ANALYSIS OF A CONSERVATION PDE WITH DISCONTINUOUS FLUX: A MODEL OF SETTLER

J.-PH. CHANCELIER†, M. COHEN DE LARA†, AND F. PACARD†

Abstract. A dynamic model of the settling process in the secondary settler of a wastewater treatment plant is given by a nonlinear scalar conservation law \( c_t + \psi(x, c)_x = 0 \) for the sludge concentration \( c(t, x) \), where the flux function \( \psi(x, c) \) presents discontinuities. We analyze this PDE with emphasis both on the existence of stationary solutions and on the evolution of the shock corresponding to the rising of a sludge blanket. Theoretical and numerical simulations are compared with real data. A model with two classes of particles in interaction is introduced to take into account the thickening process: it appears to improve the fit with the data. What is more, regulation strategies of the rising of a sludge blanket in case of important water admission to the plant are proposed.

Key words. nonlinear scalar conservation laws, entropy, shocks, mathematical modeling.

AMS subject classifications. 35L65, 92A08.

1. Introduction. This paper is devoted to the study of the secondary settler in a wastewater treatment plant with emphasis on mathematical analysis but with practical applications in view.

In wastewater engineering literature on the settling process, one mostly finds purely experimental studies with analysis of data collected on a real plant, but also a certain number of more theoretical studies. These latter can be roughly divided into two classes. In the first class \([7, \ 13, \ 12]\), one finds the direct writing of a layered cylindrical settler model, expressing the sludge mass conservation and allowing to calculate the sludge concentration in each layer from one time step to another. This is followed by numerical simulations and comparisons with real data. Emphasis is put on the rising of the sludge blanket, a separation between sludge and clear water which is observed in practice and explained by the so called limiting solids flux theory \([2]\), valid below the feeding point of a cylindrical settler. The second class of theoretical studies comprises studies of an analytical model, here a conservation partial differential equation expressing the sludge mass conservation. Thus, one finds in \([3, \ 10]\) some mathematical properties of the concentration profile below the feeding point of a cylindrical settler as well as an equation for the rising of the sludge blanket. In \([8]\), a characterization of the steady states in a whole cylindrical settler is given, with no emphasis on the mathematical aspects.

In this paper, our approach consists in extending this latter rather mathematical point of view to a general settler, that is of general shape (with non increasing section but not necessarily constant as in the cylindrical settler), in its totality, that is including the delicate feeding point. For this, we shall establish a well posed partial differential equation modeling the whole settler regardless of its shape. What is more, as the clarification and thickening functions of the settler have been little addressed in a unified approach as noticed in \([12]\), we shall propose another mathematical model with two classes of particles for this purpose. The practical evaluation of our theoretical results is ensured thanks to a cooperation with the Water Division of the French LCPC – public applied research organization which leads and coordinates the scientific and technical work of the network of Public Works Laboratories.

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The paper is organized as follows. In Section 2, we introduce the mathematical dynamic model and detail the physical meaning of the different terms involved in each zone of the settler, namely above the surface, between the surface and the feeding point, between the feeding point and the bottom and finally under the bottom. We obtain a nonlinear scalar conservation law \( c_t + \psi(x,c)_x = 0 \), where the flux function \( \psi(x,c) \) presents discontinuities in reason of the distinct physical mechanisms in each zone. We thus briefly review the necessary mathematical background concerning nonlinear scalar conservation laws. This section ends up with a model of the whole wastewater treatment plant. Section 3 presents a unified approach of the settling and thickening processes. Two classes of particles, each with its own settling velocity, are in interaction: “small” particles flocculate into “big” ones with a simple rule to take into account the thickening phenomena. In Section 4, we give a precise description of the steady states in the settler. One of the conditions found (the one for the existence of a steady state with sludge blanket at the feeding point) corresponds to the one given by the limiting solids flux theory for cylindrical settlers. Our results provide a mathematical basis for this latter theory as well as a general extension. Mathematical proofs about existence and uniqueness of stationary solutions are found in Section B of the appendix. The analysis is delicate because, as noted above, we deal with a nonlinear scalar conservation law where the flux function depends not only on the unknown but also on the \( x \)-coordinate and presents discontinuities. Section 5 is devoted to the sludge blanket description. We will propose an ordinary differential equation that reproduces the dynamic behavior of the position of a shock between pure water and concentrated sludge. This makes possible for us to formulate certain control laws to stabilize this sludge blanket at a prescribed depth. Section 6 contains various numerical simulations as well as comparisons with real data. We end this paper with a series of open questions in Section 7 where we raise and try to evaluate the possible applications of our results in conception and design.

2. A mathematical model of the settling process. We first give here a short description of how a wastewater treatment plant works. To be more specific, the Figure 2.1 is that of a small plant, as the one in Serent where experiments are carried out by the LCPC.

After its entry in a primary settler to eliminate big materials, waste water is poured into a huge basin where it is re-oxygenated and submitted to bacteriological action. In this biological reactor, the role of the bacteria is to transform the soluble organic material into biomass, called sludge. After this treatment, the water and sludge fall by gravity into the settler by an introduction point located more or less at the middle of the settler. Part of the sludge and water is pumped out of the settler at the bottom and the remaining flow (which should only consist of water) drops out into the environment. A portion of the pumped sludge is injected back into the biological reactor, because of its biological properties, and the rest is driven out of the treatment plant.

2.1. A scalar dynamic mathematical model. The basic equation to write is one of mass conservation for the sludge in the settler. For simplicity, we will assume that all the quantities involved only depend on the depth \( x \) and time \( t \). Let us emphasize the fact that the \( x \)-axis is directed downward.

2.1.1. Description of the settler. We consider a settler whose geometry is described by the only mean of the depth \( x \), directed downward (see Figure 2.2 for a conical settler sketch) :
1. $x = 0$ is the depth of the top of the settler,
2. $x_f$ is the depth of the feeding point, where sludge from the aeration tank is introduced,
3. $x_b$ is the total depth of the settler,
4. $A(x)$ is the sectional area of the settler, assumed to be non increasing.

Concerning the volumes flows, we denote by :
1. $Q_e$ the effluent flow at the surface,
2. $Q_f$ the input flow at the feeding point,
3. $Q_o$ the output flow at the bottom.

2.1.2. Main assumptions on the settling process. The object under study is the density of sludge mass $c(t,x)$ at time $t$ and depth $x$. We make the following assumptions concerning its evolution :
1. the mass transport is only due to advection and not to diffusion,
2. above the surface and below the bottom of the settler, the advection is only due to the water velocity, supposed to be vertical,
3. between the surface and the bottom, that is in the settler itself, there is an additional speed due to gravity (settling velocity $v$) which we assume to depend only on the concentration $c$, that is $v = v(c)$,
4. this latter settling velocity is an analytical function, rapidly decreasing as well as all its derivatives, satisfying
\[ v'(c) < 0, \quad \forall \ c > 0 \] (2.1)
5. the introduction of sludge in the settler is punctual and located at the feeding point with feeding concentration $c_f$ and positive input flow ($Q_f > 0$),
6. water pours out of the settler at the surface ($Q_e \geq 0$).

These assumptions call for some comments. Diffusion is neglected to focus on the shock phenomenon rather than fine fitting with real data. Above the surface and below the bottom of the settler, the velocity of the water is reasonably vertical in a small layer. It is a classical assumption that the settling velocity depends only upon the concentration (see [3]). It is natural to require that this function be decreasing.
for the more the particles are concentrated the greater the viscosity forces are. The growth and analyticity assumptions are technical (the first implies in particular that $v(c)$ goes quickly enough towards 0 at infinity).

Now, it is very classical to write the following conservation law

\begin{equation}
A(x) \frac{\partial c}{\partial t} + \frac{\partial}{\partial x} (A(x)cV(t, x, c)) = s(t, x),
\end{equation}

where, by our assumptions,

1. $V(t, x, c) = -Q_e(t)/A(0)$ for $x < 0$, since, above the surface, the advection is only due to the vertical velocity of the water,

2. $V(t, x, c) = -Q_e(t)/A(x) + v(c)$ for $0 < x < x_f$, by the additional speed due to gravity,

3. $V(t, x, c) = Q_o(t)/A(x)$ for $x_f < x < x_b$,

4. $V(t, x, c) = Q_o(t)/A(x_b)$ for $x_b < x$,

5. $s(t, x) = Q_f(t)c_f(t)\delta(x - x_f)$ is the source term, representing the sludge input in the settler.

From now on, we extend continuously the function $A(x)$ by $A(x) \equiv A(0)$ for $x < 0$ and $A(x) \equiv A(x_b)$ for $x > x_b$, so that we extend equation (2.2) to the whole real line.

2.2. A dynamic model of the settler. It is easy to notice that the speed function $V$ is discontinuous at the points 0 (the top of the settler), $x_f$ (the feeding point) and $x_b$ (the bottom of the settler). We shall now smooth this function to
avoid technical difficulties. These discontinuities of the speed correspond to some infinite acceleration and a more physical model would assume that this acceleration is progressive over some small interval of space.

Let us denote by \( \delta_\varepsilon(x) \) a regular positive function whose support is included in \([-\varepsilon, \varepsilon]\), with \( \delta_\varepsilon(x) > 0 \) in \((-\varepsilon, \varepsilon)\), and whose integral over \( \mathbb{R} \) is 1. This is an approximation of the Dirac distribution and, using this function, we build

\[
H_\varepsilon(x) = \int_{-\varepsilon}^{\varepsilon} \delta_\varepsilon(s) ds
\]

which is an approximation of the Heaviside function \( H \) defined by

\[
H(x) = 0 \quad \text{if} \quad x < 0 \quad \text{and} \quad H(x) = 1 \quad \text{if} \quad x > 0
\]

Now, we smooth the function \( V(t, x, c) \) by

\[
V_\varepsilon(t, x, c) = H_\varepsilon(x - x_f)Q_o(t)c - H_\varepsilon(x_f - x)Q_e(t)c + H_\varepsilon(x)H_{\varepsilon z}(x_b - x)A(x)cv(c)
\]

where \( \varepsilon = (\varepsilon_1, \varepsilon_2) \). We choose to smooth the source term function \( s(t, x) \) in the same way by

\[
s_\varepsilon(t, x) = Q_f(t)c_f(t)\delta_\varepsilon(x - x_f) = Q_f(t)c_f(t) \frac{\partial H_\varepsilon}{\partial x}(x - x_f)
\]

If we assume that all flows are stationary, equation (2.2) can therefore be rewritten using the above functions as

\[
A(x) \frac{\partial c}{\partial t} + \frac{\partial}{\partial x} \psi_\varepsilon(x, c) = 0, \quad \forall x \in \mathbb{R}
\]

where the flux function \( \psi_\varepsilon \) is given by

\[
\psi_\varepsilon(x, c) = (H_{\varepsilon_1}(x - x_f)Q_o - H_{\varepsilon_1}(x_f - x)Q_e)c
+ H_{\varepsilon z}(x)H_{\varepsilon z}(x_b - x)A(x)cv(c) - H_{\varepsilon_1}(x - x_f)c_fQ_f
\]

The parameters \( \varepsilon_1 \) and \( \varepsilon_2 \) are two smoothing parameters and a solution of equation (2.2) must be understood as the limit (if it exists) of solutions of (2.7) when those two parameters tend to 0.

### 2.3. Some recalls on nonlinear scalar conservation laws.

It is well known that there are some difficulties to define a solution of a nonlinear scalar conservation law as (2.7) (see [11]). In [4, 5], the notion of generalized solution is introduced and existence and uniqueness is proved under large assumptions. What is more, a generalized solution is proved to be the limit of viscous solutions, that is the limit of solutions of the equation with a small additional diffusion term. When the generalized solution is a piecewise smooth function, it is characterized as satisfying (2.7) in the sense of distributions and an additional criterion which takes the following simple form in the one dimensional case. We denote

\[
\sigma(x, c_1, c_2) = \frac{\psi_\varepsilon(x, c_1) - \psi_\varepsilon(x, c_2)}{A(x)(c_1 - c_2)}.
\]

**Proposition 2.1 (Kruzkov).** There exists a unique generalized solution of (2.7) with bounded measurable initial condition. If this generalized solution is a piecewise smooth function \( c(t, x) \), it is characterized by the two following conditions:
1. \(c(t,x)\) satisfies (2.7) in the sense of distributions,

2. in the neighborhood of a point \((t_0,x_0)\) of a discontinuity curve \(t \rightarrow x(t)\), the following inequalities are satisfied

\[
\sigma(x_0, c(t_0, x_0^0), c) \leq \sigma(x_0, c(t_0, x_0^+), c(t_0, x_0^-)) \leq \sigma(x_0, c(t_0, x_0^-), c),
\]

for all \(c\) between \(c(t_0, x_0^-)\) and \(c(t_0, x_0^+)\).

Proof. In [4], existence and uniqueness is proved under assumptions which are easily satisfied by \(\psi_\epsilon(x,c)\) since this latter function is smooth, as well as \(A(x)\), and that \(v(c)\) and all its derivatives are rapidly decreasing.

Definition 2.2. An entropic solution of (2.7) is a piecewise smooth function satisfying the two previous conditions of Proposition 2.1.

There exists a constructive method for solutions of (2.7) based on the method of characteristics. When \(\psi_\epsilon\) does not explicitly depend upon the variable \(x\), a characteristic of equation (2.7) is a curve \(\Gamma\) of the plane \(t,x\) along which \(c(t,x)\) has a constant value and it can be shown that \(\Gamma\) is a straight line. Here, this definition does not fit and we introduce the following one.

Definition 2.3. For any solution of (2.7), we will say that a curve \(\Gamma\) in the space \(t,x\) is a characteristic if \(\psi_\epsilon(x,c(t,x))\) is constant on \(\Gamma\).

We can easily prove the following lemma.

Lemma 2.4. A curve \(\Gamma\) parameterized by \(t \mapsto (t,x(t))\) satisfying the differential equation

\[
\frac{dx}{dt} = \frac{1}{A(x(t))} \frac{\partial \psi_\epsilon}{\partial c}(x(t),c(t,x(t)))\, ,\quad x(t_0) = x_0
\]

is the characteristic of (2.7) issued from the point \((t_0,x_0)\).

We now give some classical definition and lemma about shocks.

Definition 2.5. If \(c\) is a solution of (2.7), a shock is a curve \(C\) of the space \(t,x\) which consists of points of discontinuity of the function \(c\).

Lemma 2.6. We denote by \(c^-\) the concentration just above the shock and \(c^+\) the concentration just below the shock. If we assume that the shock curve is parameterized by \(t\), the slope \(\sigma\) of this curve is given by the Rankine-Hugoniot relation [11]:

\[
\sigma = \frac{\psi_\epsilon(x,c^+) - \psi_\epsilon(x,c^-)}{A(x)(c^+ - c^-)}.
\]

2.4. A model of the whole plant. The dynamic model (2.7) of the settler has to be completed with a mathematical description of the aeration tank. Our main assumptions on the whole plant are the following:

1. the concentration in the aeration tank is uniform and thus equal to the feeding concentration of the settler,

2. sludge in the aeration tank is produced by bacteriological action at a constant rate \(m_i\) per unit of time,

3. the volume \(\Omega\) of water and sludge in the aeration tank is stationary,

4. the volume of water and sludge in the settler is stationary,

5. a part \(Q_r\) of the output flow \(Q_o\) is recycled in the aeration tank while the rest \(Q_w\) is taken out of the plant.
As seen in Figure 2.1, the dynamic of $c_f$ is given by a mass conservation balance

$$\Omega \dot{c}_f = m_i + c(t, x_b)Q_r - Q_f c_f$$

where $c(t, x_b)$ is the concentration at the bottom of the settler. The volume balance in the aeration tank and in the settler imply

$$Q_f = Q_i + Q_r \quad \text{and} \quad Q_f = Q_e + Q_o$$

The complete model of both aeration tank and settler is therefore given by:

$$\begin{cases}
0 &= A(x) \frac{\partial c}{\partial t} + \frac{\partial}{\partial x} \psi(x, c) \\
Q_o &= Q_r + Q_w \\
\Omega \dot{c}_f &= m_i + c(t, x_b)Q_r - Q_f c_f \\
Q_f &= Q_i + Q_r \\
Q_f &= Q_e + Q_o
\end{cases}$$

(2.9)

3. A model with two classes of particles. As will be seen in the numerical simulations, the dynamic model (2.7) provides concentration profiles with good monotonicity but with overestimated steepness above the feeding point and underestimated steepness below the feeding point. This was noticed in an equivalent model and attributed to the growing size of sludge particles from surface to bottom [7]. More generally, this phenomenon results from a limited attention to the thickening process as noted in [12]. These authors propose a new form for the settling velocity to take into account the dispersion in the particles sizes. However, they still keep only one class of particles and the settling velocity remains a function of the concentration.

Here, we propose another form of dynamical model which should capture this previously unmodeled phenomenon of size growth. We consider that particles in the settler are divided in two classes, the “small” ones with concentration $c_s$ and the “big” ones with concentration $c_b$. Two effects are now taken into account: big particles have a more important settling velocity than the small ones and small particles have a tendency to become big ones by aggregation.

Our main assumptions for this new model are the following:

1. concerning the transport phenomenon, small and big particles obey the same law as in Section 2, but each class with its own settling velocity, denoted respectively by $v_s(c_s + c_b)$ and $v_b(c_s + c_b)$.
2. these latter functions are of the form $v_s(c) = \lambda_s v(c)$ and $v_b(c) = \lambda_b v(c)$ with $\lambda_s < \lambda_b$,
3. the number of small particles decays by collision, these latter being in first approximation proportional to $c_s \times c_s$, that is proportional to opportunities of meeting,
4. the decay of the number of small particles is counterbalanced by the growth of the number of big particles.

The form of the settling velocities $v_s(c)$ and $v_b(c)$ is justified by the observation in [9] that the settling velocity for a given class of particles is a linear function of their cross-sectional diameter. The expression of the decay of the number of small particles as being proportional to $c_s \times c_s$ is inspired by the classical two-species mathematical models of Volterra (see [1] for instance).
Now, defining $\psi^s_c$ and $\psi^b_c$ as follows

$$
\begin{align*}
\psi^s_c(x, c_s, c_b) &= (H_{e_1}(x-x_f)Q_0 - H_{e_1}(x_f - x)Q_e)c_s \\
+ \lambda_s H_{e_2}(x)H_{e_2}(x_b - x)A(x)c_s v(c_s + c_b) - H_{e_1}(x-x_f)c_f Q_f \\
\psi^b_c(x, c_s, c_b) &= (H_{e_1}(x-x_f)Q_0 - H_{e_1}(x_f - x)Q_e)c_b \\
+ \lambda_b H_{e_2}(x)H_{e_2}(x_b - x)A(x)c_b v(c_s + c_b) - H_{e_1}(x-x_f)c_f Q_f
\end{align*}
$$

we obtain the following mathematical model:

$$
\begin{align*}
\forall \ x \in \mathbb{R} & \quad \begin{cases}
A(x) \frac{\partial c_s}{\partial t} + \frac{\partial}{\partial x} \psi^s_c(x, c_s, c_b) - kc_s^2 = 0 \\
A(x) \frac{\partial c_b}{\partial t} + \frac{\partial}{\partial x} \psi^b_c(x, c_s, c_b) + kc_b^2 = 0
\end{cases} \quad \text{and} \quad c = c_s + c_b
\end{align*}
$$

The analytical study of (3.2) is not easy. This is why we shall validate it by numerical simulations in Section 6.

4. Sludge concentration steady states in the settler.

4.1. Mathematical characterization. Here we must define what we understand by steady state or stationary solution.

**Definition 4.1.** We will say that $c(x)$ is a stationary solution (resp. an entropic stationary solution) of (2.7) if the function $c(x)$ is a solution (resp. an entropic stationary solution) of (2.7).

The following lemma is slightly more than a re-phrasing of Definition 2.2.

**Lemma 4.2.** The function $c(x)$ is an entropic stationary solution of (2.7) if and only if

1. there exists a constant $\psi_0$ such that

$$
\psi_c(x, c(x)) = \psi_0, \quad \forall \ x \in \mathbb{R}
$$

2. the discontinuity points of $x \mapsto c(x)$ are isolated,

3. at any discontinuity point $x_0$ of $c(x)$, the following inequalities are satisfied

$$
\sigma(x_0, c(x_0^+), c) \leq 0 \leq \sigma(x_0, c(x_0^-), c)
$$

for all $c$ between $c(x_0^+)$ and $c(x_0^-)$.

**Remark 4.3.** Let us notice that, for $\psi_0 \in \mathbb{R}$ given, the characterization of an entropic solution of (2.7) mostly amounts to the analysis of the following implicit equation in $(x, c)$

$$
(H_{e_1}(x-x_f)Q_0 - H_{e_1}(x_f - x)Q_e)c \\
+ H_{e_2}(x)H_{e_2}(x_b - x)A(x)cv(c) - H_{e_1}(x-x_f)c_f Q_f = \psi_0
$$

In order to simplify our study, we shall assume that $\epsilon_1$ is so small that the area $A(x)$ can be considered as constant in $[x_f - \epsilon_1, x_f - \epsilon_1]$.

In the specific case of a cylindrical settler, the so called limiting solids flux theory [2] states that “the solids inflow to the clarifier may not exceed the permeability of any layer of the sludge blanket in order to avoid sludge from accumulating in the clarifier and hence raising the level of sludge blanket surface” [6]. With this heuristic statement, one obtains graphical conditions on the curve $c \mapsto Acv(c)$ to ensure that the sludge blanket does not rise.
In the following theorem which characterizes the steady states in the settler, the discussion depends on a quantity $\Phi_l$ that we identify with the so called limiting flux. This done, we observe that the practice which consists in having the sludge inflow lower than the limiting flux to prevent the sludge blanket from rising corresponds to the theoretical result asserting the existence of a steady state with sludge blanket at the feeding point. In fact, we provide a mathematical basis for the heuristic use of the limiting solids flux theory and, what is more, give precise definitions, conditions and descriptions of steady states which are not limited to the case of a cylindrical settler. The case when the sludge inflow is greater than the limiting flux is also investigated.

**Definition 4.4.** For a given input sludge concentration $c_f$ and output flow $Q_o$, we define

\[ \Phi(c) = A(x_f)cv(c) + Q_o c \]  

and the limiting flux

\[ \Phi_l = \inf_{c > c_f} (A(x_f)cv(c) + Q_o c) = \inf_{c > c_f} \Phi(c) \geq 0 \]  

If $Q_o > 0$, this infimum is achieved for $c^* \in [c_f, +\infty)$.

**Remark 4.5.** By definition, the following inequality is satisfied:

\[ A(x_f)c_f v(c_f) - Q_e c_f \geq \Phi_l - Q_f c_f \]  

The first term represents the flux just above the feeding point and, when negative, we expect to see sludge accumulate above the feeding point and then pour out. This is indeed what happens but rather as a consequence that, in this latter case, the second term $\Phi_l - Q_f c_f$ is negative. In fact, we shall see in the following theorem that $\Phi_l - Q_f c_f$ is the right parameter for the description of steady states.

The proof of the following theorem can be found in Section B where it is noticed that the necessary and sufficient conditions given which ensure the existence of an entropic stationary solution of (2.7) do not depend on $\epsilon_1$ and $\epsilon_2$. Moreover, the stationary solution itself does not depend on $\epsilon_1$ and $\epsilon_2$, on the intervals $(\epsilon_2, x_f - \epsilon_1) \cup (x_f + \epsilon_1, x_b - \epsilon_2)$. This allows one to pass to the limit in a sequence of solutions when the parameters $\epsilon_1$ and $\epsilon_2$ both tend to 0, and obtain a solution to the limit problem almost everywhere on $[0, x_b]$. Thus, in what follows, by unique we mean uniqueness for this limit problem.

**Theorem 4.6.** Assume that $Q_f > 0$ and $Q_o > 0$.

1. If $\Phi_l > Q_f c_f$, then $A(x_f)c_f v(c_f) - Q_e c_f > 0$ and there exists a unique entropic stationary solution of (2.7). This solution has a sludge blanket at the feeding point and is increasing below the feeding point with possible discontinuities. The concentration is lower than $c_f$ just below the feeding point and its value $c(x_f^+)$ is given by

\[ c(x_f^+) = \max\{c, \Phi(c) = Q_f c_f\} \]

2. If $\Phi_l = Q_f c_f$, entropic stationary solutions of (2.7) are described by a one-parameter family $(c_p)_{p \in (0, x_f)}$. Each solution $c_p$ has a sludge blanket at depth $p$. Then, this solution is continuous and decreasing up to the feeding point and is increasing below it with possible discontinuities. The concentration is greater or equal to $c_f$ right above the feeding point and greater or equal to this former value just below it. More precisely, the discussion is organized as follows.
where is the unique solution of (2.7). This solution does not present a sludge blanket and is such that the output flow at the surface is given by \( -\psi_0 = Q_0 c_f - \Phi_1 > 0 \). This solution \( c \) is continuous and decreasing up to the feeding point and is increasing below it with possible discontinuities. The concentration is greater or equal to \( c_f \) right above the feeding point and greater or equal to this former value just below it. More precisely, the discussion is organized as follows.

1. If \( \Phi_1 < Q_0 c_f \), then \( Q_e > 0 \) and there is a unique entropic stationary solution of (2.7). This solution does not present a sludge blanket and is such that the output flow at the surface is given by \( -\psi_0 = Q_0 c_f - \Phi_1 > 0 \). This solution \( c \) is continuous and decreasing up to the feeding point and is increasing below it with possible discontinuities. The concentration is greater or equal to \( c_f \) right above the feeding point and greater or equal to this former value just below it. More precisely, the discussion is organized as follows.

2. If \( \Phi_1 = Q_0 c_f \), then \( Q_e c_f > 0 \) and

\[
c_f < c_p(x_f^-) < c_p(x_f^+)\]

2-1. If the infimum \( \Phi_1 \) of \( \Phi \) is not achieved for \( c = c_f \), then \( A(x_f) c_f v(c_f) - Q_e c_f > 0 \) and

\[
c_f < c_p(x_f^-) < c_p(x_f^+)^\star\]

where \( c_p(x_f^+) \) is the largest argument for which \( \Phi \) achieves its infimum \( \Phi_1 \), then

\[
c_f = c_p(x_f^-) < c_p(x_f^+)^\star\]

2-2. If the infimum \( \Phi_1 \) of \( \Phi \) is achieved for \( c = c_f \), then \( A(x_f) c_f v(c_f) - Q_e c_f = 0 \) and

2-2-1. if there exists another point than \( c = c_f \) for which \( \Phi \) achieves its infimum \( \Phi_1 \), then

\[
c_f = c_p(x_f^-) < c_p(x_f^+)\]

2-2-2. if there exists no other point than \( c = c_f \) for which \( \Phi \) achieves its infimum \( \Phi_1 \), then

\[
c_f = c_p(x_f^-) = c_p(x_f^+)\]

3. If \( \Phi_1 < Q_0 c_f \), then \( Q_e > 0 \) and there is a unique entropic stationary solution of (2.7). This solution does not present a sludge blanket and is such that the output flow at the surface is given by \( -\psi_0 = Q_0 c_f - \Phi_1 > 0 \). This solution \( c \) is continuous and decreasing up to the feeding point and is increasing below it with possible discontinuities. The concentration is greater or equal to \( c_f \) right above the feeding point and greater or equal to this former value just below it. More precisely, the discussion is organized as follows.

3-1. If the infimum \( \Phi_1 \) of \( \Phi \) is not achieved for \( c = c_f \), then \( A(x_f) c_f v(c_f) + Q_e c_f - \Phi_1 > 0 \) and

\[
c_f < c(x_f^-) < c(x_f^+)^\star\]

where \( c(x_f^+) \) is the largest argument for which \( \Phi \) achieves its infimum \( \Phi_1 \) and \( c(x_f^-) \) is the unique solution \( c \) of

\[
A(x_f) c v(c) - Q_e c + Q_0 c_f = \Phi_1
\]

3-2. If the infimum \( \Phi_1 \) of \( \Phi \) is achieved for \( c = c_f \), then \( A(x_f) c_f v(c_f) + Q_e c_f - \Phi_1 = 0 \) and

3-2-1. if there exists another point than \( c = c_f \) for which \( \Phi \) achieves its infimum \( \Phi_1 \), then

\[
c_f = c(x_f^-) < c(x_f^+)\]

where \( c(x_f^+) \) is the largest argument for which \( \Phi \) achieves its infimum \( \Phi_1 \),

3-2-2. if there exists no other point than \( c = c_f \) for which \( \Phi \) achieves its infimum \( \Phi_1 \), then

\[
c_f = c(x_f^-) = c(x_f^+)\]

 Remark 4.7. If the area \( A(x) \) is constant over an interval, then the solution, if it exists, is piecewise constant over it.

In the following corollary of Theorem B.7 in the Appendix, we point out some global relations between quantities that characterize the settler and its steady states.

**Corollary 4.8.** Assume that \( Q_f > 0 \) and \( Q_o > 0 \). Let \( c_{\text{top}} \) and \( c_{\text{bottom}} \) denote the respective values of a steady state concentration at the top of the settler and at its bottom.
1. If $\Phi_l \geq Q_fc_f$, any entropic stationary solution of (2.7) (with sludge blanket) is such that

$$c_{\text{top}} = 0 \quad \text{and} \quad c_{\text{bottom}} = \frac{Q_f}{Q_o}c_f$$

In particular, the limiting flux $\Phi_l$ is greater or equal to the bottom output flow $Q_o c_{\text{bottom}} = Q_f c_f$ and the bottom concentration $c_{\text{bottom}}$ is greater or equal to the input sludge concentration $c_f$.

2. If $\Phi_l < Q_fc_f$, any entropic stationary solution of (2.7) (without sludge blanket) is such that

$$c_{\text{top}} = \frac{Q_f c_f - \Phi_l}{Q_e} \quad \text{and} \quad c_{\text{bottom}} = \frac{\Phi_l}{Q_o}$$

In particular, the feeding input flow $Q_f c_f$ is equal to the sum of the surface output flow $Q_e c_{\text{top}}$ and the limiting flux $\Phi_l$. This latter is also equal to the bottom output flow $Q_o c_{\text{bottom}}$.

These properties of $\Phi_l$ justify the terminology of “limiting flux”.

4.2. Pseudo-stationary solution. To give a simple mathematical description of the evolution of the sludge blanket in the next section, it is appropriate to introduce functions which are “almost” stationary solutions in the following sense.

**Definition 4.9.** The function $c(x)$ is said to be an entropic pseudo-stationary solution of (2.7) if and only if

1. there exists a constant $\psi_0$ such that

$$\psi_\epsilon(x, c(x)) = \psi_0, \quad \forall x \in (\epsilon_2, +\infty)$$

2. at any discontinuity point $x_0 \in (\epsilon_2, +\infty)$ of $c(x)$, the following inequalities are satisfied

$$\sigma(x_0, c(x_0^+), c) \leq 0 \leq \sigma(x_0, c(x_0^-), c)$$

for all $c$ between $c(x_0^-)$ and $c(x_0^+)$. This definition is to be compared with Lemma 4.2 where $\forall x \in \mathbb{R}$ in (4.1) is replaced here by $\forall x \in (\epsilon_2, +\infty)$. In some sense, a pseudo-stationary solution of (2.7) is a stationary solution of (2.7) up to a narrow layer under the surface.

It should be emphasized that, contrarily to the stationary solutions case, when such a pseudo-stationary solution exists, $\psi_0$ in (4.8) is not necessarily given by $Q_e$ and $c_f$ as in Theorem 4.6 and its sign is undetermined a priori.

5. The sludge blanket. In this section, we study the evolution of the depth $p$ of the sludge blanket which marks the separation between sludge and clear water. It appears that $p$ satisfies an ordinary differential equation.

5.1. Evolution of the depth of the sludge blanket above the feeding point. Here, we shall exhibit a specific solution of (2.7) with a sludge blanket located above the feeding point $x_f$ at a depth $p(t)$ satisfying a certain ordinary differential equation.

**Proposition 5.1.** Assume that the concentration $c_f$ and the flow rates $Q_e, Q_o$ are fixed. If $c_S$ is an entropic pseudo-stationary solution of (2.7), there exists an entropic solution $c(t, x)$ of (2.7) given by

$$c(t, x) = H(x - p(t))c_S(x)$$
where \( p(t) \) satisfies the following ordinary differential equation

\[
(5.2) \quad \dot{p} = -\frac{\psi_0}{A(p)c_S(p)} = -\frac{Q_e}{A(p)} + v(c_S(p))
\]

Here, \( \psi_0 \) is the sludge flux associated with the pseudo-stationary solution \( c_S \) in (4.8).

**Proof.** If \( c(t, x) \) is given by (5.1), we have

\[
\begin{cases}
\psi_t(x, c(t, x)) = \psi_s(x, c_S(x)) = \psi_0 & \text{if } x < p(t) \\
\psi_t(x, c(t, x)) = \psi_s(x, 0) = 0 & \text{if } x > p(t)
\end{cases}
\]

that is \( \psi_t(x, c(t, x)) = \psi_0 H(p(t) - x) \) and therefore

\[
A(x) \frac{\partial c}{\partial t} + \frac{\partial}{\partial x} \psi_t(x, c) = A(x)(-\dot{p} \delta_{x-p(t)}) - \psi_0 \delta_{x-p(t)}
\]

\[
= (A(x)p + \psi_0) \delta_{x-p(t)}
\]

\[
= (A(p)p + \psi_0) \delta_{x-p(t)} = 0
\]

as soon as \( p(t) \) satisfies (5.2). Furthermore, \( c(t, x) \) is an entropic solution since, as long as \( p \in (x_2, x_f - \epsilon_1) \), we have

\[
\sigma(p, c_S(p), c) = -\frac{Q_e}{A(p)} + \frac{c_S(p)v(c_S(p)) - cv(c)}{c_S(p) - c}
\]

\[
\sigma(p, c_S(p), 0) = -\frac{Q_e}{A(p)} + v(c_S(p)) \quad (= \dot{p})
\]

\[
\sigma(p, 0, c) = -\frac{Q_e}{A(p)} + v(c)
\]

and the Kruzkov criterion of Proposition 2.1 is satisfied since \( v(c) \) is decreasing. \( \Box \)

**Remark 5.2.** It should be noted that (5.1) implies that the sludge blanket of such solutions goes up or down along a profile given by the pseudo-stationary solution \( c_S(x) \). This fact is observed in numerical simulations.

Now, assume that \( c_f, Q_e \), and \( Q_o \) are such that \( \Phi_t < Q_f c_f \). Then, by Theorem 4.6, there exists a unique entropic solution \( c_S(x) \) of (2.7), which is, for \( x \in (x_2, x_f - \epsilon_1) \), the unique solution of the implicit equation

\[
-Q_e c_S + A(x)c_S(x)v(c_S(x)) = -Q_e c_f + A(x_f)c_f v(c_f) = \psi_0 < 0
\]

Therefore, for a rising sludge blanket (\( \psi_0 < 0 \) in (4.8)), a possible model is the following algebraic-differential system

\[
(5.3) \quad \begin{cases}
\dot{p} = -\frac{Q_e}{A(p)} + v(c_S(p)) \\
-Q_e c_S + A(p)c_S(p)v(c_S(p)) = -Q_e c_f + A(x_f)c_f v(c_f)
\end{cases}
\]

For a falling sludge blanket (\( \psi_0 > 0 \) in (4.8)), the latter implicit equation does not necessarily have a unique solution and the characterization of pseudo-stationary solutions is delicate. This is a drawback for a possible model of a falling sludge blanket.
5.2. Stabilization of the sludge blanket by different control strategies.

Here, we propose different strategies to stabilize the sludge blanket with \( Q_w \), waste sludge flow, and \( Q_r \), return sludge flow, as control variables. The control laws are based upon the previous differential model (5.3) although we assumed all flows to be constant to establish this model in the former study. However, these laws will be validated by numerical simulations on the partial differential model (2.7).

Assume that the settler is in a state where the sludge blanket is rising as in (5.3). With perturbation \( Q_i \) and control variables \( Q_w, Q_r \), our information is not complete and we still need \( Q_e \) and \( c_f \). For this, we consider the complete model (2.9) and consequently obtain a two dimensional model with state variables \( c_f \) and \( p \)

\[
\begin{align*}
\dot{p} &= \frac{\alpha(c_f, Q_f, Q_r, Q_w)}{A(p)c_S(p)} \\
\Omega c_f &= m_i + Q_r c_S(x_b) - Q_f c_f
\end{align*}
\]

and the relations

\[
\begin{align*}
\alpha(c_f, Q_f, Q_r, Q_w) &= -(Q_i - Q_w)c_S(p) + A(p)c_S(p)v(c_S(p)) \\
Q_e &= Q_i - Q_w \\
Q_o &= Q_r + Q_w \\
Q_f &= Q_o + Q_e = Q_r + Q_i
\end{align*}
\]

If the concentration \( c_S(p) \) at the sludge blanket is measured, then the control law \( Q_w = f_1(p, c_S(p), Q_i) \) given by

\[-\frac{Q_e}{A(p)} + v(c_S(p)) = -k_1(p - \bar{p})\]

that is, with perturbation \( Q_i \) and control variable \( Q_w \),

\[
Q_w = Q_i - A(p)v(c_S(p)) - k_1A(p)(p - \bar{p})
\]

is such that \( \dot{p} = -k_1(p - \bar{p}) \). Thus, we see that \( p(t) \) converges to \( \bar{p} \) and a theoretical possibility to stabilize the depth of the sludge blanket exists by using sludge extraction flow or any similar device to remove temporarily some sludge from the plant.

For the concentration in the aeration tank, we can use \( Q_r \) to stabilize \( c_f \) by a feedback law \( Q_r = f_2(p, c_S(p), c_f, Q_i, Q_w) \) given in implicit form by

\[m_i + \frac{Q_r}{Q_r + Q_w} \frac{k_1(p - \bar{p})}{A(p)c_S(p)} = (Q_r + Q_i)c_f = -k_2(c_f - \overline{c_f})\]

since then \( \Omega c_f = -k_2(c_f - \overline{c_f}) \) and \( c_f(t) \) converges to \( \overline{c_f} \) when \( k_2 > 0 \).

**Remark 5.3.** In these control laws, it must be checked that \( Q_w \geq 0 \) and \( Q_r \geq 0 \). We have assumed that the flow rate \( Q_i \), the concentration \( c_f \) and the sludge blanket depth are measured.

Numerical simulations are currently being launched. The first results are encouraging but these laws remain to be evaluated and tested in view of real experiments.
6. Numerical simulations and comparisons with experiments. To test whether the previous theoretical results offer a reasonable description of a settler functioning, we conducted numerical simulations with data corresponding to experiments performed by the LCPC (Water Division at Nantes) on a conical settler with the following characteristics.

\[ A(0) = 35.25 \text{ m}^2 \quad \text{area of the settler at the surface} \]
\[ A(x_b) = 0.5 \text{ m}^2 \quad \text{area of the settler at the bottom} \]
\[ c_f(0) = 3.46 \text{ g l}^{-1} \quad \text{initial solids concentration in the aeration tank} \]
\[ k = 0.1 \text{ kg}^{-1} \text{ m}^3 \text{ h}^{-1} \quad \text{specific rate of small particles concentration decay by collision} \]
\[ \lambda_s = 1.0 \quad \text{factor of the sludge velocity for small particles} \]
\[ \lambda_b = 5.0 \quad \text{factor of the sludge velocity for big particles} \]
\[ m_i = 2.0 \text{ kg h}^{-1} \quad \text{rate of sludge production by bacteriological action} \]
\[ \Omega = 150.0 \text{ m}^3 \quad \text{volume of the aeration tank} \]
\[ Q_i = 40.0 \text{ m}^3 \text{ h}^{-1} \quad \text{inflow to treatment plant (aeration tank)} \]
\[ Q_r = 35.0 \text{ m}^3 \text{ h}^{-1} \quad \text{return sludge flow} \]
\[ Q_w = 0.0 \text{ m}^3 \text{ h}^{-1} \quad \text{waste sludge flow} \]
\[ x_f = 2.77 \text{ m} \quad \text{depth of the feeding point} \]
\[ x_b = 4.17 \text{ m} \quad \text{total depth of the settler} \]

The gravity induced sludge velocity is given by the following expression

\[ v(c) = 14.5e^{-0.47c} \]

6.1. Steady states. Steady states profiles are difficult to obtain experimentally and this is why the Figure 6.1 only reproduces numerical simulations. One observes that the profiles generated by the two particles classes model (3.2) (model 2) are more accentuated than those generated by the one particle class model (2.7) (model 1). Data seem to indicate that, above the feeding point, the real profiles are steeper than those generated by both models but that, below the feeding point, they are comparable with those generated by the two particles classes model (3.2) (model 2) [7].

6.2. Evolution of the sludge blanket. In the Figures 6.2 and 6.3, one plots the evolution of the sludge blanket with respect to time, respectively calculated with the PDE model (2.7) and with the ODE model (5.2), as well as real data.

In Figure 6.4, one observes by the profiles calculated with the PDE model (2.7) that the sludge blanket rises along a stationary profile as noted in Remark 5.2.

7. Conclusion. In this paper, we have analyzed in detail certain mathematical properties of a dynamic model of settler. The characterization of its steady states should be of some help in questions of static dimensioning. However, this latter problem may also be grasped in a dynamic perspective. Indeed, with the control laws given in Section 5, we are supposed to maintain the sludge blanket depth and the concentration in the aeration tank at prescribed values \( \overline{p} \) and \( \overline{c_f} \). Therefore, we address the following question : for a given input flow \( Q_i \), what are the best values, if they exist, for \( \overline{p} \) and \( \overline{c_f} \) in the sense that the effluent flow \( Q_e \) and the output sludge concentration are maximum ? What is more, if we ever had to design a settler, we would even master the feeding point \( x_f \), the shape \( A(x) \) and the height \( x_b \) and study their optimal values too.
These questions and many more are currently being discussed with practitioners in the field, like research laboratories either public or industrial. A simulation software is available and allows to test various scenarios and compare them with real data or evaluate them.

Acknowledgements The authors are indebted to M. Claude JOANNIS from the LCPC for bringing this problem to their attention and for his advice. They also thank Melle Orlane BAILLARD for her help and fruitful discussions.

Appendix A. List of symbols.
Rising of the sludge blanket (PDE model)

Fig. 6.2. Height of the sludge blanket calculated with the PDE model

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Unit</th>
<th>Declaration</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A(x)$</td>
<td>$m^2$</td>
<td>area of the settler at depth $x$</td>
</tr>
<tr>
<td>$c$</td>
<td>kg m$^{-3}$, gl$^{-1}$</td>
<td>solids concentration in the settler</td>
</tr>
<tr>
<td>$c_b$</td>
<td>kg m$^{-3}$, gl$^{-1}$</td>
<td>big particles concentration in the settler</td>
</tr>
<tr>
<td>$c_f$</td>
<td>kg m$^{-3}$, gl$^{-1}$</td>
<td>solids concentration in the aeration tank</td>
</tr>
<tr>
<td>$c_s$</td>
<td>kg m$^{-3}$, gl$^{-1}$</td>
<td>small particles concentration in the settler</td>
</tr>
<tr>
<td>$c_S$</td>
<td>kg m$^{-3}$, gl$^{-1}$</td>
<td>pseudo-stationary concentration</td>
</tr>
<tr>
<td>$\epsilon_1$</td>
<td>$m$</td>
<td>smoothing parameter of the flux function around the feeding point</td>
</tr>
<tr>
<td>$\epsilon_2$</td>
<td>$m$</td>
<td>smoothing parameter of the flux function around the surface and the bottom</td>
</tr>
<tr>
<td>$H$</td>
<td></td>
<td>Heaviside function</td>
</tr>
<tr>
<td>$H_c$</td>
<td></td>
<td>approximation of the Heaviside function</td>
</tr>
<tr>
<td>$k$</td>
<td>kg$^{-1}$ m$^3$ h$^{-1}$</td>
<td>specific rate of small particles concentration decay by collision</td>
</tr>
<tr>
<td>$\lambda_s$</td>
<td></td>
<td>factor of the sludge velocity for small particles</td>
</tr>
<tr>
<td>$\lambda_b$</td>
<td></td>
<td>factor of the sludge velocity for big particles</td>
</tr>
<tr>
<td>$m_i$</td>
<td>kg h$^{-1}$</td>
<td>rate of sludge production by bacteriological action</td>
</tr>
<tr>
<td>$\Omega$</td>
<td>m$^3$</td>
<td>volume of the aeration tank</td>
</tr>
<tr>
<td>$p$</td>
<td>$m$</td>
<td>depth of the sludge blanket</td>
</tr>
<tr>
<td>$\psi(x, c)$</td>
<td>kg m$^{-2}$ h$^{-1}$</td>
<td>flux function</td>
</tr>
<tr>
<td>$\psi_0$</td>
<td>kg m$^{-2}$ h$^{-1}$</td>
<td>total flux in steady state</td>
</tr>
<tr>
<td>$Q_e$</td>
<td>m$^3$ h$^{-1}$</td>
<td>effluent flow</td>
</tr>
<tr>
<td>$Q_i$</td>
<td>m$^3$ h$^{-1}$</td>
<td>inflow to treatment plant (aeration tank)</td>
</tr>
<tr>
<td>$Q_f$</td>
<td>m$^3$ h$^{-1}$</td>
<td>inflow to settler</td>
</tr>
<tr>
<td>$Q_o$</td>
<td>m$^3$ h$^{-1}$</td>
<td>output flow</td>
</tr>
<tr>
<td>$Q_r$</td>
<td>m$^3$ h$^{-1}$</td>
<td>return sludge flow</td>
</tr>
<tr>
<td>$Q_w$</td>
<td>m$^3$ h$^{-1}$</td>
<td>waste sludge flow</td>
</tr>
<tr>
<td>$v(c)$</td>
<td>m h$^{-1}$</td>
<td>gravity induced sludge velocity</td>
</tr>
<tr>
<td>$x_f$</td>
<td>$m$</td>
<td>depth of the feeding point</td>
</tr>
<tr>
<td>$x_b$</td>
<td>$m$</td>
<td>total depth of the settler</td>
</tr>
</tbody>
</table>
Appendix B. Proofs of the main theorems.

B.1. Assumptions and new formulation of the theorem. For the proof of Theorem 4.6, we make a renormalization of the different quantities characterizing the settler. We introduce:

\[
\begin{align*}
  h(x) &= H_{e_1}(x - x_f) \\
  u &= \frac{c}{c_f} \\
  h_1 &= \frac{Q_e}{Q_f} \\
  V(u) &= A(x_f)v(c)/Q_f \\
  \alpha(x) &= A(x)H_{e_2}(x)H_{e_2}(x_b - x)/A(x_f)
\end{align*}
\]  

(B.1)

**Remark B.1.** It should be noted that we do not treat the case where \( Q_f = 0 \), that is when no sludge enters the settler.

Since we focus on entropic stationary solutions, the following definition will be useful.

**Definition B.2.** For \( h_0 \in \mathbb{R} \), let us define \( F(x, u) = (h(x) - h_1)u + \alpha(x)uV(u) - h(x) + h_0 \). We say that a function \( x \in \mathbb{R} \mapsto u(x) \) is an entropic solution of the implicit equation

\[
F(x, u) \equiv (h(x) - h_1)u + \alpha(x)uV(u) - h(x) + h_0 = 0
\]

(B.2)
Fig. 6.4. Rising of the sludge blanket along a steady state profile

if

1. \( \forall x \in \mathbb{R}, u(x) \geq 0 \) and \( F(x, u(x)) = 0 \),
2. the discontinuity points of \( x \mapsto u(x) \) are isolated,
3. at any discontinuity point \( \bar{x} \), we have \( F(\bar{x}, v)(u(\bar{x}^+) - u(\bar{x}^-)) \geq 0 \), for all \( v \) between \( u(\bar{x}^-) \) and \( u(\bar{x}^+) \).

With the new notations, equation (B.2) is just equation (4.3) with the correspondence

\[
(B.3) \quad h_0 = -\psi_0/(c_f Q_f)
\]

Thus, by Lemma 4.2, \( u(x) \) is an entropic solution of the implicit equation (B.2) if and only if \( c(x) = c_f u(x) \) is is an entropic stationary solution of (2.7) with \( \psi_0 = -h_0 c_f Q_f \) in (4.1).

We re-formulate the assumptions on the velocity function \( c \mapsto v(c) \) in terms of the function \( u \mapsto V(u) \). The analyticity assumption is technical and mainly used to have isolated zeros for some functions appearing in the proof.

Assumption B.3. The function \( V(.) \) is assumed to be an analytical, non negative function, rapidly decreasing as well as all its derivatives, satisfying \( V'(u) < 0 \) for all \( u > 0 \). When needed, we still denote by \( V \) its extension to a left neighborhood of \( u = 0 \).
Moreover, for technical reasons, we assume from now on that \( \alpha(x) \) is not constant on any open subset of \( [-\epsilon_2, x_f - \epsilon_1] \cup [x_f + \epsilon_1, x_b + \epsilon_2] \). If this is not the case, we can always reduce to this assumption because on such an open subset equation (B.2) does not explicitly depend upon \( x \). More precisely, if \( \alpha(x) \) is constant on some interval \( (x_1, x_2) \), we define the operator \( T \) by: given a function \( w \) defined on \( \mathbb{R} \), \( T(w)(x) = w(x) \) for all \( x \leq x_1 \) and \( T(w)(x) = w(x - (x_2 - x_1)) \) for all \( x \geq x_1 \). If \( u \) is an entropic solution of (B.2), it is easily checked that \( T(u) \) is an entropic solution of (B.2), where we have replaced \( \alpha \) by \( T(\alpha) \).

**Assumption B.4.** The function \( \alpha(.) \) in (B.1) is smooth, non negative and has the following features:

- \( x \leq -\epsilon_2, \) \( \alpha \) is constant \( (\alpha(x) \equiv 0) \),
- \( x \in (-\epsilon_2, \epsilon_2), \) \( \alpha \) is positive but its sense of variation is undetermined,
- \( x \in [\epsilon_2, x_f - \epsilon_1], \) \( \alpha \) is positive and strictly decreasing,
- \( x \in [x_f - \epsilon_1, x_f + \epsilon_1], \) \( \alpha \) is constant \( (\alpha(x) \equiv 1) \),
- \( x \in [x_f + \epsilon_1, x_b + \epsilon_2], \) \( \alpha \) is positive and strictly decreasing,
- \( x \geq x_b + \epsilon_2, \) \( \alpha \) is constant \( (\alpha(x) \equiv 0) \).

Since \( \epsilon_1 \) and \( \epsilon_2 \) are two smoothing parameters (see Paragraph 2.2), we introduce the following definition.

**Definition B.5.** We say that an entropic solution \( x \in \mathbb{R} \Rightarrow u(x) \) of the implicit equation (B.2) is

(i) unique if any other entropic solution of (B.2) coincides with it outside their (isolated) discontinuity points,

(ii) \( \epsilon \)-unique if any other entropic solution of (B.2) coincides with it outside their (isolated) discontinuity points and outside \( [-\epsilon_2, \epsilon_2] \) and \( [x_f - \epsilon_1, x_f + \epsilon_1] \).

With these notations and definitions, the formulation of Theorem 4.6 is as follows. We first introduce convenient definitions.

**Definition B.6.** For \( h_1 \in [0, 1] \) fixed, we note

(B.4) \[ \phi(u) = uV(u) + (1 - h_1)u \]

and since this latter function is non negative, we define

(B.5) \[ \phi_1 = \inf_{u > 1} (uV(u) + (1 - h_1)u) = \inf_{u > 1} \phi(u) \]

Moreover, for \( h_0 \in \mathbb{R} \), let

(B.6) \[ r(h_0) = \inf_{u > 1} \frac{(1 - h_1)(u - 1) + uV(u) - h_1 + h_0}{u - 1} = \inf_{u > 1} \frac{\phi(u) - (1 - h_0)}{u - 1} \]

**Theorem B.7.** Assume that \( h_1 \in [0, 1] \).

1. If \( \phi_1 \geq 1 \), there exists an entropic solution of the implicit equation (B.2) if and only \( h_0 = 0 \).

   1.1. If \( \phi_1 > 1 \), then \( V(1) - h_1 > 0 \) and there exists a unique solution of the implicit equation (B.2) for \( h_0 = 0 \). This solution \( u \) is zero for \( x < x_f - \epsilon_1 \) and is increasing below the feeding point with possible discontinuities. It satisfies

\[ 0 < u((x_f + \epsilon_1)^+) < 1 \]
1. If $\phi_1 = 1$, entropic solutions of the implicit equation (B.2) for $h_0 = 0$ are described by a one-parameter family \((u_\pi)_{\pi \in (0, x_f)}\). Each solution $u_\pi$ is zero for $x < p$, is continuous and decreasing up to $x_f - \epsilon_1$ and is increasing below $x_f + \epsilon_1$ with possible discontinuities.

For the values in a neighborhood of $x_f$, two cases may be distinguished.

1.1. If the infimum of $\phi$ is not achieved for $u = 1$, then $V(1) - h_1 > 0$ and

$$1 < u_\pi((x_f - \epsilon_1)^-) < u_\pi((x_f + \epsilon_1)^+)$$

1.2. If the infimum of $\phi$ is achieved for $u = 1$, then $V(1) - h_1 = 0$ and

$$1 = u_\pi((x_f - \epsilon_1)^-) \leq u_\pi((x_f + \epsilon_1)^+)$$

The value $u_\pi((x_f + \epsilon_1)^+)$ is the largest argument for which $\phi$ achieves its infimum and the latter inequality is strict if and only there exists another point than $u = 1$ for which $\phi$ achieves its infimum.

The values at the top and at the bottom are given by:

$$u(-\epsilon_2) = 0 \quad \text{and} \quad u(x_b + \epsilon_2)^+ = \frac{1}{1 - h_1}$$

2. If $\phi_1 < 1$, then $h_1 \neq 0$ and there exists an entropic solution of the implicit equation (B.2) for a unique $h_0^* \in (0, 1]$ given by $h_0^* = 1 - \phi_1$. This solution is continuous and decreasing up to $x_f - \epsilon_1$ and is increasing below $x_f + \epsilon_1$ with possible discontinuities.

For the values in a neighborhood of $x_f$, two cases may be distinguished.

2.1. If the infimum of $\phi_1$ is not achieved for $u = 1$, then $V(1) - h_1 + h_0^* > 0$ and

$$1 < u((x_f - \epsilon_1)^-) < u((x_f + \epsilon_1)^+)$$

2.2. If the infimum of $\phi_1$ is achieved for $u = 1$, then $V(1) - h_1 + h_0^* = 0$ and

$$1 = u((x_f - \epsilon_1)^-) \leq u((x_f + \epsilon_1)^+)$$

The value $u((x_f + \epsilon_1)^+)$ is the largest argument for which $\phi$ achieves its infimum and the latter inequality is strict if and only there exists another point than $u = 1$ for which $\phi$ achieves its infimum.

The values at the top and at the bottom are given by:

$$u(-\epsilon_2) = \frac{1 - \phi_1}{h_1} \quad \text{and} \quad u(x_b + \epsilon_2)^+ = \frac{\phi_1}{1 - h_1}$$

**Proof.** We show how Theorem B.7 can be proved by means of the three Theorems B.8, B.9 and B.10 as well as Lemma B.11, stated in the next subsection.

1. If $\phi_1 \geq 1$, let $h_0 \in \mathbb{R}$ be such that there exists an entropic solution of the implicit equation (B.2). We shall prove that necessarily $h_0 = 0$.

First, we know by Theorem B.10 that $h_0 \in [0, 1]$. We shall now show that if $h_0 \in (0, 1]$, this leads to a contradiction.

Since $h_0 > 0$, we have

$$V(1) - h_1 + h_0 > V(1) - h_1 + 1 - 1 = \phi(1) - 1 \geq \phi_1 - 1 \geq 0$$
Thus, if \( r(h_0) > 0 \), there cannot exist an entropic solution of the implicit equation (B.2) because of point 1 in Theorem B.8 and this contradicts the assumption on \( h_0 \).

If \( r(h_0) = 0 \), then by (B.15), we necessarily have \( h_0 = 1 - \phi_t \). But this latter real number is non positive so that \( h_0 \leq 0 \) and this is contradictory with \( h_0 \in (0,1] \).

Now we show that, when \( h_0 = 0 \), there exists an entropic solution of the implicit equation (B.2).

1-1. If \( \phi_t > 1 \), we have \( r(0) \geq 0 \) by (B.12). Moreover, by (B.15), we cannot have \( r(0) = 0 \) and thus \( r(0) > 0 \). Then, we use point 21 in Theorem B.8 since \( V(1) - h_1 = \phi(1) - 1 \geq \phi_t - 1 > 0 \).

1-2. If \( \phi_t = 1 \), two cases must be distinguished.

1-2-1. If \( \phi(1) > \phi_t = 1 \), then \( r(0) = 0 \) and \( V(1) - h_1 > 0 \) by (B.13).

Then, we use point 21 in Theorem B.9.

1-2-2. If \( \phi(1) = \phi_t = 1 \), then \( V(1) - h_1 = 0 \) and two cases must be distinguished.

If \( \phi'(1) = 0 \) or if \( \phi'(1) > 0 \) and there exists another point than \( u = 1 \) where \( \phi \) achieves its infimum, then \( r(0) = 0 \) by (B.16). Then, we use point 22 in Theorem B.9.

If \( \phi'(1) > 0 \) and there exists no other point than \( u = 1 \) where \( \phi \) achieves its infimum, then \( r(0) > 0 \) by (B.16). Then, we use point 22 in Theorem B.8.

For the values at the top and at the bottom, they are the solutions \( u \) of \( F(x,u) = 0 \) in (B.18) and (B.22).

2. If \( \phi_t < 1 \), then \( h_1 \neq 0 \) because when \( h_1 = 0 \) we have \( \phi(u) = uV(u) + u > u \geq 1 \) for \( u > 1 \) and thus \( \phi_t \geq 1 \). Let \( h_0 \in \mathbb{R} \) be such that there exists an entropic solution of the implicit equation (B.2). We shall prove that necessarily \( h_0 = h_0^* = 1 - \phi_t \).

First, we know by Theorem B.10 that \( h_0 \in [0,1] \). If \( h_0 \in (0,h_0^*) \), we have \( \phi_t = 1 - h_0^* < 1 - h_0 \) and thus \( r(h_0) < 0 \) by (B.12), which contradicts Theorem B.10. On the other hand, and by the same argument, if \( h_0 \in (h_0^*,1] \), we have \( r(h_0) > 0 \) and

\[
V(1) - h_1 + h_0 > V(1) - h_1 + h_0^* = \phi(1) - 1 + h_0^* = \phi(1) - \phi_t \geq 0
\]

This contradicts point 1 in Theorem B.8.

Now we show that, when \( h_0 = h_0^* \), there exists an entropic solution of the implicit equation (B.2).

2-1. If \( \phi(1) > \phi_t \), then by (B.13) we have \( r(h_0^*) = 0 \) and \( V(1) - h_1 + h_0^* > 0 \) and we use point 11 in Theorem B.9.

2-2. If \( \phi(1) = \phi_t \), then \( V(1) - h_1 + h_0^* = 0 \) and two cases must be distinguished.

2-2-1. If \( \phi'(1) = 0 \) or if \( \phi'(1) > 0 \) and there exists another point than \( u = 1 \) where \( \phi \) achieves its infimum, then \( r(h_0^*) = 0 \) by (B.16). Then, we use point 12 in Theorem B.9.

2-2-2. If \( \phi'(1) > 0 \) and there exists no other point than \( u = 1 \) where \( \phi \) achieves its infimum, then \( r(h_0^*) > 0 \) by (B.16). Then, we use point 1 in Theorem B.8.

For the values at the top and at the bottom, they are the solutions \( u \) of \( F(x,u) = 0 \) in (B.18) and (B.22).

**B.2. Main results.** From now on, we assume that \( h_1 \in [0,1] \) (see (B.1)). The three following theorems give the answers to the existence and uniqueness properties of entropic solutions of the implicit equation (B.2). We shall prove them in the following subsections.
Theorem B.8. Assume that the following inequality is satisfied

\[ 1 - h_1 + \inf_{u > 1} \left\{ \frac{uV(u) - h_1 + h_0}{u - 1} \right\} > 0 \]  

Then \( V(1) - h_1 + h_0 \geq 0 \).

1. If \( h_0 \in (0, 1] \), there exists an entropic solution of (B.2) if and only if

\[ V(1) - h_1 + h_0 = 0 \]

Moreover, this solution is unique, continuous and decreasing up to \( x_f - \epsilon_1 \), constant and equal to \( u(x) = 1 \) for all \( x \in [x_f - \epsilon_1, x_f + \epsilon_1] \), and is increasing below \( x_f + \epsilon_1 \) with possible discontinuities.

2. If \( h_0 = 0 \), then

2-1. If \( V(1) - h_1 > 0 \), there exists an \( \epsilon \)-unique entropic solution \( u \) of (B.2), where \( u(x) = 0 \) for \( x < x_f - \epsilon_1 \), \( u \) is increasing below \( x_f + \epsilon_1 \) with possible discontinuities and satisfies \( 0 < u((x_f + \epsilon_1)^+) < 1 \).

2-2. If \( V(1) - h_1 = 0 \), there exists a one parameter continuous family of entropic solutions of (B.2) \( (u_{\epsilon})_{\epsilon \in (x_f - \epsilon_1, 1]} \), where \( u_{\epsilon}(x) = 0 \) for \( x < \epsilon \), then \( u_{\epsilon} \) is continuous and decreasing up to \( x_f - \epsilon_1 \), is increasing below \( x_f + \epsilon_1 \) with possible discontinuities and satisfies \( u_{\epsilon}(x_f - \epsilon_1^-) = u_{\epsilon}(x_f + \epsilon_1)^+ = 1 \).

Proof. See (B.12) and

1. see Lemma B.33 for the existence and Lemma B.28 for the nonexistence,

2. see Lemma B.34.

The limit case is treated in the next theorem.

Theorem B.9. Assume that

\[ 1 - h_1 + \inf_{u > 1} \left\{ \frac{uV(u) - h_1 + h_0}{u - 1} \right\} = 0 \]

1. If \( h_0 \in (0, 1] \) and \( (h_0, h_1) \neq (1, 1) \), there exists an \( \epsilon \)-unique entropic solution of (B.2). This solution \( u \) is continuous and decreasing up to \( x_f - \epsilon_1 \), increasing below \( x_f + \epsilon_1 \) with possible discontinuities. What is more, we have \( V(1) - h_1 + h_0 \geq 0 \) and

1-1. If \( V(1) - h_1 + h_0 > 0 \), the solution \( u \) satisfies \( 1 < u((x_f - \epsilon_1)^-) < u((x_f + \epsilon_1)^+) \).

1-2. If \( V(1) - h_1 + h_0 = 0 \), the solution \( u \) satisfies \( 1 = u((x_f - \epsilon_1)^-) \leq u((x_f + \epsilon_1)^+) \). The value \( u((x_f + \epsilon_1)^+) \) is the largest argument for which \( \phi \) achieves its infimum and the latter inequality is strict if and only there exists another point than \( u = 1 \) for which \( \phi \) achieves its infimum.

2. If \( h_0 = 0 \), the set of entropic solutions, restricted to the complement of \( [-\epsilon_f, \epsilon_1] \) and \( [x_f - \epsilon_1, x_f + \epsilon_1] \), is described by a one parameter family \( (u_{\epsilon})_{\epsilon \in (x_f - \epsilon_1, 1]} \), where \( u_{\epsilon}(x) = 0 \) for \( x < \epsilon \), then \( u_{\epsilon} \) is continuous and decreasing up to \( x_f - \epsilon_1 \), is increasing below \( x_f + \epsilon_1 \) with possible discontinuities. What is more, we have \( V(1) - h_1 \geq 0 \) and

2-1. If \( V(1) - h_1 > 0 \), the solution \( u_{\epsilon} \) satisfies \( 1 < u_{\epsilon}(x_f - \epsilon_1^-) < u_{\epsilon}(x_f + \epsilon_1)^+ \).

2-2. If \( V(1) - h_1 = 0 \), the solution \( u_{\epsilon} \) satisfies \( 1 = u_{\epsilon}(x_f - \epsilon_1^-) \leq u_{\epsilon}(x_f + \epsilon_1)^+ \). The value \( u((x_f + \epsilon_1)^+) \) is the largest argument for which \( \phi \) achieves its infimum and the latter inequality is strict if and only there exists another point than \( u = 1 \) for which \( \phi \) achieves its infimum.

Proof.
1. see Lemma B.35,
2. see Lemma B.37.

The nonexistence part of the results is given by the following theorem.

**Theorem B.10.** If \( h_0 \notin [0, 1] \) or if

\[
B.10 \quad h_0 \in [0, 1] \quad \text{and} \quad 1 - h_1 + \inf_{u \geq 1} \left\{ \frac{uV(u) - h_1 + h_0}{u - 1} \right\} < 0
\]

then there are no entropic solutions of (B.2).

**Proof.** see Lemma B.27.

The existence results that we shall prove are summarized in the following table.

<table>
<thead>
<tr>
<th>Condition</th>
<th>( 1 - h_1 + \inf_{u \geq 1} \left{ \frac{uV(u) - h_1 + h_0}{u - 1} \right} )</th>
<th>(&lt; 0)</th>
<th>(= 0)</th>
<th>(&gt; 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( h_0 \notin [0, 1] )</td>
<td>no solution</td>
<td>no solution</td>
<td>no solution</td>
<td></td>
</tr>
<tr>
<td>( h_0 \in (0, 1] ) and ( V(1) - h_1 + h_0 = 0 )</td>
<td>no solution</td>
<td>solution</td>
<td>solution</td>
<td></td>
</tr>
<tr>
<td>( h_0 \in (0, 1] ) and ( V(1) - h_1 + h_0 \neq 0 ) and ( (h_0, h_1) \neq (1, 1) )</td>
<td>no solution</td>
<td>solution</td>
<td>no solution</td>
<td></td>
</tr>
<tr>
<td>( h_0 = 0 ) and ( V(1) - h_1 = 0 )</td>
<td>no solution</td>
<td>solution</td>
<td>solution</td>
<td></td>
</tr>
<tr>
<td>( h_0 = 0 ) and ( V(1) - h_1 \neq 0 )</td>
<td>no solution</td>
<td>solution</td>
<td>solution</td>
<td></td>
</tr>
</tbody>
</table>

The proofs of the three previous theorems are very technical and we have decomposed them in a certain number of steps.

**B.3. Preliminary results.** In the following lemma, we focus on the links between the function \( r(h_0) \) in (B.6), whose sign determines the existence or not of solutions, and the “limiting flux” \( \phi_l \) in (B.5), currently used by practitioners.

**Lemma B.11.** For any \( h_0 \in \mathbb{R} \), we have

\[
B.11 \quad r(h_0) \leq 1 - h_1
\]

and the function \( h_0 \mapsto r(h_0) \) is increasing. Moreover, if \( h_1 \in [0, 1) \), the following properties are satisfied.

\[
B.12 \quad r(h_0) \geq 0 \iff \phi_l \geq 1 - h_0 \Rightarrow V(1) - h_1 + h_0 \geq 0
\]

\[
B.13 \quad r(h_0) = 0, \ V(1) - h_1 + h_0 > 0 \iff \phi_l > 1 - h_0
\]

\[
B.14 \quad r(h_0) = 0, \ V(1) - h_1 + h_0 = 0 \Rightarrow \phi_l = 1 - h_0
\]

\[
B.15 \quad r(h_0) = 0 \Rightarrow h_0 = 1 - \phi_l
\]

As for the limit case where \( \phi_l = 1 - h_0 \) (and then \( V(1) - h_1 + h_0 = 0 \)), we have :

\[
B.16 \begin{cases}
\phi(1) = \phi_l = 1 - h_0, & \phi'(1) = 0 \Rightarrow r(h_0) = 0 \\
\phi(1) = \phi_l = 1 - h_0, & \phi'(1) > 0, \quad \exists u > 1, \phi(u) = \phi(1) \Rightarrow r(h_0) = 0 \\
\phi(1) = \phi_l = 1 - h_0, & \phi'(1) > 0, \quad \phi(u) > \phi(1), \forall u > 1 \Rightarrow r(h_0) > 0
\end{cases}
\]
Proof. Since \(\lim_{u \to +\infty} u V(u) = 0\), then \(\lim_{u \to +\infty} \frac{u V(u) + h_0 - h_1}{u - 1} = 0\) and thus
\[
\inf_{u > 1} \frac{(1 - h_1)(u - 1) + u V(u) - h_1 + h_0}{u - 1} = 1 - h_1 + \inf_{u > 1} \frac{u V(u) - h_1 + h_0}{u - 1} \leq 1 - h_1
\]
The function \(h_0 \mapsto r(h_0)\) is clearly increasing by the formula (B.6).

To prove (B.12), it suffices to note that, by (B.6), we have
\[
r(h_0) \geq 0 \iff \forall u > 1, \phi(u) - (1 - h_0) \geq 0 \iff \phi_l = \inf_{u > 1} \phi(u) \geq 1 - h_0
\]
In this case, we also have
\[
V(1) + 1 - h_1 = \phi(1) \geq \phi_l \geq 1 - h_0 \quad \text{and thus} \quad V(1) - h_1 + h_0 \geq 0
\]
For the case \(r(h_0) = 0\), we shall use the following easy property
\[
(B.13) u^* > 1,0 = r(h_0) = \frac{\phi(u^*) - (1 - h_0)}{u^* - 1} \iff \exists u^* > 1,1 - h_0 = \phi_l = \phi(u^*)
\]
Now, if \(r(h_0) = 0\) and \(V(1) - h_1 + h_0 > 0\), then by B.12, we have \(r(h_0) = 0\) and
\[
V(1) + 1 - h_1 = \phi(1) > 1 - h_0.
\]
Thus, by (B.6), the infimum value 0 of \(\frac{\phi(u) - (1 - h_0)}{u - 1}\) is not achieved for \(u = 1\), neither at \(u = +\infty\) since there \(\phi(u) = u V(u) + (1 - h_1)u \simeq (1 - h_1)u\) with \(1 - h_1 > 0\). Thus, \(\phi(1) > \phi_l = 1 - h_0\) by the previous equivalence. For the reverse implication in (B.13), the same argument applies and this proves (B.13).

For (B.14), we note that in this case, we have
\[
1 - h_0 = V(1) + 1 - h_1 = \phi(1) \geq \phi_l \geq 1 - h_0
\]
and the conclusion follows. The implication (B.15) is a straightforward consequence of (B.13) and (B.14). The last implications (B.16) are left to the reader. \(\square\)

By (B.1), we see that \(h(x) \equiv 0\) for \(x \leq x_f - \epsilon_1\) and \(h(x) \equiv 1\) for \(x \geq x_f + \epsilon_1\).
This, coupled with Assumption B.4, explains the following lemma.

**Lemma B.12.** On each of the following closed intervals, the smooth function \(F(x, u)\) takes each of the following expressions:
1. For \(x \leq -\epsilon_2\)
\[
(F.18) \quad F(x, u) = -h_1 u + h_0
\]
2. For \(x \in [-\epsilon_2, x_f - \epsilon_1]\),
\[
(F.19) \quad \begin{cases} F(x, u) = -h_1 u + \alpha(x) u V(u) + h_0 \\
\frac{\partial F}{\partial x} = \alpha'(x) u V(u) \\
\frac{\partial F}{\partial u} = -h_1 + \alpha(x)(V(u) + u V'(u)) \end{cases}
\]
3. For \(x \in [x_f - \epsilon_1, x_f + \epsilon_1]\),
\[
(F.20) \quad \begin{cases} F(x, u) = (h(x) - h_1) u + u V(u) - h(x) + h_0 \\
\frac{\partial F}{\partial x} = h'(x) (u - 1) \\
\frac{\partial F}{\partial u} = h(x) - h_1 + V(u) + u V'(u) \end{cases}
\]
4. for \( x \in [x_i + \epsilon_1, x_b + \epsilon_2] \),
\[
F(x, u) = (1 - h_1)u + \alpha(x)uV(u) - 1 + h_0
\]
(B.21)
\[
\frac{\partial F}{\partial x} = \alpha'(x)uV(u)
\]
\[
\frac{\partial F}{\partial u} = 1 - h_1 + \alpha(x)(V(u) + uV'(u))
\]

5. for \( x \geq x_b + \epsilon_2 \),
\[
F(x, u) = (1 - h_1)u - 1 + h_0
\]
(B.22)

The following definition and technical lemma will be useful in the sequel.

**Definition B.13.** We note by \( Z \) the zeros of \( F(x, u) \) in \( \mathbb{R} \times [0, +\infty) \) and \((\pi, \overline{\pi}) \in Z\) is said to be

(i) **stable** if \( F(\pi, v)(v - \overline{\pi}) \leq 0 \) for all \( v \) in some neighborhood of \( \overline{\pi} \) in \([0, +\infty)\),
(ii) **unstable** if \( F(\pi, v)(v - \overline{\pi}) \geq 0 \) for all \( v \) in some neighborhood of \( \overline{\pi} \) in \([0, +\infty)\),
(iii) a jump point if \( v \mapsto F(\pi, v)(v - \overline{\pi}) \) changes sign in every neighborhood of \( \overline{\pi} \) in \([0, +\infty)\).

**Lemma B.14.** Let \((\pi, \overline{\pi}) \in Z\).

1. If \((\pi, \overline{\pi})\) is stable, then \( \frac{\partial F}{\partial u}(\pi, \overline{\pi}) \leq 0 \) and if \( \frac{\partial F}{\partial u}(\pi, \overline{\pi}) < 0 \), then \((\pi, \overline{\pi})\) is stable,
2. If \((\pi, \overline{\pi})\) is unstable, then \( \frac{\partial F}{\partial u}(\pi, \overline{\pi}) \geq 0 \) and if \( \frac{\partial F}{\partial u}(\pi, \overline{\pi}) > 0 \), then \((\pi, \overline{\pi})\) is unstable,
3. If \((\pi, \overline{\pi})\) is a jump point, then \( \frac{\partial F}{\partial u}(\pi, \overline{\pi}) = 0 \).

If \((h_0, h_1) \neq (0, 0)\) and \((h_0, h_1) \neq (1, 1)\), then the set
\[
S = \{(\pi, \overline{\pi}) \in \mathbb{R} \times [0, +\infty), F(\pi, \overline{\pi}) = \frac{\partial F}{\partial u}(\pi, \overline{\pi}) = 0\}
\]
and the set of jump points consist of isolated points of \( \mathbb{R} \times [0, +\infty) \).

*Proof.* The first part of the lemma being clear, we only show that \( S \) consists of isolated points of \( \mathbb{R} \times [0, +\infty) \).

- \( \overline{\pi} \leq -\epsilon_2 \). By (B.18), we have \( S \cap (-\infty, -\epsilon_2] \times [0, +\infty) = \emptyset \) if and only if \((h_0, h_1) \neq (0, 0)\).
- \( \overline{\pi} \in (-\epsilon_2, x_f - \epsilon_1] \). By (B.19), we have
\[
(\pi, \overline{\pi}) \in S \quad \iff \quad -h_1\overline{\pi} + \alpha(\pi)\overline{\pi}V(\overline{\pi}) + h_0 = 0, -h_1 + \alpha(\pi)(V(\overline{\pi}) + \overline{\pi}V'(\overline{\pi})) = 0
\]
\[
\iff \left\{ \begin{array}{l}
\pi > 0, \alpha(\pi) = \frac{h_1(\pi - h_0)}{\overline{\pi}V(\overline{\pi})}, h_1\pi^2V'(\pi) - h_0(V(\pi) + \pi V'(\pi)) = 0 \\
or \\
\pi = 0, h_0 = 0, \alpha(\pi) = \frac{h_1}{V(0)}
\end{array} \right.
\]
The function \( \rho(u) = h_1\pi^2V'(\pi) - h_0(V(\pi) + \pi V'(\pi)) \) being analytic and non zero \((\rho(0) \neq 0 \text{ if } h_0 = 0 \text{ and } \rho(1) \neq 0 \text{ if } h_0 = 0)\), the set \( U \) of its zeros consists of isolated points of \([0, +\infty)\). With \( X = \{x \in (-\epsilon_2, x_f - \epsilon_1), \exists u \in U, \alpha(x) = \frac{(h_1u - h_0)}{uV(\overline{\pi})}\}, \) we see that \( S \cap (-\epsilon_2, x_f - \epsilon_1] \times [0, +\infty) \subset X \times U \cup \{\alpha^{-1}(\frac{h_1}{V(0)})\} \) (if this latter point exists). The set \( X \cup \{\alpha^{-1}(\frac{h_1}{V(0)})\} \) consists of isolated points since \( \alpha \) is strictly decreasing on \((-\epsilon_2, x_f - \epsilon_1] \).
\( \bar{x} \in [x_f - \epsilon_1, x_f + \epsilon_1] \). By (B.20), we have

\[
(\bar{x}, \bar{u}) \in S \iff \begin{cases} 
\bar{x} = 1, h_1 - V(1) - h_0 = 0, h(\bar{x}) = h_1 - V(1) - V'(1) \\
\bar{x} \neq 1, h(\bar{x}) = h_1 - \bar{u}V(\bar{x}) - V'(\bar{x}) = \frac{h_1 - \bar{u}V(\bar{x}) - h_0}{\bar{x} - 1} 
\end{cases}
\]

The function \( \rho(u) = (u - 1)(h_1 - uV(u) - V'(u)) - h_1 + uV(u) + h_0 \) being analytic and non zero (\( \rho(0) = V(0) + h_0 - h_1 \) and \( \rho(1) = V(1) + h_0 - h_1 \) cannot both be zero), the set \( U \) of its zeros consists of isolated points of \([0, +\infty)\).

With \( X = \{ x \in [x_f - \epsilon_1, x_f + \epsilon_1], \exists \bar{x} \in U, h(x) = h_1 - uV(u) - V'(u) \} \), we see that \( S \cap [x_f - \epsilon_1, x_f + \epsilon_1] \times [0, +\infty) \subset X \times U \cup \{(h^{-1}(1 - V(1) - V'(1)), 1)\} \) (if this latter point exists). The set \( X \cup \{(h^{-1}(1 - V(1) - V'(1)), 1)\} \) consists of isolated points since \( h \) is strictly decreasing on \([x_f - \epsilon_1, x_f + \epsilon_1]\).

\( \bar{x} \in (x_f + \epsilon_1, x_b + \epsilon_2) \). By (B.21), we have

\[
(\bar{x}, \bar{u}) \in S \iff \begin{cases} 
\bar{x} > 0, \alpha(\bar{x}) = \frac{(h_0 - 1)}{\bar{u}V'(\bar{x})} \\
((1 - h_1)\bar{x} - 1 + h_0)\bar{u}V'(\bar{x}) - (1 - h_0)V(\bar{x}) = 0 
\end{cases}
\]

where \( \bar{x} > 0 \) because \( \frac{\partial F}{\partial \bar{x}}(\bar{x}, 0) = 1 - h_1 + \alpha(\bar{x})V(0) \geq 1 - h_1 > 0 \). The function \( \rho(u) = \{(1 - h_1)u - 1 + h_0\}uV'(u) - (1 - h_0)V(u) \) being analytic and non zero (\( \rho(0) \neq 0 \) if \( h_0 \neq 1 \) and \( \rho(1) \neq 0 \) if \( h_0 = 1 \)), the set \( U \) of its zeros consists of isolated points of \([0, +\infty)\).

With \( X = \{ x \in [x_f + \epsilon_1, x_b + \epsilon_2], \exists \bar{x} \in U, \alpha(x) = \frac{(h_0 - 1)}{\bar{u}V'(\bar{x})} \} \), we see that \( S \cap (x_f + \epsilon_1, x_b + \epsilon_2) \times [0, +\infty) \subset X \times U \). The set \( X \) consists of isolated points since \( \alpha \) is strictly decreasing on \([x_f + \epsilon_1, x_b + \epsilon_2]\), so that, being bounded, it is finite and we write \( X = \{ \bar{x}_1, \ldots, \bar{x}_n \} \) where \( \bar{x}_1 < \cdots < \bar{x}_n \).

By (B.22), we have \( S \cap [x_b + \epsilon_2, +\infty) \times [0, +\infty) = \emptyset \) if and only if \( \frac{(h_0, h_1)}{(1, 1)} \neq (1, 1) \).

We now study \( Z \) in two regions.

Above the feeding point \( x_f \)

**Lemma B.15.** In the region \( x \leq x_f - \epsilon_1 \), the set \( Z \) of zeros of \( F(x, u) \) is given as follows.

1. If \( h_0 \in (0, 1] \) and \( h_1 \in (0, 1] \), \( Z \cap (-\infty, x_f - \epsilon_1] \times [0, +\infty) \) consists of one branch \( x \mapsto (x, \bar{u}(x)) \) for \( x \leq x_f - \epsilon_1 \). Moreover, \( \bar{u}(x) > 0 \) and every point of this curve is stable.

2. If \( h_0 \in (0, 1] \) and \( h_1 = 0 \), \( Z \cap (-\infty, x_f - \epsilon_1] \times [0, +\infty) = \emptyset \).

3. If \( h_0 = 0 \) and \( h_1 = V(\bar{x}) \) for some \( \bar{x} > 0 \), \( Z \cap (-\infty, x_f - \epsilon_1] \times [0, +\infty) \) consists of two distinct branches for \( \epsilon_2 < x \leq x_f - \epsilon_1 : x \mapsto (x, 0) \) and \( x \mapsto (x, \bar{u}(x)) \). Moreover, every point of this latter curve is stable.

4. If \( h_0 = 0 \) and \( h_1 = 0 \), \( Z \cap (-\infty, x_f - \epsilon_1] \times [0, +\infty) = (-\infty, -\epsilon_2] \times [0, +\infty) \cup (-\epsilon_2, x_f - \epsilon_1] \times \{0\} \). Moreover, every point of the branch \( x \mapsto (x, 0) \) for \( \epsilon_2 < x \leq x_f - \epsilon_1 \) is unstable.
Proof. We start with a preliminary estimate. We claim that

\begin{equation}
(B.24) \text{or} \quad \begin{cases} 
  h_0 > 0, F(x, \overline{u}) = 0, \overline{u} > 0, \overline{x} \leq x_f - \epsilon_1 \\
  h_0 = 0, F(x, \overline{u}) = 0, \overline{u} > 0, -\epsilon_2 < \overline{x} \leq x_f - \epsilon_1 
\end{cases} \implies \frac{\partial F}{\partial u}(x, \overline{u}) < 0
\end{equation}

Indeed, let \((x, \overline{u})\) be such that \(\overline{x} \leq x_f - \epsilon_1\) and \(F(x, \overline{u}) = 0\). By (B.19), we have

\[\alpha(x)\overline{u}V(\overline{u}) = h_1 \overline{u} - h_0\]

Thus, still by (B.19) and since in any case \(h_0 > 0\) or \(\alpha(x)\overline{u}V'(\overline{u}) < 0\), we have the estimate

\[\frac{\partial F}{\partial u}(x, \overline{u}) = -h_1 + \alpha(x)V(\overline{u}) + \alpha(x)\overline{u}V'(\overline{u}) = -\frac{h_0}{\overline{u}} + \alpha(x)\overline{u}V'(\overline{u}) < 0\]

Now, we only prove the first point, the others being left to the reader. By (B.19), for all \(x \leq x_f - \epsilon_1\) we have \(F(x, 0) = h_0 > 0\) and \(\lim_{u \to +\infty} F(x, u) = -\infty\) since \(\lim_{u \to +\infty} uV(u) = 0\) by Assumption B.3. Thus, the function \(u \mapsto F(x, u)\) crosses zero at least once and also at most once by the estimate \(\partial F/\partial u < 0\) which holds at any zero of \(F(x, u)\). This defines a unique \(\tilde{u}(x)\) such that \(F(x, \tilde{u}(x)) = 0\). What is more, the estimate \(\partial F/\partial u(x, \tilde{u}(x)) < 0\) ensures stability of the zero \((x, \tilde{u}(x))\) and also continuity of \(x \mapsto \tilde{u}(x)\) by the implicit function theorem.

Below the feeding point \(x_f\)

The following lemma is an easy consequence of (B.21) and (B.22).

Lemma B.16. In the region \(x \geq x_f + \epsilon_1\), the set \(Z\) of zeros of \(F(x, u)\) is given as follows.

1. If \(h_0 \in [0, 1]\) and \(h_1 \in [0, 1]\), \(Z \cap [x_f + \epsilon_1, +\infty) \times [0, +\infty)\) consists of one branch \(x \mapsto (x, u_b)\), where \(u_b = (1 - h_0)/(1 - h_1)\), for \(x \geq x_b + \epsilon_2\). Moreover, every point of this curve is unstable.
2. If \(h_0 \in [0, 1]\) and \(h_1 = 1\), \(Z \cap [x_f + \epsilon_1, +\infty) \times [0, +\infty)\) is empty for \(x \geq x_b + \epsilon_2\).
3. If \(h_0 = 1\) and \(h_1 = 1\), \(Z \cap [x_f + \epsilon_1, +\infty) \times [0, +\infty)\) consists of one branch \(x \mapsto (x, 0)\) for \(x_f + \epsilon_1 < x < x_b + \epsilon_2\). Moreover, every point of this curve is unstable.

B.4. Critical points and parameterization. This section is devoted to the local analysis of the set of zeros of \(F(x, u)\).

Proposition B.17. Assume that \(h_0 \in [0, 1]\) and that (B.7) is satisfied. Then \(V(1) - h_1 + h_0 \geq 0\) and

1. If \(V(1) - h_1 + h_0 > 0\), then \(Z \cap (-\epsilon_2, +\infty) \times (0, +\infty)\) is locally parameterized by \(u\) or by \(x\),
2. If \(V(1) - h_1 + h_0 = 0\), then \(Z \cap (-\epsilon_2, +\infty) \times (0, +\infty)\) is locally parameterized by \(u\) or by \(x\), except at one point \((x_c, u_c) \in (x_f - \epsilon_1, x_f + \epsilon_1) \times \mathbb{R}^+\) given by \(x_c = h^{-1}(h_0 - V'(1))\), \(u_c = 1\).

Proof. By (B.12), we have \(V(1) - h_1 + h_0 \geq 0\). Moreover, by (B.7) and (B.20) we have

\begin{equation}
(B.25) \quad F(x_f + \epsilon_1, u) = (u - 1) \left(1 - h_1 + \frac{h_0 - h_1 + uV(u)}{u - 1}\right) > 0 \quad \forall\ u > 1
\end{equation}

Let \((\overline{x}, \overline{u})\) be such that \(\overline{u} > 0\) and \(F(\overline{x}, \overline{u}) = 0\). We want to know if the set of zeros of \(F(x, u) = 0\) can be locally parameterized by \(x\) or \(u\) near \((\overline{x}, \overline{u})\).
\( \bar{\varphi} \leq - \epsilon_2 \). By (B.18), \( \mathcal{Z} \) is locally parameterized by \( x \) if and only if \( h_1 \neq 0 \). This happens to be necessarily the case when \( V(1) - h_1 + h_0 = 0 \) because then \( h_1 = V(1) + h_0 \geq V(1) > 0 \).

\( \bar{\varphi} \in (-\epsilon_2, x_f - \epsilon_1) \). By (B.24), we have \( \frac{\partial F}{\partial u}(\bar{\varphi}, \bar{\pi}) < 0 \) since \( \bar{\pi} > 0 \). Thus, \( \mathcal{Z} \) can be locally parameterized by \( x \) near \( (\bar{\varphi}, \bar{\pi}) \) as a consequence of the implicit function theorem.

\( \bar{\varphi} \in (x_f - \epsilon_1, x_f + \epsilon_1) \). Here, we focus on the critical points of \( F \), namely the solutions

\( (\bar{\varphi}, \bar{\pi}) \in (x_f - \epsilon_1, x_f + \epsilon_1) \times (0, +\infty) \) of \( F(\bar{\varphi}, \bar{\pi}) = 0, \frac{\partial F}{\partial u}(\bar{\varphi}, \bar{\pi}) = 0 \) and \( \frac{\partial F}{\partial \bar{\pi}}(\bar{\varphi}, \bar{\pi}) = 0 \). Thanks to (B.20) and since \( h'(\bar{\varphi}) \neq 0 \), \( (\bar{\varphi}, \bar{\pi}) \) is a critical point of \( F \) if and only if

\[
(\text{B.26}) \quad \bar{\pi} = 1, -h_1 + V(1) + h_0 = 0 \quad \text{and} \quad h(\bar{\varphi}) = h_1 - V(1) - V'(1)
\]

Thus, \( V(1) - h_1 + h_0 = 0 \), there are no critical points of \( F \) in \( (x_f - \epsilon_1, x_f + \epsilon_1) \times (0, +\infty) \) and we conclude that, in this region, \( \mathcal{Z} \) can be locally parameterized by \( x \) or by \( u \).

On the other hand, if \( V(1) - h_1 + h_0 = 0 \), then (B.7) gives

\[
1 - h_1 + \lim_{u \to 1} \left\{ \frac{uV(u) - V(1)}{u - 1} \right\} \geq 1 - h_1 + \inf_{u > 1} \left\{ \frac{uV(u) - V(1)}{u - 1} \right\} > 0
\]

and therefore,

\[
(\text{B.27}) k'(1) - h_1 + h_0 = 0 \quad \text{and} \quad (\text{B.7}) \quad \Rightarrow 1 - h_1 + V(1) + V'(1) > 0
\]

Thus, \( h_1 - V(1) - V'(1) < 1 \) but also \( h_1 - V(1) - V'(1) = h_0 - V'(1) > 0 \) since \( h_0 \geq 0 \) and \( V'(1) < 0 \). As a result, since \( h \) is one-to-one from \( (x_f - \epsilon_1, x_f + \epsilon_1) \) to \( (0, 1) \), there exists a unique \( u_c \in (x_f - \epsilon_1, x_f + \epsilon_1) \) such that \( h(u_c) = h_1 - V(1) - V'(1) \). By (B.26), there exists a unique critical point of \( F(x, u) = 0 \) on \( (x_f - \epsilon_1, x_f + \epsilon_1) \times (0, +\infty) \) given by \( u_c = 1 \) and \( x_c = h^{-1}(h_0 - V'(1)) \).

\( \bar{\varphi} = x_f + \epsilon_1 \). This case has to be treated separately from the previous one because \( h'(\bar{\varphi}) = 0 \) and thus \( \frac{\partial F}{\partial u}(\bar{\varphi}, \bar{\pi}) = 0 \) and \( \frac{\partial F}{\partial \bar{\pi}}(\bar{\varphi}, \bar{\pi}) \) is undetermined. Before all, we notice that \( \bar{\pi} \in (0, 1] \) by (B.25). The first case is that when \( \bar{\pi} = 1 \). Since \( h(\bar{\varphi}) = 1 \) and thanks to (B.20) and (B.27), we get \( \frac{\partial F}{\partial u}(\bar{\varphi}, \bar{\pi}) > 0 \) and thus \( \mathcal{Z} \) can be parameterized by \( x \) near \( (x_f + \epsilon_1, 1) \).

The second case is that when \( \bar{\pi} \in (0, 1) \). Here, even if \( \frac{\partial F}{\partial \bar{\pi}}(\bar{\varphi}, \bar{\pi}) = 0 \), we shall prove that \( u \) is a good parameter for \( \mathcal{Z} \) near \( (\bar{\varphi}, \bar{\pi}) \). By (B.2) for the expressions of \( F(x, u) \) and \( F(\bar{\varphi}, \bar{\pi}) \) (with \( \alpha(\bar{\varphi}) = 1, h(\bar{\varphi}) = 1 \)), we can write

\[
F(x, u) - F(\bar{\varphi}, \bar{\pi}) = (h(x) - 1)(u - 1) + (\alpha(x) - 1)uV(u) + (uV(u) - \bar{\pi}V(\bar{\pi}) + (1 - h_1)(u - \bar{\pi}))
\]

Since \( F(\bar{\varphi}, \bar{\pi}) = 0 \) and by (B.20) and (B.21), this gives

\[
(\text{B.28}) \quad \begin{cases} 
F(x, u) = (h(x) - 1)(u - 1) - \varphi(u) & \forall x \in [x_f - \epsilon_1, x_f + \epsilon_1] \\
F(x, u) = (\alpha(x) - 1)uV(u) - \varphi(u) & \forall x \in [x_f + \epsilon_1, x_f + \epsilon_2]
\end{cases}
\]

where \( \varphi(u) = uV(u) - \bar{\pi}V(\bar{\pi}) + (1 - h_1)(u - \bar{\pi}) \). Since \( V(\cdot) \) is analytical (and non constant because \( \varphi'(0) = V(0) + 1 - h_1 > V(0) > 0 \), the zeros
of $\varphi$ are isolated and there exists a neighborhood $U$ of $\varpi$ in $(0,1)$ where $\varpi$ is the only zero of $\varphi$. If $U$ is small enough, $F(x,u) = 0$ can be solved continuously in $x$ on both $(x_f - \epsilon_1,x_f + \epsilon_1) \times \{u \in U \mid \varphi(u) > 0\}$ and $(x_f + \epsilon_1,x_b + \epsilon_2) \times \{u \in U \mid \varphi(u) < 0\}$. Indeed, on the one hand, for $(x,u) \in (x_f - \epsilon_1,x_f + \epsilon_1) \times (0,1)$, we have by \eqref{B.28}

$$F(x,u) = 0 \iff \varphi(u) > 0 \quad \text{and} \quad h(x) = 1 + \frac{\varphi(u)}{u-1}$$

where we used the fact that $h(x) < 1$ for $x \in (x_f - \epsilon_1,x_f + \epsilon_1)$. The assertion follows since $h(x)$ is strictly increasing on $(x_f - \epsilon_1,x_f + \epsilon_1)$. On the other hand, for $(x,u) \in (x_f + \epsilon_1,x_b + \epsilon_2) \times (0,\infty)$, we have by \eqref{B.28}

$$F(x,u) = 0 \iff \varphi(u) < 0 \quad \text{and} \quad \alpha(x) = 1 + \frac{\varphi(u)}{uV(u)}$$

where we used the fact that $\alpha(x) < 1$ for $x \in (x_f + \epsilon_1,x_b + \epsilon_2)$. The assertion follows since $\alpha(x)$ is strictly decreasing on $(x_f + \epsilon_1,x_b + \epsilon_2)$. Therefore, $u$ is a good parameter for $Z$ near $(x_f,\varpi)$.

Proposition B.18. Let $Z_0$ be the set of zeros $(x,\varpi)$ of $F(x,u)$ with $\varpi = 0$.

1. If $h_0 \in (0,1)$, then $Z_0$ is reduced to one point $(x_f,0)$, where $x_f \in (x_f - \epsilon_1,x_f + \epsilon_1)$ is given by $h(\varpi) = 0$. What is more, $Z$ is parameterized by $u$ near this point.

2. If $h_0 = 1$, $Z_0 = (-\infty,-\epsilon_2] \times \{0\} \cup [x_f + \epsilon_1,\infty) \times \{0\}$.

2-1. If $h_1 \in (0,1)$, $Z$ is parameterized by $x$ near any point of $(x_f + \epsilon_1,\infty) \times \{0\}$.

2-2. If $h_1 \in [0,1)$, $Z$ coincides with $[x_f + \epsilon_1,\infty) \times \{0\}$ in a neighborhood of $(x_f + \epsilon_1,0)$. Therefore, $Z$ is not parameterized by $x$ near $(x_f + \epsilon_1,0)$ and there are no elements of $Z$ near this latter point with x-coordinate less than $x_f + \epsilon_1$.

Proof. We just sketch the proofs.

1. By \eqref{B.20}, we find that $\frac{\partial F}{\partial u}(x,0) = -h'(\varpi) < 0$ and $Z$ is parameterized by $u$.

2. By \eqref{B.2} and if $h_1 \in [0,1)$, we have

$$\frac{\partial F}{\partial u}(x,0) = h(x) - h_1 + \alpha(x)V(0) > 0 \quad \text{and} \quad F(x,0) = 1 - h(x) \geq 0$$

for $x \in (h^{-1}(h_1),\infty) \supset [x_f + \epsilon_1,\infty)$. This proves that $Z$ and $[x_f + \epsilon_1,\infty) \times \{0\}$ coincide in a neighborhood of any point of this latter set. In particular, $Z$ is parameterized by $x$ near any point of $(x_f + \epsilon_1,\infty) \times \{0\}$. \qed
B.5. Discontinuities of entropic solutions. In this section, we focus upon the possible discontinuities of an entropic solution of (B.2).

The following proposition on the conservation of stability along an entropic solution is made possible both by the entropy conditions of Definition B.2 and by the specific form of the function \( F(x, u) \) in (B.2).

**Proposition B.19.** Let \( x \mapsto u(x) \) be an entropic solution of (B.2) and let \( \pi \) be given. Assume that \( x \mapsto u(x) \) has a discontinuity at \( x = \pi \) and that \( Z \) can locally be parameterized by \( x \) or \( u \) near the points \((\pi, u(\pi^-))\) and \((\pi, u(\pi^+))\).

If \((x, u(x))\) is stable for all \( x < \pi \) in some neighborhood of \( \pi \) then \((x, u(x))\) is also stable for all \( x > \pi \) in some neighborhood of \( \pi \). What is more, the point \((\pi, u(\pi^-))\) is a jump point and \((\pi, u(\pi^+))\) is stable.

**Lemma B.20.** Let \( x \mapsto u(x) \) be an entropic solution of (B.2) having a discontinuity at \( x = \pi \). Assume that \((x, u(x))\) is stable for all \( x < \pi \) in some neighborhood of \( \pi \).

1. If, near the point \((\pi, u(\pi^+))\), either \( Z \) can be parameterized by \( x \) or \( Z \) is given by some continuous function \( u \mapsto s^+(u) \) such that \( s^+(u) - \pi \) does not keep a constant sign in any neighborhood of the point \( u(\pi^+) \), then \((x, u(x))\) is also stable for \( x > \pi \) in some neighborhood of \( \pi \).

2. If, near the point \((\pi, u(\pi^-))\), \( Z \) is given by some continuous function \( v \mapsto s^+(v) \) such that \( s^+(v) - \pi \) keeps a constant sign for \( v \neq u(\pi^-) \) in certain neighborhood of the point \( u(\pi^-) \), then this sign can only be positive.

**Proof.** We decompose the proof into several steps.

**Step 1.** We start with a technical result around which the proof is organized.

Assume that \( x \mapsto u(x) \) is an entropic solution of (B.2) having a discontinuity at \( x = \pi \) and that \( Z \) can locally be parameterized by \( x \) or \( u \) near the points \((\pi, u(\pi^-))\) and \((\pi, u(\pi^+))\). We claim that if there exist \( x_s > \pi \) and \((v_1, v_s)\) neighborhood of \( u(\pi^+) \) such that

\[
(W_{29}) \quad \begin{cases} \begin{align*}
W_+ &= \{(x, v) \in (\pi, x_s) \times (v_1, v_s) \mid v > u(x)\} \\
W_- &= \{(x, v) \in (\pi, x_s) \times (v_1, v_s) \mid v < u(x)\}
\end{align*} \end{cases}
\]

are non empty connected sets having no zeros of \( F(x, u) \), then \((x, u(x))\) is stable for all \( x \in (\pi, x_s) \) (as well as \((\pi, u(\pi^+))\)).

For the proof of the claim, we distinguish two cases. If \( u(\pi^+) > u(\pi^-) \), there exists \( v_1^i \) such that \( \text{sup}(v_1, u(\pi^-)) < v_1^i < u(\pi^+) \) and that \( F(\pi, v) \neq 0 \) for \( v \in (v_1^i, u(\pi^+)) \) since the zeros of \( v \mapsto F(\pi, v) \) are isolated by analyticity. Now, on the one hand, the closure \( W^- \) of the set \( W_- \) contains the segment \( (\pi) \times (v_1^i, u(\pi^+)) \) where we know by Definition B.2 that \( F(x, u) \) is non negative and therefore positive by our choice of \( v_1^i \). On the other hand, the function \( F(x, u) \) has no zeros on the connected set \( W_- \) and therefore keeps a constant sign. Combining both results, we conclude that \( F(x, u) > 0 \) for \((x, u) \in W_- \). Then, we necessarily have \( F(x, u) < 0 \) for \((x, u) \in W_+ \), because otherwise the whole curve \( \{(x, u(x)) \mid x \in (\pi, x_s)\} \) would consist of minima of \( F(x, u) \) and therefore of critical points of \( F(x, u) \): this cannot be since these latter are isolated points when \( Z \) can locally be parameterized by \( x \) or \( u \) near the points \((\pi, u(\pi^-))\) and \((\pi, u(\pi^+))\) (see Lemma B.14). To end up, let us notice that for \( x \in (\pi, x_s) \), the zero \((x, u(x))\) is stable since for \( v_s > u(x) \), we have \((x, v) \in W_+ \) and thus \( F(x, u) < 0 \) and for \( v_s < u(x) \), we have \((x, v) \in W_- \) and thus \( F(x, u) > 0 \). Concerning \((\pi, u(\pi^+))\), its stability is ensured by the sign of \( F(x, u) \) on \( W^- \) and \( W_+ \).

If \( u(\pi^+) < u(\pi^-) \), we prove in the same way that \( F(x, u) > 0 \) for \((x, u) \in W_- \) and
F(x, u) < 0 for (x, u) ∈ W_+, thus also concluding to stability for (x, u(x)), x ∈ (x, x_*).}}}\left) {(as well as for (x, u(x))).}

\textbf{Step 2.} Here, we claim that if Z can be parameterized by x near the point \( (x, u(x)) \), then the entropic solution is stable for \( x > \tau \) in some neighborhood of \( \tau \).

For the proof of the claim, we exhibit \( x_* > \tau \) and \( (v_1, v_2) \) neighborhood of \( u(\tau^+) \) as in Step 1. Since Z can be parameterized by x near the point \( (x, u(x)) \), then this parameterization coincides with \( x \mapsto u(x) \) on a rectangle \([\tau, x_*] \times (v_1, v_2)\). Picking up any non empty interval \([\tau, x_*] \subset [\tau, x_*] \), we have \( u([\tau, x_*]) = [v_1, v_2] \subset (v_1, v_2)\).

With \( x_* = x_* \) and \( (v_1, v_2) = (v_1, v_2) \), it is easily seen by this last inclusion that \( W_+ \) and \( W_\tau \) defined in \((B.29)\) are arcwise connected. What is more, they contain no zeros of \( F(x, u) \) because, by the parameterization, there is only one zero with x-coordinate \( x \), namely \((x, u(x))\) by our assumption on the parameterization. Thanks to the result of Step 1, this answers our claim.

\textbf{Step 3.} Here, we claim that if Z near the point \( (x, u(x)) \) is given by some continuous function \( u \mapsto s^+(u) \) such that \( s^+(u) - \tau \) does not keep a constant sign in any neighborhood of the point \( u(\tau^+) \), then the entropic solution is stable for \( x > \tau \) in some neighborhood of \( \tau \).

For the proof of the claim, we exhibit \( x_* > \tau \) and \( (v_1, v_2) \) neighborhood of \( u(\tau^+) \) as in Step 1. For this, we first prove that there exists a right neighborhood of \( \tau \) upon which \( x \mapsto u(x) \) is injective, and therefore one-to-one on its image set, with inverse \( v \mapsto s^+(v) \). Then, we relate the monotonicity of \( x \mapsto u(x) \) to the sign of \( u \mapsto s^+(u) - \tau \).

Let \((x_1^1, x_2^1) \times (v_1^1, v_2^1)\) be an open rectangle containing \((x, u(x)) \) upon which Z is given by some continuous function \( u \mapsto s^+(u) \). Since \( x \mapsto u(x) \) is continuous for \( x \geq \tau \) in some neighborhood of \( \tau \), there exists \( x_* > \tau \) such that \([x, x_*] \subset (x_1^1, x_2^1)\) and \( u([x, x_*]) = [v_1^1, v_2^1] \subset (v_1^1, v_2^1)\). If \( v \in [v_1^1, v_2^1] \), there is only one \( x \in [x, x_*] \) such that \( u(x) = v \) because, by the parameterization, there is only one zero of \( F(x, u) \) with u-coordinate \( v \) in the rectangle \((x_1^1, x_2^1) \times (v_1^1, v_2^1)\), namely \((s^+(v), v)\). This proves that \( x \mapsto u(x) \) is one-to-one from \([x, x_*] \times [v_1^1, v_2^1]\) with inverse \( v \mapsto s^+(v) \).

Let us study the monotonicity of \( x \mapsto u(x) \). First, we can always restrict \((v_1^1, v_2^1)\) so that \( u(\tau^+) \) is the only \( v \in (v_1^1, v_2^1) \) such that \( s^+(v) - \tau = 0 \) since \( u(\tau^+) \) is an isolated zero of \( v \mapsto F(\tau, v) \) by analyticity of this latter function. For \( v \in (u(\tau^+), v_1^1) \), \( s^+(v) - \tau \) cannot be zero and hence keeps a constant sign. By our assumption on the necessary previous reasoning of the sign of \( v \mapsto F(\tau, v) \), this latter keeps the opposite constant sign for \( v \in (v_1^1, u(\tau^+)) \). If \( s^+(v) - \tau > 0 \) (resp. < 0) for \( v \in (u(\tau^+), v_1^1) \), then the bijection \( v \mapsto s^+(v) \) is necessary strictly increasing (resp. decreasing) on \([u(\tau^+), v_1^1]\) and \( x \mapsto u(x) \) is strictly increasing (resp. decreasing) on \([\tau, x_*]\).

For the choice of \( x_* > \tau \) and \((v_1, v_2) \) neighborhood of \( u(\tau^+) \) as in Step 1, we distinguish two cases. Suppose that \( s^+(v) - \tau > 0 \) for \( v \in (u(\tau^+), v_1^1) \). Choosing \( x_* = x_*^2 \) and \((v_1, v_2) = (v_1^1, u(x_*^2))\), it is easily seen that \( W_+ \) defined in \((B.29)\) are arcwise connected because \( x \mapsto u(x) \) is increasing on \([x, x_*^2]\). What is more, these sets contain no zeros of \( F(x, u) \). Indeed, assume that there exists \((x, v) \in W_+ \) such that \( x < x_* \). On the one hand, since \((x, v) \in W_+ \), it would be such that \( u(x) < v < u(x_*^2) \) and thus \( s^+(u(x)) = x < s^+(v) < x_*^2 \). On the other hand, since \((x, v) \in Z \) and \((x, v) \in W_+ \subset (x_1^1, x_2^1) \times (v_1^1, v_2^1)\), we know by our assumption on the parameterization, that necessarily \( x = x_* \). This contradiction shows that \( W_+ \) contains no zeros of \( F(x, u) \). Now, assume that there exists \((x, v) \in W_+ \cap Z \). As previously, we necessarily have \( x = x_* \). If \( u(\tau^+) \leq v \), the previous reasoning applies and also leads to a contradiction. If \( u(\tau^+) \geq v \), then
since \( v > v_1 \), we know that \( s^+(v) - \overline{\pi} < 0 \) since this last expression must have an opposite sign as the one on \( v_1, u(\overline{\pi}) \). This gives \( x = s^+(v) < \overline{\pi} \) and also leads to a contradiction since \( (x, v) \in W_- \subset (\overline{\pi}, x_2^1) \times (v_1^1, u(x_2^1)) \).

When \( s^+(v) - \overline{\pi} < 0 \) for \( v \in (u(\overline{\pi}^+), v_1^1) \), we choose \( x_3 = x_5^1 \) and \( (v_3, v_5) = (u(x_5^1), v_1^1) \) and prove in the same way that \( W_+ \) and \( W_- \) are non-empty connected sets having no zeros of \( F(x, u) \).

This completes the proof of our claim.

**Step 4.** Assume that, near the point \( (\pi, u(\overline{\pi}^+)) \), \( Z \) is given by some continuous function \( v \mapsto s^+(v) \) such that \( s^+(v) - \overline{\pi} \) keeps a constant sign for \( v \neq u(\overline{\pi}^+) \) in certain neighborhood of the point \( u(\overline{\pi}^+) \). In this case, this sign can only be positive because otherwise there would be no zeros with \( x \)-coordinate greater than \( \overline{\pi} \) and this would contradict the existence of an entropic solution of (B.2) for \( x > \overline{\pi} \).

**Lemma B.21.** Let \( x \mapsto u(x) \) be an entropic solution of (B.2) having a discontinuity at \( x = \overline{\pi} \). Assume that \( (x, u(x)) \) is stable for all \( x < \overline{\pi} \) in some neighborhood of \( \overline{\pi} \).

Then if, near the point \( (\overline{\pi}, u(\overline{\pi}^-)) \), \( Z \) can be parameterized by \( x \) or by \( u \), \( Z \) is necessarily given by some continuous function \( v \mapsto s^-(v) \) such that \( s^-(v) - \overline{\pi} < 0 \) for \( v \neq u(\overline{\pi}^-) \) in certain neighborhood of the point \( u(\overline{\pi}^-) \).

**Proof.** The following result for \( x < \overline{\pi} \) is proved in the same way as the one in Step 1 of the previous lemma.

Assume that \( x \mapsto u(x) \) is an entropic solution of (B.2) having a discontinuity at \( x = \overline{\pi} \). We claim that if there exist \( x_i < \overline{\pi} \) and \( (v_i, v_s) \) neighborhood of \( u(\overline{\pi}^-) \) such that

\[
\begin{align*}
W_+^i &= \{ (x, v) \in (x_i, \overline{\pi}) \times (v_i, v_s) \mid v > u(x) \} \\
W_-^i &= \{ (x, v) \in (x_i, \overline{\pi}) \times (v_i, v_s) \mid v < u(x) \}
\end{align*}
\]

are non-empty connected sets having no zeros of \( F(x, u) \), then \( (x, u(x)) \) is unstable for all \( x \in (x_i, \overline{\pi}) \).

Then, it can be proven as in Steps 2, 3 and 4 of the previous lemma (and using the previous result instead of that of Step 1) that, near the point \( (\overline{\pi}, u(\overline{\pi}^-)) \), \( Z \) is necessarily given by some continuous function \( v \mapsto s^-(v) \) such that \( s^-(v) - \overline{\pi} < 0 \) for \( v \neq u(\overline{\pi}^-) \) in certain neighborhood of the point \( u(\overline{\pi}^-) \).

What is more, the point \( (\overline{\pi}, u(\overline{\pi}^-)) \) is a jump point.

**Remark B.22.** The same reasoning with \( x \) changed in \(-x\) leads to the following result about unstable points.

Let \( x \mapsto u(x) \) be an entropic solution of (B.2) having a discontinuity at \( x = \overline{\pi} \).

Assume that \( (x, u(x)) \) is unstable for all \( x > \overline{\pi} \) in some neighborhood of \( \overline{\pi} \).

Then if, near the point \( (\overline{\pi}, u(\overline{\pi}^-)) \), \( Z \) can be parameterized by \( x \) or by \( u \), \( Z \) is necessarily given by some continuous function \( v \mapsto s^+(v) \) such that \( s^+(v) - \overline{\pi} > 0 \) for \( v \neq u(\overline{\pi}^+) \) in certain neighborhood of the point \( u(\overline{\pi}^+) \).

To end up, the following lemma holds because of the special structure of \( F(x, u) \) in (B.2).

**Lemma B.23.** For any \( \overline{\pi} \in \mathbb{R} \), there is no \( \overline{\pi}^-, \overline{\pi}^+ \in [0, +\infty) \) satisfying all the following conditions:

1. \( (\overline{\pi}, \overline{\pi}^-) \in Z \) and \( (\overline{\pi}, \overline{\pi}^+) \in Z \) with \( \overline{\pi}^- \neq \overline{\pi}^+ \),
2. \( Z \) is given by some continuous function \( u \mapsto s^-(u) \leq \overline{\pi} \) near the point \( (\overline{\pi}, \overline{\pi}^-) \),
3. \( Z \) is given by some continuous function \( u \mapsto s^+(u) \geq \overline{\pi} \) near the point \( (\overline{\pi}, \overline{\pi}^+) \).
4. \( F(\overline{\pi}, \nu)(\overline{\pi}^+ - \overline{\pi}^-) \geq 0 \), for all \( \nu \) between \( \overline{\pi}^- \) and \( \overline{\pi}^+ \).

**Proof.** Assume that there exist \( \overline{\pi}, \overline{\pi}^-, \overline{\pi}^+ \in [0, +\infty) \) satisfying all the previous conditions.

First, we prove that there exists an open interval \((v_i^-, v_j^+)\) containing both \( \overline{\pi}^- \) and \( \overline{\pi}^+ \) such that

\[
F(\overline{\pi}, \nu)(\overline{\pi}^+ - \overline{\pi}^-) \geq 0 \quad \forall \nu \in (v_i^-, v_j^+)
\]

For this, we follow the construction done at Step 3 in Lemma B.20 and exhibit two open rectangles, \((x_i^-, x_j^+) \times (v_i^-, v_j^+)\) containing \((\overline{\pi}, \overline{\pi}^-)\) and \((x_i^+, x_j^+) \times (v_i^+, v_j^+)\) containing \((\overline{\pi}, \overline{\pi}^+)\), such that

\[
Q_i^+ = \{(x, \nu) \in (x_i^+, x_j^+) \times (v_i^+, v_j^+) \mid x > s^-(\nu)\} \quad \text{and} \quad Q_i^- = \{(x, \nu) \in (x_i^+, x_j^+) \times (v_i^+, v_j^+) \mid x < s^+(\nu)\}
\]

are non empty connected sets having no zeros of \( F(x, u) \). Therefore, \( F(x, u) \) keeps a constant sign (non negative or non positive) on each of their closures \( Q_i^+ \) and \( Q_i^- \). Having taken care that \((v_i^-, v_j^+)\) is small enough so that \( s^-(\nu) \leq \overline{\pi} \) for \( \nu \in (v_i^-, v_j^-)\), we deduce from this inequality that \( Q_i^- \) contains the segment \( (\overline{\pi}) \times (v_i^-, v_j^-) \). Thus, \( F(x, u) \) keeps a constant sign on this segment and the same holds for the segment \((\overline{\pi}) \times (v_i^+, v_j^+)\). This sign is the same because each of these segments intersects the segment joining \((\overline{\pi}, \overline{\pi}^-)\) to \((\overline{\pi}, \overline{\pi}^+)\), where \( F(x, u) \) keeps a constant sign by point 4. This ends our first claim.

Given this preliminary result, we can discuss according to the position of \( \overline{\pi} \) and exhibit a contradiction in each case.

\( \overline{\pi} \leq -\epsilon_2 \). By (B.18), we get \( F(\overline{\pi}, u) = -h_1 \nu + h_0 \) and the only case to study is when \( h_0 = h_1 = 0 \) because of point 1. But then points 2 and 3 would not be satisfied.

\( \overline{\pi} \in (-\epsilon_2, x_f - \epsilon_1) \). Using the result of Lemma B.15, we see that there is only one zero of \( u \mapsto F(\overline{\pi}, u) \) when \( h_0 \neq 0 \). If \( h_0 = 0 \), we know by (B.24) that \( \frac{\partial F}{\partial u}(\overline{\pi}, \nu) < 0 \) at \( \overline{\pi} = \overline{\pi}^- \) or at \( \overline{\pi} = \overline{\pi}^+ \) since one of them is different from 0. Thus \( Z \) is parameterized by \( x \) near the point \((\overline{\pi}, \overline{\pi})\) and this contradicts either point 2 or point 3.

\( \overline{\pi} \in (x_f - \epsilon_1, x_f + \epsilon_1) \). By (B.20), the function \( x \mapsto F(x, \nu)(\overline{\pi}^+ - \overline{\pi}^-) \) has derivative \( h'(x)(v - 1)(\overline{\pi}^+ - \overline{\pi}^-) \) where \( h'(x) > 0 \). Thus, if there exists a neighborhood \( U \) of \( u(\overline{\pi}^-) \) such that, for \( \nu \in U \), we have \((v - 1)(\overline{\pi}^+ - \overline{\pi}^-) < 0 \), there could be no zeros of \( F(x, u) \) with \( u \)-coordinate in \( U \) and \( x \)-coordinate smaller than \( \overline{\pi} \), because of the inequality (B.30). This would contradict point 2. Therefore, we have \((\overline{\pi}^- - 1)(\overline{\pi}^+ - \overline{\pi}^-) \geq 0 \). By the same reasoning in the neighborhood of \( \overline{\pi}^+ \), we get \((\overline{\pi}^- - 1)(\overline{\pi}^+ - \overline{\pi}^-) \leq 0 \). Subtracting these two last inequalities yields \((\overline{\pi}^+ - \overline{\pi}^-)^2 \leq 0 \), which contradicts point 1.

\( \overline{\pi} \in [x_f + \epsilon_1, x_f + \epsilon_3] \). By (B.21), the function \( x \mapsto F(x, \nu) \) has a negative right-derivative at \( x = \overline{\pi} \). If \( F(\overline{\pi}, \nu) \) were non positive for \( \nu \) in a neighborhood \( U_+ \) of \( \overline{\pi}^+ \), there could be no zeros of \( F(x, u) \) with \( u \)-coordinate in \( U_+ \) and \( x \)-coordinate greater than \( \overline{\pi} \) and this would contradict point 3. Therefore, by inequality (B.30), we obtain that \( \overline{\pi}^+ > \overline{\pi}^- \) and that \( F(\overline{\pi}, \nu) \) is positive for \( \nu \) in a right-neighborhood \( U_- \) of \( \overline{\pi}^- \). On the other hand, \( F(x, \nu) \) is negative for \( x \) in a right-neighborhood \( X_- \) of \( \overline{\pi} \) because \( F(\overline{\pi}, \nu) = 0 \) and \( \frac{\partial F}{\partial u}(\overline{\pi}, \nu) < 0 \). Therefore, by continuity of \( F(x, u) \), any segment joining a point of \((\overline{\pi}) \times U_- \) to a point of \((\overline{\pi}) \times X_- \) must contain a zero of \( F(x, u) \). Since \( X_- \) is a right-neighborhood of \( \overline{\pi} \), this would contradict point 2.
By (B.22), we get
\[ x > x_b + \epsilon_2. \]
By (B.2), we get
\[ F(\tilde{x}, u) = (1-h_1)u + 1 - h_0 \]
and the only case to study is when \( h_0 = h_1 = 1 \) because of point 1. But then points 2 and 3 would not be satisfied.

**Remark B.24.** The same reasoning with \( x \) changed in \(-x\) leads to the following result about unstable points.

Let \( x \mapsto u(x) \) be an entropic solution of (B.2) having a discontinuity at \( x = \overline{x} \).
If \( (x, u(x)) \) is unstable for all \( x > \overline{x} \) in some neighborhood of \( \overline{x} \) then \( (x, u(x)) \) is also unstable for all \( x < \overline{x} \) in some neighborhood of \( \overline{x} \). What is more, the point \( (\overline{x}, u(\overline{x}^-)) \) is a jump point and \( (\overline{x}, u(\overline{x}^+)) \) is unstable.

The case without discontinuity is simpler and can be proven with the same tools, so that it is left to the reader.

**Proposition B.25.** Let \( x \mapsto u(x) \) be an entropic solution of (B.2) and let \( \overline{x} \) be given. Assume that \( x \mapsto u(x) \) has no discontinuity at \( x = \overline{x} \) and \( Z \) can locally be parameterized by \( x \) or \( u \) near the point \( (\overline{x}, u(\overline{x})) \). Then, if \( (x, u(x)) \) is stable for all \( x < \overline{x} \) in some neighborhood of \( \overline{x} \), \( (x, u(x)) \) is also stable for all \( x > \overline{x} \) in some neighborhood of \( \overline{x} \). What is more, the point \( (\overline{x}, u(\overline{x})) \) is stable.

Likewise, if \( (x, u(x)) \) is unstable for all \( x > \overline{x} \) in some neighborhood of \( \overline{x} \), \( (x, u(x)) \) is also unstable for all \( x > \overline{x} \) in some neighborhood of \( \overline{x} \). What is more, the point \( (\overline{x}, u(\overline{x})) \) is unstable.

**B.6. Non existence results.** Lemma B.26. If \( h_0 \notin [0, 1] \), there is no entropic solution of (B.2).

**Proof.** For \( x < -\epsilon_2 \), equation (B.2) reduces to \( h_1 u = h_0 \) and for \( x > x_b + \epsilon_2 \) to \((1-h_1)u = 1 - h_0\). The lemma follows since an entropic solution is non negative.

The following lemma deals with the nonexistence result stated in Theorem B.10.

**Lemma B.27.** If (B.10) is satisfied, then there is no entropic solution of (B.2).

**Proof.** The idea of the proof is as follows. By the inequality (B.10), the set \( Z \) is not totally connected. Then, if there exists an entropic solution \( x \mapsto u(x) \), we will show that the points \((x, u(x))\) for \( x < x_f - \epsilon_1 \) and \((x, u(x))\) for \( x > x_b + \epsilon_2 \) do not belong to the same connected component of the set \( Z \). Therefore, a discontinuity must occur and we will show that this contradicts the entropic conditions of Definition B.2.

Before starting, note that, by (B.10) and (B.20), there exists \( \tilde{u} > 1 \) such that
\[ F(x_f + \epsilon_1, \tilde{u}) = (\tilde{u} - 1) \left( 1 - h_1 + \frac{h_0 - h_1 + \tilde{u} V(\tilde{u})}{\tilde{u} - 1} \right) < 0 \]

If \( h_1 = 1 \), then \( h_0 \neq 1 \). Indeed, otherwise, (B.10) would give \( \inf_{u > 1} \frac{V(u)}{u-1} < 0 \) and this would contradict \( u V(u) \geq 0 \). By Lemma B.16, we conclude that there is no solution of (B.2).

Now, we assume that \( h_1 \neq 1 \). For \( x > x_b + \epsilon_2 \), we have \( F(x, u) = (1-h_1)u - 1 + h_0 \) by (B.22). Thus, since \( h_1 < 1 \), we have \( F(x, u) \geq 0 \) for \( x \) and \( u \) large enough and we introduce the arcwise connected component \( \Omega_1 \) of \( Z_+ = \{(x, u) \in \mathbb{R} \times [0, +\infty) | F(x, u) \geq 0 \} \) containing the point \((+\infty, +\infty)\). We first prove that \( \Omega_1 \) is included in an upper-right quartant.

Let \( \tilde{u} > 1 \) satisfying (B.31) be given. We know by (B.20) that the function \( x \mapsto F(x, \tilde{u}) \) is increasing on \([x_f - \epsilon_1, x_f + \epsilon_1]\) and, by (B.21), decreasing on \([x_f + \epsilon_1, +\infty)\). Since \( F(x_f + \epsilon_1, \tilde{u}) < 0 \), this leads to \( F(x, \tilde{u}) \leq 0 \) for all \( x \geq x_f - \epsilon_1 \) and \( F(x, \tilde{u}) \geq 0 \) for all \( u \geq \tilde{u} \). Indeed, \( F(x_f - \epsilon_1, \tilde{u}) < 0 \) and, at any zero of \( u \mapsto F(x_f - \epsilon_1, u) \), we would have a negative slope by (B.24), which cannot be. These two estimates along half-lines imply that \( \Omega_1 \subset \{(x, u) | x > x_f - \epsilon_1, u > \tilde{u} \} \).
Now, suppose that there exists some entropic solution \( x \mapsto u(x) \) of (B.2). Since we assumed \( h_1 \neq 1 \), we know by Lemma B.16 that for all \( x > x_l + \varepsilon_2 \), \( (x, u(x)) = (x, (1 - h_0)/(1 - h_1)) \) and that this half-line contains all the zeros of \( F(x, u) \) for \( x \) large enough. Therefore, \( (x, u(x)) \in \partial \Omega_1 \) for \( x > x_l + \varepsilon_2 \) and the set \( \{ x \in \mathbb{R} : \forall x' \geq x, (x', u(x')) \in \Omega_1 \} \) is not empty. Let us denote by \( \overline{\pi} \) its infimum. It is clear that \( \overline{\pi} \leq x_l + \varepsilon_2 \) and that \( (\overline{\pi}, u(\overline{\pi}^+)) \in \Omega_1 \). What is more, \( \overline{\pi} \geq x_f - \varepsilon_1 \) since \( (x, u(x)) \notin \Omega_1 \) for \( x < x_f - \varepsilon_1 \) by the inclusion \( \Omega_1 \subset \{(x, u)/x > x_f - \varepsilon_1, u > \pi\} \).

The rest of the proof is devoted to demonstrating that the existence of \( \overline{\pi} \) leads to a contradiction. It relies on the following property of \( \overline{\pi} \): there is no arc (continuous path) where \( F(x, v) \geq 0 \) joining \( (\overline{\pi}, u(\overline{\pi}^+)) \) to \( (\overline{\pi}, u(\overline{\pi}^-)) \). Indeed, if it were so, we would first have that \( (\overline{\pi}, u(\overline{\pi}^-)) \in \Omega_1 \) since \( (\overline{\pi}, u(\overline{\pi}^+)) \in \Omega_1 \) and this last set is arcwise connected. Then, by choosing a left-neighborhood of \( \overline{\pi} \) upon which \( x \mapsto u(x) \) is continuous (we recall that an entropic solution has isolated discontinuities), we would be able to connect these latter points, satisfying \( F(x, u(x)) = 0 \), to \( (\overline{\pi}, u(\overline{\pi}^-)) \in \Omega_1 \).

By the same argument, they would thus also belong to \( \Omega_1 \) and this would contradict the definition of \( \overline{\pi} \).

As a direct consequence of this result, there must be a discontinuity of \( x \mapsto u(x) \) at \( x = \overline{\pi} \). If \( u(\overline{\pi}^+) > u(\overline{\pi}^-) \), then we know by Definition B.2 that \( F(\overline{\pi}, v) \geq 0 \) for all \( v \in [u(\overline{\pi}^-), u(\overline{\pi}^+)] \). Therefore, the whole segment \( \{ \overline{\pi} \} \times [u(\overline{\pi}^-), u(\overline{\pi}^+)] \), where \( F(x, v) \geq 0 \), joins \( (\overline{\pi}, u(\overline{\pi}^+)) \) to \( (\overline{\pi}, u(\overline{\pi}^-)) \) and this contradicts the above-mentioned property of \( \overline{\pi} \).

Thus \( u(\overline{\pi}^-) > u(\overline{\pi}^+) \) and we first show that there exists some \( v_0 \in (u(\overline{\pi}^+), u(\overline{\pi}^-)) \) such that \( F(x_f + \varepsilon_1, v_0) < 0 \) by discussing according to the position of \( \overline{\pi} \). In the case \( \overline{\pi} < x_f + \varepsilon_1 \), the functions \( x \mapsto F(x, u(\overline{\pi}^+)) \) are both (strictly) increasing on \( [\overline{\pi}, x_f + \varepsilon_1] \) by the expression of their derivatives in (B.20) (we recall that \( u(\overline{\pi}^-) - 1 > u(\overline{\pi}^+) - 1 > \bar{u} - 1 > 0 \). Since \( F(\overline{\pi}, u(\overline{\pi}^+)) = 0 \), this leads to \( F(x, u) \geq 0 \) on both segments \( [\overline{\pi}, x_f + \varepsilon_1] \times \{ u(\overline{\pi}^+) \} \). If \( F(x, u) \) were non negative on the segment \( \{ x_f + \varepsilon_1 \} \times [u(\overline{\pi}^+), u(\overline{\pi}^-)] \), then the three segments just defined would make an arc where \( F(x, v) \geq 0 \) and joining \( (\overline{\pi}, u(\overline{\pi}^+)) \) to \( (\overline{\pi}, u(\overline{\pi}^-)) \). Since this cannot be, there exists \( v_0 \in (u(\overline{\pi}^+), u(\overline{\pi}^-)) \) such that \( F(x_f + \varepsilon_1, v_0) < 0 \). In the case \( \overline{\pi} \geq x_f + \varepsilon_1 \), the functions \( x \mapsto F(x, u(\overline{\pi}^+)) \) are both decreasing on \( [x_f + \varepsilon_1, \overline{\pi}] \) by the expression of their derivatives in (B.20) and an analogous reasoning yields the desired result.

Since there exists \( v_0 \in (u(\overline{\pi}^+), u(\overline{\pi}^-)) \) such that \( F(x_f + \varepsilon_1, v_0) < 0 \), then \( x \mapsto F(x, u) \) having its maximum at \( x = x_f + \varepsilon_1 \) on \( [x_f - \varepsilon_1, +\infty) \) (by (B.20), (B.21) and (B.22)), we also have \( F(x, v_0) < 0 \) for all \( x \geq x_f - \varepsilon_1 \). Since \( (\overline{\pi}, u(\overline{\pi}^+)) \) belongs to the arcwise connected set \( \Omega_1 \), there exists a continuous path \( \gamma \) in \( \Omega_1 \) from this latter point to the point \( (+\infty, +\infty) \). As \( \Omega_1 \subset \{(x, u)/x > x_f - \varepsilon_1, u > \bar{u}\} \) and \( u(\overline{\pi}^-) < v_0 \), there exists \( (x, v) \) on the path \( \gamma \), hence in \( \Omega_1 \), such that \( x > x_f - \varepsilon_1 \) and \( v = v_0 \). On the one hand, we have \( F(x, v) \geq 0 \) since \( (x, v) \in \Omega_1 \) and on the other hand \( F(x, v) < 0 \) since \( x > x_f - \varepsilon_1 \) and \( v = v_0 \). This last contradiction ends the proof of the lemma.

We now prove the nonexistence part of Theorem B.8 in the case where \( h_0 \in (0, 1] \).

Lemma B.28. Assume that the following conditions hold

\[
h_0 \in (0, 1] \quad \text{and} \quad 1 - h_1 + \inf_{u > 1} \left\{ \frac{uV(u) - h_1 + h_0}{u - 1} \right\} > 0 \quad \text{and} \quad V(1) - h_1 + h_0 \neq 0
\]

Then, there is no entropic solution of (B.2).

Proof. First, let us note that \( h_1 \neq 1 \) because, by (B.11) and (B.7), we find \( 0 < 1 - h_1 \).
Now, let \( x \mapsto u(x) \) be an entropic solution of (B.2). For \( x < x_f - \epsilon_1 \), \( (x, u(x)) \) is a stable point (by Lemma B.15) and for \( x > x_b + \epsilon_2 \), \( (x, u(x)) \) is an unstable point (Lemma B.16). Thus, the set \( \{ x \in \mathbb{R} , \forall x' \leq x , (x', u(x')) \} \) is not empty and we denote by \( \mathcal{Z} \) its supremum. It is clear that \( x_f - \epsilon_1 \leq \mathcal{Z} \leq x_b + \epsilon_2 \).

Assume that \( h_0 \neq 1 \). Then, under the assumptions of the lemma, the set \( \mathcal{Z} \) is locally parameterized by \( x \) or \( u \) around any \( (\mathcal{Z}, \mathcal{P}) \in \mathcal{Z} \) thanks to Propositions B.17 and B.18 (point 1). If the solution is continuous at \( x = \mathcal{Z} \), we conclude by Proposition B.25 that \( (x, u(x)) \) is stable for all \( x \geq \mathcal{Z} \) in some neighborhood of \( \mathcal{Z} \). This contradicts the definition of \( \mathcal{Z} \). If the solution is discontinuous at \( x = \mathcal{Z} \), we conclude by Proposition B.19, that \( (x, u(x)) \) is stable for all \( x > \mathcal{Z} \) in some neighborhood of \( \mathcal{Z} \). This also contradicts the definition of \( \mathcal{Z} \).

Assume that \( h_0 = 1 \). Then, under the assumptions of the lemma, the set \( \mathcal{Z} \) is locally parameterized by \( x \) or \( u \) around any \( (\mathcal{Z}, \mathcal{P}) \in \mathcal{Z} \), except the point \( (x_f + \epsilon_1, 0) \), thanks to Propositions B.17 and B.18 (point 2 since \( h_1 \neq 1 \)). Thus, if \( \mathcal{Z} \neq x_f + \epsilon_1 \), the previous arguments for \( h_0 \neq 1 \) apply and lead to a contradiction. Thus \( \mathcal{Z} = x_f + \epsilon_1 \).

But then, on the one hand, we have \( F(\mathcal{Z}, u) = (1 - h_1)u + uV(u) > 0 \) if \( u > 0 \) so that \( (\mathcal{Z}, 0) \) is the only zero of \( F(x, u) \) with \( x \)-coordinate \( x = \mathcal{Z} \) and there may be no isolated discontinuity of the solution at \( x = \mathcal{Z} \). On the other hand, \( x \mapsto u(x) \) cannot be continuous at \( x = \mathcal{Z} = x_f + \epsilon_1 \) because there are no elements of \( \mathcal{Z} \) near the point \( (x_f + \epsilon_1, 0) \) with \( x \)-coordinate less than \( x_f + \epsilon_1 \) by point 2 in Proposition B.18. This last contradiction ends the proof. \( \square \)

**B.7. Lemmas to build entropic solutions and ensure uniqueness.** In this section, we exhibit a general construction method of entropic solutions and provide two uniqueness results.

**Lemma B.29.** Let \( I \subset \mathbb{R} \) be an interval and assume that the following assumptions are satisfied.

1. There exists an increasing mapping \( x \in I \mapsto w(x) \in (0, +\infty] \) with isolated points of discontinuity \( \mathcal{Z} \) such that, for all \( x \in I \), \( F(\mathcal{Z}, v) > 0 \) for \( v \in [w(\mathcal{Z}^-), w(\mathcal{Z}^+)] \) (which may be reduced to a single point if \( w \) is continuous at \( x = \mathcal{Z} \)) and there exists \( v \in [0, w(x)] \) satisfying \( F(x, v) = 0 \).

2. \( \frac{DF}{dx}(x, v) \leq 0 \), for all \( (x, v) \) such that \( x \in I \) and \( v \leq w(x) \).

3. The set \( \mathcal{Z} \cap I \times [0, +\infty) \) is locally parameterized by \( x \) or \( u \).

Then, the function \( u(x) = \inf \{ v \in [0, w(x)] \} \) is well defined and is an increasing entropic solution of (B.2). Moreover, each point \( (x, u(x)) \) is either unstable or a jump point.

**Proof.** First of all, it is clear that \( u(x) \) is well defined and that we have for all \( x \in I \) \( (x, u(x)) \in \mathcal{Z} \).

What is more, \( x \mapsto u(x) \) is an increasing function. Indeed, let \( x_1 < x_2 \) be elements of \( I \). Since \( x \in I \mapsto w(x) \in (0, +\infty] \) is increasing, we have \( w(x_1) < w(x_2) \leq w(x) \) for all \( x \in I \cap [x_1, +\infty) \). Therefore, we have \( \frac{DF}{dx}(x, u(x_1)) \leq 0 \) for all \( x \in I \) so that \( 0 = F(x_1, u(x_1)) > F(x_2, u(x_1)) \). Since \( u(x_1) < u(x_2) \), we conclude that \( u(x_1) < u(x_2) \). Now, if \( x \in I \) is given, the point \( (x, u(x)) \) is not stable (and therefore is either unstable or a jump point) because we have \( F(x, v) > 0 \) if \( v \in (u(x), w(x)) \).

We now show that the discontinuity points of \( x \mapsto u(x) \) are isolated. By Lemma B.14, the set \( \mathcal{S} \) defined in (B.23) consists of isolated points of \( \mathbb{R} \times [0, +\infty) \) and
thus $X_1 = \{ x \in \mathbb{R}, \exists u \in [0, +\infty), (x, u) \in S \}$ consists of isolated points of $\mathbb{R}$. If $X_2$ denotes the set of isolated points of discontinuity of $x \mapsto w(x)$, then $\mathbb{R} \cap (X_1 \cup X_2)$ is formed of the union of open intervals. On each of these intervals $J$, it can be shown by the implicit functions theorem that $Z$ consists of a finite number of branches $x \in J \mapsto u_1(x), \ldots, x \in J \mapsto u_m(x)$ which do not intersect, so that we can always suppose that $u_1(x) \leq \cdots \leq u_m(x)$. Since $x \mapsto w(x)$ is continuous on $J$, there exists $j \in \{1, \ldots, m\}$ such that $u_j(x) < w(x) < u_{j+1}(x)$ and thus, by definition of $u(x)$, we see that $u(x) = u_j(x)$ for $x \in J$. This proves that discontinuity points of $x \mapsto u(x)$, if any, necessarily belong to the set $\mathbb{R} \cap (X_1 \cup X_2)$ and are therefore isolated points.

Now, let $\overline{J}$ be an isolated discontinuity point of $x \mapsto u(x)$. We have $F(x, v) > 0$ for $v \in (u(x), w(x)]$, so that by continuity of $x \mapsto u(x)$ and $x \mapsto w(x)$ for $x$ in a left-neighborhood of $\overline{J}$, we get $F(x, v) > 0$ for $v \in [u(\overline{J}), w(\overline{J})]$. On the other hand, we know that $F(x, v) > 0$ for $v \in [w(\overline{J}^-), w(\overline{J}^+)]$. Thus all the conditions of Definition B.2 are satisfied.

Changing $x$ in $-x$ and $u$ in $-u$ in the proof of Lemma B.29 yields the following result.

**Lemma B.30.** Let $I \subset \mathbb{R}$ be an interval and assume that the following assumptions are satisfied.

1. There exists an increasing mapping $x \in I \mapsto w(x) \in (0, +\infty]$ with isolated points of discontinuity $x$ such that, for all $x \in I$, $F(x, v) > 0$ for $v \in [w(x), w(x)]$ (which may be reduced to a single point if $w$ is continuous at $x = x$) and there exists $v \geq w(x)$ satisfying $F(x, v) = 0$.

2. $\frac{\partial F}{\partial v}(x, v) \geq 0$, for all $(x, v)$ such that $x \in I$ and $v \geq w(x)$.

3. The set $Z \cap I \times [0, +\infty)$ is locally parameterized by $x$ or $u$.

Then, the function

$$u(x) = \sup \{ v \in [w(x), +\infty), \forall u' \in [w(x), v) : F(x, u') > 0 \}$$

is well defined and is an increasing entropic solution of (B.2). Moreover, each point $(x, u(x))$ is either stable or a jump point.

**Lemma B.31.** Assume that $(h_0, h_1) \neq (1, 1)$ and $(h_0, h_1) \neq (0, 0)$. If there exists $x_p \in \mathbb{R}$ such that

1. $Z \cap (-\infty, x_p) \times [0, +\infty)$ is locally parameterized by $x$ or $u$,

2. there exists $\alpha < x_p$ such that $Z \cap (-\infty, \alpha) \times [0, +\infty)$ consists of a curve $x \mapsto (x, u_0(x))$ where every point $(x, u_0(x))$ is either stable or a jump point,

then any entropic solution of (B.2) is uniquely defined on $(-\infty, x_p)$, except possibly at discontinuity points.

**Proof.** Let $u_1$ and $u_2$ be two entropic solution of (B.2) and let

$$J = \{ x < x_p, \forall x' \in (-\infty, x), u_1(x') = u_2(x') \}$$

and $(x', u_1(x')) = (x', u_2(x'))$ is either stable or a jump point except possibly at discontinuity points of the solutions.

Then, by assumption, $(-\infty, \alpha) \subset J$ so that $x_1 = \sup J \leq x_p$ is well defined. We shall prove by contradiction that $x_1 = x_p$.

For this, we suppose that $x_1 < x_p$ and start with a series of remarks. Discontinuity points of $u_1$ and $u_2$ being isolated, there exists a left neighborhood of $x_1$ where $u_1$ and $u_2$ coincide and thus

$$u_1(x^-_1) = u_2(x^-_1)$$

By Definition B.2, we have for $i = 1, 2$,

$$F(x_1, v)(u_1(x^+_i) - u_i(x^-_i)) \geq 0$$

(B.32)
for all \( v \) between \( u_i(x_i^-) \) and \( u_i(x_i^+) \). What is more, we know by Lemma B.14 that jump points are isolated so that \((x', u_i(x')) = (x', u_2(x'))\) is stable for \(x'\) in a left neighborhood of \(x_i\). Thus, by Propositions B.19 and B.25, \((x_i, u_1(x_i^-))\) and \((x_i, u_2(x_i^+))\) are stable points.

Now, assume that either \(u_1\) or \(u_2\) is continuous at \(x = x_i\). If it is \(u_1\), then \(u_1(x_i^-) = u_1(x_i^+)\) but \(u_1(x_i^-) \neq u_2(x_i^+)\) because otherwise, the definition of \(x_i\) would contradict the assumption that \(Z \cap (-\infty, x_p) \times [0, +\infty)\) is locally parameterized by \(x\) or \(u\). Thus \(u_2(x_i^+) \neq u_1(x_i^-) = u_2(x_i^-)\) and \(u_2\) is discontinuous at \(x = x_i\). Then we find, on the one hand, that \((x_i, u_2(x_i^-))\) is a jump point by Proposition B.19 and, on the other hand, that \((x_i, u_1(x_i^-))\) is stable by Proposition B.25. This contradicts the equality (B.32).

Therefore, \(u_1\) and \(u_2\) are both discontinuous at \(x = x_i\). By Proposition B.19, 
\[
(x_i, u_1(x_i^-)) = (x_i, u_2(x_i^-))
\]
is a jump point so that for \(i = 1, 2\) and for \(v_i\) close to \(u_i(x_i^-)\) between \(u_i(x_i^-)\) and \(u_i(x_i^+)\), \(F(x_i, v_i)\) and \(F(x_i, v_2)\) have the same sign (non zero). Thus, if we multiply (B.33) for \(i = 1\) and \(v = v_1\) by (B.33) for \(i = 2\) and \(v = v_2\), we get
\[
(u_1(x_i^+) - u_1(x_i^-))(u_2(x_i^+) - u_2(x_i^-)) \geq 0
\]
so that \(u_1(x_i^+)\) and \(u_2(x_i^+)\) are on the same side of \(u_1(x_i^-) = u_2(x_i^-)\). We necessarily have \(u_1(x_i^+) = u_2(x_i^+)\) because otherwise, for instance \(u_1(x_i^+)\) would be strictly located between \(u_2(x_i^+)\) and \(u_1(x_i^-)\) where, by (B.33), \(F(x_i, v)\) keeps a constant sign. This would contradict the fact that \((x_i, u_1(x_i^-))\) is stable.

Thus \(u_1(x_i^+) = u_2(x_i^+)\) and, by definition of \(x_i\), there can be no parameterization of \(Z\) in \(x\) in a neighborhood of \((x_i, u_1(x_i^-)) = (x_i, u_2(x_i^-))\). Therefore, there exists a parameterization \(v \mapsto s(v)\) and, by the Step 3 of Lemma B.20, there exists a right neighborhood of \(x_i\) where both \(u_1\) and \(u_2\) are injective with inverse \(v \mapsto s(v)\). Thus, for \(x'\) in this right neighborhood, we have \(u_1(x') = u_2(x')\) and \((x', u_1(x')) = (x', u_2(x'))\) is stable by Proposition B.19. This contradicts the definition of \(x_i\) and the proof is completed.

The following lemma is proved in the same way.

**Lemma B.32.** Assume that \((h_0, h_1) \neq (1, 1)\) and \((h_0, h_1) \neq (0, 0)\). If there exists \(x_p \in \mathbb{R}\) such that
1. \(Z \cap (x_p, +\infty) \times [0, +\infty)\) is locally parameterized by \(x\) or \(u\),
2. there exists \(a > x_p\) such that \(Z \cap (a, +\infty) \times [0, +\infty)\) consists of a curve \(x \mapsto (x, u(x))\) where every point \((x, u(x))\) is either unstable or a jump point, then any entropic solution of (B.2) is uniquely defined on \((x_p, +\infty)\), except possibly at discontinuity points.

**B.8. Existence and uniqueness results.** We now prove the existence and uniqueness parts of the theorems. We recall that \(h_1 \in [0, 1]\).

**Lemma B.33.** Assume that inequality (B.7) is satisfied and that the following conditions hold
\[
h_0 \in (0, 1) \quad \text{and} \quad V(1) - h_1 + h_0 = 0
\]
Then, there exists a unique entropic solution of (B.2). This solution is continuous and decreasing up to \(x_f - \epsilon_1\), constant and equal to \(u(x) = 1\) for all \(x \in [x_f - \epsilon_1, x_f + \epsilon_1]\), and is increasing below \(x_f + \epsilon_1\) with possible discontinuities.

**Proof.** First, let us note that \(h_1 \neq 1\) because by (B.11) and (B.7), we find \(0 < 1 - h_1\). Moreover, \(h_0 \neq 1\) since \(h_0 = h_1 - V(1) < h_1 \leq 1\).
By (B.20), the assumptions can be rewritten as

\[
\begin{align*}
F(x, 1) &= -h_1 + V(1) + h_0 = 0, \quad \forall \ x \in [x_f - \epsilon_1, x_f + \epsilon_1] \\
\inf_{u_1 > 1} \frac{F(x_f + \epsilon_1, u)}{u - 1} &= 1 - h_1 + \inf_{u_1 > 1} \left( \frac{h_0 - h_1 + uV(u)}{u - 1} \right) > 0
\end{align*}
\]  

(B.34)

Moreover, we recall that, by Proposition B.17, \( \mathcal{Z} \) has a unique point \((x_c, 1)\) where it cannot be parameterized by \( x \) or \( u \) and that \( x_c = h^{-1}(h_0 - V'(1)) \in (x_f - \epsilon_1, x_f + \epsilon_1) \).

First, we exhibit an entropic solution, in particular with Lemma B.29. Then, by Lemmas B.31 and B.32, we prove uniqueness for \( x < x_c \) and \( x > x_c \). We end up by noticing that the solution is continuous at \( x = x_c \).

\[ x \in (-\infty, x_f - \epsilon_1] \]. By Lemma B.15, \( \mathcal{Z} \cap (-\infty, x_f - \epsilon_1] \times [0, +\infty) \) consists of a continuous curve \( x \mapsto (x, \bar{u}(x)) \), of which every point is stable. We thus take \( u(x) = \bar{u}(x) \) and notice that \( u(x_f - \epsilon_1) = \bar{u}(x_f - \epsilon_1) = 1 \) since \( F(x_f - \epsilon_1, 1) = -h_1 + V(1) + h_0 = 0 \). What is more, this solution is increasing as seen from the estimate (B.24) and from (B.19).

\[ x \in [x_f - \epsilon_1, x_f + \epsilon_1) \]. We take \( u(x) = 1 \). Up to now, the solution defined is continuous, hence satisfies all the conditions of Definition B.2.

\[ x \in [x_f + \epsilon_1, +\infty] \]. We define the solution \( u \) as in Lemma B.29 with \( I = [x_f + \epsilon_1, +\infty) \) and \( \bar{w}(x) = +\infty \). It is continuous at \( x = x_f + \epsilon_1 \) since, on the one hand, \( u(x_f + \epsilon_1) \leq u((x_f + \epsilon_1)^+) \) because \( u \) is increasing and, on the other hand, the largest zero of \( v \mapsto F(x_f + \epsilon_1, v) \) is \( v = 1 \) by (B.34). We recall that, by construction, \((x, u(x))\) is unstable.

Thus defined, \( u \) is an entropic solution of (B.2). By Lemma B.31, it is uniquely determined on \((-\infty, x_c)\) since, on the one hand, \( \mathcal{Z} \cap (-\infty, x_c) \times [0, +\infty) \) is locally parameterized by \( x \) or \( u \) by Proposition B.17 and, on the other hand, if \( x \in [x_f - \epsilon_1, x_c) \), we have \( \frac{\partial F}{\partial u}(x, 1) = h(x) - h_1 + V(1) + V'(1) = h(x) - h(x_c) < 0 \) and \((x, u(x))\) is a stable zero by Lemma B.14. Likewise, the entropic solution \( u \) is uniquely determined on \( (x_c, +\infty) \) since, on the one hand, \( \mathcal{Z} \cap (x_c, +\infty) \times [0, +\infty) \) is locally parameterized by \( x \) or \( u \) by Proposition B.17 and, on the other hand, if \( x \in (x_c, x_f + \epsilon_1] \), we have \( \frac{\partial F}{\partial u}(x, 1) = h(x) - h(x_c) > 0 \) and \((x, u(x))\) is an unstable zero by Lemma B.14.

To end up, we notice that the entropic solution, thus defined, is continuous at \( x = x_c \).

We finally turn to the case \( h_0 = 0 \). The main result is given in the next lemma.

**Lemma B.34.** Assume that \( h_0 = 0 \) and that inequality (B.7) is satisfied. Then \( V(1) - h_1 \geq 0 \) and

1. if \( V(1) - h_1 > 0 \), there exists an \( \epsilon \)-unique entropic solution \( u \) of (B.2), where \( u(x) = 0 \) for \( x < x_f - \epsilon_1 \), \( u \) is increasing below \( x_f + \epsilon_1 \) with possible discontinuities and \( u \) satisfies \( 0 < u((x_f + \epsilon_1)^+) < 1 \).

2. if \( V(1) - h_1 = 0 \), there exists a one parameter continuous family of entropic solutions of (B.2) \((u_\tau)_{\tau \in (\epsilon_2, x_f - \epsilon_1)}\), where \( u_\tau(x) = 0 \) for \( x < \tau \), then \( u_\tau \) is continuous and decreasing up to \( x_f - \epsilon_1 \), is increasing below \( x_f + \epsilon_1 \) with possible discontinuities and satisfies \( u_\tau((x_f - \epsilon_1)^-) = u_\tau((x_f + \epsilon_1)^+) = 1 \).

**Proof.** First, note that by (B.12), we have \( V(1) - h_1 \geq 0 \). We also have \( h_1 \neq 1 \), because otherwise (B.7) would give \( \inf_{u_1 > 1} \frac{uV(u)-1}{u-1} > 0 \) and (B.11) would give \( \inf_{u_1 > 1} \frac{uV(u)-1}{u-1} \leq 0 \).

1. If \( V(1) - h_1 > 0 \), it is easy to see, as in Lemma B.33, that all the conditions of Definition B.2 are satisfied for the following \( x \mapsto u(x) \).
Let us define $w(x) = 1$ for $x \in [\epsilon_2, x_f + \epsilon_1)$ and $w(x) = +\infty$ for $x \geq x_f + \epsilon_1$. For $x \in [\epsilon_2, x_f + \epsilon_1)$, we have

$$F(x, w(x)) = F(x, 1) = -h_1 + \alpha(x)V(1) \geq -h_1 + V(1) > 0$$

and for $x \geq x_f + \epsilon_1$, we have $F(x, w(x)) = +\infty > 0$. What is more, $F(x_f + \epsilon_1, v) > 0$ for $v > 1$ by (B.34) and the Lemma B.29 applies. Thus defined, $u$ is continuous and decreasing up to $x = \epsilon_2$ since for $x \in [\epsilon_2, x_f - \epsilon_1]$ we have $u(x) = 0$ because

$$F(x_v, v) = v(-h_1 + \alpha(x)V(v)) \geq v(-h_1 + V(1)) > 0 \quad \forall v > 0$$

What is more, $u$ is increasing and such that $u((x_f + \epsilon_1)^+) < 1$ since $F(x_f + \epsilon_1, v) > 0$ for $v > 1$ by (B.34) and $F(x_f + \epsilon_1, 1) = V(1) - h_1 > 0$.

2. If $V(1) - h_1 = 0$, the remarks at the beginning of Lemma B.33 are still valid with $h_0 = 0$ and, moreover, we have $h_1 = V(1) = 1$. If $\pi \in (\epsilon_2, x_f - \epsilon_1)$ is given, the one parameter continuous family of entropic solutions of (B.2) is defined as follows.

$$x \in (-\infty, \pi], \quad \bar{\pi}(x) = 0$$

The last two lemmas give the proof of Theorem B.9.

**Lemma B.35.** Assume that $h_0 \in (0, 1], (h_0, h_1) \neq (1, 1)$ and that (B.9) is satisfied. Then, there exists an $\epsilon$-unique entropic solution of (B.2). This solution $u$ is continuous and decreasing up to $x_f - \epsilon_1$, increasing below $x_f + \epsilon_1$ with possible discontinuities. What is more, we have $V(1) - h + h_0 \geq 0$ and

1. if $V(1) - h - h_0 > 0$, the solution is unique and satisfies $1 < u((x_f - \epsilon_1)^-) < u((x_f + \epsilon_1)^+)$,

2. if $V(1) - h + h_0 = 0$, the solution satisfies $1 = u((x_f - \epsilon_1)^-) \leq u((x_f + \epsilon_1)^+)$.

The value $u((x_f + \epsilon_1)^+)$ is the largest argument for which $\phi$ achieves its infimum, so that the latter inequality is strict if and only there exists another point than $u = 1$ for which $\phi$ achieves its infimum.

**Proof.** First, we note that $V(1) - h + h_0 \geq 0$ by (B.12) and that $h_1 \neq 0$ because of (B.9).

Then, we show that the zeros of $u \mapsto F(x_f + \epsilon_1, u)$ for $u \geq 1$ form a non empty finite set. Indeed, on the one hand, we have by (B.20)

$$F(x_f + \epsilon_1, u) = (1 - h_1)(u - 1) + h_0 - h_1 + uV(u) = \phi(u) - (1 - h_0)$$

so that $\lim_{u \to +\infty} F(x_f + \epsilon_1, u) = +\infty$. On the other hand, we have by (B.15) $\phi_1 = 1$ and thus $\inf_{u \geq 1} F(x_f + \epsilon_1, u) = 0$. Thus, the zeros of $u \mapsto F(x_f + \epsilon_1, u)$ in $[1, +\infty)$ exist
and are bounded: they are therefore in finite number, being isolated by analyticity. Let \( n_\alpha \geq 1 \) denote the smallest zero and \( n_\alpha \geq n_\beta \) the largest one.

1. Assume that \( V(1) - h_1 + h_0 > 0 \). We have \( F(x_f + \epsilon_1, 1) = - h_1 + V(1) + h_0 > 0 \) and thus \( n_\alpha > 1 \).

2. Assume now that \( V(1) - h_1 + h_0 = 0 \). We have \( F(x_f + \epsilon_1, 1) = - h_1 + V(1) + h_0 = 0 \) and thus \( n_\alpha = 1 \). Moreover, \( h_0 \neq 1 \) since \( h_0 = h_1 - V(1) < h_1 \leq 1 \).

As in the proof of Lemma B.33. In addition we see that \( u(x_f - \epsilon_1) > 1 \) since \( F(x_f - \epsilon_1, 1) = - h_1 + V(1) + h_0 > 0 \).

\[ \lim_{x \to +\infty} F(x, v) = -\infty \quad \text{for} \quad x \in [x_f - \epsilon_1, x_f + \epsilon_1]. \]

By (B.21), we see that \( F(x, 1) = - h_1 + V(1) + h_0 > 0 \) and

\[ \lim_{x \to +\infty} F(x, v) = -\infty \quad \text{for} \quad x \in [x_f - \epsilon_1, x_f + \epsilon_1]. \]

With \( u(x) = 1 \), we define an increasing \( u(x) > 1 \) thanks to Lemma B.30. There is no discontinuity at \( x = x_f - \epsilon_1 \) since \( u((x_f - \epsilon_1)^+) \geq 1 \) is a zero of \( v \mapsto F(x_f - \epsilon_1, v) \) and that we know by Lemma B.15 that it is unique, thus equal to \( u((x_f - \epsilon_1)^-) = u(x_f - \epsilon_1) \).

Till now, we have \( 1 < u((x_f - \epsilon_1)^-) \leq u((x_f + \epsilon_1)^-) \).

\[ x \in [x_f + \epsilon_1, +\infty). \]

By (B.21), we see that \( F(x, v) > 0 \) for \( x > x_f + \epsilon_1 \) and \( v \) large enough. With \( w(x) = +\infty \), we define an increasing \( u(x) \) thanks to Lemma B.29. Now, in case of a discontinuity at \( x = x_f + \epsilon_1 \), we show that the conditions of Definition B.2 are satisfied so that \( u \) defined above is an entropic solution of (B.2).

Indeed, by construction, \( 1 < u((x_f + \epsilon_1)^-) = n_\alpha \leq n_\alpha = u((x_f + \epsilon_1)^+) \)

and that we define \( F(x_f + \epsilon_1, u) \geq 0 \) for all \( u \in [1, +\infty) \) \( u((x_f + \epsilon_1)^), u(x_f + \epsilon_1^+) \) since \( \inf_{u>1} \frac{F(x_f + \epsilon_1, u)}{u-1} = 0 \).

To prove the inequalities in point 1, note that we have

\[ 1 < u((x_f - \epsilon_1)^-) \leq u((x_f + \epsilon_1)^-) \leq u((x_f + \epsilon_1)^+) \]

so that it suffices to show that \( \alpha = u((x_f - \epsilon_1)^-) \) is not a zero of \( F(x_f + \epsilon_1, v) \). This is the case since:

\[ F(x_f + \epsilon_1, \alpha) = (1 - h_1)\alpha + \alpha V(\alpha) - 1 + h_0 \]

\[ > (1 - h_1)\alpha + \alpha V(\alpha) - 1 + h_0 + (1 - \alpha) \]

\[ = - h_1 \alpha + \alpha V(\alpha) + h_0 \]

\[ = F(x_f - \epsilon_1, \alpha) = 0 \]

The inequality in point 2 is easily seen to be satisfied, where by construction \( u((x_f + \epsilon_1)^+) \) is the largest zero of \( F(x_f + \epsilon_1, u) = \phi(u) - (1 - h_0) \), that is the largest argument for which \( \phi \) achieves its infimum \( \phi(1) = 1 - h_0 \).

Remark B.36. For \( x \in [x_f - \epsilon_1, x_f + \epsilon_1] \), there is not necessarily uniqueness and a one parameter family of solutions can be defined as sketched below.

We have

\[ 1 - h_1 + \lim_{u \to 1} \frac{h_0 - h_1 + u V(u)}{u - 1} = 1 - h_1 + V(1) + V'(1) \geq \inf_{u>1} \frac{h_0 - h_1 + u V(u)}{u - 1} = 0 \]

so that \( 0 < h_0 - V'(1) \leq 1 \) and we define \( x_c = h^{-1}(h_0 - V'(1)) \in (x_f - \epsilon_1, x_f + \epsilon_1] \). Furthermore, since \( F(x, 1) = - h_1 + V(1) + h_0 = 0 \), we choose \( u(x) = 1 \).

\[ x \in (x_c, x_f + \epsilon_1]. \]

This case is void if \( x_c = x_f + \epsilon_1 \), that is \( h_0 - V'(1) = 1 \). Otherwise, given some \( \tau \in (x_c, x_f + \epsilon_1) \) we define \( u(x) = 1 \) for \( x \in (x_c, \tau) \) and \( u(x) \) as
in Lemma B.30 with a \( w(x) \) given by the analysis of \( Z \) in the neighborhood of the non-degenerate critical point \( (x_\ast, 1) \) of \( F(x,u) \).

**Lemma B.37.** Assume that \( h_0 = 0 \) and that (B.9) is satisfied. Then we have \( V(1) - h_1 \geq 0 \) and \( 1 + V'(1) \geq 0 \) and

1. if \( V(1) - h_1 > 0 \), the set of entropic solutions, restricted to the complement of \([-\epsilon_2, \epsilon_2]\) is described by a one parameter family \((u_\pi)_{\pi \in (\epsilon_2, x_f - \epsilon_1)}\), such that \( 1 < u_\pi((x_f - \epsilon_1)^-) < u_\pi((x_f + \epsilon_1)^+)\).

2. if \( V(1) - h_1 = 0 \), the set of entropic solutions, restricted to the complement of \([-\epsilon_2, \epsilon_2]\) and \([x_f - \epsilon_1, x_f + \epsilon_1]\), is described by a one parameter family \((u_\pi)_{\pi \in (\epsilon_2, x_f - \epsilon_1)}\), such that \( 1 = u_\pi((x_f - \epsilon_1)^-) \leq u_\pi((x_f + \epsilon_1)^+)\).

The value \( u_\pi((x_f + \epsilon_1)^+) \) is the largest argument for which \( \phi \) achieves its infimum, so that the latter inequality is strict if and only there exists another point than \( u = 1 \) for which \( \phi \) achieves its infimum.

**In both cases,** any solution \( u_\pi \) is such that \( u_\pi(x) = 0 \) for \( x < \pi \), \( u_\pi \) is continuous and decreasing up to \( x - \epsilon_1 \), is increasing below \( x + \epsilon_1 \) with possible discontinuities.

**Proof.** The proof is very similar to the proof of the former lemma and, thereby, we only give the steps of the construction of the solution.

We start by noting that \( F(x_f + \epsilon_1, u) = \phi(u) - 1 \) and \( \phi_1 = 1 \).

1. Assume that \( V(1) - h_1 > 0 \). Note that, by (B.15), we have \( \phi(1) > \phi_1 = 1 \). Moreover, we have \( h_1 \not= 0 \) because otherwise we would have \( \phi(u) = uV(u) + u > u \) and \( r(0) = \inf_{u>1} \frac{\phi(u) - 1}{u - 1} \) could not be zero.

Now, let \( \pi \in (\epsilon_2, x_f - \epsilon_1) \) be given.

\[ x \in (-\infty, \pi) \text{ or } x \in [\pi, x_f - \epsilon_1] \text{.} \]

We have \( h_1 < V(1) \) and since \( u \mapsto V(u) \) is decreasing towards 0, there exists \( \pi > 1 \) such that \( h_1 = V(\pi) \). Then, using the result of Lemma B.15, there exists a continuous curve \( x \mapsto (x, \bar{u}(x)) \) in \( Z \), of which every point is stable, and we take \( u_\pi(x) = \bar{u}(x) \). In addition, we see that \( u(x_f - \epsilon_1) > 1 \) since \( F(x_f - \epsilon_1, 1) = -h_1 + V(1) > 0 \).

2. Assume now that \( V(1) - h_1 = 0 \). As in Lemma B.35, we have \( 0 < -V'(1) \leq 1 \) and we define \( x_c = h^{-1}(-V'(1)) \in (x_f - \epsilon_1, x_f + \epsilon_1) \).

If \( \pi \in (\epsilon_2, x_f - \epsilon_1) \) is given, the one parameter continuous family of entropic solutions of (B.2) is defined as follows.

\[ x \in (-\infty, \pi) \text{ or } x \in [\pi, x_f - \epsilon_1] \text{.} \]

Since \( h_1 = V(1) \), then, using the result of Lemma B.15, there exists a continuous curve \( x \mapsto (x, \bar{u}(x)) \) in \( Z \), of which every point is stable, and we take \( u_\pi(x) = \bar{u}(x) \). In addition, we see that \( u(x_f - \epsilon_1) = 1 \) since \( F(x_f - \epsilon_1, 1) = -h_1 + V(1) = 0 \).

\[ x \in (x_f - \epsilon_1, x_f + \epsilon_1) \text{.} \] As in the second case of the proof of Lemma B.35, that is \( u(x) = 1 \).

\[ x \in (x_f + \epsilon_1, +\infty) \text{.} \] As in the second case of the proof of Lemma B.35. □

**B.9. Final remarks about the smoothing of flux discontinuities.** We notice that the necessary and sufficient conditions given in Theorems B.8 and B.9 which ensure the existence of an entropic solution of (B.2) do not depend on \( \epsilon_1 \) and \( \epsilon_2 \). Moreover it is easy to see that the stationary solution itself does not depend on \( \epsilon_1 \) and \( \epsilon_2 \), on the intervals \((\epsilon_2, x_f - \epsilon_1) \cup (x_f + \epsilon_1, x_0 - \epsilon_2)\). This allows one to pass
to the limit in a sequence of solutions when the parameters $\epsilon_1$ and $\epsilon_2$ both tend to 0, and obtain a solution to the limit problem almost everywhere on $[0,x_b]$.

REFERENCES