but with an extra factor \( \alpha^2 \). We have in particular interaction terms similar to the ones considered in (4.49) but with \( Z(x_j)Z(x_i) \) in place of the products \( w_0, jw_0, i \). Because we have \( Z(x) \sim e^{-a_p|x|}, \) as \( |x| \to \infty \) with \( a_p \geq \frac{3}{2} \), we can neglect them in this case.

To calculate the right hand side of (4.53) we use the fact that \( Z \) is the principal eigenfunction of \( \partial_{x_j}^2 + g_p'(w(x_j)) \). This gives immediately

\[
\int_{R} X_{\alpha, j}^*(\chi_j S(w))X_{\alpha, j}^* Z_j \, dx_j \sim (\alpha^2 e_j''(\alpha z_j) + \lambda_1 e_j(\alpha z_j))(\int_{R} \chi Z^2 \, dx).
\]

Formula (4.51) follows after dividing by \( \alpha^2 \). The proof of the Lipschitz property is left to the reader. \( \square \)

Now we recall that from our considerations in section 2.3 it follows that problem (4.51) is solvable in exponentially decaying class if in addition to (4.40) the following
consequently their contribution is relatively smaller. Denoting where the neglected parts are of order $O(\Lambda^j z_j)$ we will now show that $4.54$ leads to conditions on $\Delta$. We have (see section 9 for details):

$$Y_j = \int_R \int_R X_{\alpha,j}^*(k_j Z(x_j)) \cos(\sqrt{\lambda_j z_j}) \, dx_j \, dz_j = 0,$$

$$\int_R \int_R X_{\alpha,j}^*(k_j Z(x_j)) \sin(\sqrt{\lambda_j z_j}) \, dx_j \, dz_j = 0.$$

We will now show that $4.54$ leads to conditions on $\Delta_\pm$, and $\tau_\pm$. Let us denote the first integral above by $Y_j$. We have (see section 9 for details):

$$Y_j = \int_R \int_R X_{\alpha,j}^*(\chi_j S(u_j)) \cos(\sqrt{\lambda_j z_j}) \, dx_j \, dz_j + O(\alpha^{1+\nu_1}),$$

where $u_j$ is defined in (3.21). With the notation (3.13)–(3.14) we get

$$X_{\alpha,j}^*(S(u_j)) \sim [\partial^2 z_j + \partial z_j] u_j + g_p(u_j),$$

where the neglected terms give at the end contributions of order $O(\alpha^{1+\nu_1})$ to $Y_j$.

It is not hard to see that, after neglecting lower order terms, (c.f. considerations in section 5, (5.1) and also section 9)

$$[\partial^2 z_j + \partial z_j] u_j + g_p(u_j) \sim \alpha^2 [\varepsilon_+^{\tau_+} w_{+,j} + \varepsilon_-^{\tau_-} w_{-j} - w_{0,j} + \varepsilon_-^{\tau_-} w_{-j}]$$

$$+ 2\alpha [\varepsilon_+^{\tau_+} \partial z_j w_{+j} + \varepsilon_-^{\tau_-} \partial z_j w_{-j}]$$

$$= \alpha^2 [\varepsilon_+^{\tau_+} (w_{+j} - w_{0,j}) + \varepsilon_-^{\tau_-} (w_{-j} - w_{0,j})]$$

$$+ 2\alpha [\varepsilon_+^{\tau_+} \partial z_j w_{+j} + \varepsilon_-^{\tau_-} \partial z_j w_{-j}].$$

We note that by (3.3) we have:

$$\partial z_j w_{\pm,j} \sim \sqrt{\lambda_j} Z[\delta \pm,j \sin(\sqrt{\lambda_j} z_j) + \tau_{\pm,j} \cos(\sqrt{\lambda_j} z_j)],$$

$$w_{\pm,j} - w_{0,j} \sim Z[\delta_{\pm,j} \cos(\sqrt{\lambda_j} z_j) + \tau_{\pm,j} \sin(\sqrt{\lambda_j} z_j)],$$

where the neglected parts are of order $O_{C^\infty(R^2)}(\delta_{\pm,j}^2 + \tau_{\pm,j}^2)(\cosh x_j)^{-1}$ and consequently their contribution is relatively smaller. Denoting

$$\Theta_{\pm,j} = [\delta_{\pm,j} \cos(\sqrt{\lambda_j} z_j) + \tau_{\pm,j} \sin(\sqrt{\lambda_j} z_j)], \quad \zeta_0 = \int_R \chi Z^2,$$

we calculate:

$$Y_j \sim \zeta_0 \int_R [\alpha^2 \varepsilon_+^{\tau_+} \Theta_{+j} + 2\alpha \varepsilon_-^{\tau_-} \Theta'_{+j}] \cos(\sqrt{\lambda_j} z_j) \, dz_j$$

$$+ \zeta_0 \int_R [\alpha^2 \varepsilon_-^{\tau_-} \Theta_{-j} + 2\alpha \varepsilon_-^{\tau_-} \Theta'_{-j}] \cos(\sqrt{\lambda_j} z_j) \, dz_j$$

$$= \sqrt{\lambda_j} \zeta_0 (\tau_{+j} - \tau_{-j}).$$

Similar calculations can be done for the second integral in (4.54). Denoting it by $A_j$ we can summarize our considerations as follows:

**Lemma 4.4.** With the notation introduced above it holds:

$$Y_j = \sqrt{\lambda_j} \zeta_0 (\tau_{+j} - \tau_{-j}) + O(\alpha^{1+\nu_1}),$$

$$A_j = \sqrt{\lambda_j} \zeta_0 (\delta_{+j} - \delta_{-j}) + O(\alpha^{1+\nu_1}).$$
For future references we will denote:

\[
\tilde{Y}_j = Y_j - \sqrt{\lambda_1} \zeta_0 (\tau_{+,j} - \tau_{-,j}),
\]
\[
\tilde{A}_j = A_j - \sqrt{\lambda_1} \zeta_0 (\delta_{+,j} - \delta_{-,j}),
\]

and \( Y = (\tilde{Y}_1, \ldots, \tilde{Y}_k) \) and \( A = (\tilde{A}_1, \ldots, \tilde{A}_k) \).

4.8. Solution of the reduced system. We will now complete the proof of Theorem 1.1. To this end we have to solve the following system of equations (see Proposition 4.5, Proposition 4.6 and Lemma 4.4):

(4.58) \[ c_p (h + v)'' + N(h + v) = P(v, h, e, \delta, \tau), \]

(4.59) \[ e'' + \frac{\lambda_1}{\alpha^2} e = Q(v, h, e, \delta, \tau), \]

(4.60) \[
\begin{align*}
\sqrt{\lambda_1} \zeta_0 (\tau_+ - \tau_-) &= Y(v, h, e, \delta, \tau), \\
\sqrt{\lambda_1} \zeta_0 (\delta_+ - \delta_-) &= A(v, h, e, \delta, \tau).
\end{align*}
\]

Proposition 4.7. System (4.58)–(4.60) has a 2k parameter family of solutions in the sense that for each choice of \( k \) components of the vector \((\delta_-, \delta_+) \) \( \in \mathbb{R}^{2k} \), and \( k \) components of the vector \((\tau_-, \tau_+) \) \( \in \mathbb{R}^{2k} \) this system has a solution for the remaining 2k components of \((\delta_-, \delta_+), (\tau_-, \tau_+) \) and the functions \( v, h, e \).

Proof. First we choose \( \kappa_i, \mu \in (0,1) \), and \( 0 < \sigma < \min\{p, 2, 1\} \) in such a way that

\[
\min\{1 - \mu, 2\kappa_1 - \mu, 2\kappa_2 - \mu, 1 + \kappa_4 - \mu, \kappa_4 + \kappa_4 - \mu, \frac{3}{4} \sigma - \mu\} = \nu_1 > \max\{\kappa_i\}.
\]

Second we fix \( k \) components of \((\delta_-, \delta_+) \) \( \in \mathbb{R}^{2k} \). For simplicity we assume that the fixed components correspond to the lower ends of the bump lines, however it is easy to see that any combination of \( k \) ends will do. We will denote them by \( \delta_- \). Similarly we fix \( \tau_- \). We assume that the fixed vectors satisfy

(4.61) \[ ||\delta|| + ||\tau|| \leq \frac{1}{2} \alpha^{1+q_4}. \]

(c.f. (3.19)). Now we will solve the system by a fixed point argument following the three steps below.

Step 1. We fix \( \tilde{v}, \tilde{h}, \tilde{\delta}, \tilde{\delta}, \tilde{\tau} \) satisfying, respectively, (3.1), (3.3), (3.18) and (4.61). We set \( \delta_+ = \delta + \delta_- \) and \( \tilde{\tau}_+ = \tau + \tau_- \) and use these functions and parameters, together with \( \delta_-, \tau_- \) to calculate the right hand sides of the equations (4.58)–(4.60) above. We observe that these functions satisfy the assertions of Proposition 4.5, Proposition 4.6 and Lemma 4.4. In particular they are Lipschitz as functions of \( \tilde{h} \) and \( \tilde{e} \) continuous as functions of \( \tilde{v} \) and \( \delta_-, \tilde{\tau}_+ \).

Step 2. Next, we use the Banach contraction mapping theorem to solve (4.58)–(4.60) for \( \tilde{h} \) and \( \tilde{e} \). We observe that as a result we get the following system:

\[
\begin{align*}
c_p (h + v)'' + N(h + v) &= P(v, h, e, \delta, \tau, \delta_+, \tau_+, \tilde{\delta}, \tilde{\tau}), \\
\sqrt{\lambda_1} \zeta_0 \tau &= Y(v, h, e, \delta, \tau, \delta_+, \tau_+, \tilde{\delta}, \tilde{\tau}), \\
\sqrt{\lambda_1} \zeta_0 \delta &= A(v, h, e, \delta, \tau, \delta_+, \tau_+, \tilde{\delta}, \tilde{\tau}).
\end{align*}
\]
Using the theory developed in section 2 we find in addition that
\begin{equation}
\|H\|_{C^2_0(\mathbb{R};\mathbb{R}^k)} \leq C\|P\|_{C^0_0(\mathbb{R};\mathbb{R}^k)} \leq C\alpha^{\nu_1},
\end{equation}
\begin{equation}
\|\mathcal{E}\|_{C^2_0(\mathbb{R};\mathbb{R}^k)} \leq C\alpha^2\|Q\|_{C^0_0(\mathbb{R};\mathbb{R}^k)} \leq C\alpha^{2+\nu_1},
\end{equation}
and that \(v, \delta, \tau\) satisfy
\begin{equation}
\|v\| \leq C\alpha^{\nu_1},
\end{equation}
\begin{equation}
\|\delta\| + \|\tau\| \leq C\alpha^{1+\nu_1}.
\end{equation}

Step 3. We notice that the map
\[
(v, \delta, \tau) : \mathcal{E} \times \mathbb{R}^k \times \mathbb{R} \to \mathcal{E} \times \mathbb{R}^k \times \mathbb{R},
\]
\[
(\tilde{v}, \tilde{\delta}, \tilde{\tau}) \to (v, \delta, \tau),
\]
is continuous and, because of the choice of \(\nu_1\) and (4.63), we can use Brower’s theorem to find a fixed point of this map. In summary we obtain a solution to (4.58)–(4.60) as claimed. \(\Box\)

We recall that in the statement of Theorem (1.1) we assert the existence of \(4k\) parameter family of solutions. So far we have only demonstrated a \(2k\) parameter family of solutions of the system (4.58)–(4.60) however the missing \(2k\) parameters are easy to find. Indeed at the beginning of our considerations we have chosen a solution of the Toda system (1.9) represented by \(f\). Of course this solution depends on \(2k\) real parameters representing its initial conditions. These, together with the choice of \(2k\) Dancer parameters give the \(4k\) parameter family of solutions.

5. Proof of Propositions 4.1 and 4.2

5.1. Evaluation of the error in the case of two bump lines. In order to make the argument more transparent we will consider the special case of two bump lines i.e. \(k = 2\) Recall that we have \(g_p(t) = -t + t_p, p > 2\). Let us consider the error restricted to the set:
\[
U_1 := \{x_1 + x_2 \leq 0\} \cap V_2.
\]
In this set it is convenient to write (with the notation (3.21)):
\[
S(u) = \Delta(u_1 + u_2) + g_p(u_2) + |\Delta + g_p'(u_1 + u_2)|e_1Z_1 + e_2Z_2)
\]
\[
+ g_p(u_1 + w_2 + e_1Z_1 + e_2Z_2) - g_p(u_1) - g_p(u_2) - g_p(u_1 + tw_2)(e_1Z_1 + e_2Z_2).
\]
To estimate the first term we notice that using Taylor’s expansion we get
\[
w^p_j = \Xi_{+,j}w^p_{+,j} + \Xi_{0,j}w^p_{0,j} + \Xi_{-,j}w^p_{-,j}
\]
\[
+ (w_{0,j} + \Xi_{+,j}(w_{+,j} - w_{0,j}) + \Xi_{-,j}(w_{-,j} - w_{0,j}))^p
\]
\[
- \Xi_{+,j}(w_{0,j} + (w_{+,j} - w_{0,j}))^p - \Xi_{0,j}w^p_{0,j} + \Xi_{-,j}(w_{0,j} + (w_{-,j} - w_{0,j}))^p
\]
\[
= \Xi_{+,j}w^p_{+,j} + \Xi_{0,j}w^p_{0,j} + \Xi_{-,j}w^p_{-,j}
\]
\[
+ \mathcal{C}_0^\rho(u_1)((|\delta_{+,j}|^2 + |\tau_{+,j}|^2)(\cosh x_j)^{-2}(\cosh z_j)^{-\theta_\alpha}),
\]
since the 0th and the 1st order term in \((w_{\pm,j} - w_{0,j})\) in the two middle lines cancel out, and the the quality \(w^p_j = w^p_{\pm,j}\) holds whenever \(\Xi_{\pm,j} = 1\). Using this and denoting by \(P_j\) the differential operator \((\Delta - \partial^2_{x_j} - \partial^2_{z_j})\) we can write:

\[
E_1 = \sum_{j=1}^{2} [P_j(\Xi_{+,j}w_{+,j}) + P_j(\Xi_{0,j}w_{0,j}) + P_j(\Xi_{-,j}w_{-,j})]
\]

\[
+ 2 \sum_{j=1}^{2} \left[ \partial_{x_j} \Xi_{+,j} \partial_{x_j} w_{+,j} + \partial_{z_j} \Xi_{0,j} \partial_{x_j} w_{0,j} + \partial_{z_j} \Xi_{-,j} \partial_{z_j} w_{-,j} \right]
\]

\[
+ 2 \sum_{j=1}^{2} \left[ \partial^2_{x_j} \Xi_{+,j} w_{+,j} + \partial^2_{x_j} \Xi_{0,j} w_{0,j} + \partial^2_{z_j} \Xi_{-,j} w_{-,j} \right]
\]

\[
+ \mathcal{O}_c(\|z_\pm,j\|^2 + |\tau_{\pm,j}|)(\cosh x_j)^{-2}(\cosh z_j)^{-\theta_\alpha}.
\]

We observe that the term involving \(P_j\) above is, because of (3.15), of order \(\alpha^2\) and in addition it decays in \(x_j\) and \(z_j\) exponentially, like \((\cosh x_j)^{-\sigma}(\cosh z_j)^{-\theta_\alpha}\) for any \(\sigma < 1\). In making this claim we use of the asymptotic form of the Dancer solution and also estimates (3.3), (3.19). Similar estimate can be proven for the two following terms since for example we have

\[
\partial_{x_j} \Xi_{+,j}(\alpha z_j) = \alpha \Xi'_{+,j}(\alpha z_j), \quad \partial^2_{x_j} \Xi_{+,j}(\alpha z_j) = \alpha \Xi''_{+,j}(\alpha z_j)
\]

and \(\delta_{\pm,j}, \tau_{\pm,j} \sim \alpha^1 + \kappa^4\), while on the other hand \(\partial_{z_j} w_{0,j} = 0\). Thus we get:

\[
\|E_1\|_{C^{\alpha,\nu}(U_1)} \leq C\alpha^2.
\]

The second term denoted by \(E_2\) above satisfies an estimate of the same type by (3.18). We observe also that for any \(\sigma < 1\):

\[
\|E_1\|_{C^{\alpha,\nu}(U_1 \cap V^\sigma)} \leq C\alpha^{2 + \frac{3}{2} \sigma},
\]

since in \(U_1 \cap V^\sigma\) we have \(|x_1| \geq \frac{1}{2} \log \frac{1}{\sigma}\). It is important that in (5.3) we take the exponential weight in the norm only in the \(z_1\) direction. Again, same estimate is true for \(E_2\).

Finally, we estimate the term denoted by \(E_3\). It is not hard to see that the leading order in \(E_3\) comes from the first three terms in its definition and thus we have:

\[
E_3 \sim g_p(w_1 + w_2) - g_p(w_1) - g_p(w_2)
\]

\[
= p_w^{p-1} w_2 - w_2^p + \frac{p(p-1)}{2} (\zeta w_1 + (1 - \zeta)w_2)^{p-2}w_2^2
\]

\[
\sim p_w^{p-1} w_2,
\]

with some \(\zeta \in (0,1)\). The last relation is easily justified, since in \(U_1\) we have \(w_1 \gg w_2\). We need to consider the product \(p_w^{p-1} w_2\). We use (3.9) to express \(x_2\) in terms of \(x_1\) to get, as \(z_1 \to \pm\infty\):

\[
x_2 = x_1 + (a_{\pm,1} - a_{\pm,2})z_1 - 2 \log \frac{1}{2} + (x_1 + z_1)\mathcal{O}(\alpha^2) + \mathcal{O}(1),
\]

where the coefficients \(a_{\pm,j}\) satisfy \(|a_{\pm,j} - a_{\pm,j-1}| \geq \theta\). From this we find:

\[
|w_1 w_2| \leq C\alpha^2 (\cosh x_1)^{\alpha^2} (\cosh z_1)^{-\theta \alpha + \alpha^2},
\]
with some $c > 0$. In all we have then, with $0 < \sigma < p - 2$:

\begin{equation}
|w_1^{-1}w_2| \leq C\alpha^2(cosh x_1)^{-\sigma}(cosh z_1)^{-\theta\alpha},
\end{equation}

hence, with $\theta < \vartheta$,

\begin{equation}
\|E_3\|_{C^0_{\varsigma}(U_1)} \leq C\alpha^2.
\end{equation}

From this (4.16) it follows in the set $U_1 \cap supp\chi$. Exactly same argument can be carried out in the set

\begin{equation}
U_2 := \{x_1 + x_2 > 0\} \cap V_2.
\end{equation}

It is also easy to see from the above considerations that $S(w)$ is continuous as a function of its parameters.

To conclude (4.18) restricted to the set $U_1 \cap V_1^o$ we observe that as $x_1 > \frac{3}{4} \log \frac{1}{\alpha}$ in $U_1 \cap V_1^o$ from (5.5) we get:

\begin{equation}
\|w_1^{-1}w_2\|_{C^0_{\varsigma}(U_1 \cap V_1^o)} \leq C\alpha^{2 + \frac{3}{2}\sigma}.
\end{equation}

Finally in the complement of $U_1 \cup U_2$ in $V_1$ we have for instance the following terms to estimate for each $j = 1, 2$:

\begin{equation}
\partial^2_{x_j} (X_{\alpha,j}^* \eta_k)(X_{\alpha,j}^* w_j) + 2\partial x_j (X_{\alpha,j}^* \eta_k)\partial x_j (X_{\alpha,j}^* w_j) \leq Ce^{-|x_j|} = C e^{-\sigma|x_j|} e^{-(1-\sigma)|x_j|}.
\end{equation}

In the support of $\partial^2_{x_j} (X_{\alpha,j}^* \eta_k), \partial x_j (X_{\alpha,j}^* \eta_k)$ we have

\begin{equation}
|x_j| \geq \frac{\varsigma}{2\alpha} \sqrt{1 + |z_j|^2},
\end{equation}

hence we can estimate, with some constants $C_1, C_2$ depending on $\sigma$ and $\varsigma$:

\begin{equation}
e^{-(1-\sigma)|x_j|} \leq e^{-\frac{C_1}{\alpha}e^{-\frac{C_2}{\alpha}|x_j|}} \leq C\alpha^2 e^{-\theta \alpha |x_j|},
\end{equation}

provided that $\alpha$ is taken sufficiently small. It follows from this:

\begin{equation}
|\partial^2_{x_j} (X_{\alpha,j}^* \eta_k)(X_{\alpha,j}^* w_j)| + |2\partial x_j (X_{\alpha,j}^* \eta_k)\partial x_j (X_{\alpha,j}^* w_j)| \leq C\alpha^2 e^{-\sigma|x_j|} e^{-\theta \alpha |x_j|}.
\end{equation}

We obtain (4.18) noting that in $V_1^o$ we have

\begin{equation}
|x_j| \geq \frac{3}{4} \log \frac{1}{\alpha}.
\end{equation}

(Notice that the estimate (4.18) does not carry any weight in $x_j$).

To show the Lipschitz property (4.17) we observe that the dependence on the function $h$ appears in the expression for the operator $P_j$ above and also in the nonlinearity because of the formula (3.9), through terms of order $\alpha^{n_2}$. In particular the leading order term for $S(\cdot, h^{(1)}) - S(\cdot, h^{(2)})$ comes from estimating an expression similar to (5.4). This gives the factor $\alpha^2$ in the first line in the estimate (4.17). As for the the Lipschitz dependence on $e$ we observe that the leading behavior of $S(\cdot, e^{(1)}) - S(\cdot, e^{(2)})$ comes form the linear term (in $e$) denoted above by $E_2$. Thus the second part of the estimate (4.17) follows. We omit somewhat tedious details. Again using (5.8) we conclude (4.19).
5.2. The error in the general case. In the general case \( S(w) \), i.e. when \( k > 2 \) and \( p > 2 \) we consider the following subsets of \( \mathbb{R}^2 \):

\[
U_j := \{x_j + x_{j-1} \geq 0\} \cap \{x_j + x_{j+1} \leq 0\} \cap V_2,
\]

\[
U_1 := \{x_1 \leq 0\} \cap \{x_1 + x_2 \leq 0\} \cap V_2,
\]

\[
U_k := \{x_k + x_{k-1} \geq 0\} \cap \{x_k \geq 0\} \cap V_2.
\]

Since, by (3.23), \( w = \bar{w} \) in \( V_2 \) we can write

\[
S(w) = \sum_{j=1}^{k} \chi_{U_j} S(\bar{w}) + S((1 - \eta_k)\bar{w}),
\]

where \( \chi_{U_j} \) denotes the characteristic function of the set \( U_j \).

We fix a \( j \) and consider the error restricted to the set \( U_j \). Setting for convenience \( g_p(t) = -t + t^p_+ \) and using the notation (3.21) we have in \( U_j \):

\[
S(\bar{w}) = \underbrace{\Delta w_j + g_p(w_j)}_{E_{1,j}} + \underbrace{\Delta (e_j(\alpha z_j)Z_j) + g_p'(w_j)(e_j(\alpha z_j)Z_j)}_{E_{2,j}}
\]

\[
+ \sum_{i \neq j} \Delta w_i + g_p(w_i) + \sum_{i \neq j} \Delta (e_i(\alpha z_i)Z_j) + g_p'(w_i)(e_i(\alpha z_i)Z_i)
\]

\[
+ g_p\left( \sum_{i=1}^{k} w_i + e_i(\alpha z_i)Z_j \right) - \sum_{i=1}^{k} g_p(w_i) - \sum_{i=1}^{k} g_p'(w_i)(e_i(\alpha z_i)Z_i).
\]

All the components above can be estimated using the same argument as in the case of two lines noting that the error due to the interactions between the bump lines is the biggest when the closest neighbors are considered. Another observation is that in the Taylor expansion of the nonlinear function \( g_p(w) \), \( p > 2 \) around \( w_j \) all components with powers higher than 2 give rise to terms that are negligible. We leave the details to the reader.

6. The background equation: Proof of Lemma 4.1

Let us consider first the following problem:

\[
(\Delta - 1)\psi = h, \quad \text{in } \mathbb{R}^2,
\]

where \( h \in C^0(\mathbb{R}^2) \) is such that

\[
\|h(\cosh z_j)^{\theta\alpha}\|_{C^0(\mathbb{R}^2)} < \infty,
\]

for \( j = 1, \ldots, k \) (here \( z_j = z_j(z) \) via (3.2)). Since by assumption \( h \in C^0(\mathbb{R}^2) \) as well, by maximum principle and elliptic regularity theory we get the existence of a unique solution \( \psi \) such that

\[
\|\psi\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\|h\|_{C^0(\mathbb{R}^2)}.
\]

We will now prove:

\[
\|\psi(\cosh z_j)^{\theta\alpha}\|_{C^2(\mathbb{R}^2)} \leq C\|h(\cosh z_j)^{\theta\alpha}\|_{C^0(\mathbb{R}^2)}.
\]
As we will see \((4.21)\) will follow from this. Using \((3.2)\) we see that functions of the form:

\[
\psi_{\theta \alpha, \nu} = (\cosh z_j)^{\theta \alpha} + \nu \left[ \cosh\left(\frac{x}{2}\right) + \cosh\left(\frac{z}{2}\right) \right],
\]

with \(\nu \geq 0\) and \(\alpha\) sufficiently small are positive supersolutions for \(\Delta - 1\) in \(\mathbb{R}^2\). In fact:

\[
(\Delta - 1) \psi_{\theta \alpha, \nu} \leq -\frac{1}{4} \psi_{\theta \alpha, \nu}.
\]

Considering now the function

\[
\omega_{\theta \alpha, \nu, M} = M \| h(\cosh z_j)^{\theta \alpha} \|_{C^{0,\mu}(\mathbb{R}^2)} \psi_{\theta \alpha, \nu} - \psi,
\]

where \(M\) large is to be chosen, we get:

\[
(\Delta - 1) \omega_{\theta \alpha, \nu, M} \leq - M \frac{1}{4} \| h(\cosh z_j)^{\theta \alpha} \|_{C^{0,\mu}(\mathbb{R}^2)} \psi_{\theta \alpha, \nu} + h
\]

\[
+ \| h(\cosh z_j)^{\theta \alpha} \|_{C^{0,\mu}(\mathbb{R}^2)} (\cosh z_j)^{-\theta \alpha}
\]

\[
\leq 0.
\]

By letting \(\nu \to 0\) we get the upper bound:

\[
\psi(\cosh z_j)^{\theta \alpha} \leq C \| h(\cosh z_j)^{\theta \alpha} \|_{C^{0,\mu}(\mathbb{R}^2)}.
\]

The lower bound and the rest of the proof of \((6.13)\) follow by a straightforward argument and are left to the reader.

Next we need to examine the size of the function \(Q\) and also its dependence on \(\phi\) and \(h\), \(e\) and other parameters. We will now assume \(\phi\) to be given and of finite \(C^{2,\mu}_{\sigma, \theta \alpha}(\mathbb{R}^2)\) norm. We will show that

\[
\| Q(\cosh z_j)^{\theta \alpha} \|_{C^{0,\mu}(\mathbb{R}^2)} \leq C \alpha^{2+\frac{3}{4}\sigma} + C \alpha^{\frac{3}{4}\sigma} \sum_{j=1}^{k} \| X^{*}_{\alpha, \theta \alpha, j} \phi_j \|_{C^{0,\mu}_{\sigma, \theta \alpha}(\mathbb{R}^2)}.
\]

from which, using \((6.13)\) the required estimate will follow. In the remainder of the proof we will use the fact that in the \(\text{supp} \ Q\) we have

\[
|x_j| \geq \frac{3}{4} \log \frac{1}{\alpha},
\]

(c.f. \((5.8)\)) to estimate terms whose norm (including the exponential weight in \(x_j\)) is bounded (see for example the proof of estimate \((4.18)\)). We observe that the first term on the right hand side above comes from \((1 - \sum_{i=1}^{k} \rho_i) S(\omega)\) and has already been estimated in \((4.18)\). To estimate the remaining terms involved in \(Q\) we observe that they depend on the functions \(\phi\) and \(\psi\), see \((4.10)\). For example, using the fact that the derivatives of the functions \(\rho_j\) are supported in the set where

\[
\frac{3}{4} \log \frac{1}{\alpha} \leq |x_j| \leq \log \frac{1}{\alpha},
\]

we get for all \(j = 1, \ldots, k\)

\[
\| (L_{\phi_j} \rho_j) - \rho_j L(\phi_j) ) (\cosh z_j)^{\theta \alpha} \|_{C^{0,\mu}(\mathbb{R}^2)} \leq C \alpha^{\frac{3}{4}\sigma} \| X^{*}_{\alpha, \theta \alpha, j} \phi_j \|_{C^{2,\mu}_{\sigma, \theta \alpha}(\mathbb{R}^2)}.
\]
Finally we will use (5.8) and the fact that $\|L - \Delta + 1 = pw^{p-1}$ with $p > 2$ to get:

$$\left\|\left[1 - \sum_{i=1}^{k} \rho_i (L - \Delta + 1)\psi\right](\cosh z_j)^{\sigma}\right\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{2\frac{\gamma}{\alpha}}\psi(\cosh z)^{\sigma}\right\|_{C^{2,\mu}(\mathbb{R}^2)}.$$ 

Summarizing, we have found:

$$\|Q(\phi, \psi)(\cosh z_j)^{\sigma}\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{2\frac{\gamma}{\alpha}}\psi(\cosh z)^{\sigma}\right\|_{C^{2,\mu}(\mathbb{R}^2)}$$

(6.14)

$$+ C\alpha^{2\frac{\gamma}{\alpha}}\left[\sum_{j=1}^{k} \|X^*_{\alpha,j}\phi_j\|_{C^{2,\mu}(\mathbb{R}^2)}\right].$$

Now assuming that $\phi$ is given, using (6.11)–(6.13) and a standard fixed point argument we find a $\psi = \psi(\phi)$ that satisfies (4.13). Moreover we have:

$$\|\psi(\phi)(\cosh z_j)^{\sigma}\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{2\frac{\gamma}{\alpha}}\left[\sum_{j=1}^{k} \|X^*_{\alpha,j}\phi_j\|_{C^{2,\mu}(\mathbb{R}^2)}\right].$$

(6.15)

Since the function $Q(\phi, \psi)$ is a uniform contraction (as a function of $\psi$) and it is continuous (as function of its parameters, assuming of course that $\phi$ is continuous), we conclude that $\psi$ is a continuous function of $v, h, e, \delta_\pm, \tau_\pm$. It is also easy to see that $\psi(\phi)$ is Lipschitz as a function of $\phi$ and in fact we have:

$$\|\psi(\phi^{(1)}) - \psi(\phi^{(2)})\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\alpha^{2\frac{\gamma}{\alpha}}\left[\sum_{j=1}^{k} \|X^*_{\alpha,j}(\phi^{(1)}_j - \phi^{(2)}_j)\|_{C^{2,\mu}(\mathbb{R}^2)}\right].$$

(6.16)

Final estimate in Lemma 4.1, namely (4.23) follows from (4.19).

**Remark 6.1.** We observe that a slight modification of the proof of (6.13) gives

$$\|\psi(\cosh z)^{\sigma}\|_{C^{2,\mu}(\mathbb{R}^2)} \leq C\|h\|_{C^{0,\mu}(\mathbb{R}^2)}.$$ 

(6.17)

In the case at hand we have, with $\sigma_* < \sigma, \theta_* < \theta$

$$\|Q\|_{C^{0,\mu}(\mathbb{R}^2)} \leq C\alpha^{2\frac{\gamma}{\alpha} - \sigma_*}(1 + k)\left[\sum_{j=1}^{k} \|X^*_{\alpha,j}\|_{C^{2,\mu}(\mathbb{R}^2)}\right],$$

because of (4.38). Therefore when $\phi$ is the true solution of (4.4) we get:

$$\|Q\|_{C^{0,\mu}(\mathbb{R}^2)} \leq C\alpha^{2\frac{\gamma}{\alpha} - \sigma_*}(1 + k),$$

which is the same type of estimate as (4.37).

7. **A priori estimates and invertibility of the basic linear operator**

7.1. **Non-degeneracy of the homoclinic.** In this section we will consider first the following linearized operator

$$L_0(\phi) = \phi_{xx} + g'_p(w)\phi, \quad g'_p(w) = pw^{p-1} - 1.$$ 

We recall some well known facts about $L_0$. First notice that $L_0(w') = 0$ i.e. has one dimensional kernel. Second we observe that

$$\lambda_1 = \frac{1}{4}(p - 1)(p + 3), \quad Z = \frac{w^{(p+1)/2}}{\sqrt{\int_R w^{p+1}}},$$
correspond, respectively, to principal eigenvalue and eigenfunction of $L_0$. Except for $\lambda_1 > 0$ and $\lambda_2 = 0$ the rest of the spectrum of $L_0$ is negative. This means in particular that there exists a positive constant $\gamma_0$ such that

$$\langle L_0(\phi), \phi \rangle \geq \gamma_0 \|\phi\|_{L^2(R)}^2,$$  

whenever

$$\langle \phi, w' \rangle = 0 = \langle \phi, Z \rangle.$$  

From (7.1) it also follows that there exists a $\gamma > 0$ such that:

$$\langle L_0(\phi), \phi \rangle \geq \gamma (\|\phi_x\|_{L^2(R)}^2 + \|\phi\|_{L^2(R)}^2),$$  

As another consequence of these facts we observe that problem

$$L_0(\phi) - \xi^2 \phi = h,$$  

is uniquely solvable whenever $\xi \neq \pm \sqrt{\lambda_1}$, 0 for $h \in L^2(R)$. Actually, rather standard argument, using comparison principle and the fact that $L_0$ is of the form

$L_0(\phi) = \phi_{xx} - \phi + q(x)\phi, \quad |q(x)| \leq Ce^{-c|x|},$

can be used to show that whenever $h$ is for instance a compactly supported function then the solution of (7.3) is an exponentially decaying function.

Let us consider now the basic linearized operator

$L(\phi) = L_0(\phi) + \phi_{zz},$

defined in the whole plane $(x, z) \in R^2$. Using (7.1) we get that

$$\langle L(\phi), \phi \rangle \geq \gamma_0 \|\phi\|_{L^2(R^2)}^2,$$  

whenever

$$\int_R \phi(x, z)w'(x) dx = 0 = \int_R \phi(x, z)Z(x) dx, \quad \text{for all } z.$$  

Equation $L(\phi) = 0$, has 3 obvious bounded solutions

$w'(x), \quad Z(x)\cos(\sqrt{\lambda_1}z), \quad Z(x)\sin(\sqrt{\lambda_1}z).$

Our first result shows that converse is also true.

**Lemma 7.1.** Let $\phi$ be a bounded solution of the problem

$$L(\phi) = 0 \quad \text{in } R^2.$$  

Then $\phi(x, z)$ is a linear combination of the functions $w'(x), Z(x)\cos(\sqrt{\lambda_1}z)$, and $Z(x)\sin(\sqrt{\lambda_1}z).$

**Proof.** Let assume that $\phi$ is a bounded function that satisfies

$$\phi_{zz} + \phi_{xx} + (pw^{p-1} - 1)\phi = 0.$$  

Let us consider the Fourier transform of $\phi(x, z)$ in the $z$ variable, $\hat{\phi}(x, \xi)$ which is by definition the distribution defined as

$$\langle \hat{\phi}(x, \cdot), \mu \rangle_R = \langle \phi(x, \cdot), \hat{\mu} \rangle_R = \int_R \phi(x, \xi)\hat{\mu}(\xi) d\xi,$$

where $\mu(\cdot)$ is any smooth rapidly decreasing function. Let us consider a smooth rapidly decreasing function $\psi$ of the two variables $(x, \xi)$. Then from equation (7.6) we find

$$\int_R \langle \hat{\phi}(x, \cdot), \psi_{xx} - \xi^2\psi + (pw^{p-1} - 1)\psi \rangle_R^2 dx = 0.$$
Let $\varphi(x)$ and $\mu(\xi)$ be smooth and compactly supported functions such that
\[ \{ \sqrt{\lambda_1}, -\sqrt{\lambda_1}, 0 \} \cap \text{supp}(\mu) = \emptyset. \]
Then we can solve the equation
\[ \psi_{xx} - \xi^2 \psi + (pw^{p-1} - 1)\psi = \mu(\xi)\varphi(x), \quad x \in \mathbb{R}, \]
uniquely for a smooth, rapidly decreasing function $\psi(x,\xi)$ such that $\psi(x,\xi) = 0$ whenever $\xi \notin \text{supp}(\mu)$. We conclude that
\[ \int_{\mathbb{R}} \langle \hat{\varphi}(x,\cdot),\mu\rangle \varphi(x) \, dx = 0, \]
so that for all $x \in \mathbb{R}$, $\langle \hat{\varphi}(x,\cdot),\mu\rangle = 0$, whenever $\{ \sqrt{\lambda_1}, -\sqrt{\lambda_1}, 0 \} \cap \text{supp}(\mu) = \emptyset$, in other words
\[ \text{supp}(\hat{\varphi}(x,\cdot)) \subset \{ \sqrt{\lambda_1}, -\sqrt{\lambda_1}, 0 \}. \]
By distribution theory we find that $\hat{\varphi}(x,\cdot)$ is a linear combination (with coefficients depending on $x$) of derivatives up to a finite order of Dirac masses supported in $\{ \sqrt{\lambda_1}, -\sqrt{\lambda_1}, 0 \}$. Taking inverse Fourier transform, we get that
\[ \varphi(x,z) = p_0(z,x) + p_1(z,x) \cos(\sqrt{\lambda_1}z) + p_2(z,x) \sin(\sqrt{\lambda_1}z), \]
where $p_j$ are polynomials in $z$ with coefficients depending on $x$. Since $\varphi$ is bounded these polynomials are of zero order, i.e. $p_j(z,x) \equiv p_j(x)$, and the bounded functions $p_j$ must satisfy the equations
\[ L_0(p_0) = 0, \quad L_0(p_1) - \lambda_1 p_1 = 0, \quad L_0(p_2) - \lambda_1 p_2 = 0, \]
and the desired result follows.

7.2. A priori estimates for the basic linearized operator. The linear theory used in this paper is based on a priori estimates for the solutions of the following problem
\[ (7.7) \quad L(\phi) = h, \quad \text{in } \mathbb{R}^2. \]
The results of Lemma 7.1 imply that such estimates without imposing extra conditions on $\phi$ may not exist. The form of the bounded solutions of $L(\phi) = 0$ and (7.4) suggest the following orthogonality conditions:
\[ (7.8) \quad \int_{\mathbb{R}} \phi(x,z)w'(x) \, dx = 0 = \int_{\mathbb{R}} \phi(x,z)Z(x) \, dx, \quad \text{for all } z \in \mathbb{R}. \]
With these restrictions imposed we have the following result concerning a priori estimates for this problem.

Lemma 7.2. Assuming that $\phi$ is a bounded solution of (7.7) satisfying (7.8) we have
\[ \|\phi\|_{L^\infty(\mathbb{R}^2)} \leq C\|h\|_{L^\infty(\mathbb{R}^2)}. \]
Proof. We will argue by contradiction. Assuming the opposite means that there are sequences $\phi_n, h_n$ such that
\[ \|\phi_n\|_\infty = 1, \quad \|h_n\|_\infty \to 0, \]
and
\[(7.9)\quad L(\phi_n) = h_n, \quad \text{in } \mathbb{R}^2,\]
\[(7.10)\quad \int_{\mathbb{R}} \phi_n(x,z)w_x(x)\,dx = 0 = \int_{\mathbb{R}} \phi_n(x,z)Z(x)\,dx, \quad \text{for all } z \in \mathbb{R}.\]

Let us assume that \((x_n, z_n) \in \mathbb{R}^2\) is such that \(|\phi_n(x_n, z_n)| \to 1\).

We claim that the sequence \(x_n\) is bounded. Indeed, if not, using the fact that \(L\phi = \Delta\phi - \phi + O(e^{-c|x|})\phi\) and employing elliptic estimates we find that the sequence of functions \(\tilde{\phi}_n(x,z) = \phi_n(x_n + x, z_n + z)\), converges, up to a subsequence, locally uniformly to a solution \(\tilde{\phi}\) of the equation
\[\Delta\tilde{\phi} - \tilde{\phi} = 0, \quad \text{in } \mathbb{R}^2,\]
whose absolute value attains its maximum at \((0,0)\). This implies \(\tilde{\phi} \equiv 0\), so that \(x_n\) is indeed bounded. Let now
\[\tilde{\phi}_n(x,z) = \phi_n(x, z_n + z).\]

Then \(\tilde{\phi}_n\) converges uniformly over compacts to a bounded, nontrivial solution \(\tilde{\phi}\) of
\[L(\tilde{\phi}) = 0, \quad \text{in } \mathbb{R}^2,\]
\[\int_{\mathbb{R}} \tilde{\phi}(x,z)w_x(x)\,dx = 0 = \int_{\mathbb{R}} \tilde{\phi}(x,z)Z(x)\,dx, \quad \text{for all } z \in \mathbb{R}.\]

Lemma 7.1 then implies \(\tilde{\phi} \equiv 0\), a contradiction and the proof is concluded. \(\square\)

Using Lemma 7.2 we can also find a priori estimates with norms involving exponential weights. When the weights involve only the \(x\) variable we have the following a priori estimates.

**Lemma 7.3.** Assuming that \(\|(\cosh x)^\sigma h\|_{C^{0,\nu}(\mathbb{R}^2)} < +\infty, \sigma \in [0,1)\), then a bounded solution \(\phi\) of \((7.7)\)–\((7.8)\) satisfies
\[(7.11)\quad \|(\cosh x)^\sigma \phi\|_{C^{2,\nu}(\mathbb{R}^2)} \leq C\|(\cosh x)^\sigma h\|_{C^{0,\nu}(\mathbb{R}^2)}.\]

**Proof.** We already know that
\[\|\phi\|_{L^\infty(\mathbb{R}^2)} \leq C\|(\cosh x)^\sigma h\|_{C^{0,\nu}(\mathbb{R}^2)}.\]

We set \(\tilde{\phi} = \phi\|(\cosh x)^\sigma h\|_{C^{0,\nu}(\mathbb{R}^2)}^{-1}\). Then we have
\[L(\tilde{\phi}) = \tilde{h}, \quad \text{where } \|(\cosh x)^\sigma \tilde{h}\|_{C^{0,\nu}(\mathbb{R}^2)} = 1,\]
and also \(\|\tilde{\phi}\|_{L^\infty(\mathbb{R}^2)} \leq C\). Let us fix a number \(R_0 > 0\) such that for \(x > R_0\) we have
\[p w^{p-1}(x) < \frac{1 - \sigma^2}{2},\]
which is always possible since \(w(x) = O(e^{-c|x|})\). For an arbitrary number \(\rho > 0\) let us set
\[\tilde{\phi}(x,z) = \rho[\cosh(z/2) + e^{\sigma x}] + Me^{-\sigma x},\]
where \(M\) is to be chosen. Then we find that,
\[L(\tilde{\phi}) \leq -\frac{M(1 - \sigma^2)}{4} e^{-\sigma x}, \quad \text{for } x > R_0.\]
Thus

$$L(\tilde{\phi}) \leq \tilde{h}, \quad \text{for } x > R_0,$$

if

$$\frac{M(1 - \sigma^2)}{4} \geq \|(\cosh x)^\sigma \tilde{h}\|_{C^0(\mathbb{R}^2)} = 1.$$ 

If we also assume

$$Me^{-\sigma R_0} \geq \|\tilde{\phi}\|_{\infty},$$

we conclude from maximum principle that \(\tilde{\phi} \leq \bar{\phi}\). Letting \(\rho \to 0\) we get (since \(M\) can be fixed independent on \(\rho\)),

$$\tilde{\phi} \leq Me^{-\sigma x}, \quad \text{for } x > 0,$$

hence

$$\phi \leq M\|(\cosh x)^\sigma h\|_{C^0(\mathbb{R}^2)}e^{-\sigma x}, \quad \text{for } x > 0.$$ 

In a similar way we obtain the lower bound

$$\phi \geq -M\|(\cosh x)^\sigma h\|_{C^0(\mathbb{R}^2)}, \quad \text{for } x > 0.$$ 

Finally, the same argument for \(x < 0\) yields

$$\|(\cosh x)^\sigma h\|_{C^2(\mathbb{R}^2)} \leq C\|(\cosh x)^\sigma h\|_{C^0(\mathbb{R}^2)}.$$ 

The required estimate now follows from local elliptic estimates and the proof is concluded. \(\square\)

When we also take into account the exponential decay in the \(z\) variable we have the following a priori estimates.

**Lemma 7.4.** There exists \(a_0 > 0\) such that assuming \(\|h(\cosh x)^\sigma(\cosh z)^a\|_{C^{0,\mu}(\mathbb{R}^2)} < +\infty, \sigma \in (0, 1), a \in [0, a_0]\), for any bounded solution \(\phi\) to problem (7.7)-(7.8) we have

$$\|(\cosh x)^\sigma(\cosh z)^a\phi\|_{C^2(\mathbb{R}^2)} \leq C\|(\cosh x)^\sigma(\cosh z)^a h\|_{C^{0,\mu}(\mathbb{R}^2)}.$$ 

**Proof.** We already know that

$$\|(\cosh x)^\sigma(\cosh z)^a h\|_{C^{0,\mu}(\mathbb{R}^2)} \leq C\|(\cosh x)^\sigma(\cosh z)^a h\|_{C^0(\mathbb{R}^2)}.$$ 

Then we may write

$$\psi(z) = \int_{\mathbb{R}} \phi^2(x, z) \, dx,$$

and differentiate twice weakly to get

$$\psi''(z) = 2\int_{\mathbb{R}} \phi^2_z \, dx + 2\int_{\mathbb{R}} \phi_{zz} \phi \, dx.$$ 

We have

$$\int_{\mathbb{R}} \phi_{zz} \phi \, dx = \int_{\mathbb{R}} \phi^2_z \, dx + \int_{\mathbb{R}} (1 - pw^{p-1})\phi^2 \, dx + \int_{\mathbb{R}} h\phi.$$ 

Because of the orthogonality conditions (7.8) we also have by (7.2) that,

$$\int_{\mathbb{R}} \phi^2_z \, dx + \int_{\mathbb{R}} (1 - pw^{p-1})\phi^2 \, dx \geq \gamma \int_{\mathbb{R}} (\phi_z^2 + \phi^2) \, dx, \quad \gamma > 0.$$ 

Hence we find that for a certain constant \(C > 0\)

$$\psi''(z) \geq \frac{\gamma}{4} \psi(z) - C \int_{\mathbb{R}} h^2(x, z) \, dx,$$
so that
\[-\psi''(z) + \frac{2}{4} \psi(z) \leq \frac{C}{\sigma} e^{-2a|z|} \|(\cosh x)^\sigma (\cosh z)^a h\|_{L^0,\mu}^2.\]

Since we also know that \(\psi\) is bounded by:
\[|\psi(z)| \leq \frac{C}{\sigma} \|(\cosh x)^\sigma (\cosh z)^a h\|_{L^0,\mu}^2,\]
we can use a barrier of the form \(\psi^+(z) = M \|h\|_{L^0,\mu}^2 e^{-2a|z|} + \rho e^{2a|z|}\), with \(M\) sufficiently large and \(\rho > 0\) arbitrary, to get the bound \(0 \leq \psi \leq \psi^+\) for \(z \geq 0\) and any \(a < \frac{\sqrt{\sigma}}{2} \equiv a_0\). A similar argument can be used for \(z < 0\). Letting \(\rho \to 0\) we get then
\[\int_{\mathbb{R}} \phi^2(x, z) \, dx \leq C_\sigma e^{-2a|z|} \|(\cosh x)^\sigma (\cosh z)^a h\|_{L^0,\mu}^2, \quad a < a_0.\]

Elliptic estimates yield that for \(R_0\) fixed and large
\[|\phi(x, z)| \leq C_\sigma e^{-a|z|} \|(\cosh x)^\sigma (\cosh z)^a h\|_{L^0,\mu}^2 \quad \text{for } |x| < R_0.\]
The corresponding estimate in the complementary region can be found by barriers. For instance in the quadrant \(\{x > R_0, z > 0\}\) we may consider a barrier of the form
\[\tilde{\phi}(x, z) = M \|(\cosh x)^\sigma (\cosh z)^a h\|_{L^0,\mu}^2 e^{-(\sigma x + ax)} + \rho e^x + \tilde{\sigma},\]
with \(\rho > 0\) arbitrarily small. Fixing \(M\) depending on \(R_0\) we find the desired estimate for \(|(\cosh x)^\sigma (\cosh z)^a \phi|\) in this quadrant by letting \(\rho \to 0\). The argument in the remaining quadrants is similar. The corresponding bound for the \(C^{2,\mu}(\mathbb{R}^2)\) weighted norm is then deduced from local elliptic estimates. This concludes the proof.

7.3. The existence result for the basic linearized operator: Proof of Proposition 4.3.

Proof. We will argue by approximations. Let us replace \(h\) in (4.24) by the function \(h(x, z) \chi_{(-R, R)}(z)\) extended \(2R\)-periodically to the whole plane. With this right hand side we can give to the problem (4.24) a weak formulation in the closed subspace \(H^1_R \subset H^1(\mathbb{R}^2)\) of functions that are \(2R\)-periodic in \(z\) and which also satisfy the orthogonality conditions in (4.24). To be more precise we say that \(\phi_R\) is a weak solution of this problem if for
\[\langle L(\phi_R), \eta \rangle := \int_{-\infty}^{\infty} \int_{-R}^{R} \nabla \psi \cdot \nabla \eta \, dx \, dz + \int_{-\infty}^{\infty} \int_{-R}^{R} (1 - pw^{p-1}) \psi \eta \, dx \, dz,\]
we have
\[\langle L(\phi_R), \eta \rangle = \int_{-\infty}^{\infty} \int_{-R}^{R} h \eta \, dx \, dz\]
for all tests functions \(\eta \in H^1(\mathbb{R}^2)\) which are \(2R\) periodic and which satisfy
\[\int_{\mathbb{R}} w'(x) \eta(x, z) \, dx = 0 = \int_{\mathbb{R}} Z(x) \eta(x, z) \, dx, \quad \text{for all } z \in (-R, R).\]

Because of the orthogonality conditions the bilinear form \(a(\psi, \eta) = \langle L(\psi), \eta \rangle\) is actually positive definite in \(H^1_R\) and consequently there exists a unique \(\phi_R \in H^1_R\) which satisfies
\[a(\phi_R, \eta) = \int_{-\infty}^{\infty} \int_{-R}^{R} h \eta \, dx \, dz, \quad \text{for all } \eta \in H^1_R.\]
Given that \( \phi_R \) satisfies the orthogonality conditions we check that also for any smooth, compactly supported in \((-R,R)\) function \( \tilde{\eta}(z) \) we have
\[
a(\phi_R, w'(x)\tilde{\eta}(z)) = 0 = \int_{-\infty}^{\infty} \int_{-R}^{R} h w'(x)\tilde{\eta}(z)\ dx\ dz,
\]
\[
a(\phi_R, Z(x)\tilde{\eta}(z)) = 0 = \int_{-\infty}^{\infty} \int_{-R}^{R} hZ(x)\tilde{\eta}(z)\ dx\ dz.
\]
This proves that \( \phi_R \) is the unique weak solution of \( L(\phi_R) = h \) in the space of \( H^1(\mathbb{R}^2) \) functions which are \( 2R \) periodic in \( z \). Letting \( R \to +\infty \) and using the uniform a priori estimates valid for the approximations completes the proof of the Proposition. \( \square \)

8. Estimates for the interaction system

This entire section is devoted to the proof of Lemma 4.2.

Proof. We will use the definition of \( X_{\alpha,j}^* k_j \) in (4.12) to estimate term by term. First we observe
\[
\|X_{\alpha,j}^* (\chi_j S(w))\|_{C^{0,\sigma}_\alpha(\mathbb{R}^2)} \leq C\alpha^2,
\]
by (4.16). Next we will consider the nonlinear (in \( \phi_j \) and \( \psi \)) term. Since by assumption
\[
\|X_{\alpha,j}^* \phi_j\|_{C^{2,\sigma}_\alpha(\mathbb{R}^2)} \leq \alpha^{\frac{3}{4}}\sigma,
\]
therefore by (4.21) we have
\[
(8.1)
\]
\[
\|X_{\alpha,j}^* (\chi_j \psi)(\cosh z_j)\|_{C^{2,\sigma}_\alpha(\mathbb{R}^2)} \leq C\alpha^{\frac{3}{4}}\sigma (\alpha^2 + \sum_{i=1}^{k} \|X_{\alpha,i}^* \phi_i\|_{C^{2,\sigma}_\alpha(\mathbb{R}^2)}) \leq C\alpha^{\frac{3}{4}}\sigma.
\]
We will now estimate the nonlinear term, for which we get:
\[
\chi_j N = N(\sum_{i=1}^{k} \rho_i \phi_i + \psi) = \chi_j N(\rho_j \phi_j + \psi),
\]
using (4.7). Let us observe that \( N \) is a ”quadratic” function of its argument. Indeed, with \( p > 2 \) we have for any \( t, s \in \mathbb{R}, \ t \geq 0: \)
\[
| (s+ t)^{p} - t^p - pt^{p-1}s | \leq C \max\{|t^{p-2}, |s|^{p-2} \}|s|^2,
\]
Then it follows:
\[
|X_{\alpha,j}^* (\chi_j N)| \leq C(|X_{\alpha,j}^* \phi_j|^2 + |X_{\alpha,j}^* (\chi_j \psi)|^2).
\]
We have in supp \( X_{\alpha,j}^* (\chi_j) \):
\[
(8.2) \quad |x_j| \leq \frac{15}{16} \log \frac{1}{\alpha},
\]
hence, by (8.1)
\[
\|(\cosh x_j)^p (\cosh z_j)^{\theta \alpha} X_{\alpha,j}^* (\chi_j \psi)\|_{C^{0,\sigma}_\alpha(\mathbb{R}^2)}^2 \leq C\alpha^{-\frac{4\sigma}{8}}\|(\cosh z_j)^{\theta \alpha} X_{\alpha,j}^* (\chi_j \psi)\|_{C^{0,\sigma}_\alpha(\mathbb{R}^2)}^2 \leq C\alpha^{-\frac{4\sigma}{8}} (\alpha^2 + \sum_{i=1}^{k} \|X_{\alpha,i}^* \phi_i\|_{C^{2,\sigma}_\alpha(\mathbb{R}^2)}^2)^2.
\]
We will begin with (4.39). We have computed the leading order of

using Lemma 4.1 we get

as long as (3.3) and (3.19) hold. Another important term comes from

Now we observe that

Using this we find:

To estimate the last term we observe that using (3.15) we get:

making use of (3.3), (3.18), (3.19). The proof of the Lipschitz property (4.33) is

standard and is omitted.

□

9. THE REDUCED PROBLEM: ERROR OF THE PROJECTIONS

In this sections we will fill in some details in the computations in section 4.7. We will begin with (4.39). We have computed the leading order of

which in particular gives rise to the Toda system, see (4.46)–(4.50). In particular we have neglected terms denoted by \( P_j(\Xi_{\pm,j}w_{\pm,j}) \), \( P_j(\Xi_{0,j}w_{0,j}) \) in (5.1). Among these lower order terms we will concentrate on one, representative term, namely, using the notation (3.13)–(3.14) and (4.56),

Now we observe that

as long as (3.3) and (3.19) hold. Another important term comes from

Using Lemma 4.1 we get

Other calculations can be done a similar way.
To see a representative term (slightly different than the ones we have seen above) in (4.40) we will recall the definition of $k_j$ (4.12) and in particular consider this component of $k_j$ that depends on the unknown function $\phi_j$ linearly, namely:

$$-X^*_{\alpha,j}(\chi_jL(\phi_j)) + (X^*_{\alpha,j} \chi_j)[\partial_{x_j}^2 + \partial_{x_j}^2 + g_p'(w_{0,j})]X^*_{\alpha,j} \phi_j.$$ 

Although perhaps not immediately obvious but rather straightforward is the following relation

$$\int_{\mathbb{R}}[-X^*_{\alpha,j}(\chi_jL(\phi_j)) + (X^*_{\alpha,j} \chi_j)[\partial_{x_j}^2 + \partial_{x_j}^2 + g_p'(w_{0,j})]X^*_{\alpha,j} \phi_j] Z(x_j) dx_j \sim \int_{\mathbb{R}} X^*_{\alpha,j}(\chi_j(g_p'(w_{0,j}) - g_p'(w_{j}))(\phi_j) Z(x_j) dx_j.$$ 

Then we get

$$\|\int_{\mathbb{R}} X^*_{\alpha,j}(\chi_j(g_p'(w_{0,j}) - g_p'(w_{j}))(\phi_j) Z(x_j) dx_j\|_{L^2_x(\mathbb{R})} \leq C\alpha^{3+\kappa_4-\mu} \leq C\alpha^{2+\nu_1}.$$ 

Let us now consider some of the terms we have neglected while considering $\gamma_j$. One of them is

$$|\int_{\mathbb{R}} \int_{\mathbb{R}} X^*_{\alpha,j}(\chi_j(g_p'(w_{0,j}) - g_p'(w_{j}))(\phi_j) Z(x_j) \cos(\sqrt{\lambda_1}z_j) dx_j dz_j| \leq C\alpha^{1+\kappa_4} \|\phi_j\|_{C^{0,\mu}_{z,\mu}(\mathbb{R}^2)} \int_{\mathbb{R}} (\cosh z)^{-\theta \alpha} dz \leq C\alpha^{2+\kappa_4}.$$ 

Another, similar in type term, is (c.f. (9.5)):

$$|\int_{\mathbb{R}} \int_{\mathbb{R}} X^*_{\alpha,j}(\chi_j(\psi z_j)^{\nu_1}(\psi z_j)^{\mu} Z(x_j) \cos(\sqrt{\lambda_1}z_j) dx_j dz_j| \leq C\|\psi(\cosh z_j)^{\theta \alpha}\|_{C^{0,\mu}_{z,\mu}(\mathbb{R}^2)} \int_{\mathbb{R}} (\cosh z)^{-\theta \alpha} dz \leq C\alpha^{1+\frac{\kappa_4}{\nu_1}}.$$ 

These terms are bounded by $\alpha^{1+\nu_1}$ for sufficiently small $\alpha$. The rest of the calculations follow the same scheme and are omitted.

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References


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