Linear and Nonlinear Aspects of Vortices:
the Ginzburg-Landau Model

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This book has expanded from our attempt to understand the geometry of Ginzburg-Landau vortices. First in trying to construct these vortices and then in trying to show that they enjoy some uniqueness properties. In addition to the fact that the result are interesting for their own sake, we hope that this book will help to promote the use of weighted spaces and conservation laws in the study of nonlinear problems arising in geometry or mathematical physics.

The Ginzburg-Landau vortices are naturally associated to variational technics, however, in the construction of such vortices we have chosen to use some gluing technics based on a fixed point argument. Gluing theorems are by now common place in many problems arising from geometry, such as minimal surfaces and constant mean curvature surfaces. One of the main ingredient is the extensive use of weighted Hölder spaces which happen to be a very powerful tool in the analysis of linear differential operators over manifolds with corners.

Among the advantages of this gluing technic is that one can obtain some deep understanding of the nonlinear partial differential equation associated with Ginzburg-Landau vortices. The interaction between the linear analysis and the nonlinearities of the problem plays a central rôle and allows us to shed new light on the geometry of vortices.

We further investigate the geometry of vortices by mean of purely nonlinear tools such as conservation laws, Pohozaev formula for conformal fields, nonlinear comparison arguments, ... Our understanding of the uniqueness type properties of Ginzburg-Landau vortices relies on a delicate analysis of the interaction between linear estimates in weighted spaces for linear operators with regular singularities and the nonlinear tools described above.

These existence and uniqueness results yield to a precise description of the moduli space of Ginzburg-Landau vortices in the strongly repulsive case. Therefore, reducing the study of this infinite dimensional problem to the study of a finite dimensional one.
Chapter 1

Qualitative Aspects of Ginzburg-Landau Equations

The Ginzburg-Landau functional has been introduced by V.L. Ginzburg and L.D. Landau in [27] as a model for superconductivity. If $\Omega$ is a domain of $\mathbb{R}^n$ which is diffeomorphic to the unit ball $B_1 \subset \mathbb{R}^n$ the functional has the following form

$$G_\kappa(u, A) := \int_\Omega |\nabla u - iAu|^2 + \frac{\kappa^2}{2} \int_\Omega (1 - |u|^2)^2 + \int_\Omega |dA - h_{ext}|^2. \quad (1.1)$$

Here the condensate wave function $u$ is defined from $\Omega$ into $\mathbb{C}$ and $A$ is a 1-form defined in $\Omega$ which represents the potential associated to the induced magnetic field $h = dA$ in the material. The quantity $|u|^2$ is nothing but the density of cooper pairs of electrons which produce the superconductivity. Finally, $h_{ext}$ denotes the external magnetic field which is applied and then which is given in the problem. The parameter $\kappa > 0$ is usually called the Ginzburg-Landau parameter.

Let us mention that, in Abelian gauge theory, this functional is defined not only on domains of $\mathbb{R}^n$ but also on manifolds, in which case the function $u$ and the 1-form $A$ have to be replaced respectively by sections and connections of a given hermitian complex line bundle over the manifold. And $\nabla u - iAu$ has to be replaced by the covariant derivative of $u$ relative to $A$.

Critical points of the functional $G$ are known as Ginzburg-Landau vortices, they solve the following system of equations

$$\begin{cases} 
- \ast d_A \ast d_A u &= \kappa^2 u (1 - |u|^2) \\
- \ast d \ast (dA - h_{ext}) &= (iu, d_A u), \quad (1.2)
\end{cases}$$

where $(,)$ is by definition the real part of the hermitian product (on $\mathbb{C}$ or on the given hermitian complex line bundle). We have set

$$d_A u := du - iAu,$$
and where \( \star \) is the Hodge star operator (see Chapter 12 for a precise definition). Granted these definitions, we obtain in local coordinates

\[
\star d_A \star d_A u = \sum_{k=1}^{2} (\partial_k - iA_k)(\partial_k - iA_k)u.
\]

Naturally, these objects can be defined in an intrinsic way, we refer to [36] for further details.

Depending on \( \kappa \) and the applied field \( h_{\text{ext}} \), a precise description of the minimizers of \( G_{\kappa} \) (or more generally a description of the critical points of \( G_{\kappa} \)) is a very rich mathematical problem which remains still open in most of the situations. Surprisingly, the set of critical points of \( G_{\kappa} \) is highly sensitive to the Ginzburg-Landau parameter \( \kappa \).

### 1.1 The integrable case

When \( \Omega = \mathbb{R}^2 \) (which we will identify with \( \mathbb{C} \)), when there is no exterior magnetic field \( h_{\text{ext}} = 0 \) and when the Ginzburg-Landau parameter is given by \( \kappa = \frac{1}{\sqrt{2}} \), solutions of (1.2) have been intensively studied by A. Jaffe and C. Taubes [36]. The value \( \kappa = 1/\sqrt{2} \) is a critical value of the Ginzburg-Landau parameter since, in this case, solutions of (1.2) whose energy \( G_{\kappa}(u, A) \) is finite, are also solutions of a first order system of equations. This is the reason why this special case is known as the integrable or self dual case. A. Jaffe and C. Taubes have given a complete description of the moduli space of Ginzburg-Landau vortices in that case. More precisely, they proved that:

(i) Given any set of points \( a_1, \ldots, a_N \in \mathbb{C}^2 \) and given any set of integers \( n_1, \ldots, n_N \) which either all belong to \( \{n \in \mathbb{Z} : n < 0\} \) or all belong to \( \{n \in \mathbb{Z} : n > 0\} \), there exists a solution such that

\[
u^{-1}(\{0\}) = \{a_1, \ldots, a_N\} \quad \text{and} \quad \deg \left( \frac{u}{|u|}, a_j \right) = n_j.
\]

(ii) Any Ginzburg-Landau vortex with finite energy is one which is described in (i).

(iii) If \( u \) and \( v \) are two Ginzburg-Landau vortices, then the following property holds

\[
\text{zeros of } u = \text{zeros of } v \quad \implies \quad u = v.
\]

Of course, in the right hand side equality, one has to take into account the multiplicity of the zeros of \( u \) or \( v \), moreover, the zeros have to be counted negatively for negative degree. The zeros are called the vortices of \( u \).
This result, which holds in the integrable case $\kappa = \frac{1}{\sqrt{2}}$, serves to illustrate to what extent is the information about a solution $u$ of the Ginzburg-Landau equation included in of the zero set of $u$.

In the literature, this integrable case is also known as the non interacting situation. Indeed, in this case, vortices can exist anywhere in the plane, namely there is no constraint on the location of the vortices and, moreover, the energy of a solution only depends on $\sum_j n_j$ the sum of the degrees at each vortex and does not depend on the location of the vortices. Concerning what happens in the other situations $\kappa < 1/\sqrt{2}$ and $\kappa > 1/\sqrt{2}$ A. Jaffe and C. Taubes have stated some conjectures (see [36] pages 58-59), but as far as we know nothing has been rigorously proved until now.

Qualitatively speaking, the case where $\kappa < 1/\sqrt{2}$ is known as the attractive case in which vortices tend to attract each other and to concentrated at one place, while the case $\kappa > 1/\sqrt{2}$ is known as the repulsive case in which vortices of the same sign tend to repulse each other. In the first case, supraconductor materials are known as Type I supraconductors while, in the second case supraconductor materials are known as Type II supraconductors.

1.2 The strongly repulsive case

In this book, we will focus our attention on the strong repulsive case which corresponds to the $\kappa \rightarrow +\infty$ (equivalently, the parameter $\varepsilon = 1/\kappa \rightarrow 0$). When $\Omega = \mathbb{C}^2$ and $h_{ext} = 0$, it has been conjectured by A. Jaffe and C. Taubes that there should exist only 3 stable critical points of (1.2). Moreover, these critical points should be axially symmetric of vorticity (topological degree) $\pm 1$ or 0. We refer to [64] and [28] for a proof of the stability of the degree $\pm 1$ axially symmetric solutions. Roughly speaking, all other vortices are sent at infinity, thus one may work on a bounded domain where vortices are be confined.

We will pay a special attention to properties like (1.3) in the strong repulsive situation and we will investigate the consequences of this property when we will describe the set of solutions in this situation where vortices are confined.

In [11], it was proposed by F. Bethuel, H. Brezis and F. Hélein to begin with the analysis of a simpler problem we describe now. Assume that $\Omega$ is a simply connected bounded domain in $\mathbb{C}$. For all $\varepsilon > 0$, we consider the non-gauge invariant Ginzburg-Landau functional

$$E_\varepsilon(u) := \int_{\Omega} |\nabla u|^2 + \frac{1}{2\varepsilon^2} \int_{\Omega} (1 - |u|^2)^2.$$ (1.4)

In order to have a relevant problem, the absence of external magnetic field has to be compensated by a prescription of a Dirichlet boundary condition $u = g$ on $\partial \Omega$, where

$$g : \partial \Omega \rightarrow S^1,$$

has some topological degree (or vorticity on the boundary) $\deg g = d > 0$. Critical points of this problem are solutions of the following semilinear elliptic
equation
\[
\begin{aligned}
\Delta u + \frac{1}{\varepsilon^2} u (1 - |u|^2) &= 0 \quad \text{in} \quad \Omega \\
u &= g \quad \text{on} \quad \partial \Omega.
\end{aligned}
\] (1.5)

Although the boundary condition is not physically relevant, this model has been very successful in isolating and in rigorously describing how
\[
\text{vorticity} \implies \text{formation of vortices}.
\]

Otherwise stated F. Bethuel, H. Brezis and F. Hélein give a proof of how a global topological obstruction generates vortices.

In [12], the corresponding gauge invariant prescribed Dirichlet boundary condition problem was considered for the functional (1.1) instead of (1.4), and it was proved that the absence of gauge invariance in the simple model above does not change qualitatively the result about the formation of vortices.

Going back to the original model (1.1) without prescribing the Dirichlet boundary data anymore, one would like to understand where this global topological obstruction should arise naturally:

Why should there be vorticity?

It was first observed by physicist, and then verified by formal computations (see for instance [85]), that if one applies a uniform external magnetic field $h_{\text{ext}}$ on a superconductor material, then as long as $h_{\text{ext}}$ is less than some critical value $H_{c_1}$, the fundamental state (that is the one which minimizes the energy) has no vorticity while if
\[
h_{\text{ext}} \geq H_{c_1} \implies \text{vorticity}.
\]
Moreover, in this later case, the number of vortices depends on the value of $h_{\text{ext}}$.

There are recent progresses in the mathematical understanding of this question. For example, some answers are sketched in [13] and [83]. It is proven in [86] that there exists a critical value of $h_{\text{ext}}$ below which the absolute minimizer of (1.1) has no vortex. Finally, in [88] and in [89], a family of stable solutions having all possible vortices has been investigated and most of the physical observations concerning the existence or the non existence of vortices are rigorously established (see also the recent contribution of [23]). Let us just observe that all the above mentioned works rely on the analysis which is carried on for the model studied by F. Bethuel, H. Brezis and F. Hélein.

We now go back to the simpler model for which we wish to study of the asymptotic behavior of solutions of (1.5) as $\varepsilon$ tends to zero. This analysis is far from being straightforward. Indeed, in the case where the boundary data $g$ has a non zero degree, then on the one hand it is well known that $g$ does not admit an interior extension $u : \Omega \longrightarrow S^1$ with finite norm in $W^{1,2}(\Omega)$ (otherwise, by standard density argument, $g$ would also admit a regular extension), on the other hand the presence in the definition of $E_\varepsilon$ of the potential
\[
\frac{1}{\varepsilon^2} \int_\Omega (1 - |u|^2)^2,
\]
somehow forces the function $u$ to take values into $S^1$. Thus, even if we restrict our attention to the case of minimizers of $E_\varepsilon$, as $\varepsilon$ tends to 0, the infimum of the Ginzburg-Landau energy tends to $+\infty$ and this makes it hard (even impossible) to deduce directly some \textit{a priori} bounds for the solutions of (1.5) in some reasonable space from which one would be able to deduce a convergence result.

Nevertheless, the following result of [11] is the first step towards the complete description of the asymptotic behavior of solutions of (1.5) as $\varepsilon$ tends to 0.

\textbf{Theorem 1.1} [11] Assume that $\Omega$ is a starshaped bounded regular domain of $\mathbb{C}$, let $\varepsilon_n$ be a sequence of positive numbers tending to zero and let $u_{\varepsilon_n}$ be a sequence of solutions of (1.5). Then there exists a subsequence (still denoted $u_{\varepsilon_n}$), there exist $N$ points $a_1,\ldots,a_N$ in $\Omega$ and $N$ integers $d_1,\ldots,d_N$ such that

$$u_{\varepsilon_n} \rightarrow u_* \quad \text{in} \quad C^{k,\alpha}_{\text{loc}}(\Omega \setminus \{a_1,\ldots,a_N\},\mathbb{C}),$$

where $u_*$ is the harmonic map defined from $\Omega \setminus \{a_1,\ldots,a_N\}$ into $S^1$ given by

$$u_* = e^{i\phi} \prod_{j=1}^{N} \left( \frac{z - a_j}{|z - a_j|} \right)^{d_j} \quad (1.6)$$

and where $\phi$ is a harmonic function determined by the condition $u_* = g$ on $\partial \Omega$.

In some sense, the loss of compactness of the sequences of critical points $u_{\varepsilon_n}$ in $W^{1,2}(\Omega)$ induces a point concentration phenomenon which is similar to the concentration compactness phenomenon for critical powers (see [49]), except that here the energy also blows up. The locus where concentration of energy occurs corresponds to the limit of zero set of the sequence $u_{\varepsilon_n}$. Let us mention that the asymptotic behavior stated above was first established for the gauge invariant Ginzburg-Landau functional restricted to axially symmetric couples $(u,A)$ (modulo Gauge action) on all of $\mathbb{C}$ by M. S. Berger and Y. Y. Chen in [9].

Among the ingredients of the proof of Theorem 1.1, the following bounds are essential :

$$|u| \leq 1 \quad \text{in} \quad \Omega, \quad (1.7)$$

$$\|\nabla u\|_{L^\infty(\Omega)} \leq \frac{c}{\varepsilon}, \quad (1.8)$$

and

$$\frac{1}{\varepsilon^2} \int_{\Omega} (1 - |u|^2)^2 \leq c. \quad (1.9)$$

The derivation of these \textit{a priori} bounds relies on the fact that the domain $\Omega$ is assumed to be starshaped. Using these bounds one can already obtain a rough picture of $u$ the solution of (1.5). Indeed, let $a_j$ be one of the points where $u$
cancels. Assume for simplicity that \( a_j = 0 \) Then (1.8) implies that \(|u| \leq 1/2\) in the ball \( B_{\varepsilon/(2c)} \), and an easy computation shows that

\[
\frac{1}{\varepsilon^2} \int_{B_{\varepsilon/(2c)}} (1 - |u|^2)^2 \geq (3/8)^2 \pi c^2.
\]

Now, because of the uniform bound (1.9), we can conclude that the zero set of \( u \) is included in a uniformly bounded number of balls of size \( \varepsilon \) and that away from these balls \(|u| \geq 1/2\) (of course the value 1/2 is arbitrary and could have been replaced by any fixed positive number strictly less than 1). The parameter \( \varepsilon \) is then the characteristic length for the concentration phenomenon. So the problem reduces to the determination of where this zero set is going to concentrate. The answer to this last question is given in the following:

**Theorem 1.2** [11] Under the assumptions of the previous Theorem, the \( N \)-tuple \((a_j)_{j=1,...,N}\) is a critical point of a function which is defined on \( \Omega^N \setminus \Delta \) and which is commonly called the renormalized energy

\[
W_g((a_j)_{j}) := -2\pi \sum_{k \neq l}^N d_k d_l \log |a_k - a_l| + \int_{\partial \Omega} \Phi \left( g \times \frac{\partial g}{\partial \tau} \right) - 2\pi N \sum_{j=1}^N R(a_j), \tag{1.10}
\]

where \( \Phi \) is the harmonic conjugate of \( u_* \), i.e. this is the unique solution of

\[
\begin{cases}
\Delta \Phi &= 2\pi \sum_{j=1}^N d_j \delta_{a_j} & \text{in } \Omega \\
\frac{\partial \Phi}{\partial \nu} &= g \times \frac{\partial g}{\partial \tau} & \text{on } \partial \Omega
\end{cases} \tag{1.11}
\]

And where, by definition

\[
\Delta := \{(a_1, \ldots, a_N) \in \Omega^N : \exists j \neq k : a_j = a_k \}.
\]

Here \( \nu \) denotes the outward unit normal to \( \Omega \) and \( \tau \) denotes the unit tangent vector to \( \partial \Omega \) such that the orthonormal basis \((\nu, \tau)\) is direct and

\[
R := \Phi - \sum_{j=1}^N \log |z - a_j|. \tag{1.12}
\]

Though this does not appear explicitly in the notation, observe that the renormalized energy depends on the choice of \((d_j)_{j=1,...,N}\). The denomination renormalized energy can be justified in the following way. Take any \( N \)-tuple of distinct points \((b_1, \ldots, b_N)\) in \( \Omega \) and construct \( \Phi \) the harmonic conjugate of the
harmonic map $v_* : \Omega \setminus \{b_1, \ldots, b_N\} \rightarrow S^1$ which has degree $d_j$ at $b_j$. Namely, $\Phi$ is a solution of (1.11) where the points $a_j$ are now replaced by $b_j$. Finally, denote $\Phi_j(z) := \log |z - b_j|$ and define $R := \Phi - \sum_j \Phi_j$ to be the regular part of $\Phi$. Clearly

$$\int_\Omega |\nabla v_*|^2 = \int_\Omega |\nabla \Phi|^2 = +\infty.$$ 

Thus, in order to define some energy one subtracts to the above integrals some infinite energy which does not depend on $\Omega$, neither on $g$ nor on the $b_j$. More precisely, the function $\Phi$ is the sum of $R$ and of all the $\Phi_j$. The expression of $W_g$ is then obtained through the following procedure:

First expand $|\nabla R + \sum_j \Phi_j|^2$, integrate each term on $\Omega$ and perform a formal integration by parts of $|\nabla \Phi_j|^2$ on $\Omega$. Now subtract to that expression the infinite quantity $- \sum_j \int_\Omega \Phi_j \cdot \Delta \Phi_j$, which is a quantity which is independent of $\Omega$, $g$ and $b_j$.

It follows from a short computation that $W_g$, defined by (1.10), is exactly given by

$$W_g((b_j)_j) = \sum_{k \neq l} \int_\Omega \nabla \Phi_k \cdot \nabla \Phi_l + 2 \int_\Omega \nabla R \cdot \sum_j \nabla \Phi_j$$

$$+ \int_\Omega |\nabla R|^2 + \int_{\partial \Omega} \sum_j \Phi_j \cdot \frac{\partial \Phi_j}{\partial \nu}$$

$$= \int_\Omega |\nabla v_*|^2 + \sum_j \int_\Omega \Phi_j \cdot \Delta \Phi_j.$$ 

The renormalized energy can also be understood as the sum of terms modeling the interaction between the different vortices $-2\pi d_k d_l \log |b_k - b_l|$ (the interaction is attractive if sign $d_k \neq$ sign $d_l$ and repulsive otherwise) and terms modeling the interaction between the boundary and each vortex (repulsive).

A very useful characterization of the renormalized energy is provided by the following:

**Proposition 1.1** [11] The $N$-tuple $(a_j)_{j=1, \ldots, N}$ is a critical point of $W_g$ if and only if for all $j = 1, \ldots, N$, the map $u_*$ defined by (1.6) can be written as

$$u_* = \left( \frac{z - a_j}{|z - a_j|} \right)^{d_j} e^{iH_j},$$

in some neighborhood of $a_j$. With

$$\nabla H_j(a_j) = 0.$$
In the particular case where the solutions $u_\varepsilon$ are not only critical points but also minimizers, the above results have been extended and made more precise by M. Struwe in [96], under the assumption that $\Omega$ is simply connected (see also the approach in [12]).

**Theorem 1.3** [96] Assume that $\Omega$ is simply connected and that $u_{\varepsilon,n}$ is a sequence of minimizers then the conclusions of Theorem 1.1 and Theorem 1.2 hold with

$$d_j = +1,$$

for all $j = 1, \ldots, N$. Furthermore, the $N$-tuple $(a_j)_{j=1}^N$ is a minimizer of $W_g$ and the following asymptotic expansion of the energy holds

$$E_{\varepsilon,n}(u_{\varepsilon,n}) = -2\pi \deg g \log \varepsilon_n + W_g((a_j)) + c \deg g + \delta(\varepsilon_n)$$

where $c$ is some universal constant and where $\delta(s) \to 0$ as $s \to 0$.

Since then, this result has been generalized to situations where the potential in the functional also depends on the point in the domain (see [7], [8] and [1]). Higher dimensional versions of this Theorem have been established in [82] and [46]. Let us mention that a similar phenomenon has been observed by R. Hardt and F.H. Lin for the $p$-energy of maps defined from domains into $S^1$, as the parameter $p$ tends to 2 (see [31], [32]). Finally, in the self-dual case, this kind of asymptotic has been investigated in [35].

So Theorem 1.1, Theorem 1.2 and Theorem 1.3 provide a complete description of the limit of Ginzburg-Landau vortices as the parameter $\varepsilon$ tends to 0. In particular, if one knows the critical points of the function $W_g$, using (1.6), one deduces almost explicitly the possible limits of the critical points of the functional $E_\varepsilon$. Stated differently, one can obtain information about the solutions of an infinite dimensional problem by first solving a finite dimensional one. The main purpose of this book is to further investigate the relation between the critical points of $E_\varepsilon$, for $\varepsilon$ small, and the critical points of $W_g$. More precisely we would like to answer the following natural question:

Assume that the $N$-tuple $(a_j)_{j=1}^N$ is a critical point of $W_g$ and consider the corresponding $u_\star$ defined in (1.6). How many branches of critical points of $E_\varepsilon$ converge to $u_\star$ as $\varepsilon$ tends to 0?

Among others, topological methods have been very efficient in providing a partial answer to this question and F.H. Lin and T.C. Lin in [45], have successfully investigated the links between the level sets of $E_\varepsilon$ and the level sets of $W_g$. In particular they have proved the:

**Theorem 1.4** [45] Assume that $(a_j)_{j=1}^N$ is a non degenerate critical point of $W_g$, where all $d_j = +1$. Then, for all $\varepsilon$ small enough, there exists $u_\varepsilon$ a critical point of $E_\varepsilon$, which converges to the corresponding $u_\star$.

Still using topological tools, L. Almeida and F. Bethuel have also obtained some existence results [3].
Although the Ginzburg-Landau equations are Euler-Lagrange equations of some variational problem, our approach to the above stated question will be based more on PDE and functional analysis tools versus topological and variational tools! For instance, one of our first task will be to recover and generalize F.H. Lin and T.C. Lin’s result by the mean of some local implicit function theorem in weighted Hölder spaces instead of topological methods.

In doing so, our motivation is twofold: First this approach will provide a very precise description of the solution and in particular of its zero set. Second, this implicit function theorem approach leads to local uniqueness results which will be the first step towards a precise description of the branches of solutions.

Before we go further into the description of the branches of solutions, we need to have a precise picture of the shape of the solutions when \( \varepsilon \) is small, \( \varepsilon \neq 0 \). Many works have been devoted to this question. Let us concentrate on the case where \( u_\varepsilon \) is a minimizer since in this case the results are more precise. We already know that, as \( \varepsilon \) tends to 0, \( u_\varepsilon \) converges strongly to \( u^* \) in \( C^k \) norm, away from the points \( a_j \). Thus, it remains to understand what happens near the set where \( u_\varepsilon \) cancels. The following result has been established by P. Bauman, N. Carlson and D. Philips:

**Theorem 1.5** [6] Assume that \( \varepsilon \) is sufficiently small. Then, for any minimizer \( u_\varepsilon \) of \( E_\varepsilon \), 0 is a regular value of \( u_\varepsilon \) moreover

\[
u^{-1}_\varepsilon (\{0\}) = \{ a_1(\varepsilon), \ldots, a_d(\varepsilon) \},
\]

where \( d = |\text{deg} g| \) and

\[|a_j(\varepsilon) - a_j| = o(1),\]

where \((a_j)_{j=1,\ldots,N}\) is a minimizer of \( W_g \).

Making use of very precise estimates of the energy of minimizers, M. Comte and P. Mironescu [21] have been able to prove that there exists \( \alpha \in (0, 1) \) such that if \((a_j)_{j=1,\ldots,N}\) is a non degenerate minimizer of \( W_g \), then

\[|a_j(\varepsilon) - a_j| = O(\varepsilon^\alpha).\]

They further conjecture that the same estimate should be true for \( \alpha = 1 \) but our construction seems to show that this conjecture a little too optimistic.

Granted the result of P. Bauman, N. Carlson and D. Philips, it is natural to look at the tangent map (or the limit profile) of \( u_\varepsilon \) at \( a_j(\varepsilon) \). More precisely, assume that \( a_j(\varepsilon) \) is a vortex of \( u_\varepsilon \) (i.e. a zero of \( u_\varepsilon \)) and assume that \( \varepsilon > 0 \) is chosen small enough so that \( B_\varepsilon(a_j(\varepsilon)) \subset \Omega \) for all \( \varepsilon \) small enough and also that \( a_j(\varepsilon) \) is the only zero of \( u_\varepsilon \) in \( B_\varepsilon(a_j(\varepsilon)) \). We then define

\[\tilde{u}_\varepsilon := u_\varepsilon(\varepsilon \cdot).\]

Clearly, this function satisfies

\[
\Delta \tilde{u}_\varepsilon + \tilde{u}_\varepsilon(1 - |\tilde{u}_\varepsilon|^2) = 0 \quad \text{in} \quad B_{\varepsilon/\varepsilon}(0)
\]

\[\tilde{u}_\varepsilon(0) = 0.\]
Moreover, it can also be proved that
\[
\int_{B_{\varepsilon/c}(0)} (1 - |\tilde{u}_{\varepsilon}|^2)^2 \leq C
\]
\[
\deg \left( \frac{\tilde{u}_{\varepsilon}}{|\tilde{u}_{\varepsilon}|}; \partial B_{\varepsilon/c}(0) \right) = +1.
\]
As \(\varepsilon \to 0\), we may pass to the limit in the above equation and obtain that the limit map \(\tilde{u}\) solves
\[
\Delta \tilde{u} + \tilde{u} (1 - |\tilde{u}|^2) = 0 \quad \text{in} \quad \mathbb{C}
\]
\[
\tilde{u}(0) = 0
\]
\[
\int_{\mathbb{C}} (1 - |\tilde{u}|^2)^2 \leq C.
\]
(1.13)

It is proven in [15] that:

**Theorem 1.6** [15] [Quantization result] Assume that \(\tilde{u}\) is a solution of
\[
\Delta \tilde{u} + \tilde{u} (1 - |\tilde{u}|^2) = 0,
\]
in \(\mathbb{C}\), such that
\[
\int_{\mathbb{C}} (1 - |\tilde{u}|^2)^2 \leq C.
\]
Then, the degree \(d\) of \(\tilde{u}/|\tilde{u}|\) at infinity is well defined and we have
\[
\int_{\mathbb{C}} (1 - |u|^2)^2 = 4 \pi d^2.
\]
In particular, if \(d = 0\), then \(u\) is constant.

It is therefore important to classify all the solutions of (1.13) since they provide the local profile of \(u_{\varepsilon}\) near each 0.

Among the possible solutions of (1.13), the axially symmetric ones are most natural to look at since, if we write \(\tilde{u} = S(r) e^{i(\theta + \theta_0)}\) for \(\theta_0 \in \mathbb{R}\), then we are left to solve a second order ordinary differential equation
\[
\begin{aligned}
\frac{d^2 S}{dr^2} + \frac{1}{r} \frac{dS}{dr} - \frac{1}{r^2} S + S (1 - S^2) &= 0 \\
S(0) &= 0 \\
\lim_{r \to \infty} S &= 1.
\end{aligned}
\]
(1.14)

It is proven by R. M. Hervé and M. Hervé that:

**Theorem 1.7** [34] There exists a unique solution of (1.14) which is defined on all \(\mathbb{R}^+\), is not constant and whose limit at \(\infty\) is 1. This solution \(S\) is strictly increasing and we have \(0 \leq S < 1\).
Moreover, it is proved in [34] that
\[ \int_{\mathbb{R}^2} (1 - S^2)^2 < \infty. \]
The study of such special solutions of (1.13) in general was undertaken in [15], where it was proven that the asymptotic behavior at infinity of any solution is the one of \( S \) (see also [90]). The question whether all solutions of (1.13) are axially symmetric remained an open problem for some years, until this problem was finally solved by P. Mironescu using a short and surprising argument which will be largely commented and generalized in Chapter 9 of this book.

**Theorem 1.8** [65] Any solution of (1.13) coincides with \( S e^{i(\theta + \theta_0)} \), where \( S \) is the solution defined in Theorem 1.6 and \( \theta_0 \in \mathbb{R} \).

To summarize, we have obtained a fairly precise picture of the minimizers of \( E_\varepsilon \) since we know that, away from the points \( a_j \), they look like \( u_\ast \) and near each zero \( a_j(\varepsilon) \) (which is known to converge to \( a_j \) with a speed at least \( \varepsilon^\alpha \) for some \( \alpha \in (0, 1) \)) they looks approximately like \( S \left( \frac{|z - a_j(\varepsilon)|}{\varepsilon} \right) e^{i \arg(z - a_j(\varepsilon))} \). There is a quantitative version of this last assertion [91], [65]: There exists a harmonic function \( \psi : \Omega \rightarrow \mathbb{R} \) such that
\[
\left\| u_\varepsilon - e^{i\psi} \prod_{j=1}^d S \left( \frac{|z - a_j(\varepsilon)|}{\varepsilon} \right) e^{i \arg(z - a_j(\varepsilon))} \right\|_{L^\infty(\Omega)} \rightarrow 0. \tag{1.15}
\]
This estimate can be complemented by other pointwise estimates [21]
\[
|\nabla u_\varepsilon| \leq \frac{c}{L} \quad |\nabla u_\varepsilon|| \leq \frac{c \varepsilon^2}{L^3} \tag{1.16}
\]
where \( L = \max(\varepsilon, \text{dist}(z, \{a_1(\varepsilon), \ldots, a_d(\varepsilon)\})) \).

Even though the asymptotics of the minimizers of the Ginzburg-Landau functional are well understood, some completely new arguments are needed in order to establish some uniqueness result which would provide a bijection between the minimizers of \( E_\varepsilon \) (or the critical points of \( E_\varepsilon \)) and the minimizers of \( W_g \) (or critical points \( W_g \)).

For example, let us consider the simple situation where \( \Omega = B_1 \) and \( g = e^{i\theta} \). In this case, it was conjectured by F. Bethuel, H. Brezis and F. Hélein that, for all \( \varepsilon > 0 \), the minimizer of \( E_\varepsilon \) is axially symmetric, i.e. has the form \( S_\varepsilon e^{i\theta} \) (such a \( S_\varepsilon \) is known to be unique [34]). When \( \varepsilon \) is large enough, the fact that the conjecture is true is a direct consequence of the convexity of the functional (see [11] Theorem VIII.7). While, in the case where \( \varepsilon \) is small, it is proven in [10] that any sequence of minimizers converges to \( u_\ast = \frac{r}{|r|} \). In other words, 0 is the unique minimum of \( W_{e^{i\theta}} \). Several works have been devoted to this seemingly simple uniqueness question [44] and [20]. We shall see that the tools

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P. Mironescu has used for proving Theorem 1.8 can be adapted to this situation and if one assumes that \( u_\varepsilon \) vanishes at 0. Hence, this already proves that \[ \text{zero } u_\varepsilon = \{0\} \implies u_\varepsilon = S_\varepsilon(r) e^{i\theta}. \] (1.17)

Unfortunately, even for a minimizer, one does not know that \( u_\varepsilon \) vanishes at 0. For the time being the most refined energy estimates which are available [66] yields \( |\text{zero } u_\varepsilon| \leq c \varepsilon^2 \), but no direct arguments are known to show \( \text{zero } u_\varepsilon = \{0\} \). However, we will see that a contraction mapping argument, which is justified by the previous observations, allows to solve Bethuel-Brezis-Hélein’s conjecture for small values of \( \varepsilon \).

This property also serves to illustrate, in the axially symmetric setting, the strength of the information which is carried by the zero set of the solution. This is in fact a special case of a much general property we are currently investigating, which should extend to the strongly interacting case on bounded domain, the property (1.3) which has been established by A. Jaffe and C. Taubes.

The case where the limit vortices can have degree \( d_j \neq \pm 1 \) seems much more complicated, let us mention the very interesting comments on this question in [69]. When \( d_j \neq \pm 1 \) it is not known if there exist limit profiles which are different from the axially symmetric one, for example, a limit profile with splitted zeros (however, if there exists a zero of degree \( d_j \), Mironescu’s result still holds and says that the solution is also axially symmetric see [65]). It is also not known if one can expect intermediate configurations, say with different zeros distant one from the other by a distance \( \gg \varepsilon \), but converging to the same limit vortex. For instance in the case where \( \Omega = B_1 \) and \( g = e^{2i\theta} \), 0 is still a critical point of \( W_{e^{2i\theta}} \) (although it is not a minimum), and there exists a unique axially symmetric solution which converges to the \( u_* = e^{2i\theta} \). Is the axially symmetric solution the only one converging to \( e^{2i\theta} \)? Formal computations seem to militate for the existence of a critical point having 3 vortices of degree \( +1 \) at the vertices of a (small) equilateral triangle centered at 0 and one of degree \( -1 \) close to this center. Therefore, it is far from being impossible that limit vortices of higher degree \( (d_j \neq \pm 1) \) induces a degeneracy of the number of solutions associated to some critical point of \( W_g \).

### 1.3 The existence result

In this book, we will restrict ourselves to the case where the degree of the limit vortices is either \( +1 \) or \( -1 \). In Chapters 3 through 7, we construct solutions using some fixed point Theorem for contraction mappings. Precisely we prove the:

**Theorem 1.9** Assume that \( d_j = \pm 1, \) for all \( j = 1, \ldots, N \). Let \( (a_j)_{j=1,\ldots,N} \) be a nondegenerate critical point of \( W_g \). Then, there exists \( \varepsilon_0 > 0 \) and, for all \( \varepsilon \in (0, \varepsilon_0) \), there exists a solution \( u_\varepsilon \) of (1.5) with exactly \( N \) isolated zeros \( a_j(\varepsilon) \) such that.
(i) \( \lim_{\varepsilon \to 0} a_j(\varepsilon) = a_j \),
(ii) \( \lim_{\varepsilon \to 0} u_\varepsilon = u_* \) in \( C^{2,\alpha}_{\text{loc}}(\Omega \setminus \{a_1, \ldots, a_N\}) \),

where \( u_* \) is given by (1.6).

In particular, we recover and slightly generalize Theorem 1.4 of F.H. Lin and T.C. Lin, which has been obtained first by variational methods. As a byproduct of our construction, we obtain more precise informations about the critical points we construct. For example, we know the exact number of zeros and also the rate of convergence of the zeros towards the \( a_j \).

**Proposition 1.2** For all \( \alpha \in (0, 1) \) there exists a constant \( c_\alpha > 0 \) independent of \( \varepsilon \in (0, \varepsilon_0) \) such that
\[
|a_j(\varepsilon) - a_j| \leq c_\alpha \varepsilon^\alpha.
\] (1.18)

We also have the following information about what is usually called the density of area

**Proposition 1.3** There exists \( \varepsilon_0 > 0, \sigma > 0 \) and \( c > 0 \) depending only on \( \Omega \) and \( g \) such that for all \( \varepsilon \in (0, \varepsilon_0) \)
\[
det(\nabla u_\varepsilon) \geq c \frac{\varepsilon^2}{\sup(\varepsilon^4, r^4)},
\]
in each \( B_\sigma(a_j(\varepsilon)) \), where \( r := |z - a_j(\varepsilon)| \).

But probably the most important byproduct of this approach is a local uniqueness property for the solutions we have constructed, in a space which will be made precise later on. This will be one of the main tool for proving our main uniqueness result. Let us now sketch the strategy of the proof of Theorem 1.9.

The proof of Theorem 1.9 relies on the use of a fixed point theorem, for contraction mappings, in weighted Hölder spaces. Such an approach is strongly influenced by the work of the first author and R. Mazzeo [54] concerning the existence of complete conformally flat metrics with constant scalar curvature on \( S^n \setminus \{p_1, \ldots, p_N\} \) for any set of points \( p_1, \ldots, p_N \in S^n \).

The general strategy is fairly natural and standard. We first construct \( \varepsilon \to \tilde{u}_\varepsilon \) a one parameter family of approximate solutions of equation (1.5) by patching together several concentrated profiles \( S(r/\varepsilon) e^{i\theta} \) centered at the limit vortices \( a_j \), together with the limit solution \( u_* \). Naturally, as \( \varepsilon \) tends to 0, the quality of the approximate solution improves. Then, we study the mapping properties of \( \mathcal{L}_\varepsilon = DN_{\tilde{u}_\varepsilon} \), the linearized Ginzburg-Landau operator about \( \tilde{u}_\varepsilon \), where we have set
\[
\mathcal{N}(u) := \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2).
\]

Finally, the solution is obtained as a perturbation of \( \tilde{u}_\varepsilon \), applying some contraction mapping argument.
Even if one can prove that, for \( \epsilon \) small enough, the linearized operator \( L_\epsilon \) is bijective when defined between standard spaces (e.g. \( L_\epsilon : W^{2,2} \rightarrow L^2 \) or \( L_\epsilon : C^{2,\alpha} \rightarrow C^{0,\alpha} \)), the norm of its inverse blows up as \( \epsilon \) tends to 0. Reducing to nothing the hopes to find the solution using such a crude approach.

Therefore, we have to be more careful and our first task will be to identify what is responsible for this loss of invertibility as \( \epsilon \) tends to 0. The main observation is that, if one performs a space dilation about one of the blow-up points at the scale which makes appear the limit profile (in our case \( 1 \rightarrow 1/\epsilon \)), the rescaled linearized operator \( L_\epsilon \) converges to \( L \) the linearized operator about the limit profile solution of the limit equation (in our case \( \Delta u + u(1 - |u|^2) = 0 \) on all \( \mathbb{C} \)). This limit equation has a group of invariance (in our case, the isometries of the plane) and the infinitesimal action of this group gives rise to Jacobi fields of \( L \) which, in some sense, are responsible for the loss of invertibility of \( L_\epsilon \) as \( \epsilon \) tends to 0. This loss of invertibility is by now well understood since a similar phenomenon appears in many other problems, for instance in the theory of critical points at infinity for the Yamabe problem and its generalizations in [78], [79], [80], [4] or in the construction of constant mean curvature surfaces in [37], [55] or also in the singular Yamabe problem in [87], [54].

In \( \mathbb{C} \), there exists of course an infinite dimensional space of solutions of \( Lw = 0 \), but only a 3 dimensional space of solutions coming from geometric perturbation of the radially symmetric solution \( Se^{i\theta} \):

- \( \Phi^0 \) which corresponds to the action of rotations,
- \( \Phi^+1 \) which corresponds to the action of translation along \( x \)-axis,
- \( \Phi^-1 \) which corresponds to the action of translation along \( y \)-axis.

Somehow surprisingly, these Jacobi fields generate exactly the space of solutions of the homogeneous problem \( Lw = 0 \) which are bounded in \( L^\infty \) norm.

Now, performing the dilation backward, we consider the \( 3N \) following maps \( \Phi^0(\frac{z-a_j}{\epsilon}) \), \( \Phi^+1(\frac{z-a_j}{\epsilon}) \) and \( \Phi^-1(\frac{z-a_j}{\epsilon}) \) which (after multiplication by a cutoff function \( \eta \) away from the \( a_j \) in order to ensure these functions are equal to 0 on \( \partial \Omega \)) are responsible for the norm of the inverse of \( L_\epsilon \) to blow up.

We also have to deal with an additional problem which comes from the fact that, when we correct \( \tilde{u}_\epsilon \) by adding to it a function \( v \) to obtain an exact solution to (1.5), we would like to preserve the zero set of \( \tilde{u}_\epsilon \). Since at each vortex \( |\tilde{u}_\epsilon| \sim |z - a_j| \), we can only allow perturbations \( v \) satisfying, for some \( \mu > 1 \)

\[
|v| \leq c|z - a_j|^\mu,
\]

(1.19)
at each vortex.

We set \( \Sigma := \{a_1, \ldots, a_N\} \). In order to overcome the second problem, it is fairly natural to work on the weighted Hölder space \( C^{2,\alpha}_\mu(\overline{\Omega} \setminus \Sigma) \), which can roughly be defined to be the space of \( C^{2,\alpha} \) functions which are bounded by \( |z - a_j|^\mu \) near each \( a_j \). Unfortunately, on this space, the operator \( L_\epsilon \) is not
surjective anymore, but has finite dimensional cokernel. This is the reason why we will work with a finite dimensional extension of this weighted Hölder space by adding at least the space $K_\varepsilon$ which is generated by the functions $\Phi^0(\frac{z-a_j}{\varepsilon})$ and $\Phi^{\pm 1}(\frac{z-a_j}{\varepsilon})$ near each $a_j$. Notice, and this is very important, that none of the functions in the finite dimensional extension belong to the weighted space $C^{2,\alpha}_{\mu}(\Omega \setminus \Sigma)$ provided $\mu > 1$ since they behave like $|z-a_j|^0$ or like $|z-a_j|^1$ near each $a_j$.

When $\mu \in (1, 2)$, we will prove that the cokernel of $L_\varepsilon$ is exactly $6N$ dimensional. And, beside the above mentioned functions, we also have to consider the extension $H_\varepsilon$ generated by $3N$ other functions $\Psi^0(\frac{z-a_j}{\varepsilon})$ and $\Psi^{\pm 2}(\frac{z-a_j}{\varepsilon})$ (see Chapter 3). We can now consider the space

$$C^{2,\alpha}_{\mu}(\Omega \setminus \Sigma) \oplus H_\varepsilon \oplus K_\varepsilon.$$ 

We will prove that $L_\varepsilon$ is an isomorphism from this space into $C^{0,\alpha}_{\mu-2}(\Omega \setminus \Sigma)$, but again, as $\varepsilon$ tends to zero, the norm of its inverse blows up. This time this phenomenon can be identified explicitly and geometrically since it is intimately connected with the definition of the finite dimensional extension mentioned above.

There is still one phenomenon we have not taken into account yet: the scaling effect. This phenomenon is related to the fact that $u$ is not scalar but vector (complex) valued and that the modulus $|u_\varepsilon|$ and the phase of $u_\varepsilon$ behave in a different way as $\varepsilon$ tends to zero.

Given the precise description of the convergence of $u_\varepsilon$, critical point of the Ginzburg-Landau functional, towards $u_*$ (which is established for minimizers but should also hold for critical points in general) and given estimates like (1.16), it follows that the characteristic length for $|u_\varepsilon|$ is $\varepsilon$ although the characteristic length for the phase is 1 (more details will be given in Chapter 4). This is one of the reasons why, we have to distinguish the perturbations of the modulus of $\tilde{u}_\varepsilon$ from the perturbations of the phase of $\tilde{u}_\varepsilon$ and instead of considering the linearized operator acting on $C^{k,\alpha}_{\mu}(\Omega \setminus \Sigma) \oplus H_\varepsilon \oplus K_\varepsilon$, we will work with $E^{k,\alpha}_{\mu}(\Omega \setminus \Sigma) \oplus H_\varepsilon \oplus K_\varepsilon$ which an altered version of the above defined spaces in which the real part and imaginary parts of the functions have distinct behaviors. The exact definition of this space will be given in Chapter 4. With these definitions, we will prove that

$$\tilde{L}_\varepsilon : E^{2,\alpha}_{\mu}(\Omega \setminus \Sigma) \oplus H_\varepsilon \oplus K_\varepsilon \longrightarrow E^{0,\alpha}_{\mu-2}(\Omega \setminus \Sigma), \quad (1.20)$$

is uniformly invertible independently of $\varepsilon$, provided $\mu \in (1, 2)$.

In the final step of the construction of the solution $u_\varepsilon$, we will perturb $\tilde{u}_\varepsilon$ by adding to it a function $w \in E^{2,\alpha}_{\mu} \oplus H_\varepsilon$ and then by composing the result with the action of a diffeomorphisms on the domain whose infinitesimal action on $\tilde{u}_\varepsilon$ generates exactly $K_\varepsilon$ (namely, small rotations around each $a_j$, small translations in a neighborhood of each $a_j$). An application of a standard fixed point theorem for contraction mappings will ensure both the existence and the local uniqueness of a solution of $\Delta u + \frac{\bar{u}}{\varepsilon^2} (1 - |u|^2) = 0$ in a neighborhood of $\tilde{u}_\varepsilon$. 

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To conclude, let us mention two open problems:

It should be possible to apply our construction not only when the vortices have degrees $d_j \in \{\pm 1\}$ but also for arbitrary degrees $d_j \in \mathbb{Z}$. For the time being (at least) one preliminary result seems to be missing. Indeed, let $u_d = e^{id\theta} S_d(r)$ denote the radially symmetric solution of

$$N(u) := \Delta u + u(1 - |u|^2) = 0,$$

in $\mathbb{C}$, which tends to $e^{id\theta}$ at $\infty$. Let $L_d = dN_{u_d}$ denote the linearized operator about $u_d$, we would like to say that the only bounded solutions of $L_d w = 0$ which are defined in $\mathbb{C}$ are linear combinations of the following three Jacobi fields $i u_d$, $\partial_x u_d$ and $\partial_y u_d$. Unfortunately we have not been able to prove such a result.

It is known that $L_d$ necessary has negative eigenvalues (see [2]) but a better understanding of the behavior of the eigenvalues close to 0 as $\varepsilon$ tends to 0 is still missing.

Another direction were the above described technique could be successful is the construction of solutions of the Gross-Pitaevskii equation which models the dynamic of quantum fluids

$$i \frac{\partial u}{\partial t} - \Delta u = u(1 - |u|^2). \quad (1.21)$$

As a first approximation, the dynamic of the vortices can be taken to be

$$\frac{da_j}{dt}(t) = -\nabla_j^\perp W((a_j)), \quad (1.22)$$

where

$$\nabla_j^\perp W := \left( \frac{\partial W}{\partial a_j^2} - \frac{\partial W}{\partial a_j^1} \right),$$

when working in the adiabatic limit case (i.e. $|a_k - a_l| \simeq 1/\varepsilon$). The problem would be to apply a result like (1.20) in this time dependent situation in order to construct the solutions announced in [70].

### 1.4 Uniqueness results

In the second part of the book, we explain how the local uniqueness result induces a complete description of the branches of solutions converging to a limit configuration all of whose vortices have degree $\pm 1$.

One of the key points is the following property which is proved in Chapter 10. Let $v$ be one of the solutions constructed in the first half of the book and assume that $d_j = \pm 1$ for all $j$, let $u$ be any solution of (1.5), then the following holds

$$\text{zeros of } u = \text{zeros of } v \implies u = v, \quad (1.23)$$
provided $\varepsilon$ and $\|u - u_*\|_{W^{1,1}}$ are sufficiently small (the smallness which is necessary only depends on $\Omega$ and $g$). This property is exactly the generalization of (1.17) to the non necessarily axially symmetric setting. In order to prove (1.23), we first establish some generalized Pohozaev formula, where the usual conformal vector field is replaced by more general vector fields which we call $\rho$-conformal vector field (see Chapter 9). This yields some important estimates concerning the difference between $u$ and $v$ in weighted Sobolev space (see Chapter 11).

In general though, we do not know a priori that the two solutions $u$ and $v$, which do converge to the same $u_*$, have the same zeros, we only know that their respective zeros converge to the same limit vortices. Nevertheless, using the approach we have used to prove property (1.23), we obtain an information about the closeness between $u$ and $v$. Mixing this upper bound and the local uniqueness result we have already established for $v$, we get that $u = v$. This establishes one of the main results proved in this book:

**Theorem 1.10** Let $\varepsilon_n$ be a sequence tending to 0 and $u_n, u'_n$ be critical points of $E_{\varepsilon_n}$ which converge to the same $u_*$. Assume that all $d_j$ are equal to $\pm 1$ and that $(a_j)_{j=1,\ldots,N}$, the set of singularities of $u_*$, is a nondegenerate critical point of $W_g$. Further assume that there exists $c_0 > 0$ such that

$$
\frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{4 \varepsilon^2} \int_{\Omega} (1 - |u|^2)^2 \leq c_0 \log 1/\varepsilon_n,
$$

holds for $u_n$ and $u'_n$. Then, for $n$ large enough, $u_n = u'_n$.

Our technics give the same result in the simpler situation where $\deg g = 0$ and where there are no vortices in the limit: assume that $u_n$ and $u'_n$ converge to the same $u_*$ which has no vortex, then for $n$ large enough $u_n = u'_n$.

In fact our result is slightly stronger since we will prove that, for $n$ large enough, $u_n = u'_n$ is equal to the solution given by Theorem 1.9. Thus, we have as a byproduct a nice description of all the critical points of the Ginzburg-Landau functional which are sufficiently close to their limit $u_*$.

**Corollary 1.1** Let $\varepsilon_n$ be a sequence tending to 0 and let $u_n$ be a sequence of critical points of $E_{\varepsilon_n}$. Assume that the zeros of $u_n$ converge, as $n$ tends to $+\infty$, to $(a_j)_{j=1,\ldots,N}$ which is a nondegenerate critical point of $W_g$ and also that all $d_j = \pm 1$. Further assume that there exists $c_0 > 0$ such that

$$
\frac{1}{2} \int_{\Omega} |\nabla u_n|^2 + \frac{1}{4 \varepsilon^2} \int_{\Omega} (1 - |u_n|^2)^2 \leq c_0 \log 1/\varepsilon_n.
$$

Then, for $n$ large enough, $u_n$ has exactly $N$ zeros $(a_j(\varepsilon_n))_{j=1,\ldots,N}$ moreover the following bound holds

$$
|a_j(\varepsilon_n) - a_j| \leq \varepsilon^\alpha \quad \forall j = 1, \ldots, N,
$$

where $\alpha \in (0,1)$. 

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In the particular case of minimizers the result can be stated in the following way

**Corollary 1.2** Assume that all minimizers of $W_g$ are nondegenerate. Let us denote by $\mathcal{A}$ the set of all minimizers of $W_g$. Then, for all $1 > \alpha > 0$, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, the distance between the zero set of a minimizer $u_{\varepsilon}$ of $E_{\varepsilon}$ and $\mathcal{A}$ is bounded by $\varepsilon^\alpha$.

Another interesting consequence of the above Theorem is concerned with the sign of the area density

**Corollary 1.3** Under the assumptions of Corollary 1.1. There exist some constants $\varepsilon_0 > 0$ and $\sigma > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $z \in B_\sigma(a_j)$

$$\det(\nabla u_{\varepsilon}) \geq 0.$$  

From our main result, we may also deduce the following result in the case where the minimizers of $W_g$ is unique.

**Corollary 1.4** Assume that all $d_j = +1$, that $(a_j)_{j=1,...,N}$ is the unique minimizer of $W_g$ and further assume that it is a nondegenerate critical point of $W_g$. Then there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ there exists one and only one minimizer $u_{\varepsilon}$ of $E_{\varepsilon}$.

In the case where $\text{deg} g = 0$, that is when there are no vortices in the limit, the uniqueness of the minimizer for $\varepsilon$ small was proved by D. Ye and F. Zhou in [100].

Finally in the special case where $\Omega = B_1$ and $g = e^{i\theta}$ we are able to solve Brezis’s conjecture for $\varepsilon$ sufficiently small

**Corollary 1.5** Any critical point of the Ginzburg-Landau functional on the unit disk which is equal to $e^{i\theta}$ on the boundary is, for $\varepsilon$ sufficiently small, the axially symmetric solution.

In the last Chapter of the book we extend the use of the techniques developed above to the case of the gauge invariant functional. The combination of a Pohozaev type argument and estimates in weighted Sobolev spaces which arise in a crucial matter yields to the following result:

**Theorem 1.11** Let $(u, A)$ be a solution of the free Yang-Mills-Higgs equations (1.2) in $\mathbb{C}$, with $h_{\text{ext}} = 0$, verifying the energy upper-bound

$$G_{\kappa}(A, u) \leq 2 \pi \log \kappa + c_0, \quad (1.24)$$

for some constant $c_0 > 0$ independent of $\kappa$. Further assume that

$$u(0) = 0.$$

Then, for $\kappa$ large enough, $(u, A)$ is gauge equivalent to one of the two axially symmetric solutions of degree $+1$ and $-1$. 

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Notice that similar technics were used to make some progresses toward the A. Jaffe and C. Taubes conjectures in the strong repulsive case: $\kappa \gg 1$:

**Theorem 1.12** [84] For $\kappa$ sufficiently large, the only homotopy classes containing a minimizer of the Abelian Yang-Mills-Higgs functional (1.1) in $\mathbb{C}$ are the $0$ and $\pm 1$ classes, moreover any global minimizer is one of the axially symmetric solutions (modulo gauge invariance) described in [9].

As a final remark let us mention that all the above techniques can be modified to prove a result corresponding to Theorem 1.10 for $(u_n, A_n)$ and $(u'_n, A'_n)$, minimizers of the gauge invariant functional (1.1), on a compact surface, for a given $\mathbb{C}$-bundle. More precisely, adapting the different steps of our proof of Theorem 1.10 to the gauge invariant situation by the mean of the arguments developed in Chapter 12, the following statement can be proved:

**Theorem 1.13** Let $\varepsilon_n$ be a sequence tending to 0 and $(u_n, A_n)$, $(u'_n, A'_n)$ be two solutions of the free Yang-Mills-Higgs equations (1.2), $h_{ext} = 0$, on a compact surface with no boundary for a given $\mathbb{C}$-bundle. Assume these two solutions verify the energy upper-bound

$$G_{1/\varepsilon_n} \leq c_0 \log 1/\varepsilon_n,$$

and that they both converge to a singular section and connection $(u_*, A_*)$ whose singularities is a non-degenerate critical point of the renormalized energy $W$ given in [75], with multiplicities $\pm 1$. Then for $n$ large enough $(u_n, A_n)$ and $(u'_n, A'_n)$ are gauge equivalent.

In other words, we have a complete description of the moduli space of solution of the Abelian Yang-Mills-Higgs equations on a surface in the generic strongly repulsive case, starting from the critical points of the function $W$. 
Chapter 1
Chapter 2

Elliptic Operators in Weighted Hölder Spaces

Given a regular open subset $\Omega$ of $\mathbb{R}^n$ and given $a_1, \ldots, a_N \in \Omega$, we define some weighted spaces as the set of functions defined in $\Omega \setminus \{a_1, \ldots, a_N\}$ which decay or blow up near each puncture $a_i$ at most at a certain prescribed rate. Then, we proceed to the investigation of the mapping properties of some class of elliptic operators which are defined between these spaces. The material and the results of this chapter are well known and one can find a thorough study of this problem in [63], [50], [51] and also in [52]. Therefore, in this first Chapter, our intention is not to give a complete description of the theory of elliptic operators between weighted spaces but rather to expose, in a simple setting, the technics which will be extensively used in the subsequent Chapters. In particular, the proofs we give all rely on very simple tools which seem more flexible and which can be easily adapted to other contexts (such a simple approach was probably originated in [18]). Moreover, we hope that this introduction will help the interested reader to adapt the theory to his own problem.

We will assume that the reader is familiar with the basic theory of the Laplacian in the framework of Hölder and Sobolev spaces (as it can be found, for example, in [26]). In order to illustrate our purpose, we will concentrate on the case of the Laplacian in some bounded open subset of $\mathbb{R}^n$, since we believe that the reader will be more familiar with the arithmetic in this simple case.

We end this Chapter by three, somehow academic, applications of this theory in nonlinear settings.

2.1 The function spaces

In all this section, and even through all this chapter, $\Omega$ will be a regular open subset of $\mathbb{R}^n$, $n \geq 2$. And $\Sigma := \{a_1, \ldots, a_N\}$ will be a finite set of points in $\Omega$. 

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Let us chose $\sigma > 0$ in such a way that, if $a_i \neq a_j \in \Sigma$, then $B_{2\sigma}(a_i)$ and $B_{2\sigma}(a_j)$ are disjoint and both included in $\Omega$. For all $s \in (0, \sigma)$, we define
\[
A_s := \{ x \in \Omega : \text{dist}(x, \Sigma) \in [s, 2s] \}.
\] (2.1)

We also set
\[
\Omega_\sigma := \{ x \in \Omega : \text{dist}(x, \Sigma) > \sigma \}.
\] (2.2)

Finally, we chose some "regularized" distance function to the set $\Sigma$. This regularized distance function is a function $d \in C^\infty_{\text{loc}}(\Omega \setminus \Sigma)$, which satisfies the following properties:

(i) $d(x) := \text{dist}(x, \Sigma)$ for all $x \in \Omega \setminus \Omega_\sigma$.

(ii) $d(x) \geq \sigma$ for all $x \in \Omega_\sigma$.

Given $k \in \mathbb{N}$, $\alpha \in [0, 1)$ and given a regular subset $\omega \subset \mathbb{R}^n$, the space $C^{k, \alpha}(\omega)$ denotes the usual Hölder space, which is endowed with the usual Hölder norm $\| \cdot \|_{k, \alpha, \omega}$. We refer to [26] for a precise definition of these notions.

For any function $w \in C^k_{\text{loc}}(\overline{\Omega} \setminus \Sigma)$, we consider the family of semi-norms indexed by $s \in (0, \sigma)$
\[
[w]_{k, \alpha, s} := \sum_{j=0}^k s^j \sup_{A_s} |\nabla^j w| + s^{k+\alpha} \sup_{x,y \in A_s} \frac{|\nabla^k w(x) - \nabla^k w(y)|}{|x-y|^{\alpha}}.
\] (2.3)

We are now in a position to define the weighted Hölder spaces we will work with in this chapter:

**Definition 2.1** Given $k \in \mathbb{N}$, $\alpha \in [0, 1)$ and $\nu \in \mathbb{R}$, the space $C^{k, \alpha}_\nu(\Omega \setminus \Sigma)$ is defined to be the set of functions $w \in C^k_{\text{loc}}(\overline{\Omega} \setminus \Sigma)$ for which the following norm is finite
\[
\|w\|_{C^{k, \alpha}_\nu(\Omega \setminus \Sigma)} := \|w\|_{C^k_{\text{loc}}(\overline{\Omega} \setminus \Sigma)} + \sup_{s \in (0, \sigma)} s^{-\mu} [w]_{k, \alpha, s}.
\] (2.4)

For example, for all $k \in \mathbb{N}$, $\alpha \in [0, 1)$ and all $\nu \in \mathbb{R}$, the function
\[
x \in \overline{\Omega} \setminus \Sigma \mapsto d^\nu(x)
\]
belongs to $C^{k, \alpha}_\nu(\overline{\Omega} \setminus \Sigma)$.

With a slight abuse of notation, we will write for some vector valued function $w$ (having values in $\mathbb{R}^m$, $m \geq 1$) that $w \in C^{k, \alpha}_\nu(\overline{\Omega} \setminus \Sigma)$ if all the coordinate functions of $w$ belong to $C^{k, \alpha}_\nu(\overline{\Omega} \setminus \Sigma)$.

**Remark 2.1** More generally, one can define analogously some weighted spaces for which the weight parameter $\nu$ is different at each puncture $a_i$.

As a first property of these spaces, we have the following:
Lemma 2.1 Endowed with the above defined norm, the space $C^{k,\alpha}_{\nu}(\overline{\Omega} \setminus \Sigma)$ is a Banach space.

We now give a short list of useful properties which allow one to do calculus in these spaces and also to determine easily whether or not a function belongs to $C^{k,\alpha}_{\nu}(\overline{\Omega} \setminus \Sigma)$.

Proposition 2.1 The following properties hold.

(i) Assume that $w \in C^{k+1,\alpha}_{\nu}(\overline{\Omega} \setminus \Sigma)$, then $\nabla w \in C^{k,\alpha-1}_{\nu}(\overline{\Omega} \setminus \Sigma)$.

(ii) Assume that $w \in C^{k+1,0}_{\nu}(\overline{\Omega} \setminus \Sigma)$ and that $\nabla w \in C^{k,0}_{\nu-1}(\overline{\Omega} \setminus \Sigma)$, then, for all $\alpha \in [0,1)$, the function $w$ belongs to $C^{k,\alpha}_{\nu}(\overline{\Omega} \setminus \Sigma)$.

(iii) Assume that, for $i = 1,2$, $w_i \in C^{k,\alpha}_{\nu_i}(\overline{\Omega} \setminus \Sigma)$, then $w_1 w_2 \in C^{k,\alpha}_{\nu_1+\nu_2}(\overline{\Omega} \setminus \Sigma)$ and

$$\|w_1 w_2\|_{C^{k,\alpha}_{\nu_1+\nu_2}} \leq c \|w_1\|_{C^{k,\alpha}_{\nu_1}} \|w_2\|_{C^{k,\alpha}_{\nu_2}},$$

for some constant $c > 0$ independent of $w_1$ and $w_2$.

(iv) Assume that $w \in C^{k,\alpha}_{\nu}(\overline{\Omega} \setminus \Sigma)$ and further assume that $w > 0$ in $\overline{\Omega} \setminus \Sigma$, then, for all $p > 0$, $w^p \in C^{k,\alpha}_{\nu}(\overline{\Omega} \setminus \Sigma)$ and in addition

$$\|w^p\|_{C^{k,\alpha}_{\nu}} \leq c \|w\|^p_{C^{k,\alpha}_{\nu}},$$

for some constant $c > 0$ which does not depend on $w$.

When working in weighted Hölder spaces, the classical compactness embedding Theorem in Hölder spaces takes the following form:

Lemma 2.2 Assume that $k + \alpha < k' + \alpha'$ and that $\nu < \nu'$, then the embedding

$$I : C^{k',\alpha'}_{\nu'}(\overline{\Omega} \setminus \Sigma) \longrightarrow C^{k,\alpha}_{\nu}(\overline{\Omega} \setminus \Sigma)$$

is compact.

Very frequently, we will have to consider the subspace of functions which vanish on the boundary of $\Omega$. This is the reason why we define

$$C^{k,\alpha}_{\nu,\overline{\Omega} \setminus \Sigma} := \{ w \in C^{k,\alpha}_{\nu}(\overline{\Omega} \setminus \Sigma) : w = 0 \text{ on } \partial\Omega \}.$$  \hspace{1cm} (2.5)

We end up this section with some important remark. This remark is intended to shed light on our definition of weighted Hölder spaces and show that, in some sense, our definition is the natural one.

Remark 2.2 Instead of considering the spaces defined in Definition 2.2, we could also have considered the spaces

$$\check{C}^{k,\alpha}_{\nu}(\overline{\Omega} \setminus \Sigma) := \{ w^p \pi : w \in C^{k,\alpha}(\overline{\Omega}) \}.$$
A closer look at this definition shows that \( \hat{C}^k_{\nu}(\Omega \setminus \Sigma) \) is a proper subspace of \( C^k_{\nu}(\Omega \setminus \Sigma) \). For example, the function
\[
x \in \Omega \setminus \Sigma \rightarrow d^e \nabla d,
\]
belongs to the later space but does not belong to the former. This example shows that the spaces \( \hat{C}^k_{\nu}(\Omega \setminus \Sigma) \) are not well behaved since, for example, property (i) of the previous Proposition does not hold for them.

### 2.2 Mapping properties of the Laplacian

In this section, we consider the Laplacian as the simplest example of some elliptic operator acting on the weighted Hölder spaces we have just defined.

Let us denote by \( \phi_j, j \in \mathbb{N} \), the set of eigenfunctions of the Laplacian on \( S^{n-1} \) with corresponding eigenvalues \( \lambda_j \), that is
\[
\Delta_{S^{n-1}} \phi_j = -\lambda_j \phi_j.
\]
We assume that the sequence of eigenvalues is increasing \( \lambda_j \leq \lambda_{j+1} \), and that the eigenfunctions are normalized by
\[
\int_{S^{n-1}} \phi_j^2 d\theta = 1.
\]
Notice that we will always assume that the eigenvalues are counted with multiplicity, namely
\[
\lambda_0 = 0, \quad \lambda_1 = \ldots = \lambda_n = n - 1, \quad \lambda_{n+1} = 2n, \ldots
\]
In polar coordinates the Laplacian in \( \mathbb{R}^n \) reads
\[
\Delta = \partial^2_{rr} + \frac{n-1}{r} \partial_r + \frac{1}{r^2} \Delta_{S^{n-1}}. \tag{2.6}
\]
If we project the operator \( \Delta \) over the eigenspace spanned by \( \phi_j \), we obtain the operator
\[
L_j w := \partial^2_{rr} + \frac{n-1}{r} \partial_r w - \frac{\lambda_j}{r^2} w. \tag{2.7}
\]
If \( w \in C^{2,\alpha}(B_1 \setminus \{0\}) \) is a solution of \( \Delta w = 0 \) in \( B_1 \setminus \{0\} \), we may write the eigenfunction decomposition of \( w \) as
\[
w(r, \theta) = \sum_{j \geq 0} w_j(r) \phi_j(\theta).
\]
Then, the function \( w_j \), which is a solution of \( L_j w_j = 0 \) in \((0,1] \), is a linear combination of two linearly independent solutions of \( L_j w = 0 \). Such two solutions are given by
\[
w_j^+ := r^{\gamma_j^+} \quad \text{and} \quad w_j^- := r^{\gamma_j^-}, \tag{2.8}
\]
when \( n \geq 3 \) and \( j \geq 0 \) (or when \( n = 2 \) and \( j \geq 1 \)), and by
\[
\begin{align*}
  w^+_0 & := 1 \\
  \tilde{w}_j & := -\log r,
\end{align*}
\]
when \( n = 2 \) and \( j = 0 \). Here, we have set
\[
\gamma^\pm_j = \frac{2 - n}{2} \pm \sqrt{\left(\frac{2 - n}{2}\right)^2 + \lambda_j}. 
\tag{2.9}
\]

The key observation is that the coefficients \( \gamma^\pm_j \) give, at 0, all the possible asymptotic behaviors of the solutions of the homogeneous problem \( \Delta w = 0 \) in the punctured ball \( B_1 \setminus \{0\} \).

**Definition 2.2** The numbers \( \gamma^\pm_j \) are the indicial roots of the Laplacian at the origin.

We can be more explicite, since
\[
\ldots < \gamma^-_n = \ldots \gamma^-_1 = 1 - n < \gamma^-_0 = 2 - n \leq \gamma^+_0 = 0 < \gamma^+_1 = \ldots = \gamma^+_n = 1 < \ldots
\]
In particular, the set of indicial roots of the Laplacian at the origin is the set of all integers except \(-1, -2, \ldots, 3 - n\).

More generally, we can also consider elliptic partial differential operators with "regular singularities". Namely, operators of the form
\[
\mathcal{L} = \Delta + \frac{c}{r^2},
\]
where the function \( c \) is assumed to be smooth in \( B_1 \). In this case, the formula (2.9) for the indicial roots of \( \mathcal{L} \) has to be replaced by
\[
\gamma^\pm_j = \frac{2 - n}{2} \pm \sqrt{\left(\frac{n - 2}{2}\right)^2 + c(0) - \lambda_j} \in \mathbb{C}. 
\tag{2.10}
\]
Again, all these coefficients do determine, at 0, all the possible asymptotic behaviors of the solutions of the homogeneous problem \( \mathcal{L} w = 0 \) in \( B_1 \setminus \{0\} \).

Notice that the indicial roots only depend on the value of the function \( c \) at the puncture 0.

Granted the definition of the weighted Hölder spaces, we easily see that, for all \( k \geq 2, \alpha \in [0, 1) \) and \( \nu \in \mathbb{R} \),
\[
\Delta : C^{k,\alpha}_{\nu,\mathcal{D}}(\overline{\Omega} \setminus \Sigma) \to C^{k-2,\alpha}_{\nu-2}(\overline{\Omega} \setminus \Sigma), 
\tag{2.11}
\]
is a well defined bounded linear operator. Similarly, if the function \( c \) is smooth in \( \overline{\Omega} \), then
\[
\mathcal{L} : C^{k,\alpha}_{\nu,\mathcal{D}}(\overline{\Omega} \setminus \Sigma) \to C^{k-2,\alpha}_{\nu-2}(\overline{\Omega} \setminus \Sigma)
\]

\[
\begin{array}{c}
  w \\
  \Delta w + \frac{c}{d^2} w,
\end{array}
\]

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is also a well defined and bounded operator.

The main problem we are interested in can be stated as follows: For which values of the parameter $\nu$ the above defined operators are injective? surjective? isomorphisms?
In the case where the operator is not surjective, what is the dimension of the cokernel of the operator? And, in the case where the operator is not injective, what is the dimension of the kernel of the operator? How does this depend on $\nu$?

In the remaining of this section, we will give an answer to these questions in the case of the Laplacian. We will take this opportunity to introduce some useful technics which, later on, will be applied to analyze the Ginzburg-Landau vortices. Since we confine ourselves to study the Laplacian, some of the proofs below can be simplified. Nevertheless, we have chosen not to give the most efficient proofs but the proofs which can be easily adapted to the analysis of Ginzburg-Landau vortices.

2.2.1 Rescaled Schauder estimates

To go further in the investigation of the mapping properties of our operators, we will extensively use what we call "rescaled Schauder estimates". The starting point is the well known result [26]:

**Lemma 2.3** Assume that $\alpha \in (0, 1)$ is fixed. There exists a constant $c > 0$ such that, if $w \in C^{2,\alpha}(B_1)$ satisfies $\Delta w = f$ in $B_1$ and if $f \in C^{0,\alpha}(B_1)$, then

$$
\|w\|_{C^{2,\alpha}(B_1/2)} \leq c \left( \|w\|_{L^\infty(B_1)} + \|f\|_{C^{0,\alpha}(B_1)} \right).
$$

From now on, we will always assume that $\alpha \in (0, 1)$ and shall do it without further comment. Using the above Lemma, we immediately obtain the following local result:

**Corollary 2.1** Assume that $f \in C^{k-2,\alpha}(\overline{B_1 \setminus \{0\}})$ and that $w$ solves $\Delta w = f$ in $B_1 \setminus \{0\}$. Further assume that

$$
\|r^{-\nu} w\|_{L^\infty(B_1)} \leq c \|r^{2-\nu} f\|_{L^\infty(B_1)},
$$

for some constant $c > 0$ which does not depend on $f$, nor on $w$. Then, there exists $c' > 0$ which does not depend on $f$, nor on $w$, such that

$$
\|w\|_{C^{k,\alpha}} \leq c' \|f\|_{C^{k-2,\alpha}},
$$

where the norm of $w$ is taken in $\overline{B_{1/2} \setminus \{0\}}$, while the norm of $f$ is taken in $\overline{B_1 \setminus \{0\}}$. 

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Proof: Let us choose \( x_0 \in B_{1/2} \setminus \{0\} \). We set \( R = |x_0|/2 \) and define the function \( v \) in \( B_1 \) by

\[
v(x) := w(x_0 + Rx).
\]

We have \( \Delta v = g \) in \( B_1 \), where by definition \( g(x) := R^2 f(x_0 + Rx) \). Furthermore, the effect of the scaling yields

\[
\|g\|_{C^{0,\alpha}(B_1)} \leq c R^\nu \|f\|_{C^{0,\alpha}_{\nu - 2}},
\]

as well as

\[
\|v\|_{L^\infty(B_1)} \leq c R^\nu \|f\|_{C^{0,\alpha}_{\nu - 2}}.
\]

Using (2.12), the last estimate becomes

\[
\|v\|_{L^\infty(B_1)} \leq c R^\nu \|f\|_{C^{0,\alpha}_{\nu - 2}}.
\]

We may now apply Lemma 2.3 to obtain

\[
\|v\|_{2,\alpha,B_{1/2}} \leq c R^\nu \|f\|_{C^{0,\alpha}_{\nu - 2}}.
\]

Performing the scaling backward, we conclude that

\[
\sum_{j=0}^2 R^j \sup_{B_{R/2}(x_0)} |\nabla^j w| + R^{2+\alpha} \sup_{x,y \in B_{R/2}(x_0)} \frac{|\nabla^2 w(x) - \nabla^2 w(y)|}{|x-y|^\alpha} \leq c \|f\|_{0,\alpha,\nu - 2}.
\]

This estimate, together with a simple covering argument, yields

\[
\|w\|_{C^{2,\alpha}_{\nu}} \leq c' \|f\|_{C^{0,\alpha}_{\nu - 2}},
\]

where the norm of \( w \) is taken in \( B_{1/2} \setminus \{0\} \). The proof of the result is therefore complete when \( k = 2 \). The general case, when \( k \geq 2 \), follows easily by induction.

In other words, in order to obtain an inequality such as (2.13), it is sufficient to prove (2.12) since all the other estimates are consequences of Lemma 2.2.

The previous local result has a global counterpart, which can be stated as follows:

Corollary 2.2 Assume that \( f \in C^{k-2,\alpha}_{\nu - 2}(\Omega \setminus \Sigma) \) and that \( w \) solves \( \Delta w = f \) in \( \Omega \setminus \Sigma \), with \( w = 0 \) on \( \partial \Omega \). Further assume that

\[
\|d^{\nu} w\|_{L^\infty(\Omega)} \leq c \|d^{2-\nu} f\|_{L^\infty(\Omega)},
\]

for some constant \( c > 0 \) which does not depend on \( f \), nor on \( w \). Then, there exists \( c' > 0 \) which does not depend on \( f \), nor on \( w \), such that

\[
\|w\|_{C^{k,\alpha}} \leq c' \|f\|_{C^{k-2,\alpha}_{\nu - 2}}.
\]
We will also need the following classical result [26]:

**Lemma 2.4** There exists a constant $c > 0$ such that, if $w \in C^2,\alpha(B_1 \setminus B_{1/2})$ and $f \in C^{0,\alpha}(B_1 \setminus B_{1/2})$ satisfy
\[
\begin{cases}
\Delta w = f & \text{in } B_1 \setminus B_{1/2} \\
w = 0 & \text{on } \partial B_1.
\end{cases}
\]
Then,
\[
\|\nabla w\|_{L^\infty(B_{1/2} \setminus B_{1/4})} \leq c \left( \|w\|_{L^\infty(\partial B_{1/2})} + \|f\|_{L^\infty(B_1 \setminus B_{1/2})} \right),
\]
for some constant $c > 0$ which does not depend on $f$, nor on $w$.

### 2.2.2 Mapping properties of the Laplacian in the injectivity range

We determine for which values of the weight parameter $\nu$, the operator
\[
\Delta : C^2,\alpha_{\nu}(\Omega \setminus \Sigma) \longrightarrow C^0,\alpha_{\nu-2}(\Omega \setminus \Sigma),
\]
is injective. Once this is done, we look more closely at the mapping properties of $\Delta$ when the weight parameter is in the injectivity range.

To begin with, let us prove the following simple result:

**Proposition 2.2** Assume that $\nu > 2 - n$, then the operator $\Delta$ defined from $C^2,\alpha_{\nu}(\Omega \setminus \Sigma)$ into $C^0,\alpha_{\nu-2}(\Omega \setminus \Sigma)$ is injective.

**Proof**: We argue by contradiction and assume that, for some $\nu > 2 - n$, the operator $\Delta$ is not injective. There would exist a function $w \in C^2,\alpha_{\nu}(\Omega \setminus \Sigma)$ such that $\Delta w = 0$ in $\Omega \setminus \Sigma$ and $w = 0$ on $\partial \Omega$.

Let $a_i \in \Sigma$ be given, for the sake of simplicity in the notations we assume that $a_i = 0$. We define $\tilde{w}$ to be the unique (smooth) solution of
\[
\begin{cases}
\Delta \tilde{w} = 0 & \text{in } B_\sigma \\
\tilde{w} = w & \text{on } \partial B_\sigma.
\end{cases}
\]
Let us write the eigenfunction decomposition of $w - \tilde{w}$ at 0 as
\[
w - \tilde{w} = \sum_{j \geq 0} w_j \phi_j.
\]
Since $w \in C^2,\alpha_{\nu}(\Omega \setminus \Sigma)$ and since, by the maximum principle, $\tilde{w}$ is bounded by $\sup_{\partial B_\sigma}|w|$ in $B_\sigma$, we see that, for all $r \in (0, \sigma]$
\[
|w_j(r)| \leq c_n \left( \|w\|_{C^k,\alpha_{\nu}} \sigma^{\nu} + \sup_{\partial B_\sigma}|w| \right), \quad (2.14)
\]
where $c_n := |S^{n-1}|^{1/2}$.

Now, observe that in $B_\sigma$, $w_j$ is a linear combination of the two functions $w_j^+$ and $w_j^-$ which have been defined in (2.8). Moreover, $w_j(\sigma) = 0$ by assumption. Thus, there exists $\alpha_j \in \mathbb{R}$ such that

$$w_j = \alpha_j \left( \left( \frac{r}{\sigma} \right)^{\gamma_j^+} - \left( \frac{r}{\sigma} \right)^{\gamma_j^-} \right).$$

However, since $\gamma_j^- \leq 2 - n < \nu$, we see, using (2.14), that necessarily $w_j \equiv 0$. In other words, we have proved that $w = \hat{w}$ in $B_\sigma$. Therefore, $w$ is a smooth solution of $\Delta w = 0$ in $\Omega$ with $w = 0$ on $\partial\Omega$ and, as a consequence, $w \equiv 0$. $\square$

We now discuss the dimension of the cokernel of the Laplacian, when the weight parameter is chosen in the injective range, namely $\nu > 2 - n$. To do so, we need the following local result:

**Proposition 2.3** Assume that $\nu > 2 - n$ and that $\nu$ is not an indicial root, i.e. $\nu \notin \{\gamma_j^\pm : j \in \mathbb{N}\}$. Let $j_0 \in \mathbb{N}$ be the least index for which $\nu < \gamma_{j_0}^+$. Then, there exists $c > 0$ such that, for all $f \in C^0_{\nu-2}(\overline{B_1} \setminus \{0\})$, there exists a unique $w \in C^2_{\nu}(\overline{B_1} \setminus \{0\})$ solution of

$$\Delta w = f \quad \text{in} \quad B_1 \setminus \{0\},$$

with the boundary data given by

$$w \in \text{Span}\{\phi_j : j = 0, \ldots, j_0 - 1\} \quad \text{on} \quad \partial B_1.$$  

In addition, $\|w\|_{C^2_\nu} \leq c \|f\|_{C^0_{\nu-2}}$.

**Proof:** We first concentrate our attention on the existence of $w$. To this aim we consider the eigenfunction decomposition of $f$

$$f = \sum_{j \geq 0} f_j \phi_j,$$

and we look for $w$ of the form

$$w = \sum_{j \geq 0} w_j \phi_j.$$

We see that, for all $j \geq 0$, we have to solve the ordinary differential equation $L_j w_j = f_j$ in $(0,1]$, with the boundary condition $w_j(1) = 0$ for all $j \geq j_0$. We distinguish two cases.

**Case 1.** Assume that $j < j_0$ (naturally, if $j_0 = 0$ this case is void). Here we have an explicit expression for the solution $w_j$ which is given by

$$w_j = r^{\gamma_j^+} \int_0^r s^{1-2n-2\gamma_j^+} \int_0^s t^{n-1+\gamma_j^+} f_j \, dt \, ds.$$
Notice that this expression is well defined since $0 \leq \gamma^+_j < \nu$, for all $j < j_0$. In addition, it is an easy exercise to obtain the estimate
\[
\sup_{(0,1)} |r^{-\nu} w_j| \leq c_j \sup_{(0,1)} |r^{2-\nu} f_j|,
\]
for some constant $c_j > 0$ only depending on $j$. Thus, we have
\[
\sup_{B_1 \setminus \{0\}} \left| r^{-\nu} \sum_{j=0}^{j_0-1} w_j \phi_j \right| \leq c_{j_0} \sup_{B_1 \setminus \{0\}} |r^{2-\nu} f_j|, \tag{2.15}
\]
for some constant $c_{j_0} > 0$, only depending on $j_0$.

**Case 2.** Assume that $j \geq j_0$. This time, for each $j$, we also have an explicit formula for $w_j$. Namely,
\[
w_j = -r^{\gamma^+_j} \int_r^1 s^{1-n-2\gamma^+_j} \int_0^s t^{n-1+\gamma^+_j} f_j \, dt \, ds,
\]
and, as above, we observe that this expression is well defined since $2 - n < \nu$ and $0 \leq \gamma^+_j$. Furthermore, using the fact that $\nu \leq \gamma^+_j$, we can estimate
\[
\sup_{(0,1)} |r^{-\nu} w_j| \leq c_j \sup_{(0,1)} |r^{2-\nu} f_j|,
\]
for some constant $c_j > 0$ only depending on $j$.

At this point, let us observe that, for $j \geq j_0$, the existence and estimate for $w_j$ could have been obtained using the method of sub- and supersolutions once an appropriate barrier function is constructed. Notice that
\[
L_j r^\nu = (\nu (n-2+\nu) - \lambda_j) r^{\nu-2}.
\]
By assumption $\nu > 2 - n$ and $\nu < \gamma^+_j$, for all $j \geq j_0$. Hence,
\[
\nu(n-2+\nu) - \lambda_j < 0, \quad \text{for all} \quad j \geq j_0.
\]
This, together with the fact that $L_j$ satisfies the maximum principle, implies that the function
\[
r \in (0,1] \longrightarrow (\lambda_j - \nu(n-2+\nu))^{-1} \left( \sup_{(0,1)} |s^{2-\nu} f_j| \right) r^\nu,
\]
can be used as a barrier function to obtain both the existence and estimate for $w_j$.

In order to simplify the notations, we set
\[
f'' := \sum_{j \geq j_0} f_j \phi_j.
\]
So far, we know that for all $J > j_0$ there exists some constant $c_J > 0$, which is independent of $f$ but may a priori depend on $J$, such that

$$\sup_{B_1 \setminus \{0\}} \left| r^{-\nu} \sum_{j=j_0}^{J} w_j \phi_j \right| \leq c_J \sup_{B_1 \setminus \{0\}} |r^{2-\nu} f'|.$$

(2.16)

We claim that the constant $c_J$ does not depend on $J$. Assuming that we have already proven this claim, we can pass to the limit $J \to +\infty$ and define

$$w'' := \sum_{j \geq 0} w_j \phi_j,$$

which is clearly a solution of our problem. In addition, (2.15) together with (2.16) imply that

$$\sup_{B_1 \setminus \{0\}} |r^{-\nu} w| \leq c \sup_{B_1 \setminus \{0\}} |r^{2-\nu} f|.$$

In order to finish the proof of the Proposition, it is enough use the rescaled Schauder estimates of Corollary 2.1. The uniqueness of $w$ follows from Proposition 2.2.

Hence, it remains to prove the claim. We argue by contradiction and assume that the claim is not true. There would exist a sequence $J_i \geq j_0$ tending to $\infty$, a sequence of functions

$$f_i'' = \sum_{j=j_0}^{J_i} f_j^j \phi_j,$$

and a sequence of functions

$$w_i'' = \sum_{j=j_0}^{J_i} w_j^j \phi_j,$$

solutions of $\Delta w_i'' = f_i''$ in $B_1 \setminus \{0\}$, with $w_i'' = 0$ on $\partial B_1$ such that

$$\sup_{B_1 \setminus \{0\}} |r^{-\nu} w_i''| = 1,$$

(2.17)

while

$$M_i := \sup_{B_1 \setminus \{0\}} |r^{2-\nu} f_i''|,$$

tends to $0$ as $i$ tends to $\infty$.

Using (2.17), we get the existence of $x_i \in B_1 \setminus \{0\}$ such that

$$|x_i|^{-\nu} |w_i''(x_i)| \geq 1/2.$$

Now, it follows from Lemma 2.4 that the sequence $(|x_i|)_{i \geq 0}$ stays bounded away from $1$. Therefore, up to a subsequence, we may assume that the sequence $x_i$ converges to $x_\infty$ in $B_1$. We set $R_i := 1/|x_i|$, 

$$\tilde{w}_i''(x) := |x_i|^{-\nu} w_i''(|x_i| x),$$

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and
\[ \tilde{f}_i''(x) := |x_i|^{2-\nu} f_i''(|x_i| x). \]

Notice that we still have
\[ \sup_{B_{R_i} \setminus \{0\}} |r^{-\nu} \tilde{w}_i''| = 1, \quad \text{and} \quad \sup_{B_{R_i} \setminus \{0\}} |r^{2-\nu} \tilde{f}_i''| = M_i. \]

Let us now distinguish a few cases according to the value of \( x_\infty \).

1 - Assume that \( x_\infty \neq 0 \). We set \( R_\infty := 1/|x_\infty| \). Using Schauder’s estimates together with Ascoli-Arzela Theorem, and after the extraction of subsequences if this is necessary, we may pass to the limit in \( \Delta \tilde{w}_i'' = \tilde{f}_i'' \) and obtain a solution of
\[ \Delta w'' = 0, \]
in \( B_{R_\infty} \setminus \{0\} \) with \( w'' = 0 \) on \( \partial B_{R_\infty} \). We also know that
\[ \sup_{B_{R_\infty} \setminus \{0\}} |r^{-\nu} w''| \leq 1, \quad (2.18) \]
and that \( w'' \in \text{Span}\{\phi_j : j \geq j_0\} \). In addition, \( w'' \) is not identically equal to 0. Indeed, for all \( i \geq 0 \), we have \( |\tilde{w}_i''(x_i/|x_i|)| \geq 1/2 \) and, passing to the limit, we get
\[ \sup_{B_{R_\infty} \setminus \{0\}} |w''| \geq 1/2. \quad (2.19) \]

We consider the eigenfunction decomposition of \( w'' \)
\[ w'' = \sum_{j \geq 0} w_j \phi_j. \]

Now, for all \( j \geq j_0 \), there exists \( \alpha_j \in \mathbb{R} \) such that
\[ w_j = \alpha_j \left( \frac{r}{R_\infty} \right)^{\gamma_j^+} - \left( \frac{r}{R_\infty} \right)^{\gamma_j^-}. \]

However, since
\[ \gamma_j^- < \nu, \]
we see, using (2.18), that necessarily \( w_j \equiv 0 \). Hence, \( w'' \equiv 0 \) which is a contradiction.

2 - Assume that \( x_\infty = 0 \). In this case, and after the extraction of subsequences if this is necessary, we may pass to the limit in \( \Delta \tilde{w}_i'' = \tilde{f}_i'' \) and obtain a solution of
\[ \Delta w'' = 0, \]
in \( \mathbb{R}^n \setminus \{0\} \). Furthermore, we know that
\[ \sup_{\mathbb{R}^n \setminus \{0\}} |r^{-\nu} w''| \leq 1, \]

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and also that $w'' \in \text{Span}\{\phi_j : j \geq j_0\}$. Again, $w''$ is not identically equal to 0. We consider the eigenfunction decomposition of $w''$. This time, $w_j$ has to be a linear combination of $r^{\gamma_j^+}$ and $r^{\gamma_j^-}$ and has to be bounded by a constant times $r^\nu$ on all $\mathbb{R}$. However, since $\gamma_j^- < \nu < \gamma_j^+$, we see that necessarily $w_j \equiv 0$. Hence, $w'' \equiv 0$ which is the desired contradiction.

Since we have ruled out every possible case, the proof of the claim is complete. \hfill \Box

**Remark 2.3** In dimension $n = 2$ much softer arguments can be used to reach the conclusion in Case 2. Indeed, for all $j \geq j_0$, we have already shown how to prove the estimate

$$\sup_{(0,1]} |r^{-\nu} w_j| \leq (\lambda_j - \nu(n - 2 + \nu))^{-1} \sup_{(0,1]} |r^{2-\nu} f_j|. \quad (2.20)$$

But, in dimension 2, all eigenfunctions $\phi_j$ are uniformly bounded and

$$\sum_{j \geq 1} \lambda_j^{-1},$$

converges. Hence, the right hand side of (2.20) is summable and we directly obtain

$$\sup_{B_1 \setminus \{0\}} \left| r^{-\nu} \sum_{j \geq j_0} w_j \phi_j \right| \leq c \sup_{B_1 \setminus \{0\}} |r^{2-\nu} f|.$$

For all $i = 1, \ldots, N$, we will denote by $(r_i, \theta_i)$ the spherical coordinates about $a_i$, namely

$$r_i = |x - a_i| \quad \text{and} \quad \theta_i := \frac{x - a_i}{|x - a_i|}.$$

Thanks to the result of the previous Proposition, we can now prove the global result:

**Proposition 2.4** Assume that $\nu > 2 - n$ and that $\nu \notin \{\gamma_j^\pm : j \in \mathbb{N}\}$. Let $j_0 \in \mathbb{N}$ be the least index for which $\nu < \gamma_{j_0}^+$. Then,

$$\Delta : C^{2,\alpha}_{\nu}(\overline{\Omega} \setminus \Sigma) \rightarrow C^{0,\alpha}_{\nu-2}(\overline{\Omega} \setminus \Sigma),$$

is injective and the dimension of its cokernel is given by $N j_0$, where we recall that $N$ is the cardinal of $\Sigma$.

**Proof:** Since we have assumed that $\nu > 2 - n$, we see that

$$C^{0,\alpha}_{\nu-2}(\overline{\Omega} \setminus \Sigma) \subset L^1(\Omega).$$

Therefore, we already know from standard results [14] that there exists a weak solution of $\Delta w = f$ in $\Omega$ with $w = 0$ on $\partial\Omega$ such that $w \in W^{1,p}(\Omega)$, for all $p < \frac{n}{n-1}$. Furthermore, standard regularity results imply that $w \in C^{k,\alpha}_{\text{loc}}(\overline{\Omega} \setminus \Sigma)$. 33
Remark 2.4 Observe that, for more general operators, the above argument can no longer be applied since usually there is no available result in the literature that ensures the existence of \( w \), to overcome this difficulty one can for example use the "domain decomposition method" as will be described in the proof of Proposition 2.6.

We now analyze more carefully the behavior of \( w \) near any of the \( a_i \in \Sigma \).

For the sake of simplicity in the notations we assume  that \( a_i = 0 \). Thanks to Proposition 2.3, we know that there exists a unique \( \tilde{w} \) solution of

\[
\begin{cases}
\Delta \tilde{w} = f & \text{in } B_\sigma \setminus \{0\} \\
\tilde{w} - w \in \text{Span}\{\phi_0, \ldots, \phi_{j_0-1}\} & \text{on } \partial B_\sigma,
\end{cases}
\]

which belongs to \( C^{2,\alpha}_\nu(B_\sigma \setminus \{0\}) \). In particular, there exist \( \lambda_0, \ldots, \lambda_{j_0-1} \in \mathbb{R} \) such that

\[
w - \tilde{w} = \sum_{j=0}^{j_0-1} \lambda_j \sigma^n r^j \gamma^j_{\nu} \phi_j \quad \text{on } \partial B_\sigma.
\]

We set

\[
\hat{w} = \tilde{w} + \sum_{j=0}^{j_0-1} \lambda_j r^j \gamma^j_{\nu} \phi_j.
\]

By definition, we have \( \Delta(w - \hat{w}) = 0 \) in \( B_\sigma \setminus \{0\} \) and \( w - \hat{w} = 0 \) on \( \partial B_\sigma \). Since \( w - \hat{w} \in W^{1,p}(B_\sigma) \), for all \( p < n/(n-1) \), the singularity at 0 is removable and we conclude that \( w = \hat{w} \). In order to make the distinction between the different points \( a_i \), we set \( (\lambda_1, \ldots, \lambda_{j_0-1}) := (\lambda_1, \ldots, \lambda_{j_0-1}) \).

Thanks to the above analysis, we can define a continuous linear mapping

\[
\Lambda : f \in C^{0,\alpha}_\nu(\Omega \setminus \Sigma) \longrightarrow (\lambda_i^j)_{i,j} \in \mathbb{R}^{j_0N},
\]

where the indices \( i, j \) satisfy \( 1 \leq i \leq N \) and \( 0 \leq j < j_0 - 1 \). In other words, the mapping \( \Lambda \) identifies the first \( j_0 \) coefficients of the expansion of \( w \) near each \( a_i \).

Clearly, a function \( f \) belongs to the image of \( C^{0,\alpha}_\nu(\Omega \setminus \Sigma) \) by \( \Delta \) if and only if all the coefficients \( \lambda_j^j \) are equal to 0, i.e. if and only if \( \Lambda(f) = 0 \). This already proves that the dimension of the cokernel of \( \Delta \) is bounded by \( N j_0 \).

In order to prove that this dimension is indeed \( N j_0 \), it remains to show that \( \Lambda \) is surjective. We give the proof of this fact when \( n \geq 3 \). Minor modifications are needed to handle the case \( n = 2 \). Let us assume that \( \zeta \) is smooth function with compact support in \([1/2, 1]\), which is positive in \((1/2, 1)\). For all \( j \geq 0 \) and for all \( \varepsilon \leq \sigma \), we set

\[
v_j := -\varepsilon \gamma^j - 2 \int_0^{2\sigma} s^{1-n-2\gamma^j} \int_0^s t^{n-1+\gamma^j} \zeta(t/\varepsilon) \, dt \, ds,
\]

and

\[
\alpha_j := -\int_0^{+\infty} s^{1-n-2\gamma^j} \int_0^s t^{n-1+\gamma^j} \zeta(t) \, dt \, ds.
\]
Notice that
\[ \lim_{\varepsilon \to 0} v_j = \alpha_j \neq 0. \] (2.21)

Given \( \lambda := (\lambda^i_j)_{i,j} \in \mathbb{R}^{Nj_0} \), we now define, for all \( \varepsilon \in (0, 1] \), the functions
\[ W_\varepsilon(\lambda) = \sum_{i=1}^N \eta(r_i) \left( \sum_{j=0}^{j_0-1} \frac{\lambda^i_j}{\alpha_j} v_j(r_i) r_j^{\gamma_j^+} \phi_j(\theta_i) \right), \]
and
\[ F_\varepsilon(\lambda) = \sum_{i=1}^N \eta(r_i) \left( \sum_{j=0}^{j_0-1} \frac{\lambda^i_j}{\alpha_j} \zeta(r_i/\varepsilon) \phi_j(\theta_i) \right), \]
where \( \eta \) is a cutoff function identically equal to 1 in \([0, \sigma]\) and equal to 0 in \([2\sigma, +\infty)\) and where \((r_i, \theta_i)\) are the spherical coordinates about \( a_i \).

Obviously \( \Delta W_\varepsilon - F_\varepsilon \), defined in \( \Omega \setminus \Sigma \), is supported in \( A_\sigma \) and there exists a constant \( c \), independent of \( \lambda \), such that
\[ \| \Delta W_\varepsilon - F_\varepsilon \|_{0,0,\nu} \leq c \varepsilon^{n-2} \| \lambda \|. \]

Therefore
\[ \| \Lambda(\Delta W_\varepsilon - F_\varepsilon) \| \leq c \varepsilon^{n-2} \| \lambda \|. \]

It readily follows from (2.21) and from the definition of \( W_\varepsilon \) that
\[ \lim_{\varepsilon \to 0} \Lambda(\Delta W_\varepsilon(\lambda)) = \lambda. \]

We define the linear mapping \( J_\varepsilon \) by
\[ J_\varepsilon : \mathbb{R}^{Nj_0} \to \mathbb{R}^{Nj_0} \]
\[ \lambda \mapsto \Lambda(F_\varepsilon(\lambda)). \]

Collecting these results, we find that \( \lim_{\varepsilon \to 0} J_\varepsilon \) is the identity. Hence, for \( \varepsilon \) small enough, \( J_\varepsilon \) is invertible and \( \Lambda \) is surjective. \( \diamond \)

As a consequence we may now state the :

**Corollary 2.3** Assume that \( \nu > 2 - n \) and that \( \nu \notin \{ \gamma_j^\pm : j \in \mathbb{N} \} \). Let \( j_0 \in \mathbb{N} \) be the least index for which \( \nu < \gamma_{j_0}^+ \). Then
\[ \Delta : \mathcal{C}^{2,0,\alpha}_{\psi,D}(\overline{\Omega} \setminus \Sigma) \to \mathcal{C}^{0,\alpha}_{\psi-2}(\overline{\Omega} \setminus \Sigma), \]
is Fredholm of index
\[ \text{Index} = -Nj_0. \]
As in the previous proof, we let \( \eta \) be a cutoff function identically equal to 1 in \([0, \sigma]\) and equal to 0 in \([2\sigma, +\infty)\). We define

\[
K_{j_0} := \text{Span} \{ \eta(r_i) r_i^{\nu_j} \phi_j(\theta_i) : j = 0, \ldots, j_0 - 1, \ i = 1, \ldots, N \},
\]

where \((r_i, \theta_i)\) are the spherical coordinates about \(a_i\). Granted these definitions, we can restate the last result as:

**Corollary 2.4** Assume that \( \nu > 2 - n \) and that \( \nu \notin \{ \gamma_j^\pm : j \in \mathbb{N} \} \). Let \( j_0 \in \mathbb{N} \) be the least index for which \( \nu < \gamma_{j_0}^+ \). Then

\[
\Delta : C^2_{\nu, \Omega} \oplus K_{j_0} \longrightarrow C^0_{\nu - 2, \Omega},
\]

is an isomorphism.

### 2.2.3 Mapping properties of the Laplacian in the surjectivity range

Paralleling what we have done in the previous section, we determine for which values of the weight parameter \( \nu \), the operator

\[
\Delta : C^2_{\nu, D}(\Omega \setminus \Sigma) \longrightarrow C^0_{\nu - 2}(\Omega \setminus \Sigma),
\]

is surjective. Once this is done, we look more closely at the mapping properties of \( \Delta \) when the weight parameter is in the surjectivity range.

**Proposition 2.5** Assume that \( \nu < 0 \) and that \( \nu \notin \{ \gamma_j^\pm : j \in \mathbb{N} \} \). Let \( j_0 \in \mathbb{N} \) be the least index for which \( \nu > \gamma_{j_0}^- \). Then, there exists \( c > 0 \) such that, for all \( f \in C^0_{\nu - 2}(\Omega \setminus \{0\}) \) there exists \( w \in C^2_{\nu}(\Omega \setminus \{0\}) \) solution of

\[
\begin{cases}
\Delta w = 0 & \text{in } B_1 \setminus \{0\} \\
w = 0 & \text{on } \partial B_1.
\end{cases}
\]

Furthermore, \( \|w\|_{2, \nu, \Omega} \leq c \|f\|_{0, \nu - 2}. \)

**Proof:** The proof of this result is nearly identical to the proof of Proposition 2.4. Here also, we consider the eigenfunction decomposition of the function \( f \)

\[
f = \sum_{j \geq 0} f_j \phi_j,
\]

and look for \( w \) of the form

\[
w = \sum_{j \geq 0} w_j \phi_j.
\]

We see that, for all \( j \geq 0 \), we have to solve \( L_j w_j = f_j \) in \((0, 1]\), with the boundary condition \( w_j(1) = 0 \).
Case 1. If $j < j_0$ (if $j_0 = 0$, then this case is void). Here, we choose

$$w_j = r^{\gamma_j^+} \int_r^1 s^{1-n-2\gamma_j^+} \int_s^1 l^{n-1+\gamma_j^+} f_j(t) \, dt \, ds.$$  

It is then a simple exercise to see that, first this expression is well defined since $\nu < \gamma_j^-$ and $\gamma_j^+ + \gamma_j^- = 2 - n$. Moreover,

$$\sup_{B_1 \setminus \{0\}} \left| r^{-\nu} \sum_{j=0}^{j_n-1} w_j \phi_j \right| \leq c_{j_0} \sup_{B_1 \setminus \{0\}} |r^{2-\nu} f_j|,$$

for some constant $c_{j_0} > 0$, only depending on $j_0$.

Case 2. If $j \geq j_0$. This time for each $j$ there also exists some explicit formula which is given by

$$w_j(r) = -r^{\gamma_j^+} \int_r^1 s^{1-n-2\gamma_j^+} \int_s^1 l^{n-1+\gamma_j^+} f_j(t) \, dt \, ds,$$

and, using the fact that $\gamma_j^- = 2 - n - \gamma_j^+ < \nu$ together with $\nu < 0 \leq \gamma_j^+$, we obtain that

$$\sup_{(0,1]} |r^{-\nu} w_j| \leq c_j \sup_{(0,1]} |r^{2-\nu} f_j|,$$

for some constant only depending on $j$.

Again, the existence and estimate for $w_j$, for $j \geq j_0$, could have been obtained using the method of sub- and supersolutions once an appropriate barrier function is constructed. To this aim, simply observe that

$$L_j r^{\nu} = (\nu(n-2+\nu) - \lambda_j) r^{\nu-2}.$$  

By assumption $\nu < 0$ and $\gamma_j^- < \nu$, for all $j \geq j_0$. Hence,

$$\nu(n-2+\nu) - \lambda_j < 0, \quad \text{for all} \quad j \geq j_0.$$  

This, together with the fact that $L_j$ satisfies the maximum principle, implies that the function

$$r \in (0,1] \rightarrow (\lambda_j - \nu(n-2+\nu))^{-1} \left( \sup_{(0,1]} |s^{2-\nu} f_j| \right) r^{\nu},$$

can again be used as a barrier function to obtain both the existence and estimate for $w_j$.

We keep the notation

$$f'' := \sum_{j \geq j_0} f_j \phi_j.$$
It now remains to prove that there exists a constant \( c > 0 \) which does not depend on \( f'' \) such that

\[
\sup_{B_1 \setminus \{0\}} |r^{-\nu} \sum_{j \geq j_0} w_j \phi_j| \leq c \sup_{B_1 \setminus \{0\}} |r^{2-\nu} f''|.
\]

The proof of this fact is similar to the proof of the corresponding point in Proposition 2.4, therefore we omit it. \( \square \)

Using the result of the above Proposition, we can now prove the following global result:

**Proposition 2.6** Assume that \( \nu < 0 \) and that \( \nu \not\in \{\gamma_j^\pm : j \in \mathbb{N}\} \). Let \( j_0 \in \mathbb{N} \) be the least index for which \( \nu > \gamma - j_0 \). Then,

\[
\Delta : C^{2,\alpha}_{\nu, D}(\overline{\Omega} \setminus \Sigma) \rightarrow C^{0,\alpha}_{\nu, -2}(\overline{\Omega} \setminus \Sigma),
\]

is surjective and the dimension of its kernel is given by \( N j_0 \).

**Proof:** The proof relies on a domain decomposition method. Recall that, by definition \( \Omega_\sigma := \{ x \in \Omega : \text{dist}(x, \Sigma) > \sigma \} \). For the sake of simplicity, we restrict our attention to the case where \( n \geq 3 \) since minor modifications are needed to handle the 2 dimensional case.

**Step 1:** We shall introduce what we call the “exterior Dirichlet to Neumann map”. Let \( \Psi = (\psi_1, \ldots, \psi_N) \) be a \( N \)-tuple of functions in the space \( \Pi_{i=1}^N C^{2,\alpha}(\partial B_\sigma(a_i)) \). Given these boundary data, we define \( w_\Psi \in C^{2,\alpha}(\Omega_\sigma) \) to be the (unique) solution of

\[
\begin{cases}
\Delta w = 0 & \text{in } \Omega_\sigma \\
w = 0 & \text{on } \partial \Omega \\
w = \psi_i & \text{on } \partial B_\sigma(a_i) \text{ for all } i = 1, \ldots, N.
\end{cases} \tag{2.22}
\]

The “exterior Dirichlet to Neumann map” is defined by

\[
S(\Psi) := \sigma \left( \partial r_i w_\Psi |_{\partial B_\sigma(a_1)}, \ldots, \partial r_N w_\Psi |_{\partial B_\sigma(a_N)} \right). \tag{2.23}
\]

Where \( r_i = |x - a_i| \). It is easy to check that

\[
S : \Pi_{i=1}^N C^{2,\alpha}(\partial B_\sigma(a_i)) \rightarrow \Pi_{i=1}^N C^{1,\alpha}(\partial B_\sigma(a_i)),
\]

is a well defined linear bounded operator. Moreover, it is well known that this operator is elliptic, with principal symbol \(-|\xi|\). In order to have a better understanding of the operator \( S \), let us assume that \( a_i = 0 \) and consider the eigenfunction decomposition of \( \psi \in C^{2,\alpha}(\partial B_\sigma) \)

\[
\psi = \sum_{j \geq 0} \alpha_j \phi_j.
\]
Let \( v_\psi \) be the unique solution of \( \Delta v = 0 \) in \( \mathbb{R}^n \setminus \overline{B_2} \) which decays at \( \infty \) and which satisfies \( v = \psi \) on \( \partial B_2 \). Naturally, we have the explicit formula
\[
v_\psi = \sum_{j \geq 0} \alpha_j \left( \frac{r}{\sigma} \right)^{\gamma_j} \phi_j.
\]
And thus, we obtain
\[
\sigma \partial_r v_\psi |_{\partial B_2} = \sum_{j \geq 0} \gamma_j^{-} \alpha_j \phi_j.
\]
The fact that each coefficient of \( \psi \) is multiplied by \( \gamma_j^{-} \sim -\lambda_j^{1/2} \) reflects the fact that the operator \( \psi \longrightarrow \sigma \partial_r v_\psi |_{\partial B_2} \) is elliptic of principal symbol \( -|\xi| \).

Now, for all \( \Psi \in \Pi_{i=1}^N C^{2,\alpha}(\partial B_\sigma(a_i)) \), we may then define an "auxiliary Dirichlet to Neumann" map by
\[
\bar{S}(\Psi) := \sigma (\partial_r w_{\psi_1} |_{\partial B_\sigma(a_1)}, \ldots, \partial_r w_{\psi_N} |_{\partial B_\sigma(a_N)}).
\] (2.24)
This time, \( \bar{S} \) is decoupled (i.e. is in diagonal form). Let us now compare \( S \) and \( \bar{S} \). Let \( \eta \) be some cutoff function identically equal to 1 in \( B_{3\sigma}/2 \) and equal to 0 outside \( B_{2\sigma} \). We set
\[
v_\Psi := w_\Psi - \sum_{i=1}^N \eta(\cdot - a_i) v_{\psi_i},
\]
where \( w_\Psi \) has been defined above. Obviously \( \Delta v_\Psi \) is supported in \( \Omega_{3\sigma/2} \setminus \Omega_{2\sigma} \) and using classical regularity theory for the Laplacian, we find that \( \Delta v_\Psi \) is bounded by a constant times \( \|\Psi\|_{C^{2,\alpha}} \), in any \( C^{k,\alpha}(\Omega), k \geq 0 \). Hence,
\[
S - \bar{S} : \Pi_{i=1}^N C^{2,\alpha}(\partial B_\sigma(a_i)) \longrightarrow \Pi_{i=1}^N C^{1,\alpha}(\partial B_\sigma(a_i)),
\]
is compact. We conclude that \( S \) is just a compact perturbation of \( \bar{S} \).

Step 2: We now come to the definition of the "interior Dirichlet to Neumann map". Again, we let \( \Psi = (\psi_1, \ldots, \psi_N) \) be a \( N \)-tuple of functions in \( \Pi_{i=1}^N C^{2,\alpha}(\partial B_\sigma(a_i)) \). This time, we define \( w_{\psi_i} \) to be the unique solution of
\[
\begin{cases}
\Delta w_{\psi_i} = 0 & \text{in } B_\sigma(a_i) \\
w_{\psi_i} = \psi_i & \text{on } \partial B_\sigma(a_i),
\end{cases}
\] (2.25)
Then the "interior Dirichlet to Neumann map" is defined by
\[
T(\Psi) := \sigma (\partial_r w_{\psi_1} |_{\partial B_\sigma(a_1)}, \ldots, \partial_r w_{\psi_N} |_{\partial B_\sigma(a_N)}).
\] (2.26)
Again,
\[
T : \Pi_{i=1}^N C^{2,\alpha}(\partial B_\sigma(a_i)) \longrightarrow \Pi_{i=1}^N C^{1,\alpha}(\partial B_\sigma(a_i)),
\]
is a well defined linear bounded operator. And this operator is elliptic, with principal symbol \( |\xi| \).
In this simple case, we even have some explicit representation of $T$. Indeed, assume for simplicity that $a_i = 0$, if we decompose $\psi$ over eigenfunctions
\[ \psi = \sum_{j \geq 0} \alpha_j \phi_j \]
Then, $w_\psi$ is explicitly given by
\[ w_\psi = \sum_{j \geq 0} \alpha_j \left( \frac{r}{\sigma} \right) \gamma_j^+ \phi_j. \]
And thus
\[ \sigma \partial_r w_\psi |_{\partial B_\sigma} = \sum_{j \geq 0} \alpha_j \gamma_j^+ \phi_j. \]
Again, the fact that each coefficient of $\psi$ is multiplied by $\gamma_j^+ \sim \lambda_j^{1/2}$ reflects the fact that the operator $T$ is elliptic of principal symbol $|\xi|$.

**Step 3**: We claim that
\[ S - T : \Pi_{i=1}^N C^{2,\alpha}(\partial B_\sigma(a_i)) \longrightarrow \Pi_{i=1}^N C^{1,\alpha}(\partial B_\sigma(a_i)) \]
is an isomorphism.
To begin with, let us notice that $S - T$ is clearly a isomorphism. Indeed, this operator acts diagonally on $N$-tuples and, in terms of the eigenfunctions $\phi_j$, the $i^{th}$ component is determined by
\[ \sum_{j \geq 0} \alpha_j \phi_j \longrightarrow \sum_{j \geq 0} (\gamma_i^- - \gamma_i^+) \alpha_j \phi_j. \]
Notice that $\gamma_i^- - \gamma_i^+ \neq 0$ and $\gamma_i^- - \gamma_i^+ \sim -2\lambda_i^{1/2}$. In particular, this operator is Fredholm of index 0. Hence, $S - T$, which is just a compact perturbation of $S - T$, is also Fredholm of index 0. In order to prove that $S - T$ is an isomorphism, we only need to prove that it is injective.
Assume that $\Psi \in \Pi_{i=1}^N C^{2,\alpha}(\partial B_\sigma(a_i))$ satisfies
\[ (S - T)(\Psi) = 0. \]
Now, we can solve $\Delta w = 0$ with the set of boundary data given by $\psi$ on each $\partial B_\sigma(a_i)$, first in $\Omega_\sigma$ (with exterior boundary data equal to 0), and then on each $B_\sigma(a_i)$. We obtain a function $w$ which is globally defined in $\Omega$, which satisfies $\Delta w = 0$ in $\Omega_\sigma$ and in each $B_\sigma(a_i)$ and satisfies $w = 0$ on $\partial\Omega$. Furthermore, the condition $(S - T)(\Psi) = 0$ implies that $w$ is a $C^1$ function. Thus, $w$ is a weak solution of $\Delta w = 0$ in all $\Omega$. Necessarily, $w = 0$ which in turn implies that $\Psi = 0$. This ends the proof of the injectivity of $S - T$.

**Step 4**: We can now complete the proof of Proposition 2.6. For all $f \in \mathcal{C}^{0,\alpha}_{\nu - 2}(\Pi \setminus \Sigma)$, we define $w_{ext}$ to be the unique solution of
\[
\begin{cases}
\Delta w_{ext} = f & \text{in } \Omega_\sigma \\
w_{ext} = 0 & \text{on } \partial\Omega_\sigma.
\end{cases}
\]
And, for all \( i = 1, \ldots, N \), we also define \( w_{\text{int},i} \) to be the solution of

\[
\begin{align*}
\Delta w_{\text{int},i} &= f \quad \text{in } B_\sigma(a_i) \setminus \{a_i\} \\
w_{\text{int},i} &= 0 \quad \text{on } \partial B_\sigma(a_i),
\end{align*}
\]

which is given by Proposition 2.5. Notice that \( w_{\text{int},i} \in C^{2,\alpha}_\nu(B_\sigma(a_i) \setminus \{a_i\}) \). In addition, there exists some constant \( c > 0 \), such that

\[
\|w_{\text{int},i}\|_{C^{2,\alpha}_\nu} \leq c \|f\|_{C^{0,\alpha}_\nu},
\]

\[
\|w_{\text{ext}}\|_{C^{2,\alpha}} \leq c \|f\|_{C^{0,\alpha}},
\]

where all the norms are taken over the sets where the functions are defined.

To complete the construction, we now look for a solution of

\[
\Delta w_{\text{ker}} = 0 \quad \text{in } \Omega \setminus \cup_i \partial B_\sigma(a_i),
\]

which is continuous in \( \Omega \setminus \Sigma \) and such that \( w_{\text{ker}} = 0 \) on \( \partial \Omega \). In addition, we want to be able to choose \( w_{\text{ker}} \) in such a way that the function

\[
w := \begin{cases} 
  w_{\text{ext}} + w_{\text{ker}} & \text{in } \Omega, \\
  w_{\text{int},i} + w_{\text{ker}} & \text{in } B_\sigma(a_i), \quad \forall i = 1, \ldots, N,
\end{cases}
\]

is a \( C^1 \) function in all \( \Omega \setminus \Sigma \). Furthermore, we want the restriction of \( w_{\text{ker}} \) to any \( B_\sigma(a_i) \) to belong to \( C^{2,\alpha}_\nu(B_\sigma(a_i) \setminus \{a_i\}) \). Assuming we have already constructed \( w_{\text{ker}} \), we see that \( \Delta w = f \) in \( \Omega \setminus \Sigma \) with \( w = 0 \) on \( \partial \Omega \) will be a solution of our problem. Hence, this will end the construction of a right inverse for \( \Delta \).

In order to build \( w_{\text{ker}} \), we must check that we can find a solution of (2.29), which is continuous through every \( \partial B_\sigma(a_i) \) and for which the discontinuity of \( \partial_r w_{\text{ker}} \) through \( \partial B_\sigma(a_i) \) is equal to \( (\partial_r w_{\text{ext}} - \partial_r w_{\text{int},i})|_{\partial B_\sigma(a_i)} \). Since solutions of (2.29) are parameterized by their values on \( \cup_i \partial B_\sigma(a_i) \), this problem reduces to find \( \Psi \) solution of the equation

\[
(S - T)(\Psi) = -((\partial_r w_{\text{ext}} - \partial_r w_{\text{int},1})|_{\partial B_\sigma(a_1)}) \cdots \quad \cdots \quad (\partial_r w_{\text{ext}} - \partial_r w_{\text{int},N})|_{\partial B_\sigma(a_N)}).
\]

Since the existence of \( \Psi \) follows from Step 3, the properties of \( w_{\text{ker}} \) then follow from the construction of \( T \). The proof of the surjectivity of \( \Delta \) is therefore complete.

Step 5: It remains to show that the kernel of \( \Delta \) has dimension \( N j_0 \). Let \( \eta \) denotes a cutoff function identically equal to 1 in \( B_\sigma \) and identically equal to 0 outside \( B_{2\sigma} \). Given \( \lambda := (\lambda^i_j)_{i,j} \), we define

\[
w = \sum_{i=1}^N \eta(a_i) \sum_{j=0}^{j_n-1} \lambda^i_j r^i_j - \phi_j(\theta_i).
\]
Obviously, $w$ is harmonic in every $B_{\varepsilon}(a_i) \setminus \{a_i\}$ and has nonremovable singularities at each $a_i$. Moreover, the function $f := \Delta w$, defined in $\Omega \setminus \Sigma$, has support in $A_\sigma$. We extend this function by 0 at each $a_i$ and we define $v$ to be the (unique) solution of
\[
\begin{cases}
\Delta v = \Delta w & \text{in } \Omega \\
v = 0 & \text{on } \partial \Omega.
\end{cases}
\]
Then, $w - v$ belongs to $C^{2,\alpha}_2(\Omega \setminus \Sigma)$ and is harmonic in $\Omega \setminus \Sigma$. We have thus obtained a $N_{j_0}$-dimensional family of elements of the kernel of $\Delta$. Finally, it is easy to check that any element of the kernel belongs to this family. This ends the proof of the Proposition.

As a consequence we may now state the

**Corollary 2.5** Assume that $\nu < 0$ and that $\nu \notin \{\gamma_j^\pm : j \in \mathbb{N}\}$. Let $j_0 \in \mathbb{N}$ be the least index for which $\nu > \gamma_{j_0}$. Then,
\[
\Delta : C^{k,\alpha}_{\nu,D}(\Omega \setminus \Sigma) \longrightarrow C^{k-2,\alpha}_{\nu-2}(\Omega \setminus \Sigma),
\]
is Fredholm of index
\[
\text{Index} = N_{j_0}.
\]
Notice that when $n \geq 3$, if we choose $\nu \in (2 - n, 0)$, then the above operator is an isomorphism.

**Remark 2.5** When the weight parameter $\nu$ belongs to $\{\gamma_j^\pm : j \in \mathbb{N}\}$, the operator
\[
\Delta : C^{k,\alpha}_{\nu,D}(\Omega \setminus \Sigma) \longrightarrow C^{k-2,\alpha}_{\nu-2}(\Omega \setminus \Sigma),
\]
is not Fredholm since it does not even have closed range.

### 2.3 Applications to nonlinear problems

The previous framework turns out to be powerful in analyzing a certain number of nonlinear problems. The first application of this weighted spaces approach in a nonlinear context probably appeared in the work of L.A. Caffarelli, R. Hardt and L. Simon [18], [30] where minimal surfaces and hypersurfaces with isolated singularities are studied.

Later on, N. Smale has also used these spaces in [92] and [93], to produce new examples of minimal hypersurfaces with singularities. More recently, C.C. Chan [19] has also used this framework together with tools from geometric measure theory to prove the existence of minimal hypersurfaces with prescribed asymptotics at infinity. We can also mention the work of L. Mou [67], G. Liao and N. Smale [43], R. Hardt and L. Mou [33] for applications of this weighted spaces scheme to the study of harmonic maps with isolated singularities.

We now illustrate how the previous linear theory can by applied to nonlinear problems. We will only consider three examples which, in our opinion, are rather typical. All these examples are analyzed using the weighted spaces and in each of them, the crucial point is the choice of the weight parameter $\nu$.  

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2.3.1 Minimal surfaces with one catenoidal type end

As a first example, we explain how the previous framework, together with the implicit function Theorem, allows to build an infinite dimensional space of minimal surfaces with one catenoidal end (and with boundary), which are topologically equivalent to $[0, +\infty) \times S^1$. The material from this example is essentially borrowed from [54] and also from [56].

We recall that, up to a dilation and a rigid motion, a catenoid $C$ can be parameterized by

$X_0 : (s, \theta) \in \mathbb{R} \times S^1 \longrightarrow (\cosh s \cos \theta, \cosh s \sin \theta, s) \in \mathbb{R}^3$.

The normal vector field can be taken to be

$N_0 : (s, \theta) \in \mathbb{R} \times S^1 \longrightarrow \frac{1}{\cosh s} (\cos \theta, \sin \theta, -\sinh s)$.

Now, all surfaces $\tilde{C}$ close enough to $C$ can be parameterized (at least locally) as normal graphs over $C$, namely

$X = X_0 + w N_0 \in \tilde{C}$, for some scalar function $w$.

In this case, the mean curvature at the point of parameters $(s, \theta)$ is given by [54]

$H(s, \theta) = -\frac{1}{\cosh^4 s} L w + \frac{1}{\cosh s} Q \left( \frac{w}{\cosh s}, \frac{\nabla w}{\cosh s}, \frac{\nabla^2 w}{\cosh s} \right)$, \hspace{0.5cm} (2.32)

where, by definition,

$L := \cosh^2 s \left( \partial_{ss}^2 + \partial_{\theta\theta}^2 + \frac{2}{\cosh^2 s} \right)$,

is the linearized mean curvature operator about $C$ and where $Q$ is a polynomial without any constant or first order terms. In addition, all the coefficients of $Q$ only depend on $s$ and have derivatives with respect to $s$ which are bounded uniformly in $s$. To enlighten the relation between this operator and the previous analysis, let us notice that

$e^{2s} \left( \partial_{ss}^2 + \partial_{\theta\theta}^2 \right)$,

is just the Laplacian in $\mathbb{R}^2$, once the change of variables $r = e^{-s}$, for $s > 0$, is performed.

Some Jacobi fields (i.e. solutions of the homogeneous equation $L w = 0$) can be explicitly determined since they correspond to one-parameter families of minimal surfaces of which $C$ is an element. To exhibit these, first note that any smooth one-parameter family $t \rightarrow C(t)$ of minimal surfaces, with $C(0) = C$, will have differential at $t = 0$ which is a Jacobi field on $C$. For example, the one parameter family of translations along the axis of $C$ yields the Jacobi field

$\Psi^{0,+}(s, \theta) = \tanh s$. 

While, the one parameter family of dilations of $C$ yields
\[
\Psi^{0,\pm}(s, \theta) = s \tanh s - 1.
\]

Translating the axis of $C$ in the orthogonal direction, we obtain two linearly independent Jacobi fields
\[
\Psi^{1,\pm}(s, \theta) = \frac{1}{\cosh s} \cos \theta \quad \text{and} \quad \Psi^{-1,\pm}(s, \theta) = \frac{1}{\cosh s} \sin \theta,
\]
and finally, rotation of the axis of $C$ in the orthogonal direction, gives another two linearly independent Jacobi fields
\[
\Psi^{1,\pm}(s, \theta) = \left( \frac{s}{\cosh s} + \sinh s \right) \cos \theta,
\]
and
\[
\Psi^{-1,\pm}(s, \theta) = \left( \frac{s}{\cosh s} + \sinh s \right) \sin \theta.
\]

The indicial roots associated to $L$, at both $+\infty$ and $-\infty$, are given by
\[
\gamma_{j}^{\pm} = \pm j.
\]

As usual, they provide all the possible asymptotic behaviors of solutions of the homogeneous equation $L = 0$ at $\pm \infty$. Now, for all $\nu \in \mathbb{R}$ and all $S \in \mathbb{R}$, we define the weighted Hölder space
\[
C_{\nu, \alpha}^{k, \alpha}(S, +\infty) := \left\{ e^{-\nu s} w : w \in C_{loc}^{k, \alpha}([S, +\infty) \times S^1) \right\}.
\]

Notice that we recover the usual definition of $C_{\nu, \alpha}^{k, \alpha}(B_{1} \setminus \{0\})$, once we have performed the change of variables $r = e^{-s}$.

The mapping properties of $L$ can be studied as before and one proves that
\[
L : C_{\nu, \alpha}^{2, \alpha}((S, +\infty) \times S^1) \longrightarrow C_{\nu, \alpha}^{0, \alpha}((S, +\infty) \times S^1),
\]
is Fredholm provided $\nu \notin \mathbb{Z}$, is injective if $\nu > 0$, and is surjective if $\nu < 0$ and $\nu \notin \mathbb{Z}$.

We only sketch the proof of the injectivity property. Assume that $\nu > 0$ and that $w \in C_{\nu, \alpha}^{2, \alpha}((S, +\infty) \times S^1)$ solves $Lw = 0$. We decompose $w$ in Fourier series
\[
w = \sum_{j \in \mathbb{Z}} w_{j} e^{ij\theta},
\]
where the functions $w_{j}$ are complex valued.

Now, $w_{0}$ has to be a linear combination of both $\Psi^{0,\pm}$, so cannot decay exponentially unless $w_{0} \equiv 0$. Similarly $w_{-1}e^{-i\theta} + w_{1}e^{i\theta}$ has to be a linear combination of the 4 independent Jacobi fields $\Psi^{\pm 1, \pm}$. Since $w_{-1}e^{-i\theta} + w_{1}e^{i\theta}$ has to decay exponentially at $+\infty$, we readily see that this function is in fact a
linear combination of $\Psi_\pm \pm_1$. Furthermore, since $w_{-1}e^{-i\theta} + w_1e^{i\theta}$ is equal to 0 for $s = S$, it is also identically 0.

Finally, for $|j| \geq 2$, we multiply the equation $Lw = 0$ by $\cosh^{-2}s w_j e^{-ij\theta}$ and integrate by parts over $[S, +\infty) \times S^1$ to obtain

$$
\int_S^{\infty} |\partial_s w_j|^2 ds + j^2 \int_S^{\infty} |w_j|^2 ds - \int_S^{\infty} \frac{2}{\cosh^2 s} |w_j|^2 ds = 0.
$$

And, since $|j| \geq 2$, we conclude that $w_j \equiv 0$ also. This ends the proof of the injectivity of $L$ when $\nu < 0$.

Paralleling Proposition 2.3, we prove:

Proposition 2.7 Assume that $\nu > 0$ and that $\nu$ is not an indicial root, i.e. $\nu \notin \mathbb{N}$. Let $j_0 \in \mathbb{N}$ be chosen so that $j_0 - 1 < \nu < j_0$. Then, there exists $c > 0$ such that, for all $f \in C_{\nu-2}^0([S, +\infty) \times S^1)$ there exists a unique $w \in C_{\nu}^2([S, +\infty) \times S^1)$ solution of

$$
Lw = f \quad \text{in} \quad [S, +\infty) \times S^1,
$$

with boundary data given by $w(S, \cdot) \in \text{Span}\{\phi_j : j = 0, \ldots, j_0 - 1\}$. In addition, $\|w\|_{C_{\nu}^2} \leq c \|f\|_{C_{\nu-2}^0}$.

Given $S \in \mathbb{R}$, we can use the above Proposition, with $\nu \in (0, 1)$ and $j_0 = 1$, together with a straightforward application the implicit function Theorem, in order to solve

$$
\begin{align*}
\frac{\partial^3}{\partial s^3} w &= Q \left( \frac{w}{\cosh s}, \frac{\nabla w}{\cosh s}, \frac{\nabla^2 w}{\cosh s} \right) \quad \text{in} \quad [S, +\infty) \times S^1 \\
\quad w - \psi &\in \mathbb{R} \quad \text{on} \quad \{S\} \times S^1,
\end{align*}
$$

for all sufficiently small functions $\psi \in C_{2, \alpha}^0(S^1)$. Notice that $\text{Span}\{\phi_0\} = \mathbb{R}$.

Since $w$ solves (2.33), we see that the surface parameterized by $X = X_0 + w N_0$, has mean curvature 0, see (2.32). In particular, we have produced an infinite dimensional family of minimal surfaces (parameterized by the boundary data), with one end which is asymptotic to a catenoidal end.

In the past years, weighted Hölder spaces have been used to understand the moduli space of minimal surfaces in $\mathbb{R}^3$ (or constant mean curvature surfaces) [74], [57], [39] and also to build new examples of surfaces [54], [37], [99].

2.3.2 Semilinear elliptic equations with isolated singularities

We are interested in solutions of $\Delta u + u^p = 0$ with isolated singularities in some domain of $\mathbb{R}^n$, $n \geq 3$. To begin with, let us recall that, for all $p \in \left( \frac{n-2}{n-2}, \frac{n+2}{n-2} \right)$, the function $u := C |x|^{-\frac{2}{2}}$,
where \( C^{p-1} := \frac{2}{p-1} \left( n - \frac{2p}{p-1} \right) \), is a weak solution of
\[
\Delta u + u^p = 0 \quad \text{in} \quad \mathbb{R}^n,
\]
which singular at the origin (we refer to [17] and [25] for further results on
the asymptotic behavior of solutions of the above equation near an isolated
singularity).

The linearized operator about \( u \) reads
\[
\mathcal{L} := \Delta + pu^{p-1}.
\]

In polar coordinates, \( \mathcal{L} \) takes the form
\[
\mathcal{L} := \partial_{rr}^2 + \frac{n-1}{r} \partial_{r} + \frac{1}{r^2} \Delta_{S^{n-1}} + p \frac{C^{p-1}}{r^2}.
\]

As already discussed, the asymptotic behavior of solutions to \( \mathcal{L}w = 0 \) at 0 are
determined by the indicial roots of this operator. These indicial roots are given
by [53]
\[
\gamma_j^\pm := \frac{2 - n}{2} \pm \sqrt{\left( \frac{n-2}{2} \right)^2 + \lambda_j - \frac{2p}{p-1} \left( n - \frac{2p}{p-1} \right)}.
\]

Observe that, for some values of the parameter \( p \), \( \gamma_0^\pm \) could belong to \( \mathbb{C} \). Since,
\[
p \in \left( \frac{n}{n-2}, \frac{n+2}{n-2} \right),
\]
we have
\[
\ldots \leq -\frac{2}{p-1} - 1 = \gamma_1^- < -\frac{2}{p-1} < \mathcal{R}_{\gamma_0}^- \leq \frac{2 - n}{2} \leq \gamma_0^+ < \gamma_1^- \leq \ldots \quad (2.34)
\]

Moreover, it is shown in [53] that
\[
\mathcal{L} : C^{2,\alpha}_{\nu,2}(B^r_1 \setminus \{0\}) \to C^{0,\alpha}_{\nu-2}(B^r_1 \setminus \{0\}),
\]
is Fredholm provided \( \nu \notin \{ \mathcal{R}_{\gamma_j}^\pm : j \in \mathbb{N} \} \), is injective for all \( \nu > \mathcal{R}_{\gamma_0}^+ \), and is
surjective for all \( \nu < \mathcal{R}_{\gamma_0}^- \) and \( \nu \notin \{ \gamma_j^\pm : j \in \mathbb{N} \} \). In addition, if we assume
that \( \gamma_1^- < \nu < \mathcal{R}_{\gamma_0}^- \), the dimension of the kernel of \( \mathcal{L} \) is equal to 1.

In particular, we have the following :

**Proposition 2.8** Assume that \( \gamma_1^- < \nu < \mathcal{R}_{\gamma_0}^- \). Then, there exists \( c > 0 \) and
an operator
\[
G : C^{0,\alpha}_{\nu-2}(B^r_1 \setminus \{0\}) \to C^{2,\alpha}_{\nu-2}(B^r_1 \setminus \{0\}),
\]
such that, for all \( f \in C^{0,\alpha}_{\nu-2}(B^r_1 \setminus \{0\}) \), the function \( w := G(f) \) is a solution of
\[
\begin{cases}
\mathcal{L}w = f, & \text{in} \quad B^r_1 \setminus \{0\} \\
w = 0 & \text{on} \quad \partial B^r_1.
\end{cases}
\]

In addition, \( \|w\|_{C^{2,\alpha}_{\nu-2}} \leq c \|f\|_{C^{0,\alpha}_{\nu-2}} \).
If we choose $\nu \in (-\frac{2}{p-1}, \mathbb{R}^- \gamma)$ (notice that this is always possible, thanks to (2.34)), we can apply the result of Proposition 2.8, together with the implicit function Theorem, to show that, for all $x_0$ close enough to 0, there exists a one parameter family of positive weak solutions of
\[
\begin{cases}
\Delta u + u^p = 0 & \text{in } B_1 \setminus \{x_0\} \\
u = C & \text{on } \partial B_1,
\end{cases}
\]
which has a nonremovable singularity at $x_0$. Notice that the restriction $\nu > -\frac{2}{p-1}$ is needed to ensure that, near $x_0$, the solution of the above equation looks like $u = C|x - x_0|^{-\frac{2}{p-1}} + O(|x - x_0|^{\nu})$, and thus will has a nonremovable singularity at $x_0$.

For subcritical exponents, further existence results of singular solutions have been obtained by R. Mazzeo and F. Pacard [53], [73]. While, for the critical exponent, the existence results can be found in the work of R. Schoen [87], R. Mazzeo and F. Pacard [54], Y. Rebaï [77] and also in the work of R. Mazzeo, D. Pollack and K. Uhlenbeck [58] for special configuration of the singular set.

Again, weighted spaces have been used to study the moduli space of singular solutions of $\Delta u + u^p = 0$. This issue is discussed in [58] and also in [57]. Finally, still using these weighted spaces, precise asymptotics, which improve the previous results in [17], and some balancing formulæ have been established in [38].

2.3.3 Singular perturbations for Liouville equation

Here, we are interested in solutions of $\Delta u + \varrho^2 e^u = 0$ in some domain of $\mathbb{R}^2$. This equation has been extensively studied and we refer to [49], [68], [97] for precise results and further references.

For all $\varrho \in (0, 2]$, we define $\varepsilon > 0$ to be the least solution of
\[
\varrho^2 = \frac{8 \varepsilon^2}{(1 + \varepsilon^2)^2}.
\]
Then, we set
\[
\varphi_{\varepsilon} := 2 \log(1 + \varepsilon^2) - 2 \log(\varepsilon^2 + |x|^2),
\]
and we check that this is a solution of $\Delta \varphi + \varrho^2 e^{\varphi} = 0$ in $\mathbb{R}^2$. The linearized operator about $\varphi_{\varepsilon}$ reads
\[
L_{\varepsilon} w := \Delta w + \frac{8 \varepsilon^2}{(\varepsilon^2 + |x|^2)^2} w.
\]
Again, some Jacobi fields (i.e. solutions of the homogeneous equation) can be explicitly computed, since they correspond to one parameter families of solutions. For example, the Jacobi field
\[
\Psi_0(x) = \frac{\varepsilon^2 - r^2}{\varepsilon^2 + r^2},
\]
corresponds to the one parameter family of solutions \( \eta \rightarrow u(\eta x) - 2\log \eta \).

Similarly
\[
\Psi_{\epsilon}^{1}(x) = \frac{2\epsilon^2}{\epsilon^2 + \gamma^2} e^{i\theta} \quad \text{and} \quad \Psi_{\epsilon}^{-1}(x) = \frac{2\epsilon^2}{\epsilon^2 + \gamma^2} e^{-i\theta},
\]
correspond to the invariance of the equation with respect to translations.

It is proved in [5] that, for all \( \epsilon < 1 \), the operator
\[
L_{\epsilon} : C^{2,\alpha}(\overline{B_1}) \rightarrow C^{0,\alpha}(\overline{B_1}),
\]
is an isomorphism. Unfortunately, the norm of its inverse tends to \( +\infty \) as \( \epsilon \) tends to 0. To see this last fact, it is enough to notice that \( \Psi_{\epsilon}^{\pm 1} \), which is a Jacobi field, satisfies \( |\Psi_{\epsilon}^{\pm 1}| \leq 2 \) in \( B_1 \) and \( |\Psi_{\epsilon}^{\pm 1}| \leq 2\epsilon^2 \) on \( \partial B_1 \). Hence, the inverse of \( L_{\epsilon} \) cannot be bounded uniformly as \( \epsilon \) tends to 0.

In order to understand more closely the way the inverse of \( L_{\epsilon} \) blows up as \( \epsilon \) tends to 0, it is again useful to work in weighted Hölder spaces, this time, paralleling Proposition 2.3, we get

**Proposition 2.9** [5] Assume that \( 1 < \nu < 2 \). Then, there exist \( \epsilon_0 > 0 \), \( c > 0 \) and for all \( \epsilon \in (0, \epsilon_0) \), for all \( f \in C^{0,\alpha}_{\nu, -2}(\overline{B_1} \setminus \{0\}) \), there exists a unique \( w \in C^{2,\alpha}_{\nu}(\overline{B_1} \setminus \{0\}) \) solution of
\[
\begin{align*}
L_{\epsilon} w &= f, & \text{in } B_1 \setminus \{0\},
\end{align*}
\]
\[
\begin{align*}
& w \in \text{Span}\{e^{-i\theta}, 1, e^{i\theta}\} & \text{on } \partial B_1.
\end{align*}
\]
In addition, \( \|w\|_{C^{2,\alpha}} \leq c \|f\|_{C^{0,\alpha}_{\nu, -2}} \).

The key point is that, in the above result, the constant \( c > 0 \) is independent of \( \epsilon \). In other words, we have obtained, in a suitable weighted space, a right inverse for \( L_{\epsilon} \), whose norm is bounded in dependently of \( \epsilon \).

By opposition to the previous example, this time, the solutions we are interested in are not singular anymore. Nevertheless, using the weighted Hölder spaces, we are able to obtain a result which bears some features of the well known Liapunov-Schmidt decomposition. This proposition has been applied in [5] to produce multiple solutions of \( \Delta u + \rho e^u = 0 \), when the parameter \( \rho \) is small. Somehow, this can be understood as the counterpart of the theory of critical points at infinity of A. Bahri, Y. Li and O. Rey [4], [78] and [79] in a non variational framework.
Chapter 3

The Ginzburg-Landau Equation in \( \mathbb{C} \)

In this Chapter, we begin the study of Ginzburg-Landau vortices. To begin with, we define \( u_1 \), the radially symmetric solution of the Ginzburg-Landau equation in all \( \mathbb{C} \), and also \( \mathcal{L}_1 \), the linearized Ginzburg-Landau operator about \( u_1 \). Next, we carry out a careful study of all the possible asymptotic behaviors of a solution of the homogeneous equation \( \mathcal{L}_1 w = 0 \) both near the origin and near \( \infty \). This yields a classification of all bounded solutions of \( \mathcal{L}_1 w = 0 \) in \( \mathbb{C} \). This study provides a key for understanding the definition of the weighted spaces we will work with in the subsequent Chapters.

3.1 Radially symmetric solution on \( \mathbb{C} \)

We recall some well known facts about what we will call the "radially symmetric solution" of

\[
\Delta u + u (1 - |u|^2) = 0. \tag{3.1}
\]

By a radially symmetric solution, we mean a solution of (3.1) which can be written as

\[
u_1(r, \theta) := e^{i\theta} S(r). \tag{3.2}
\]

If such a solution exists, it is easy to see that the scalar function \( S \) has to be a solution of the following second order ordinary differential equation

\[
\frac{d^2 S}{dr^2} + \frac{1}{r} \frac{dS}{dr} - \frac{1}{r^2} S + S (1 - S^2) = 0. \tag{3.3}
\]

Since we want \( S \) to be a non constant bounded solution of (3.3), it is necessary to impose the following behavior at infinity

\[
\lim_{r \to +\infty} S = 1.
\]
As already mentioned, the proof of the existence of such a solution $S$, as well as some qualitative properties of the function $S$ itself, are available in [34]. The results we will need are collected in the:

**Theorem 3.1** [34] There exists a unique, non constant solution of (3.3) which is defined for all $r \geq 0$ and whose limit at $+\infty$ is 1. This solution $S$ is strictly increasing, we even have

$$\frac{dS}{dr} > 0, \quad \text{for all } r \geq 0,$$

and $0 < S < 1$, for all $r > 0$. Furthermore

$$S = 1 - \frac{1}{2r^2} + O\left(\frac{1}{r^4}\right),$$

for large $r$, and there exists some constant $\kappa > 0$ such that

$$S = \kappa r - \frac{\kappa}{8} r^3 + O(r^5),$$

for $r$ close to 0.

In addition to all these informations, we will need the following result.

**Lemma 3.1** Let us define

$$T := \frac{dS}{dr} - \frac{S}{r}.$$

Then $T < 0$ in $(0, +\infty)$.

**Proof**: Granted the asymptotic behavior of $S$ near 0 and $\infty$, one can check directly that $T < 0$ for $r > 0$ close enough to 0 or for $r$ large enough. Furthermore, since $S$ is a solution of (3.3) and since $0 < S < 1$, we have for all $r > 0$

$$\frac{dT}{dr} + 2 \frac{T}{r} = - S(1 - S^2) < 0.$$

We conclude that the function $T$ cannot achieve a positive maximum value in $(0, +\infty)$. Hence $T < 0$. \qed

### 3.2 The linearized operator about the radially symmetric solution

#### 3.2.1 Definition

For any complex valued function $u$, we define the nonlinear mapping

$$\mathcal{N}(u) := \Delta u + u(1 - |u|^2).$$

(3.4)
The linearized operator about a function \( u \) is then defined, for any complex valued function \( w \), by

\[
L \ w := \Delta w + (1 - |u|^2) w - 2 \ u (u \cdot w),
\]

where, by definition

\[
2 \ u \cdot w := \bar{u} w + w \bar{u}.
\]
is the usual scalar product in \( \mathbb{C} \).

When \( u = u_1 \), the radially symmetric solution defined above, we will denote the corresponding linearized operator by \( L_1 \). In order to study \( L_1 \), it happens to be more convenient to write any complex valued function \( w \) as

\[
w := (w_r + i w_i) e^{i \theta},
\]

where \( w_r \) and \( w_i \) are real valued functions. Granted this decomposition, it is natural to define the conjugate linearized operator by

\[
L_1 := e^{-i \theta} L_1 e^{i \theta}.
\]

(3.6)

It follows from a simple computation that

\[
L_1 (w_r + i w_i) = \left( \Delta w_r + (1 - 3 S^2) w_r - \frac{1}{r^2} w_r - \frac{2}{r^2} \partial_\theta w_i \right) + i \left( \Delta w_i + (1 - S^2) w_i - \frac{1}{r^2} w_i + \frac{2}{r^2} \partial_\theta w_r \right).
\]

(3.7)

As we will see, the study of the conjugate operator \( L_1 \) will be much easier than the study of the original linearized operator \( L_1 \).

### 3.2.2 Explicit solutions of the homogeneous problem

By definition, the Jacobi fields are solutions of the homogeneous problem \( L_1 w = 0 \). For the remaining of the analysis, it is very important to notice that we know explicitly some Jacobi fields. These Jacobi fields correspond to the invariance of (3.1) under the action of some group. The first example corresponds to the invariance of (3.1) under the one parameter group of transformation \( u \rightarrow e^{i \alpha} u \), for \( \alpha \in \mathbb{R} \). This gives us the following Jacobi field

\[
\Phi_0^1 := i e^{-i \theta} u_1 = i S.
\]

(3.8)

Using the same procedure, we also find two more Jacobi fields

\[
\Phi_{\pm 1}^1 := e^{-i \theta} \partial_x u_1
\]

\[
= \frac{1}{2} \left( \left( \frac{dS}{dr} - \frac{S}{r} \right) e^{i \theta} + \left( \frac{dS}{dr} + \frac{S}{r} \right) e^{-i \theta} \right)
\]

\[
= \frac{dS}{dr} \cos \theta - i \frac{S}{r} \sin \theta,
\]

(3.9)
\( \Phi^{-1}_1 := e^{-i\theta} \partial_y u_1 \)
\[ = \frac{i}{2} \left( - \left( \frac{dS}{dr} - \frac{S}{r} \right) e^{i\theta} + \left( \frac{dS}{dr} + \frac{S}{r} \right) e^{-i\theta} \right) \tag{3.10} \]

which correspond to the invariance of (3.1) under the action of the group of translations \( u \rightarrow u(\cdot - b) \), for \( b \in \mathbb{C} \).

### 3.3 Asymptotic behavior of solutions of the homogeneous problem

This entire subsection is devoted to the study of the asymptotic behavior of the solutions of \( L_1 w = 0 \) both at the origin and at \( \infty \). This study is a key point of our analysis since it motivates the definition of the function spaces we will work with.

#### 3.3.1 Classification of all possible asymptotic behaviors at 0

We give in the next Proposition, a complete classification of all the possible asymptotic behaviors of the solutions of \( L_1 w = 0 \) which are bounded near the origin:

**Proposition 3.1** Let \( n \in \mathbb{N} \setminus \{0\} \). Assume that

\[ w = a e^{i\theta} + b e^{-i\theta}, \]

is a solution of

\[ L_1 w = 0, \]

where \( a \) and \( b \) are complex valued functions only depending on \( r > 0 \). Then :

- either \( |w| \) is unbounded at 0 (and blows up, as \( r \) tends to 0, at least like \( r^{1-n} \), if \( n \geq 2 \) or at least like \( -\log r \), if \( n = 1 \)),

- or \( w \) is a linear combination, with real valued coefficients, of the 4 independent solutions of \( L_1 w_j^j = 0 \), for \( j = 1, \ldots, 4 \), whose behavior near the origin is given by

\[
\begin{align*}
w_1^n &= r^{n+1} (1 + \mathcal{O}(r^2)) e^{i\theta} + \mathcal{O}(r^{n+5}) e^{-i\theta}, \\
w_2^n &= i (r^{n+1} (1 + \mathcal{O}(r^2)) e^{i\theta} + \mathcal{O}(r^{n+5}) e^{-i\theta}), \\
w_3^n &= r^{n-1} (1 + \mathcal{O}(r^2)) e^{-i\theta} + \mathcal{O}(r^{n+3}) e^{i\theta}, \\
w_4^n &= i (r^{n-1} (1 + \mathcal{O}(r^2)) e^{-i\theta} + \mathcal{O}(r^{n+3}) e^{i\theta}).
\end{align*}
\]
Notice that $w_n^2 \neq i w_n^1$ and $w_n^4 \neq i w_n^3$.

In the case where $n = 0$, if we assume that $w$, is a solution of $L_1 w = 0$ which only depends on $r > 0$. Then :

- either $|w|$ is unbounded near 0 (and blows up at least like $r^{-1}$),
- or $w$ is a linear combination, with real valued coefficients, of the 2 independent solutions of $L_1 w_j^0 = 0$, for $j = 1, 2$, whose behavior near the origin is given by

$$w_1^0 = r (1 + O(r^2)) \quad \text{and} \quad w_2^0 = i r (1 + O(r^2)).$$

Notice that $w_2^0 \neq i w_1^0$.

**Proof:** Assume that $w = ae^{in\theta} + be^{-in\theta}$ is a solution of $L_1 w = 0$, with $n \geq 1$. Then, $a$ and $b$ are solutions of the following system of ordinary differential equations

$$\begin{cases}
\frac{d^2 a}{dr^2} + \frac{1}{r} \frac{da}{dr} - \frac{(n+1)^2}{r^2} a = S^2 b - (1 - 2S^2)a \\
\frac{d^2 b}{dr^2} + \frac{1}{r} \frac{db}{dr} - \frac{(n-1)^2}{r^2} b = S^2 a - (1 - 2S^2)b.
\end{cases} \quad (3.11)
$$

These ordinary differential equations are second order and the functions $a$ and $b$ are complex valued, hence the space of solutions of (3.11) is an 8-dimensional real vector space.

In order to prove the result it is sufficient to prove that there exists 8 independent solutions of this system which have the behavior described in the statement of the result, and 4 of which are bounded near 0. The proof of the existence of these solutions follows closely the proof of the classical Cauchy-Lipschitz Theorem. For example, to get the existence of the 4 independent solutions which are bounded near the origin, we rewrite the above system as

$$\begin{cases}
a = \alpha r^{n+1} + r^{n+1} \int_0^r s^{-2n-3} \int_0^s t^{n+2} (S^2 b - (1 - 2S^2)a) \, dt \, ds \\
b = \beta r^{n-1} + r^{n-1} \int_0^r s^{-2n+1} \int_0^s t^n (S^2 a - (1 - 2S^2)b) \, dt \, ds.
\end{cases} \quad (3.12)
$$

where $(\alpha, \beta) \in \mathbb{C}^2$. Obviously, any solution of (3.12) is a solution of (3.11). Now, $(\alpha, \beta)$ being fixed, we set

$$a := r^{n+1} (\alpha + v), \quad \text{and} \quad b = r^{n-1} (\beta + w).$$

Then, the existence of $(a, b)$ reduces to the existence of a fixed point $(v, w)$ for some affine mapping. Provided $r_0$ is chosen small enough, the existence of a unique fixed point $(v, w)$ in the space $C^0(0, r_0) \times C^0(0, r_0)$ follows from a standard application of the fixed point Theorem for contraction mappings, so we skip the details. Finally, the choice of the coefficients $(\alpha, \beta) \in \{(1, 0), (i, 0), (0, 1), (0, i)\}$ yields the 4 solutions $w_n^j$ which are given in the statement of the result.
Similarly, for \( n \geq 2 \), if we want to get the existence of the 4 independent solutions which blow up at the origin, it is sufficient to solve
\[
\begin{align*}
  a &= \alpha r^{-1-n} - (1-2S^2) \int_0^{r_0} t^{-n} (S^2 b - (1-2S^2) a) \, dt \, ds \\
  b &= \beta r^{-1-n} - (1-2S^2) \int_0^{r_0} t^{-n} (S^2 a - (1-2S^2) b) \, dt \, ds,
\end{align*}
\]
for \((\alpha, \beta) \in \mathbb{C}^2\). Again, for \( r_0 > 0 \) small enough, the existence of a solution of (3.13) which is defined in \((0, r_0)\), follows from a simple fixed point argument. Notice that, all the last 4 solutions do blow up at least like \( r_1^{-1-n} \), as \( r \) tends to 0. So the result is proven for all \( n \geq 2 \). When \( n = 1 \), a similar analysis leads to the existence of 4 independent solutions which blow up at least like \(-\log r\), as \( r \) tends to 0.

Finally, the case \( n = 0 \) can be treated similarly. We omit the details. \(\square\)

### 3.3.2 Classification of all possible asymptotic behaviors at \(\infty\)

Our next task will be to prove a result similar to the above one but this time for the behavior of solutions of \(L w = 0\) at \(\infty\). Before stating our result, we prove a result concerning the asymptotic behavior of some Bessel functions at \(\infty\).

**Lemma 3.2** For all \( n \geq 0 \), there exist \( r_0 > 0 \), \( C > c > 0 \) and a function \( J^+_n \), solution of
\[
\frac{d^2 J}{dr^2} + \frac{1}{r} \frac{d J}{dr} - \frac{1}{r^2} (n^2 - 2) J - 2J = 0, \quad \text{in} \quad (0, +\infty), \quad (3.14)
\]
such that
\[
e r^{-1/2} e^{\sqrt{2r}} \leq J^+_n \leq C e^{r^{-1/2} \sqrt{2r}},
\]
for all \( r > r_0 \).

For all \( n \geq 0 \) and all \( \eta > 0 \), there exist \( r_0 > 0 \), \( C_\eta > c_\eta > 0 \) and a function \( J^-_n \) solution of (3.14) in \((r_0, +\infty)\), such that
\[
e_\eta r^{-1/2-\eta} e^{-\sqrt{2r}} \leq J^-_n \leq C_\eta e^{-r^{-1/2} \eta \sqrt{2r}},
\]
for all \( r \geq r_0 \).

**Proof**: Many information about Bessel functions are available in the literature [98] and this result is a classical one. Nevertheless, we give here a short proof for the sake of completeness.

Notice that the maximum principle holds for the operator
\[
\mathcal{B} := \frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \left( 2 + \frac{1}{r^2} (n^2 - 2) \right),
\]
with
\[
\frac{d}{dr} J^+_n|_{r=r_0} = \frac{d}{dr} J^-_n|_{r=r_0} = 0.
\]
provided \( r \geq 1 \). Hence, the existence and the estimates for \( J^+_n \) are easy to obtain using the method of sub- and supersolutions once an appropriate barrier function is constructed.

For all \( \xi \in \mathbb{R} \), we compute
\[
B(\xi, e^{-\sqrt{2r}}) = -\left( \sqrt{2} \left( 1 + 2\xi \right) + (n^2 - 2 - \xi^2) \frac{1}{r} \right) r^{\xi - 1} e^{-\sqrt{2r}}.
\]

Let \( \eta > 0 \) be fixed. If we choose \( \xi := -1/2 + \eta \), the previous computation shows that the function
\[
I^+_\eta := r^{-1/2+\eta} e^{-\sqrt{2r}},
\]
is a supersolution in \( [r_0, +\infty) \), provided \( r_0 > 1 \) is chosen large enough. While, if we take \( \xi := -1/2 - \eta \), we conclude that the function
\[
I^-_\eta := r^{-1/2-\eta} e^{-\sqrt{2r}},
\]
is a subsolution in \( [r_0, +\infty) \), provided \( r_0 > 1 \) is chosen large enough. These barrier functions yield the existence and estimates for \( J^-_n \).

For all \( \xi \in \mathbb{R} \), we now compute
\[
B(\xi, e^{\sqrt{2r}}) = -\left( 2\xi \sqrt{2} + n^2 - 2 - \frac{1}{4} \right) r^{-5/2} e^{\sqrt{2r}} + \left( 4 - n^2 + \frac{1}{4} \right) \xi r^{-7/2} e^{\sqrt{2r}}.
\]

If we take \( \xi \) such that \( 2\sqrt{2} \xi > 2 - n^2 + \frac{1}{4} \), we conclude that the function
\[
\tilde{I}^+_\xi := r^{-1/2} (1 + \xi/r) e^{\sqrt{2r}},
\]
is a supersolution in \( [r_1, +\infty) \), provided \( r_1 > 1 \) is chosen large enough. While, if we take \( \xi \) such that \( 2\sqrt{2} \xi < 2 - n^2 + \frac{1}{4} \), we can conclude that the function
\[
\tilde{I}^-_\xi := r^{-1/2} (1 + \xi/r) e^{\sqrt{2r}},
\]
is a subsolution in \( [r_1, +\infty) \), provided \( r_1 > 1 \) is chosen large enough. The existence and estimates for \( J^+_n \) are now standard and left to the reader. \( \square \)

Granted the previous Lemma, we prove the :

**Proposition 3.2** Let \( n \in \mathbb{N} \setminus \{0\} \). Assume that
\[
w = a e^{i\theta} + b e^{-i\theta},
\]
is a solution of
\[
\mathcal{L}_1 w = 0,
\]
where \( a \) and \( b \) are complex valued functions only depending on \( r \). Then :
- either \( w \) is unbounded at \( +\infty \) (and blows up like \( r^n \) or like \( J_n^+ \)),
- or \( w \) decays at \( +\infty \) (at least like \( r^{-n} \) or like \( J_n^- \)).

In the case where \( n = 0 \), if \( w \) is a solution of \( L_1 w = 0 \) only depending on \( r \). If in addition \( w \) is a real valued function then, either \( w \) blows up at \( +\infty \) like \( r^n \) or decays at \( +\infty \) like \( J_n^+ \). If \( w \) is an imaginary valued function then, either \( w \) blows up at \( +\infty \) like \( \log r \) or is bounded at \( +\infty \).

**Proof**: We give the proof of the result when \( n \neq 0 \) and omit the proof of the result when \( n = 0 \) since only slight changes are needed to deal with this later case. We have already seen in the proof of the previous Proposition that the space of solutions of \( L_1(a e^{in\theta} + b e^{-in\theta}) = 0 \) is a 8-dimensional real vector space.

In order to prove the result it is sufficient to prove that there exists 8 independent solutions of this system which have the behavior described in the statement of the result, and 4 of which decay at \( +\infty \). To this aim, we define

\[
A := a + b \quad \text{and} \quad B := a - b.
\]

Using (3.11), it is a simple exercise to see that \( A \) and \( B \) satisfy

\[
\begin{align*}
\frac{d^2 A}{dr^2} + \frac{1}{r} \frac{dA}{dr} - \frac{1}{r^2} ((n^2 + 1)A + 2nB) + (1 - 2S^2)A &= S^2A \\
\frac{d^2 B}{dr^2} + \frac{1}{r} \frac{dB}{dr} - \frac{1}{r^2} ((n^2 + 1)B + 2nA) + (1 - 2S^2)B &= -S^2B.
\end{align*}
\]

(3.15)

If we take the real part of these two equations, we find that \( x := \Re A \), \( y := \Re B \),

are solutions of the system

\[
\begin{align*}
\frac{d^2 x}{dr^2} + \frac{1}{r} \frac{dx}{dr} - \frac{1}{r^2} (n^2 - 2)x - 2x &= \frac{2n}{r^2} y - 3(1 - S^2 - \frac{1}{r^2})x \\
\frac{d^2 y}{dr^2} + \frac{1}{r} \frac{dy}{dr} - \frac{n^2}{r^2} y &= \frac{2n}{r^2} x - (1 - S^2 - \frac{1}{r^2})y.
\end{align*}
\]

(3.16)

If we had taken the imaginary part of (3.15), we would have obtained that \( \tilde{x} := \Im A \) and \( \tilde{y} := \Im B \) are solution of the same system with \( x \) replaced by \( \tilde{y} \) and \( y \) replaced by \( \tilde{x} \).

In order to get the existence of 2 independent solutions of (3.16) which are bounded at \( +\infty \), we rewrite the above system as

\[
x = \alpha J_n^- - J_n^- \int_{r_0}^r (J_n^-)^{-2} t^{-1} \int_t^{+\infty} s J_n^- \left( \frac{2n}{s^2} y - 3(1 - S^2 - \frac{1}{s^2})x \right) ds dt,
\]

and

\[
y = \beta r^{-n} + r^{-n} \int_r^{+\infty} r^{2n-1} \int_t^{+\infty} s^{1-n} \left( \frac{2n}{s^2} x - (1 - S^2 - \frac{1}{s^2})y \right) ds dt,
\]

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where, \((\alpha, \beta) \in \mathbb{C}^2\). Using the fact that, for \(r\) large, we have from Theorem 3.1

\[1 - S^2 - \frac{1}{r^2} = \mathcal{O}\left(\frac{1}{r^4}\right),\]

and using the estimates of Lemma 3.2 for \(J_n^-\) (with any \(0 < \eta < 1/4\)), the existence of a solution, which is defined in \([r_0, +\infty)\) and decays at \(\infty\), follows as before from a fixed point argument. In particular, for \((\alpha, \beta) = (1, 0)\) we obtain the existence of a solution, whose behavior at \(\infty\) is given by

\[x = J_n^-(1 + \mathcal{O}(r^{-1})) \quad \text{and} \quad y = \mathcal{O}(r^{-1}) J_n^- ,\]

and, for \((\alpha, \beta) = (0, 1)\), we get another solution, whose behavior at \(\infty\) is given by

\[x = \mathcal{O}(r^{-n-1}) \quad \text{and} \quad y = r^{-n}(1 + \mathcal{O}(r^{-1})).\]

In order to get the existence of 2 independent solutions which blow up at \(+\infty\), it is enough to find solutions of the following systems

\[\begin{aligned}
&x = J_n^+ - J_n^- \int_r^{+\infty} (J_n^+)^{-2} s J_n^+ \left(\frac{2n}{s^2} y - 3(1 - S^2 - \frac{1}{s^2}) x\right) ds dt \\
y = r^n \int_{r_0}^r t^{-2n-1} \int_{r_0}^t s^{n+1} \left(\frac{2n}{s^2} x - (1 - S^2 - \frac{1}{s^2}) y\right) ds dt,
\end{aligned}\]

and

\[\begin{aligned}
&x = -J_n^+ \int_r^{+\infty} (J_n^+)^{-2} t^{-1} \int_1^t s J_n^+ \left(\frac{2n}{s^2} y - 3(1 - S^2 - \frac{1}{s^2}) x\right) ds dt \\
y = r^n - r^{-n} \int_r^{+\infty} t^{-2n-1} \int_{r_0}^t s^{n+1} \left(\frac{2n}{s^2} x - (1 - S^2 - \frac{1}{s^2}) y\right) ds dt,
\end{aligned}\]

Again, provided the constant \(r_0\) is chosen large enough, we obtain a solution whose behavior, at \(\infty\), is given by

\[x = J_n^+(1 + \mathcal{O}(r^{-1})) \quad \text{and} \quad y = \mathcal{O}(r^{-2}) J_n^+ ,\]

and another solution whose behavior, at \(\infty\), is given by

\[x = \mathcal{O}(r^{-n-2}) \quad \text{and} \quad y = r^n(1 + \mathcal{O}(r^{-2})).\]

When \(n = 0\), the equation for \(a := w\) reads

\[\frac{d^2 a}{dr^2} + \frac{1}{r} \frac{da}{dr} - \frac{1}{r^2} a = S^2 \ddot{a} - (1 - 2S^2) a,\]

and the existence of the desired solutions follow from similar arguments. \(\square\)

As a simple consequence of the former proof we have the :
Corollary 3.1 Assume that $n \neq 0$ and that $w = a e^{i n \theta} + b e^{-i n \theta}$ is a solution of $\mathcal{L}_1 w = 0$ which blows up exponentially at $+\infty$.

If both $a$ and $b$ are real valued functions only depending on $r$, then there exists a constant $c \in \mathbb{R}$ such that

$$w = c J_n^+ (\cos(n \theta) + \mathcal{O}(r^{-2})), \quad \text{for } r \text{ large enough.}$$

If both $a$ and $b$ are imaginary valued functions only depending on $r$, then there exists a constant $c \in \mathbb{R}$ such that

$$w = c J_n^+ (\sin(n \theta) + \mathcal{O}(r^{-2})), \quad \text{for } r \text{ large enough.}$$

Proof: We keep the notations of the previous proof. Since

$$a = \frac{A + B}{2}, \quad \text{and} \quad b = \frac{A - B}{2},$$

we get

$$w = A \cos(n \theta) + i B \sin(n \theta).$$

Now, if $a$ and $b$ are real valued, we have seen that

$$A = \Re A = c J_n^+ (1 + \mathcal{O}(r^{-4})) \quad \text{and} \quad B = \Re B = \mathcal{O}(r^{-2}) J_n^+,$$

for some constant $c \in \mathbb{R}$. While, if $a$ and $b$ are complex valued we have

$$B = i \Im B = i c J_n^+ (1 + \mathcal{O}(r^{-4})) \quad \text{and} \quad A = i \Im A = i \mathcal{O}(r^{-2}) J_n^+,$$

for some constant $c \in \mathbb{R}$. The result is now straightforward. \hfill \Box

As another byproduct of the proof of the previous Proposition, we also have the:

Corollary 3.2 Assume that $w$ is a real valued solution of $\mathcal{L}_1 w = 0$ which blows up exponentially at $+\infty$. Then, there exists a constant $c > 0$ such that

$$w = c J_0^+ (1 + \mathcal{O}(r^{-4})), \quad \text{for } r \text{ large enough.}$$

3.4 Bounded solution of the homogeneous problem

The main result of this section is a key result of our construction. Part of the proof of the following Theorem relies on the stability result of P. Mironescu in [64] and was communicated to us by L. Lassoued [40]:
Theorem 3.2 [64], [40]. All solutions of $L_1 w = 0$ which are defined on all $\mathbb{C}$ and bounded are linear combinations of $\Phi^0_1, \Phi^+_1$ and $\Phi^-_1$.

Proof: Let us assume that we have a function $w$ solution of $L_1 w = 0$, which is bounded and defined on all $\mathbb{C}$. We expend it into Fourier series

$$w = a_0 + \sum_{n \geq 1} (a_n e^{in\theta} + b_n e^{-in\theta}).$$

(3.17)

Our aim is to prove that $w$ is a linear combination of $\Phi^0_1, \Phi^+_1$ and $\Phi^-_1$. This will be achieved in 3 steps. In the first step, we deal with the first eigenfrequency and prove that $a_0$ is proportional to $\Phi^0_1$. Next, in the second step, we study the eigenfrequency corresponding to $e^{\pm i\theta}$ and prove that $a_1 e^{i\theta} + b_1 e^{-i\theta}$ is a linear combination of $\Phi^\pm_1$. In the third step, we prove that all other eigencomponents of $w$ are equal to 0.

Step 1. Since $w$ is bounded and solves $L_1 w = 0$, it is easy to see that $a_0$ is a bounded solution of the ordinary differential equation which is given by

$$\frac{d^2 a_0}{dr^2} + \frac{1}{r} \frac{da_0}{dr} - \frac{1}{r^2} a_0 + (1 - 2S^2) a_0 - S^2 = 0.$$ 

(3.18)

The space of solutions of (3.18) which are bounded at 0 is a 2 dimensional real vector space spanned by the functions $w^0_1$ and $w^2_0$ (which are defined in Proposition 3.1). We also know that $\Phi^0_1 := i S$ is an imaginary valued solution of $L_1 w = 0$ and, thanks to Theorem 3.1, we know that the following expansion holds

$$i S = i (\kappa r + \mathcal{O}(r^3)),$$

near the origin. Therefore, we have the identity $i S = \kappa w^2_0$. We set

$$S^* := \kappa w^1_0,$$

where $w^1_0$ is the real valued function which has been defined in Proposition 3.1. We know that the following expansion holds

$$S^* := \kappa w^3_0 = \kappa (r + \mathcal{O}(r^3)).$$

(3.19)

Thus, by definition, $S^* > 0$ for $r$ close to but not equal to 0. We now prove the:

Lemma 3.3 For all $r > 0$, $S^*(r) > 0$. Furthermore, $S^*$ blows up at $+\infty$ like $J^0_0$.

Proof: In order to prove that $S^* > 0$ for all $r > 0$, we argue by contradiction. If the result were not true, there would exist $r_0 > 0$ such that $S^* > 0$ for all $r \in (0, r_0)$,

$$S^*(r_0) = 0,$$

and

$$\frac{dS^*}{dr}(r_0) \leq 0.$$
Both \( iS \) and \( S^* \) are solutions of (3.18). Hence we obtain

\[
\frac{dS^*}{dr} S - \frac{dS}{dr} S^* = \frac{2}{r} \int_0^r S^3 S^*_t \, dt.
\]  

(3.20)

Evaluating this expression at \( r_0 \) and using the fact that \( S > 0 \), we obtain

\[
0 \geq \frac{dS^*}{dr} (r_0) S(r_0) = \frac{2}{r} \int_0^{r_0} S^3 S^*_t \, dt > 0.
\]

Which is the desired contradiction. The proof of the positivity of \( S^* \) is therefore complete.

Now that we have proven that \( S^* > 0 \), we may use once more (3.20) together with the fact that \( S \) is increasing, to obtain the inequality

\[
\frac{dS^*}{dr} S \geq \frac{2}{r} \int_0^r S^3 S^*_t \, dt.
\]

Using the fact that \( \lim_{r \to +\infty} S = 1 \), we conclude that \( S^* \) blows up faster than any power of \( r \) at \(+\infty\). Hence, using the result of Proposition 3.2, we conclude that \( S^* \) blows up like \( J_0^\ast \) at \(+\infty\). \( \Box \)

The space of solutions of (3.18) which are bounded near the origin being 2 dimensional, we have therefore proved that, up to a constant \( \Phi_0^1 \) is the only solution of (3.18) which is bounded on all \( C \).

**Step 2.** Now we treat the case \( n = 1 \), we have already seen in (3.11) that \( a_1 \) and \( b_1 \) solve the following system of ordinary differential equations

\[
\begin{align*}
\frac{d^2 a_1}{dr^2} + \frac{1}{r} \frac{da_1}{dr} - \frac{4}{r^2} a_1 + (1 - 2S^2) a_1 - S^2 b_1 &= 0 \\
\frac{d^2 b_1}{dr^2} + \frac{1}{r} \frac{db_1}{dr} + (1 - 2S^2) b_1 - S^2 a_1 &= 0.
\end{align*}
\]

(3.21)

As in the previous case, we take advantage from the fact that we know explicitly some bounded solutions of \( \mathcal{L}_1 w = 0 \). The first one is given by \( \Phi_{1'}^1 \), which we decompose as

\[
\Phi_{1'}^1 := \alpha_{1'} e^{i\theta} + \beta_{1'} e^{-i\theta}.
\]

We know from (3.5) that

\[
\alpha_1 := \frac{1}{2} \left( \frac{dS}{dr} - \frac{S}{r} \right) = -\frac{\kappa}{8} r^2 (1 + O(r^2))
\]

\[
\beta_1 := \frac{1}{2} \left( \frac{dS}{dr} + \frac{S}{r} \right) = \kappa (1 + O(r^2)),
\]

for \( r \) close to 0. Hence, it follows from Proposition 3.1 that

\[
\Phi_{1'}^1 = \kappa w_1^3 - \frac{\kappa}{8} w_1^1.
\]
Using again the result of Proposition 3.1, we now define
\[ \Psi_{1}^{+1} := \kappa w_{1}^{3} + \frac{\kappa}{8} w_{1}^{1}, \]
which also satisfies \( L_{1} \Psi_{1}^{+1} = 0 \). We decompose
\[ \Psi_{1}^{+1} := \alpha_{1}^{*} e^{i\theta} + \beta_{1}^{*} e^{-i\theta}, \]
and we see that \( (\alpha_{1}^{*}, \beta_{1}^{*}) \) is another real valued (independent) solution of (3.21).
This time we have
\[ \alpha_{1}^{*} = \frac{\kappa}{8} r^{2} (1 + O(r^{2})) \quad \text{and} \quad \beta_{1}^{*} = \kappa (1 + O(r^{2})), \quad (3.22) \]
for \( r \) close to 0.

Similarly, we know that \( \Phi_{1}^{-1} \) is a bounded solution of \( L_{1} w = 0 \), which we decompose as
\[ \Phi_{1}^{-1} := i (\alpha_{-1} e^{i\theta} + \beta_{-1} e^{-i\theta}). \]
As above, \( (i \alpha_{-1}, i \beta_{-1}) \) is an imaginary valued solution of (3.21) which is bounded for all \( r \geq 0 \). In addition
\[ \alpha_{-1} := -\frac{1}{2} \left( \frac{dS}{dr} - \frac{S}{r} \right) = -\frac{\kappa}{8} r^{2} (1 + O(r^{2})) \]
\[ \beta_{-1} := \frac{1}{2} \left( \frac{dS}{dr} + \frac{S}{r} \right) = \kappa (1 + O(r^{2})), \]
for all \( r \) close to 0. Using once more the result of Proposition 3.1, we get the identity
\[ \Phi_{1}^{-1} = \kappa w_{1}^{4} + \frac{\kappa}{8} w_{1}^{2}. \]
We define, as above
\[ \Psi_{1}^{-1} := \kappa w_{1}^{4} - \frac{\kappa}{8} w_{1}^{2}, \]
which, thanks to Proposition 3.1, solves \( L_{1} \Psi_{1}^{-1} = 0 \). We decompose
\[ \Psi_{1}^{-1} := i (\alpha_{-1}^{*} e^{i\theta} + \beta_{-1}^{*} e^{-i\theta}), \]
and we see that \( (i \alpha_{-1}^{*}, i \beta_{-1}^{*}) \) is another imaginary valued (independent) solution of (3.21). Moreover
\[ \alpha_{-1}^{*} = -\frac{\kappa}{8} r^{2} (1 + O(r^{2})), \quad \text{and} \quad \beta_{-1}^{*} = \kappa (1 + O(r^{2})), \quad (3.23) \]
for \( r \) close to 0.

We now prove the :

**Lemma 3.4** For all \( r > 0 \), \( \alpha_{1}^{*} > 0 \) and \( \beta_{1}^{*} > 0 \). In addition both functions blow up at \( +\infty \) like \( J_{1}^{+} \).
Proof: Using (3.21), we get, after having performed an integration
\[
\begin{align*}
\frac{d\alpha_1^*}{dr} \alpha_{-1} - \frac{d\alpha_{-1}}{dr} \alpha_1^* &= \frac{1}{r} \int_0^r S^2 (\beta_{-1} \alpha_1^* + \alpha_{-1} \beta_{-1}^*) \, dt \\
\frac{d\beta_{-1}^*}{dr} \beta_{-1} - \frac{d\beta_{-1}}{dr} \beta_{-1}^* &= \frac{1}{r} \int_0^r S^2 (\beta_{-1} \alpha_1^* + \alpha_{-1} \beta_{-1}^*) \, dt.
\end{align*}
\] (3.24)

We first prove that
\[
\alpha_1^* > 0, \quad \text{and} \quad \beta_{-1}^* > 0 \quad \text{for} \quad r > 0.
\]

Observe that, granted the construction of \( \alpha_1^* \) and \( \beta_{-1}^* \), these inequalities are already known to be true for \( r < 0 \). To prove the result, we argue by contradiction and assume that the result is not true. Then, there would exist \( r_0 > 0 \) such that
\[
\alpha_1^*(r_0) > 0, \quad \text{and} \quad \beta_{-1}^*(r_0) > 0 \quad \text{in} \quad (0, r_0),
\]
and, for example
\[
\alpha_1^*(r_0) = 0, \quad \text{and} \quad \frac{d\beta_{-1}^*}{dr}(r_0) \leq 0.
\]

Using the first equality in (3.24), the fact that \( \beta_{-1} > 0 \) for all \( r > 0 \) and the result of Lemma 3.1, which states that \( \alpha_{-1} > 0 \) for all \( r > 0 \), we readily find
\[
0 \geq \frac{d\alpha_1^*(r_0)}{dr} \alpha_{-1}(r_0) = \frac{1}{r_0} \int_0^{r_0} S^2 (\beta_{-1} \alpha_1^* + \alpha_{-1} \beta_{-1}^*) \, dt > 0.
\]

Which is the desired contradiction. The other case, where
\[
\beta_{-1}^*(r_0) = 0, \quad \text{and} \quad \beta_{-1}^*(r_0) \leq 0,
\]
can be treated similarly using this time the second equation of (3.24), we omit the details. The proof of the positivity of both \( \alpha_1^* \) and \( \beta_{-1}^* \) is therefore complete.

We now prove that these two functions blow up at \( +\infty \). To this aim we use once more (3.24), in order to get
\[
r (\alpha_{-1})^2 \frac{d}{dr} \left( \frac{\alpha_1^*}{\alpha_{-1}} \right) = \int_0^r S^2 (\beta_{-1} \alpha_1^* + \alpha_{-1} \beta_{-1}^*) \, dt.
\] (3.25)

Notice that the function on the right hand side is positive, increasing. Taking advantage from the fact that
\[
\alpha_{-1} = \frac{1}{2r} + O \left( \frac{1}{r^3} \right),
\]
at \( \infty \), we see that there exists \( c > 0 \) such that
\[
\frac{d}{dr} \left( \frac{\alpha_1^*}{\alpha_{-1}} \right) \geq cr,
\]
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for all \( r > 1 \). It is now easy to conclude that \( \alpha^*_1 \) blows up at \( \infty \), and, by Proposition 3.2, we even know that it blows up like \( J_{1}^+ \).

Using similar arguments, together with the fact that

\[
\beta_{-1} = \frac{1}{2r} + \mathcal{O}\left(\frac{1}{r^3}\right),
\]

at \( \infty \), we show that \( \beta^*_1 \) also blows up exponentially at \( \infty \).

The same analysis can be carried out for the imaginary part of the system (3.21). Things being almost identical to what we have already done, we will only sketch the proof of the:

**Lemma 3.5** For all \( r > 0 \), \( \alpha^*_1 < 0 \) and \( \beta^*_1 > 0 \). In addition both functions blow up at \( +\infty \) like \( J_{1}^+ \).

**Proof**: As above, using (3.21), we get, after having performed an integration,

\[
\begin{aligned}
\frac{d\alpha^*_1}{dr} \alpha_1 - \frac{d\alpha_1}{dr} \alpha^*_1 &= -\frac{1}{r} \int_0^r S^2 (\beta^*_1 \alpha_1 + \alpha^*_1 \beta_1) t dt \\
\frac{d\beta^*_1}{dr} \beta_1 - \frac{d\beta_1}{dr} \beta^*_1 &= -\frac{1}{r} \int_0^r S^2 (\beta^*_1 \alpha_1 + \alpha^*_1 \beta_1) t dt.
\end{aligned}
\]

(3.26)

This time, we want to prove that

\( \alpha^*_1 < 0 \), and \( \beta^*_1 > 0 \) for all \( r > 0 \).

Again, we already know that these inequalities are true for small \( r \). We argue by contradiction and assume that the result is not true. In this case, there would exist \( r_0 > 0 \) such that

\[
\alpha^*_1 < 0, \quad \text{and} \quad \beta^*_1 > 0 \quad \text{in} \quad (0, r_0),
\]

and, for example,

\[
\alpha^*_1 (r_0) = 0, \quad \text{and} \quad \frac{d\beta^*_1}{dr} (r_0) \geq 0.
\]

Using the first equality in (3.26), the fact that \( \beta_1 > 0 \) for all \( r > 0 \), as well as the result of Lemma 3.1, which also states that \( \alpha_1 < 0 \) for all \( r > 0 \), we readily find

\[
0 \geq \frac{d\alpha^*_1}{dr} \alpha_1 = -\frac{1}{r} \int_0^r S^2 (\beta^*_1 \alpha_1 + \alpha^*_1 \beta_1) t dt > 0,
\]

which is the desired contradiction. The other case, where

\[
\beta^*_1 (r_0) = 0, \quad \text{and} \quad \frac{d\beta^*_1}{dr} (r_0) \leq 0,
\]

can be treated similarly using this time the second equation of (3.26), we omit the details and consider the proof to be complete.
The proof of the fact that these functions blow up at $\infty$ is identical to what we have already done, therefore we omit it.

We have therefore proved that linear combinations of $\Phi_1^+$ and $\Phi_1^-$ are the only bounded solutions of $\mathcal{L}_1(a_1 e^{i\theta} + b_1 e^{-i\theta}) = 0$.

Step 3. [64], [40]. It remains to study all eigenfrequencies corresponding to $n \geq 2$. To this aim, we set

$$Q(w) := -\frac{1}{4\pi} \int_C \left( w \mathcal{L}_1 w + \bar{w} \mathcal{L}_1 w \right) r \, dr \, d\theta,$$

which is well defined for all $w \in H^1(C)$.

Let us now consider the special case where

$$w := a e^{in\theta} + b e^{-in\theta},$$

for $n \neq 0$, where $a$ and $b$ only depend on $r$. In this case, we simply have

$$Q(w) = \int_0^{+\infty} \left( \left| \frac{d}{dr} a \right|^2 + \left| \frac{d}{dr} b \right|^2 + \frac{(n+1)^2}{r^2} |a|^2 + \frac{(n-1)^2}{r^2} |b|^2 \right) r \, dr$$

$$- \int_0^{+\infty} (1 - S^2) (|a|^2 + |b|^2) r \, dr + \frac{1}{2} \int_0^{+\infty} S^2 |a + b|^2 r \, dr.$$

While, if $n = 0$ and $w := a$, we have

$$Q(w) = \int_0^{+\infty} \left( \left| \frac{d}{dr} a \right|^2 + \frac{1}{r^2} |a|^2 \right) r \, dr - \int_0^{+\infty} (1 - S^2) |a|^2 r \, dr$$

$$+ \frac{1}{2} \int_0^{+\infty} S^2 |a + a|^2 r \, dr.$$

Now, for any function

$$v := a e^{-in\theta} + b e^{in\theta},$$

we may define the auxiliary function

$$\tilde{v} := i (|a|^2 + |b|^2)^{1/2}.$$

Using Cauchy-Schwartz inequality we obtain

$$\left( \frac{d}{dr}(|a|^2 + |b|^2)^{1/2} \right)^2 \leq \left| \frac{da}{dr} \right|^2 + \left| \frac{db}{dr} \right|^2.$$

Hence

$$\int_0^{+\infty} \left( \frac{d}{dr}(|a|^2 + |b|^2)^{1/2} \right)^2 r \, dr \leq \int_0^{+\infty} \left( \left| \frac{da}{dr} \right|^2 + \left| \frac{db}{dr} \right|^2 \right) r \, dr.$$
This inequality yields
\[
Q(v) - Q(\tilde{v}) \geq \int_{0}^{+\infty} \left( n(n+2) \frac{|a|^2}{r^2} + n(n-2) \frac{|b|^2}{r^2} \right) r \, dr \\
+ \frac{1}{2} \int_{0}^{+\infty} (1-S^2) |a+b|^2 r \, dr.
\]
(3.27)

Therefore, if \( n \geq 2 \), we find
\[
Q(v) \geq Q(\tilde{v}).
\]
(3.28)

Let us now assume that \( n \geq 2 \) and that \( v := a e^{i\theta} + b e^{-i\theta} \) is a bounded solution of \( L_1 v = 0 \) in all \( \mathbb{C} \). It follows from Proposition 3.2 that \( v \in H^1(\mathbb{C}) \) and, since \( L_1 v = 0 \), we obtain
\[
Q(v) = 0.
\]

On the other hand, a result of P. Mironescu \[64\] asserts that
\[
Q(\tilde{v}) \geq 0.
\]

This, together with (3.28), yields
\[
Q(\tilde{v}) = Q(v) = 0.
\]

In particular, it follows directly from the expression of \( Q(v) - Q(\tilde{v}) \) that
\[
a = b = 0.
\]

The proof of the Theorem is therefore complete. \( \square \)

As a byproduct of the previous proof, we have the :

**Corollary 3.3** Let \( 0 \leq r_1 < r_2 \leq +\infty \) and assume that \( w \) is a bounded solution of
\[
L_1 w = 0 \quad \text{in} \quad B_{r_2} \setminus \overline{B_{r_1}},
\]
with \( w = 0 \) on \( \partial B_{r_2} \) if \( r_1 > 0 \), and on \( \partial B_{r_1} \) if \( r_2 < +\infty \). Further assume that \( w \) has no eigencomponent corresponding to the eigenfrequencies \( n = -1, 0, 1 \), in other words, \( w \) belongs to the space
\[
\text{Span}\{ h_{\pm n} e^{\pm i\theta} : n \geq 2 \}.
\]
Then, \( w = 0 \).

**Proof** : To obtain this result, it suffices to extend the solution \( w \) by 0 outside \( B_{r_2} \setminus \overline{B_{r_1}} \) and apply the arguments of Step 3 in the previous proof. \( \square \)
3.5 More solutions of the homogeneous equation

We end this Chapter with some definitions. The functions $\Phi^0$ and $\Phi^{\pm 1}$ have already been defined in (3.8), (3.9) and (3.10). In the proof of Theorem 3.2, we have defined many others solutions of the homogeneous problem $L_1 w = 0$.

In Step 1 of the proof of Theorem 3.2, we have defined

$$\Psi^0 := S^* := \kappa w_0^2.$$ 

Let us recall that the properties of this function are stated in Lemma 3.3 as well as in (3.18).

In Step 2 of the proof of Theorem 3.2, we have defined

$$\Psi^{+1} := \alpha^*_1 e^{i\theta} + \beta^*_1 e^{-i\theta} := \kappa w_1^3 + \frac{\kappa}{8} w_1^1,$$

and

$$\Psi^{-1} := i (\alpha^*_1 e^{i\theta} + \beta^*_1 e^{-i\theta}) := \kappa w_1^4 - \frac{\kappa}{8} w_1^2.$$

The properties of these functions are stated in Lemma 3.4 and in (3.23). We will also need the :

**Lemma 3.6** At $\infty$, the following expansions hold

$$\Phi^{+1} = -i \frac{S}{r} \sin \theta + O \left( \frac{1}{r^3} \right)$$

$$\Phi^{-1} = i \frac{S}{r} \cos \theta + O \left( \frac{1}{r^3} \right).$$

(3.29)

In addition, there exists some constant $c_+, c_- \in \mathbb{R} \setminus \{0\}$ such that

$$\Psi^{+1} = c_+ J_1^+ \left( \cos \theta + O \left( \frac{1}{r^3} \right) \right)$$

$$\Psi^{-1} = c_- J_1^- \left( \sin \theta + O \left( \frac{1}{r^2} \right) \right).$$

(3.30)

**Proof:** The first expansions concerning $\Phi^{+1}$ follow at once from the explicit expression for these two functions as well as from Theorem 3.1. While, the second expansions follow from Corollary 3.1. \qed
Finally, we now define
\[ \Psi_{\pm 2} := w_{2} \]
\[ := \alpha_{2}^{*} e^{2i\theta} + \beta_{2}^{*} e^{-2i\theta}, \] (3.31)
and
\[ \Psi_{1}^{-2} := w_{2}^{4} \]
\[ := i \left( \alpha_{-2}^{*} e^{2i\theta} + \beta_{-2}^{*} e^{-2i\theta} \right), \] (3.32)
which are respectively the unique solution of \( L_{\varepsilon} w = 0 \) whose behavior near 0 is given by
\[ \alpha_{2}^{*} = O(r^{5}) \quad \text{and} \quad \beta_{2}^{*} = r \left( 1 + O(r^{2}) \right), \]
respectively by
\[ \alpha_{-2}^{*} = O(r^{5}) \quad \text{and} \quad \gamma_{-2}(r, \theta) = r \left( 1 + O(r^{2}) \right). \]

There is a qualitative difference between all the different solutions of the homogeneous equation \( L_{\varepsilon} w = 0 \) which have been defined so far. Indeed, all the solutions \( \Psi_{\pm n} \) blow up at \( \infty \), while all solutions \( \Phi_{\pm n} \) are bounded.

**Remark 3.1** The inspection of Proposition 3.1 shows that all the above defined solutions constitute the collection of (linearly independent) solutions of the homogeneous equation \( L_{1} w = 0 \) which are bounded at the origin, but which are not bounded near 0 by any constant times \( r^{\mu} \), when the parameter \( \mu > 1 \).

### 3.6 Introduction of the scaling factor

For all \( \varepsilon > 0 \), we define the Ginzburg-Landau equation as
\[ \Delta u + \frac{u}{\varepsilon^{2}} (1 - |u|^{2}) = 0. \] (3.33)

Obviously,
\[ u_{\varepsilon}(z) := u_{1}(z/\varepsilon) = e^{i\theta} S(r/\varepsilon), \] (3.34)
is a solution of (3.33). For the sake of simplicity in the notations, we define
\[ S_{\varepsilon} := S(\cdot/\varepsilon). \]

We also introduce the nonlinear operator
\[ N_{\varepsilon}(u) := \Delta u + \frac{u}{\varepsilon^{2}} (1 - |u|^{2}), \] (3.35)
whose linearization about \( u_{\varepsilon} \) will be denoted by \( L_{\varepsilon} \). Again, we consider the conjugate operator
\[ \mathcal{L}_{\varepsilon} := e^{-i\theta} L_{\varepsilon} e^{i\theta}. \]
For the time being, let us just notice that, if \( w_\varepsilon \) and \( f_\varepsilon \) solve
\[
L_\varepsilon w_\varepsilon = f_\varepsilon,
\]
then
\[
w_1 := w_\varepsilon(\varepsilon \cdot), \quad \text{and} \quad f_1 := \varepsilon^2 f_\varepsilon(\varepsilon \cdot),
\]
solve \( L_1 w_1 = f_1 \). Hence, the study of \( L_\varepsilon \) reduces to the study of \( L_1 \). In particular, all the results we have obtained for \( L_1 \) can be translated for \( L_\varepsilon \).

In order to simplify the notations we set
\[
\Phi_j^\varepsilon := \Phi_j^1(\varepsilon \cdot /\varepsilon), \quad \text{and} \quad \Psi_k^\varepsilon := \Psi_k^1(\varepsilon \cdot /\varepsilon),
\]
for \( j = -1, 0, +1 \) and \( k = -2, -1, 0, +1, +2 \), which are for \( L_\varepsilon \) the Jacobi fields corresponding to the \( \Phi_j^1 \) and \( \Psi_k^1 \). For the same reason, we will also write
\[
S^\varepsilon_* := S^*_(\varepsilon \cdot /\varepsilon).
\]

Finally, for all \( \tau \in \mathbb{R} \), we may define
\[
u_{\varepsilon, \tau} := e^{i\tau} u_\varepsilon.
\]
Observe that this is also a solution of (3.33). Now, the corresponding linearized operator about \( u_{\varepsilon, \tau} \) is given by
\[
L_{\varepsilon, \tau} w := \Delta w + w \frac{u_{\varepsilon, \tau}}{\varepsilon^2} (1 - |u_{\varepsilon, \tau}|^2) - \frac{2}{\varepsilon^2} u_{\varepsilon, \tau} (u_{\varepsilon, \tau} \cdot w),
\]
and clearly depends on \( \tau \). On the other hand the conjugate operator
\[
\mathcal{L}_{\varepsilon, \tau} := e^{-i(\theta + \tau)} L_{\varepsilon, \tau} e^{i(\theta + \tau)},
\]
is equal to \( L_\varepsilon \) and thus does not depend on \( \tau \). Therefore, for all \( \tau \in \mathbb{R} \), the study of \( \mathcal{L}_{\varepsilon, \tau} \) is equivalent to the study of \( L_\varepsilon \).
Chapter 4

Mapping Properties of $\mathcal{L}_\varepsilon$

This Chapter is devoted to the study of the problem

$$\begin{cases}
\mathcal{L}_\varepsilon w = f & \text{in } B_1 \setminus \{0\} \\
w = 0 & \text{on } \partial B_1,
\end{cases} \quad (4.1)$$

for all functions $f$ defined in the punctured unit ball. We will first derive some simple estimates which come from the fact that the operator $\mathcal{L}_\varepsilon$ satisfies some maximum principle type property, away from the origin. Then, we will define the family of weighted spaces in which (4.1) will be studied. The main result of this Chapter is the construction of some well behaved right inverse for $\mathcal{L}_\varepsilon$ in the unit ball.

4.1 Consequences of the maximum principle in weighted spaces

The study of equation (4.1) partially relies on the following Propositions whose proof involves the maximum principle. The results are divided in two different sections according to the Fourier decomposition of the functions $w$ and $f$.

These results are of particular importance since they also help to understand the motivations behind the definition of the function spaces we are going to work with in the first part of the book.

4.1.1 Higher eigenfrequencies

We first consider the case where both functions $w$ and $f$ do not have any eigencomponents corresponding to the eigenfrequencies $n = -1, 0, 1$, in their Fourier decompositions. We show that $w$, solution of $\mathcal{L}_1 w = f$, is controlled, in $B_{r_1} \setminus B_{r_0}$, by $f$ in the same set and $w$ on the boundary, provided $r_0$ and $r_1$ are large enough.
Proposition 4.1 Assume that $\mu \in (1, 2)$. There exist some constants $\lambda > 0$ and $c > 0$ such that, for all $r_0 \geq \lambda$ and all $r_1 \geq 2r_0$ and for all

$$w, f \in \text{Span}\{h_{\pm n} e^{\pm in\theta} : n \geq 2\},$$

solutions of

$$\begin{cases}
\mathcal{L}_1 w = f & \text{in } B_{r_1} \setminus \overline{B_{r_0}} \\
w = 0 & \text{on } \partial B_{r_1},
\end{cases}$$

we have

$$\sup_{B_{r_1} \setminus B_{r_0}} r^{-\mu} \left( \frac{r^2}{r_0^2} |w_r| + |w_i| \right) \leq c \left( \sup_{\partial B_{r_0}} r^{-\mu} |w| + \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\mu} |f| \right),$$

where $w := w_r + iw_i$ is the decomposition of $w$ into its real and imaginary part.

The proof of this Proposition relies on the following two preliminary results.

Lemma 4.1 Assume that $\nu \in (0, 2)$. There exists a constant $c_1 > 0$ such that, for all $r_0 > 0$, $r_1 > 2r_0$ and for all

$$w, g, \tilde{g} \in \text{Span}\{h_{\pm n} e^{\pm in\theta} : n \geq 2\},$$

satisfying

$$\begin{cases}
\Delta w = g + \partial_\theta \tilde{g} & \text{in } B_{r_1} \setminus \overline{B_{r_0}} \\
w = 0 & \text{on } \partial B_{r_1},
\end{cases}$$

we have

$$\sup_{B_{r_1} \setminus B_{r_0}} r^{-\nu} |w| \leq c_1 \left( \sup_{\partial B_{r_0}} r^{-\nu} |w| + \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\nu} (|g| + |\tilde{g}|) \right).$$

Before we proceed to the proof of the Lemma, let us notice that, in the statement, the constant $c_1$ does not depend on $r_0$ nor on $r_1$.

Proof: To get the desired estimate, we first define $v$ to be the solution of

$$\begin{cases}
\Delta v = 0 & \text{in } B_{r_1} \setminus \overline{B_{r_0}} \\
v = 0 & \text{on } \partial B_{r_1},
\end{cases}$$

$$v = w \text{ on } \partial B_{r_0}.$$

The maximum principle yields the estimate

$$\sup_{B_{r_1} \setminus B_{r_0}} |v| \leq \sup_{\partial B_{r_0}} |w|,$$
which, together with the fact that we have assumed \( \nu > 0 \) implies that
\[
\sup_{B_{r_1} \setminus B_{r_0}} r^{-\nu} |v| \leq \sup_{\partial B_{r_0}} r^{-\nu} |w|.
\] (4.2)

Now, we define \( \hat{v} \) to be the solution of
\[
\begin{aligned}
\Delta \hat{v} &= g & \text{in } & B_{r_1} \setminus B_{r_0} \\
\hat{v} &= 0 & \text{on } & \partial B_{r_1} \cup \partial B_{r_0}.
\end{aligned}
\]

We decompose both \( \hat{v} \) and \( g \) into Fourier series and write
\[
g := \sum_{|n| \geq 2} g_n e^{in\theta} \quad \text{and} \quad \hat{v} := \sum_{|n| \geq 2} \hat{v}_n e^{in\theta}.
\]

For all \( |n| \geq 2 \), the function \( \hat{v}_n \) is a solution of
\[
\frac{d^2 \hat{v}_n}{dr^2} + \frac{1}{r} \frac{d \hat{v}_n}{dr} - \frac{n^2}{r^2} \hat{v}_n = g_n
\]
in \((r_0, r_1)\). In addition, we have the boundary conditions \( \hat{v}_n(r_0) = \hat{v}_n(r_1) = 0 \).

We set
\[
M := \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\nu} |g|,
\]
so that \( |g_n| \leq M r^{\nu-2} \). Since we have assumed that \( \nu \in (0, 2) \), it is easy to see that, for all \( |n| \geq 2 \), the function
\[
r \rightarrow \frac{M}{n^2 - \nu^2} r^\nu,
\]
can be used as a barrier function to obtain the estimate
\[
|\hat{v}_n| \leq \frac{M}{n^2 - \nu^2} r^\nu.
\]

Summation over \( n \) yields
\[
\sup_{B_{r_1} \setminus B_{r_0}} r^{-\nu} |\hat{v}| \leq \left( \sum_{|n| \geq 2} \frac{1}{n^2 - \nu^2} \right) \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\nu} |g|.
\] (4.3)

Finally, we define \( \tilde{v} \) to be the solution of
\[
\begin{aligned}
\Delta \tilde{v} &= \tilde{g} & \text{in } & B_{r_1} \setminus B_{r_0} \\
\tilde{v} &= 0 & \text{on } & \partial B_{r_1} \cup \partial B_{r_0}.
\end{aligned}
\]

And, thanks to what we have already proven in (4.3), we have the bound
\[
\sup_{B_{r_1} \setminus B_{r_0}} r^{-\nu} |\tilde{v}| \leq c \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\nu} |\tilde{g}|.
\]
Moreover, using rescaled Schauder’s estimates of Lemma 2.3 (as we have already done in Corollary 2.1) together with Lemma 2.4, we conclude that

\[
\sup_{B_{r_1} \setminus B_{r_0}} r^{1-\nu} |\nabla \tilde{v}| \leq c \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\nu} |\tilde{g}|.
\]

In particular

\[
\sup_{B_{r_1} \setminus B_{r_0}} r^{-\nu} |\partial_\theta \tilde{v}| \leq c \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\nu} |\tilde{g}|.
\]

This last estimate, together with (4.2) and (4.3), give the desired result since, by definition \( w = v + \tilde{v} + \partial_\theta \tilde{v} \).

Our second preliminary result reads as follows:

**Lemma 4.2** Assume that \( \nu \in \mathbb{R} \) is fixed. There exist some constants \( \lambda_1 > 0 \) and \( c > 0 \) such that, for all \( r_0 \geq \lambda_1 \) and \( r_1 > 2r_0 \), if

\[
\begin{cases}
\Delta w - \frac{1}{r^2} w + (1 - 3S^2) w = g + \partial_\theta \tilde{g} & \text{in } B_{r_1} \setminus B_{r_0} \\
w = 0 & \text{on } \partial B_{r_1}.
\end{cases}
\]

Then

\[
\sup_{B_{r_1} \setminus B_{r_0}} r^{2-\nu} |w| \leq c \left( \sup_{\partial B_{r_0}} r^{2-\nu} |w| + \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\nu} (|g| + r|\tilde{g}|) \right).
\]

**Proof:** To obtain the desired estimate we look more closely at the operator

\[
\Lambda := \Delta - \frac{1}{r^2} + (1 - 3S^2).
\]

Since \( S \) tends to 1 as \( r \) tends to +\( \infty \), there exists \( \lambda_1 > 0 \) such that the potential in this operator satisfies

\[
-\frac{1}{r^2} + (1 - 3S^2) \leq 0, \quad \text{in } [\lambda_1, +\infty).
\]

Thus, the maximum principle holds for \( \Lambda \) in \( \mathbb{C} \setminus B_{\lambda_1} \).

We now define \( v \) to be the solution of

\[
\begin{cases}
\Lambda v = g & \text{in } B_{r_1} \setminus B_{r_0} \\
v = 0 & \text{on } \partial B_{r_1} \\
v = w & \text{on } \partial B_{r_0}.
\end{cases}
\]

Let us compute

\[
\Lambda r^{\nu-2} = ((\nu - 2)^2 - 1) r^{-2} + (1 - 3S^2) r^{\nu-2}.
\]
Increasing $\lambda_1$, if this is necessary, this computation yields

$$\Lambda r^{\nu - 2} \leq -r^{\nu - 2},$$

in $\mathbb{C} \setminus B_{\lambda_1}$. We set

$$M := \sup_{\partial B_{r_0}} r^{2-\nu} |w| + \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\nu} |g|.$$

The previous computation shows that the function $z \mapsto M r^{\nu - 2}$ can be used as a barrier function for $w$ in $B_{r_1} \setminus B_{r_0}$. Hence, we obtain

$$\sup_{B_{r_1} \setminus B_{r_0}} r^{2-\nu} |v| \leq \sup_{\partial B_{r_0}} r^{2-\nu} |w| + \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\nu} |g|. \quad (4.4)$$

We finally define

$$\begin{cases}
\Lambda \tilde{v} = \tilde{g} & \text{in } B_{r_1} \setminus B_{r_0} \\
\tilde{v} = 0 & \text{on } \partial B_{r_1} \cup \partial B_{r_0}.
\end{cases}$$

We make use of (4.4), with $\nu$ replaced by $\nu - 1$, to obtain the bound

$$\sup_{B_{r_1} \setminus B_{r_0}} r^{3-\nu} |\tilde{v}| \leq c \sup_{B_{r_1} \setminus B_{r_0}} r^{3-\nu} |\tilde{g}|.$$

Since, by assumption, $r_0 \geq \lambda_1$ and $r_1 > 2r_0$, we may use Schauder’s estimates, as stated in Lemma 2.3 and in Lemma 2.4, to conclude that

$$\sup_{B_{r_1} \setminus B_{r_0}} r^{3-\nu} |\nabla \tilde{v}| \leq c \sup_{B_{r_1} \setminus B_{r_0}} r^{3-\nu} |\tilde{g}|.$$

Hence

$$\sup_{B_{r_1} \setminus B_{r_0}} r^{2-\nu} |\partial_0 \tilde{v}| \leq c \sup_{B_{r_1} \setminus B_{r_0}} r^{3-\nu} |\tilde{g}|.$$

The result follows from this last estimate together with (4.4), since by definition $w = v + \partial_0 \tilde{v}$. \qed

We are now in a position to prove Proposition 4.1, but before we do so, let us comment on the difference between the results of Lemma 4.1 and Lemma 4.2. In the first Lemma, where we are working with the operator $\Delta$ but a similar result would have been true for any operator of the form $\Delta + \frac{c}{r^2}$. The main idea is that, when we solve $\Delta w = g$, we gain a factor 2 in the weight scale since the function $g$ is bounded by $r^{\mu-2}$ while $w$ is bounded by $r^{\mu}$. A fairly reasonable way to understand this is to think that the operator $\nabla$ (respectively $\Delta$) is somehow equivalent to multiplication by $1/r$ (respectively by $1/r^2$). Then, solving $\Delta w = f$ amounts to two integrations, or equivalently a gain of two in the weight scale.
However, in the second Lemma, we are working with an operator which looks like $\Delta - 2 \circ (S)$ since $S \sim 1$ for $r$ large) but a similar result would have been true for any operator of the form $\Delta - c^2$. This time, when we solve $\Delta w - 2w = g$ we do not gain any factor in the weight scale. A fairly reasonable way to understand this is to think that the operator $\Delta$ is now equivalent to multiplication by 1, and hence $\nabla$ is equivalent to multiplication by 1. So, in this case, integrations do not produce any gain in the weight scale.

Proof of Proposition 4.1: The equations being linear we may assume that

$$\sup_{\partial B_{r_0}} r^{-\mu} |w| + \sup_{B_{r_1}\setminus B_{r_0}} r^{2-\mu} |f| = 1. \quad (4.5)$$

We shall prove that, if the parameter $\lambda$ is chosen large enough, there exists some constant $c > 0$ such that

$$\sup_{B_{r_1}\setminus B_{r_0}} r^{-\mu} \left( \frac{r^2}{r_0} |w_r| + |w_i| \right) \leq c. \quad (4.8)$$

Using the decomposition $w = w_r + i w_i$ and $f = f_r + i f_i$ where $w_r, w_i, f_r$ and $f_i$ are real valued functions, we see that $w_r$ solves

$$\Delta w_r - \frac{1}{r^2} w_r + (1 - 3S^2)w_r = \frac{2}{r^2} \partial_\theta w_i + f_r, \quad (4.6)$$

and $w_i$ solves

$$\Delta w_i = - \left( 1 - S^2 - \frac{1}{r^2} \right) w_i - \frac{2}{r^2} \partial_\theta w_i + f_i. \quad (4.7)$$

To begin with, let us prove that, if $\lambda$ is chosen large enough, there exists $c > 0$ such that

$$\sup_{B_{r_1}\setminus B_{r_0}} r^{-\mu} \left( \frac{r}{r_0} |w_r| + |w_i| \right) \leq c. \quad (4.8)$$

Observe that this is a weaker estimate than the one we ultimately want to prove.

Let us denote by

$$M_r := \sup_{B_{r_1}\setminus B_{r_0}} r^{1-\mu} |w_r| \quad \text{and} \quad M_i := \sup_{B_{r_1}\setminus B_{r_0}} r^{-\mu} |w_i|. \quad (4.9)$$

Assuming that $\lambda \geq \lambda_1$ (where $\lambda_1$ is the constant given in Lemma 4.2) and $r_0 \geq \lambda$, we may apply the result of Lemma 4.2 to (4.6), with $\nu = \mu + 1$, and obtain the inequality

$$\sup_{B_{r_1}\setminus B_{r_0}} r^{1-\mu} |w_r| \leq c \left( \sup_{\partial B_{r_0}} r^{1-\mu} |w_r| + \sup_{B_{r_1}\setminus B_{r_0}} r^{1-\mu} \left( |f_r| + \frac{2}{r} |w_i| \right) \right).$$

This, together with (4.5), shows that

$$M_r \leq c (r_0 + M_i), \quad (4.9)$$
for some constant $c > 0$.

Increasing the value of $\lambda$ if necessary, we may always assume that
\[
\left| 1 - S^2 - \frac{1}{r^2} \right| \leq \frac{1}{4c_1} r^{-2},
\]
in $[\lambda, +\infty)$, where $c_1$ is the constant which appears in the statement of Lemma 4.1. Such a choice is always possible since, by Theorem 3.1
\[
S = 1 - \frac{1}{2r^2} + O\left(\frac{1}{r^4}\right),
\]
at $\infty$. We now apply the estimate of Lemma 4.1 to (4.7), with $\nu = \mu$, to get the inequality
\[
\sup_{B_r \setminus B_{r_0}} r^{-\mu} |w_i| \leq c_1 \left( \sup_{\partial B_{r_0}} r^{-\mu} |w_i| \right.
\]
\[
+ \sup_{B_r \setminus B_{r_0}} r^{2-\mu} \left( |f_i| + \frac{1}{2c_1} |w_i| + \frac{2}{r^2} |w_r| \right).
\]
Hence, we have
\[
M_i \leq \frac{M_i}{2} + c \left( 1 + \frac{1}{r_0} M_r \right),
\]
for some constant $c > 0$.

Collecting (4.9) and (4.10), we conclude that
\[
M_i \leq c \left( 1 + \frac{1}{r_0} M_r \right) \quad \text{and} \quad M_r \leq c (r_0 + M_i).
\]
Then (4.8) follows at once by increasing the value of $\lambda$ if this is necessary. In particular, we have obtained the inequalities
\[
|w_i| \leq c_0 r^\mu, \quad \text{and} \quad |w_r| \leq c_0 \frac{r_0}{r} r^\mu.
\]
(4.11)

We define $\tilde{w}_i$ to be the solution of
\[
\begin{cases}
\Delta \tilde{w}_i = -\frac{2}{r^2} \partial_\theta w_r & \text{in } B_r \setminus B_{r_0} \\
\tilde{w}_i = 0 & \text{on } \partial B_r \cup \partial B_{r_0}.
\end{cases}
\]
(4.12)

Applying the result of Lemma 4.1 with $\nu = \mu - 1$ we obtain from (4.11) that
\[
|\tilde{w}_i| \leq c \frac{r_0}{r} r^\mu.
\]
(4.13)

Finally we defined $\hat{w}_i$ to be the solution of
\[
\begin{cases}
\Delta \hat{w}_i = -\left( 1 - S^2 - \frac{1}{r^2} \right) w_i + f_i & \text{in } B_r \setminus B_{r_0} \\
\hat{w}_i = 0 & \text{on } \partial B_r \\
\hat{w}_i = w_i & \text{on } \partial B_{r_0}.
\end{cases}
\]
(4.14)
Since we have already proven that $w_i$ is bounded by a constant times $r^\mu$, we may now apply Lemma 4.1 to obtain $|\hat{w}_i| \leq c r^\mu$ in $B_{r_1} \setminus B_{r_0}$ and then rescaled Schauder’s estimates to get

$$|\nabla \hat{w}_i| \leq c r^{\mu - 1},$$

(4.15)
in $B_{r_1} \setminus B_{2r_0}$. Obviously, $w_i = \tilde{w}_i + \hat{w}_i$. We now write (4.6) as

$$\Delta w_r = \frac{1}{r^2} w_r + (1 - 3S^2) w_r = \frac{2}{r^2} \partial_\theta \tilde{w}_i + \left( \frac{2}{r^2} \partial_\theta \hat{w}_i + f_r \right).$$

By virtue of (4.15) the last term in the right hand side is bounded by a constant times $r^{\mu - 2}$. Applying once more the result of Lemma 4.2 with $\nu = \mu$ and using (4.15), we now conclude that

$$|w_r| \leq c (1 + r_0^2 + r_0^2) r^{\mu - 2} \leq c r_0^2 r^{\mu - 2},$$

where the constant $c$ only depends on $\lambda$. The proof of the result is therefore complete.

We now give a Corollary which is nothing but the result of Proposition 4.1, for $L_\varepsilon$, in some special situation.

**Corollary 4.1** Assume that $\mu \in (1, 2)$. There exist $c > 0$ and $\lambda > 0$ such that, for all $\varepsilon \in (0, 1/(2\lambda)]$ and for all

$$w, f \in \text{Span}\{h_{\pm n} e^{\pm in\theta} : n \geq 2\},$$

solutions of

$$\begin{align*}
L_\varepsilon w &= f & \text{in } B_1 \setminus B_{\lambda \varepsilon} \\
w &= 0 & \text{on } \partial B_1,
\end{align*}$$

we have

$$\sup_{B_1 \setminus B_{\lambda \varepsilon}} r^{-\mu} \left( \frac{r^2}{\varepsilon^2} |w_r| + |w_i| \right) \leq c \left( \sup_{\partial B_{\lambda \varepsilon}} r^{-\mu} |w| + \sup_{B_1 \setminus B_{\lambda \varepsilon}} r^{2-\mu} |f| \right),$$

where $w := w_r + i w_i$ is the decomposition of $w$ into its real and imaginary part.

**Proof** : It suffices to notice that whenever $w$ solves $L_\varepsilon w = f$, then $v := w(\varepsilon \cdot)$ and $g := \varepsilon^2 f(\varepsilon \cdot)$ solve $L_1 v = g$ and apply the former result to $v$. The constant $\lambda$ is the one given in Proposition 4.1.

4.1.2 Lower eigenfrequencies

We now turn to the proof of a result which, in its spirit, is the counterpart of the previous Proposition, for lower eigenfrequencies.
Proposition 4.2 Assume that $\mu > 1$. There exist some constants $\lambda > 0$ and $c > 0$ such that, for all $r_0 \geq \lambda$ and $r_1 \geq 2r_0$, and for all solutions of
\[
\left\{ \begin{array}{l}
L_1 w = f \quad \text{in} \quad B_{r_1} \setminus \overline{B_{r_0}} \\
w_r = 0 \quad \text{on} \quad \partial B_{r_1},
\end{array} \right.
\]
we have
\[
\sup_{B_{r_1} \setminus B_{r_0}} r^{-\mu} \left( \frac{r^2}{r_0^2} |w_r| + |w_i| \right) \leq c \left( \sup_{\partial B_{r_0}} r^{-\mu} (|w| + r|\nabla w_i|) + \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\mu} |f| \right),
\]
where $w = w_r + i w_i$ is the decomposition of $w$ into its real and imaginary part.

Let us insist on the fact that on the right hand side of this estimate, a bound on $|\nabla w_i|$ on $\partial B_{r_0}$ is needed since we do not impose any boundary condition for $w_i$ on $\partial B_{r_1}$. But we do not need any bound on $|\nabla w_r|$ on $\partial B_{r_0}$, since we have $w_r = 0$ on $\partial B_{r_1}$.

The proof of this result is nearly identical to the proof of Corollary 4.1, though somehow less technical since we are only dealing with ordinary differential equations instead of partial differential equations. Nevertheless, we have to replace Lemma 4.1 by the following:

Lemma 4.3 Assume that $\mu > 1$. There exists a constant $c_2 > 0$ such that, for all $r_0 > 0$ and $r_1 \geq 2r_0$, and for all solutions of
\[
\Delta w = g \quad \text{in} \quad B_{r_1} \setminus \overline{B_{r_0}},
\]
we have
\[
\sup_{B_{r_1} \setminus B_{r_0}} r^{-\mu} |w| \leq c_2 \left( \sup_{\partial B_{r_0}} r^{-\mu} (|w| + r|\nabla w|) + \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\mu} |g| \right),
\]
Observe that, since $g$ and $w$ only have eigencomponents over two eigenfrequencies, the equation $\Delta w = g$ reduces to two ordinary differential equations.

Proof: To prove such an estimate, we first assume that $w := v e^{i\theta}$ and $g := f e^{i\theta}$ (the case were $w := v e^{-i\theta}$ and $g := f e^{-i\theta}$ can be treated similarly). Then, $v$ and $f$ solve
\[
\frac{d^2 v}{d r^2} + \frac{1}{r} \frac{d v}{d r} - \frac{1}{r^2} v = f.
\]
We set 

\[ v_0 := v(r_0), \quad \text{and} \quad \frac{dv}{dr}(r_0) := v_1. \]

We argue as in the proof of Lemma 4.1 and we define \( \tilde{v} \) to be the solution of

\[
\begin{cases}
\frac{d^2 \tilde{v}}{dr^2} + \frac{1}{r} \frac{d \tilde{v}}{dr} - \frac{1}{r^2} \tilde{v} &= 0 \quad \text{in} \quad [r_0, r_1] \\
\tilde{v}(r_0) &= v_0 \\
\frac{d \tilde{v}}{dr}(r_0) &= v_1.
\end{cases}
\]

We have explicitly

\[
\tilde{v} = \frac{1}{2} \left( (v_0 + r_0 v_1) \frac{r}{r_0} + (v_0 - r_0 v_1) \frac{r_0}{r} \right),
\]

from which it follows that

\[
\sup_{[r_0, r_1]} r^{-\mu} |\tilde{v}| \leq 2 r_0^{-\mu} (|v_0| + r_0 |v_1|).
\]

Finally, the variation of the constant formula yields

\[
v - \tilde{v} = r \int_{r_0}^{r} s^{-3} \int_{r_0}^{s} g t^2 dt.
\]

It is then easy to prove the estimate

\[
\sup_{[r_0, r_1]} r^{-\mu} |v - \tilde{v}| \leq c \sup_{[r_0, r_1]} r^{2-\mu} |g|.
\]

And the result follows at once. \( \square \)

**Proof of Proposition 4.2:** Again, the equations being linear we may assume that

\[
\sup_{\partial B_{r_0}} r^{-\mu} (|w| + r |\nabla w_i|) + \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\mu} |f| = 1.
\]

We shall prove that, if the parameter \( \lambda \) is chosen large enough, there exists some constant \( c > 0 \) such that

\[
\sup_{B_{r_1} \setminus B_{r_0}} r^{-\mu} \left( \frac{r^2}{r_0^2} |w_r| + |w_i| \right) \leq c.
\]

As in the proof of Proposition 4.1, we write

\[
\Delta w_r - \frac{1}{r^2} w_r + (1 - 3S^2) w_r = \frac{2}{r^2} \partial_\theta w_i + f_r, \quad (4.16)
\]

and

\[
\Delta w_i = - \left( 1 - S^2 - \frac{1}{r^2} \right) w_i - \frac{2}{r^2} \partial_\theta w_r + f_i. \quad (4.17)
\]
Let us denote by
\[ M_r = \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\mu} |w_r| \quad \text{and} \quad M_i = \sup_{B_{r_1} \setminus B_{r_0}} r^{-\mu} |w_i|. \]

We assume that \( \lambda \) is larger than \( \lambda_1 \) (defined in Lemma 4.3) and apply the result of Lemma 4.3 to (4.16). This gives us the inequality
\[ M_r \leq c (r_0^2 + M_i), \tag{4.18} \]
for some constant \( c > 0 \).

Let us further assume that \( \lambda \) is chosen large enough so that
\[ \left| 1 - S^2 - \frac{1}{r^2} \right| \leq \frac{1}{2c_2} r^{-2}, \]
for all \( r \geq \lambda \), where in this inequality, \( c_2 \) is the constant which appears in the statement of Lemma 4.3. We now apply the estimate of Lemma 4.3 to (4.17) and obtain the inequality
\[ M_i \leq \frac{M_i}{2} + c \left( 1 + \frac{1}{r_0^2} M_r \right), \tag{4.19} \]
for some constant \( c > 0 \).

Collecting (4.18) and (4.19), we conclude that
\[ M_i \leq c \left( 1 + \frac{1}{r_0^2} M_r \right) \quad \text{and} \quad M_r \leq c (r_0^2 + M_i). \]
The result then follows at once taking \( \lambda \) large enough. \( \square \)

Again, we give a Corollary which is nothing but the result of Proposition 4.2, for \( \mathcal{L}_\varepsilon \), in some special situation.

**Corollary 4.2** Assume that \( \mu > 1 \). There exist \( c > 0 \) and \( \lambda > 0 \) such that, for all \( \varepsilon \in (0, 1/(2\lambda)] \) and for all
\[ w, f \in \text{Span}\{h_{\pm 1} e^{\pm i\theta}\}, \]
solutions of
\[ \begin{cases} \mathcal{L}_\varepsilon w = f & \text{in} \ B_1 \setminus B_{\varepsilon \lambda} \\ w_r = 0 & \text{on} \ \partial B_1, \end{cases} \]
we have
\[ \sup_{B_1 \setminus B_{\varepsilon \lambda}} r^{-\mu} \left( \frac{r^2}{\varepsilon^2} |w_r| + |w_i| \right) \leq c \left( \sup_{\partial B_{\varepsilon \lambda}} r^{-\mu} (|w| + r|\nabla w_i|) + \sup_{B_1 \setminus B_{\varepsilon \lambda}} r^{2-\mu} |f| \right), \]
where \( w = w_r + i w_i \) is the decomposition of \( w \) into its real and imaginary part.
4.2 Function spaces

Let $\mathcal{O} \subset \mathbb{C}$ be a regular open bounded subset of $\mathbb{C}$ and $\Sigma := \{b_1, \ldots, b_k\}$ a finite set of points of $\mathcal{O}$. We now want to define some function spaces which will help us understand the mapping properties of $L_\varepsilon$ as $\varepsilon$ tends to 0.

It should be clear, from Chapter 2, that the weighted Hölder spaces $C^{k,\alpha}_\mu(\overline{\mathcal{O}} \setminus \Sigma)$ which have already been given in Chapter 2, are suited to understand the mapping properties of the operator

$$w \rightarrow \Delta w + \frac{w}{\varepsilon^2} (1 - S_2^2) - \frac{1}{r^2} w,$$

which appears in the definition of $L_\varepsilon$. Indeed, this operator behaves like the operator $\Delta - \frac{1}{r^2}$ near 0 and behaves like the operator $\Delta$ at $+\infty$, two operators for which the above weighted spaces have already proved to be well adapted. However, these spaces will not be useful for understanding the operator

$$w \rightarrow \Delta w + \frac{w}{\varepsilon^2} (1 - 3S_2^2) - \frac{1}{r^2} w.$$

While this operator still behaves like the operator $\Delta - \frac{1}{r^2}$ near 0, it behaves like the operator $\Delta - \frac{1}{2r^2}$ at $+\infty$, operator for which we already know from Lemma 4.2 that they above spaces are not well adapted at all. This, together with the result of Corollary 4.2, motivates the definition of the spaces $\tilde{C}^{k,\alpha}_{\mu,\varepsilon}(\mathcal{O} \setminus \Sigma)$, which will be given below.

We chose $\sigma > 0$ in such a way that, if $b_i \neq b_j \in \Sigma$, then $B_{2\sigma}(b_i)$ and $B_{2\sigma}(b_j)$ are disjoint and both included in $\mathcal{O}$. For all $s \in (0, \sigma)$, we define a closed subset of $\mathcal{O}$

$$A_s := \{z \in \mathcal{O} : \text{dist}(z, \Sigma) \in [s, 2s]\}.$$

We also set

$$\mathcal{O}_s := \{z \in \mathcal{O} : \text{dist}(z, \Sigma) > \sigma\}.$$

For any function $w \in C^{k,\alpha}_\mu(\overline{\mathcal{O}} \setminus \Sigma)$, we recall that we have already defined in Chapter 2 the family of semi-norms indexed by $s \in (0, \sigma)$.

$$[w]_{k,\alpha,s} := \sum_{j=0}^k s^j \sup_{A_s} |\nabla^j w| + s^{k+\alpha} \sup_{x,y \in A_s} \frac{|\nabla^k w(x) - \nabla^k w(y)|}{|x-y|^{\alpha}}. \quad (4.20)$$

We also recall that :

**Definition 4.3** Given $k \in \mathbb{N}$, $\alpha \in [0, 1)$ and $\mu \in \mathbb{R}$, the space $C^{k,\alpha}_\mu(\overline{\mathcal{O}} \setminus \Sigma)$ is defined as the set of real valued functions $w \in C^{k,\alpha}_\mu(\overline{\mathcal{O}} \setminus \Sigma)$ for which the following norm is finite

$$\|w\|_{C^{k,\alpha}_\mu(\overline{\mathcal{O}} \setminus \Sigma)} := \|w\|_{C^{k,\alpha}(\overline{\mathcal{O}} \setminus \Sigma)} + \sup_{s \in (0, \sigma)} s^{-\mu} [w]_{k,\alpha,s}. \quad (4.21)$$
The functions belonging to this space can roughly be described as the functions which are bounded by a constant times dist\((z, \Sigma)\) and whose \(k\)-th derivatives are bounded by a constant times dist\((z, \Sigma)\)^{\mu - k}.

Similarly, for any function \(w \in C^k_{\text{loc}}(\Omega \setminus \Sigma)\), we consider another family of semi-norms indexed by \(s \in (0, \sigma)\).

\[
[w]_{k, \alpha, s} := \sum_{j=0}^{k} \inf(\varepsilon, s)^j \sup_{A_s} |\nabla^j w| + \inf(\varepsilon, s)^{k+\alpha} \sup_{x,y \in A_s} \frac{|\nabla^k w(x) - \nabla^k w(y)|}{|x-y|^\alpha}.
\] (4.22)

And, we now define

**Definition 4.4** Given \(k \in \mathbb{N}, \alpha \in [0, 1), \mu \in \mathbb{R}\) and \(\varepsilon > 0\), the space \(\tilde{C}^{k, \alpha}_\mu(\overline{\Omega} \setminus \Sigma)\) is defined as the set of real valued functions \(w \in C^k_{\text{loc}}(\overline{\Omega} \setminus \Sigma)\) for which the following norm is finite

\[
\|w\|_{\tilde{C}^{k, \alpha}_\mu(\overline{\Omega} \setminus \Sigma)} := \|w\|_{C^{k, \alpha}_{\text{loc}}(\overline{\Omega})} + \sup_{s \in (0, \sigma)} \inf(\varepsilon, s)^{-k} s^{k-\mu} [w]_{k, \alpha, s}.
\] (4.23)

Observe that this space of functions depends on an additional parameter \(\varepsilon\) and though this parameter appears in the definition of the norm and in the definition of the space, it is not explicit in the notations. This time, functions belonging to this space can be described as functions which are bounded by a constant times dist\((z, \Sigma)\)^{\mu - k} and whose \(k\)-th derivatives are bounded by a constant times dist\((z, \Sigma)\)^{\mu - k}.

Finally, we define spaces of complex valued functions by

\[
E^{k, \alpha}_\mu(\overline{\Omega} \setminus \Sigma) := \tilde{C}^{k, \alpha}_\mu(\overline{\Omega} \setminus \Sigma) \oplus iC^{k, \alpha}_\mu(\overline{\Omega} \setminus \Sigma),
\] (4.24)

which is endowed with the norm

\[
\|w_r + iw_i\|_{E^{k, \alpha}_\mu(\overline{\Omega} \setminus \Sigma)} := \|w_r\|_{\tilde{C}^{k, \alpha}_\mu(\overline{\Omega} \setminus \Sigma)} + \|w_i\|_{\tilde{C}^{k, \alpha}_\mu(\overline{\Omega} \setminus \Sigma)}.
\] (4.25)

Granted the definition of \(L_\varepsilon\) which is given in (3.7), it is an easy exercise to check that

\[
L_\varepsilon : E^{2, 0}_\mu(\overline{B_1 \setminus \{0\}}) \rightarrow E^{0, 0}_{\mu - 2}(\overline{B_1 \setminus \{0\}}),
\]

is an operator whose norm is bounded independently of \(\varepsilon \in (0, 1)\).

### 4.3 A right inverse for \(L_\varepsilon\) in \(B_1 \setminus \{0\}\)

Now, we would like to obtain for \(L_\varepsilon\), a result similar to the results we have obtained for the Laplacian in Chapter 2. More precisely, we shall prove the :
Theorem 4.1 For all $\mu \in (1, 2)$, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $f \in E^{0,\alpha}_{\mu-2}(B_1 \setminus \{0\})$, there exists a unique solution of
\[
\begin{cases}
L_\varepsilon w = f & \text{in } B_1 \setminus \{0\} \\
w = 0 & \text{on } \partial B_1,
\end{cases}
\] which can be decomposed as
\[
w = v + d_0 \Phi_0^\varepsilon + \varepsilon^{-1} (d_{-1} \Phi_{-1}^\varepsilon + d_{+1} \Phi_{+1}^\varepsilon)
+ \varepsilon^\mu \eta(\varepsilon) \left( \delta_{-2} \Psi_{-2}^\varepsilon + \delta_0 \Psi_0^\varepsilon + \delta_{+2} \Psi_{+2}^\varepsilon \right),
\]
where $\eta$ is a cutoff function which is identically equal to 1 in $B_1$ and equal to 0 outside $B_2$. In addition, if we set $\bar{d} := (d_{-1}, d_0, d_{+1})$ and $\bar{\delta} := (\delta_{-2}, \delta_0, \delta_{+2})$, the linear mapping
\[
f \in E^{0,\alpha}_{\mu-2} \longrightarrow (v, \bar{d}, \bar{\delta}) \in E^{2,\alpha}_\mu \times \mathbb{R}^3 \times \mathbb{R}^3,
\]
is bounded independently of $\varepsilon \in (0, \varepsilon_0)$.

Before we proceed to the proof of this theorem, let us comment on the reason why we have chosen $\mu \in (1, 2)$. This choice is dictated by the fact that we will use this linear result to perturb an approximate solution into a solution of the Ginzburg-Landau equation. Since our approximate solution will behave at each vortex like $u_\varepsilon$ which has been defined in (3.1), and therefore decays near each vortex like $r$, it is natural to choose $\mu > 1$ so that the perturbation will decay faster. Unfortunately, with this choice, we are no longer in the range of surjectivity of the operator $L_\varepsilon$. To overcome this lack of surjectivity we have thus introduced the 6 Jacobi fields $\Phi_\varepsilon^j$ and $\Psi_\varepsilon^j$.

To begin with, we decompose $f$ and $w$ in lower and higher eigenfrequencies. More precisely, we set
\[
f = \sum_{n=0}^2 f_n + g, \quad \text{and} \quad w = \sum_{n=0}^2 w_n + v,
\]
where both $g$ and $v$ belong to $\text{Span}\{h_{\pm n} e^{\pm in\theta} : n \geq 3\}$ and where $f_n$ and $w_n$ belong to $\text{Span}\{h_{\pm n} e^{\pm in\theta}\}$, for $n = 0, 1, 2$. The proof is now divided into two parts. In the first part, we will deal with all eigenfrequencies corresponding to $|n| \geq 3$ and finally we will handle the lower eigenfrequencies.

### 4.3.1 Higher eigenfrequencies

Obviously, the function $v$ has to satisfy the equation
\[
\begin{cases}
L_\varepsilon v = g & \text{in } B_1 \setminus \{0\} \\
v = 0 & \text{on } \partial B_1.
\end{cases}
\] We prove the...
Lemma 4.4 Assume that $\mu \in (1, 2)$. There exist some constants $\varepsilon_0 > 0$ and $c > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all

$$g \in E^{\alpha, \eta}_{\mu - 2}(B_1 \setminus \{0\}) \cap \text{Span}\{h_{\pm n} e^{\pm i n \theta} : n \geq 3\},$$

there exists a unique

$$v \in E^{\alpha, \eta}_{\mu}(B_1 \setminus \{0\}) \cap \text{Span}\{h_{\pm n} e^{\pm i n \theta} : n \geq 3\},$$

solution of (4.27). In addition

$$||v||_{E^{\alpha, \eta}_{\mu}} \leq c ||g||_{E^{\alpha, \eta}_{\mu - 2}}.$$ 

Proof: The operator $L_\varepsilon$ is clearly self adjoint and, thanks to the result of Corollary 3.3, we know that it does not have any kernel in $C^{2, \alpha}(B_1 \setminus B_\rho) \cap \text{Span}\{h_{\pm n} e^{\pm i n \theta} : n \geq 3\}$, where the subscript $D$ refers to the fact that the functions have 0 boundary data (the result is even true in $C^{2, \alpha}(B_1 \setminus B_\rho) \cap \text{Span}\{h_{\pm n} e^{\pm i n \theta} : n \geq 2\}$). Therefore, for any $\rho \in (0, \lambda \varepsilon]$ (where $\lambda$ is the constant which is defined in the statement of Proposition 4.1), there exists a unique solution of

$$\begin{aligned}
L_\varepsilon v &= g \quad \text{in} \quad B_1 \setminus B_\rho \\
v &= 0 \quad \text{on} \quad \partial B_1 \cup \partial B_\rho.
\end{aligned}$$

We decompose $v$ into its real and imaginary part, writing $v := v_r + i v_i$. Standard regularity theory tells us that there exists some constant $c_{\rho, \varepsilon} > 0$ (which a priori depends in both $\varepsilon$ and $\rho$), such that

$$\sup_{B_1 \setminus B_\rho} \left( (\text{inf}(\varepsilon, r))^{-2} r^{2-\mu} |v_r| + r^{-\mu} |v_i| \right) \leq c_{\rho, \varepsilon} \sup_{B_1 \setminus B_\rho} r^{2-\mu} |g|.$$ 

We claim that, since we have chosen $\mu \in (1, 2)$, the constants $c_{\rho, \varepsilon}$ are bounded by some constant $c > 0$ which is independent of $\rho \in (0, \lambda \varepsilon]$ and $\varepsilon \in (0, \frac{1}{2\lambda}]$. Assuming that we have already proved this claim, we may pass to the limit $\rho \to 0$ and obtain the existence of $v$ solution of (4.27) such that

$$\sup_{B_1 \setminus \{0\}} \left( (\text{inf}(\varepsilon, r))^{-2} r^{2-\mu} |v_r| + r^{-\mu} |v_i| \right) \leq c \sup_{B_1 \setminus \{0\}} |z|^{2-\mu} |g|,$$

where the constant $c$ does not depend on $g$, neither on $\varepsilon$. As usual, we use both Schauder’s estimates and rescaled Schauder’s estimates to obtain the existence of some constant $c > 0$ independent of $g$ and $\varepsilon \in (0, \varepsilon_0)$ such that

$$||v||_{E^{\alpha, \eta}_{\mu}} \leq c ||g||_{E^{\alpha, \eta}_{\mu - 2}}.$$ 

To finish the proof of the Lemma, it suffices to apply Corollary 3.3 which ensures the uniqueness of the solution $v$. 

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It remains to prove the claim. The proof of this claim is very close to the proof of Proposition 2.1. Again, we argue by contradiction. If the result were not true, there would exist sequences \((\varepsilon_j)_j \geq 0 \in (0, 1/2\lambda), (\rho_j)_j \geq 0 \in (0, \lambda \varepsilon_j),\) a sequence \((g_j)_j \geq 0\) for which
\[
\sup_{B_1 \setminus B_{\rho_j}} r^{2-\mu} |g_j| = 1,
\]
and \((v_j)_j \geq 0\) the sequence of solutions of
\[
\begin{align*}
\mathcal{L}_{\varepsilon_j} v_j &= g_j \quad \text{in } B_1 \setminus \overline{B_{\rho_j}}, \\
v_j &= 0 \quad \text{on } \partial B_1 \cup \partial B_{\rho_j},
\end{align*}
\]
such that
\[
A_j := \sup_{B_1 \setminus B_{\rho_j}} ((\inf(\varepsilon_j, r))^{-2} r^{2-\mu} |v_{j,r}| + r^{-\mu} |v_{j,i}|),
\] (4.28)
tends to \(+\infty\). We set
\[
C_j := \sup_{B_{\lambda \varepsilon_j} \setminus B_{\rho_j}} r^{-\mu} |v_j|.
\]
Obviously \(A_j \geq (1 + \lambda^2) C_j\). Moreover, it follows from Corollary 4.1 that there exists a constant \(c > 0\), independent of \(j\), such that
\[
A_j \leq c \left(1 + \sup_{\partial B_{\lambda \varepsilon_j}} r^{-\mu} |v_j|\right) \leq c (1 + C_j).
\]
Since, by assumption, \(A_j\) tends to \(+\infty\), so does \(C_j\).

Now, we choose \(z_j \in \overline{B_{\lambda \varepsilon_j} \setminus B_{\rho_j}}\) in such a way that
\[
|z_j|^{-\mu} |v_j|(z_j) = C_j.
\]
We set \(r_j := \rho_j/|z_j|, \hat{r}_j := \lambda \varepsilon_j/|z_j|\) and \(\tilde{r}_j := 1/|z_j|\) and we define the sequence of rescaled functions
\[
w_j := C_j^{-1} |z_j|^{-\mu} v_j(|z_j| \cdot).
\]
By definition,
\[
\sup_{B_{\hat{r}_j} \setminus B_{r_j}} r^{-\mu} |w_j| \leq 1.
\]
Moreover, it follows from Corollary 4.1, that we also have
\[
r^{-\mu} |w_j| \leq c \quad \text{in } B_{\hat{r}_j} \setminus B_r,
\]
for some constant \(c > 0\) independent of \(j\). Finally, the function \(w_j\) solves
\[
\begin{align*}
\mathcal{L}_{\varepsilon_j/|z_j|} w_j &= \tilde{g}_j \quad \text{in } B_{\hat{r}_j} \setminus \overline{B_{r_j}}, \\
w_j &= 0 \quad \text{on } \partial B_{\hat{r}_j} \cup \partial B_{r_j},
\end{align*}
\]
where we have set
\[ \tilde{g}_j := C_j^{-1} |z_j|^{2-\mu} g_j(|z_j|). \]
Notice that by construction
\[ \sup_{B_{r_j}\setminus B_{\tilde{r}_j}} r^{2-\mu} |\tilde{g}_j| = C_j^{-1}, \]
which tends to 0 as \( j \) tends to \( +\infty \).

Up to a subsequence, we may always assume that \( (z_j)_{j \geq 0} \) converges to \( z_\infty \in \mathbb{C} \) and we set \( \tilde{r}_\infty := 1/|z_\infty| \). Similarly we may assume that \( (\varepsilon_j/|z_j|)_{j \geq 0} \) converges to \( \varepsilon_\infty \in [1/\lambda, +\infty] \) and finally that \( (r_j)_{j \geq 0} \) converges to \( r_\infty \in [0, 1) \) (notice that, thanks to Lemma 2.4, we have \( r_\infty < 1 \)). We also know that \( \tilde{g}_j \) tends to 0 in every compact subset of \( \{z \in \mathbb{C} : r_\infty < |z| < \tilde{r}_\infty \} \). We shall now distinguish two cases according to the values of \( \varepsilon_\infty \).

1 - Assume that \( \varepsilon_\infty < +\infty \). After extracting some subsequence, if this is necessary, we may assume that the sequence \( (w_j)_{j \geq 0} \) converges to some function \( w \) which is a nontrivial solution of
\[ \mathcal{L}_{\varepsilon_\infty} w = 0, \]
in the set \( \{z \in \mathbb{C} : r_\infty < |z| < \tilde{r}_\infty \} \). Furthermore, \( w = 0 \) on \( \partial B_{r_\infty} \), if \( r_\infty \neq 0 \) and also on \( \partial B_{\tilde{r}_\infty} \) if \( \tilde{r}_\infty < +\infty \). In addition, we know that \( w \in \text{Span}\{h_{\pm n}(r)e^{\pm in\theta} : n \geq 3\} \), \( w \neq 0 \) and also that, by construction, \( |w| \leq c r^\mu \).

Let us decompose \( w \) into Fourier series
\[ w := \sum_{n \geq 0} (a_n e^{in\theta} + b_n e^{-in\theta}), \]
then \( a_n e^{in\theta} + b_n e^{-in\theta} \) is also bounded by a constant times \( r^\mu \). If \( r_\infty = 0 \), we use Proposition 3.1 and the fact that \( \mu \in (1, 2) \) to conclude that \( a_n e^{in\theta} + b_n e^{-in\theta} \) is in fact bounded and even decays like \( r_n^{-1} \) at 0. Similarly, if \( \tilde{r}_\infty = +\infty \), we use Proposition 3.2 and the fact that \( \mu \in (1, 2) \), to conclude that \( a_n e^{in\theta} + b_n e^{-in\theta} \) is in fact bounded and even decays like \( r^{-n} \) at \( \infty \).

Finally, we use Theorem 3.1, when \( r_\infty = 0 \) and \( \tilde{r}_\infty = +\infty \), or Corollary 3.3 otherwise to conclude that \( w \equiv 0 \). Which is the desired contradiction.

2 - Assume that \( \varepsilon_\infty = +\infty \) and hence \( \tilde{r}_j = +\infty \). After extracting some subsequence, if this is necessary, we may assume that the sequence \( (w_j)_{j \geq 0} \) converges to some function \( w \) which is a nontrivial solution of
\[ \Delta (e^{i\theta} w) = 0, \]
in \( \mathbb{C} \setminus B_{r_\infty} \). The presence of the extra \( e^{i\theta} \) is due to the fact that we are working with the conjugate linearized operator instead of the linearized operator itself. Furthermore, \( w = 0 \) on \( \partial B_{r_\infty} \) if \( r_\infty > 0 \). In addition, we know that \( w \in \text{Span}\{h_{\pm n} e^{\pm in\theta} : n \geq 3\} \), and also that, by construction, \( |w| \leq c r^\mu \).
Again, we decompose \( w \) into Fourier series

\[
w := \sum_{|n| \geq 3} a_n e^{i n \theta},
\]

then the function \( a_n \) is also bounded by a constant times \( r^\mu \). Now, it is a simple exercise to see that

\[
a_n = \alpha_n r^{n+1} + \beta_n r^{-n-1}.
\]

for some \( \alpha, \beta \in \mathbb{R} \). Since \( |a_n| \) has to be bounded by a constant time \( r^\mu \) with \( \mu \in (1, 2) \), and taking into account the behavior of \( a_n \) at \( \infty \) we conclude that, necessarily \( \alpha_n = 0 \). Then, considering either the behavior of \( w_n \) at \( 0 \) if \( r_\infty = 0 \) or simply the fact that \( w_n(r_\infty) = 0 \) otherwise, we conclude that \( \alpha = 0 \) too. Hence \( w \equiv 0 \) and we have also obtained the desired contradiction.

Since we have obtained a contradiction in each case, the proof of the claim is complete and so is the proof of the Lemma.

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4.3.2 Lower eigenfrequencies

We are now going to prove some results similar to Lemma 4.4, for the lower eigenfrequencies. Again three cases are distinguished according to the value of the eigenfrequency.

**Eigenspace corresponding to \( n = 0 \).** This case is the easiest since we just want to solve the ordinary differential equation

\[
\frac{d^2 w_0}{dr^2} + \frac{1}{r} \frac{dw_0}{dr} - \frac{1}{r^2} w_0 + \frac{w_0}{\varepsilon^2} (1 - 2S^2_\varepsilon) - \frac{1}{\varepsilon^2} S^2_\varepsilon w_0 = f_0,
\]

in \((0, 1)\), with \( w_0(1) = 0 \). We prove the:

**Lemma 4.5** Assume that \( 1 < \mu \). There exist some constants \( \varepsilon_0 > 0 \) and \( c > 0 \) such that, for all radially symmetric function \( f_0 \in E^{0,\alpha}_\mu(B_1 \setminus \{0\}) \) and for all \( \varepsilon \in (0, \varepsilon_0) \), there exists a unique \( w_0 \), solution of \((4.29)\) which can be decomposed as

\[
w_0 = v_0 + d_0 \Phi^0_\varepsilon + \varepsilon^\mu \eta(\cdot/\varepsilon) \delta_0 \Psi^0_\varepsilon,
\]

where \( v_0 \in E^{0,\alpha}_\mu(B_1 \setminus \{0\}) \) and \( d_0, \delta_0 \in \mathbb{R} \). In addition, we have

\[
||v_0||_{E^{0,\alpha}_\mu} + |d_0| + |\delta_0| \leq c ||f_0||_{E^{0,\alpha-2}_\mu}.
\]

**Proof:** If we set \( w_0 = \alpha_0 + i\beta_0 \) and \( f_0 = g_0 + i h_0 \), the ordinary differential equation \((4.29)\) is equivalent to the system

\[
\frac{d^2 \alpha_0}{dr^2} + \frac{1}{r} \frac{d\alpha_0}{dr} - \frac{1}{r^2} \alpha_0 + \frac{\alpha_0}{\varepsilon^2} (1 - 3S^2_\varepsilon) = g_0,
\]

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\[
\frac{d^2 \beta_0}{dr^2} + \frac{1}{r} \frac{d \beta_0}{dr} - \frac{1}{r^2} \beta_0 + \frac{\beta_0}{\varepsilon^2} (1 - S^2_\varepsilon) = h_0. \tag{4.31}
\]

Since these two equations are decoupled and different in their nature, we are going to solve them independently.

Let us notice that \( S_\varepsilon \) is a solution of the second equation when \( h_0 \equiv 0 \).

Since \( S > 0 \) for all \( r > 0 \), the variation of the constant formula yields an explicit solution of (4.31). Namely

\[
\beta_0 = d_0 S_\varepsilon + S_\varepsilon \int_0^r S_\varepsilon^{-2} s^{-1} \int_0^s t S_\varepsilon h_0 dt ds, \tag{4.32}
\]

where the constant \( d_0 \) is chosen so that \( \beta_0(1) = 0 \). Using the expansions of Theorem 3.1 for \( S \) together with the fact that \( \mu > 1 \), it is an easy exercise to see that there exists some constant \( c > 0 \) such that for all \( r \in (0, 1] \)

\[
|S_\varepsilon \int_0^r S_\varepsilon^{-2} s^{-1} \int_0^s t S_\varepsilon h_0 dt ds| \leq c \left( \sup_{[0,1]} S^{2-\mu} h_0 \right) r^\mu.
\]

And, it follows from the definition of \( d_0 \) that

\[
|d_0| \leq c \|h_0\|_{E_0, \alpha}^{\mu - 2}. \]

Hence, we have solved the second ordinary differential equation and obtained the relevant estimates. Notice that the estimates for the derivatives of \( \beta_0 \) now follow by direct estimation.

We turn to the study of the first ordinary differential equation. By construction, the function \( S^*_\varepsilon \) is a solution of (4.30), when \( g_0 \equiv 0 \) and we have seen in the proof of Lemma 3.3 that \( S^*_\varepsilon > 0 \), for all \( r > 0 \). Again, we may apply the variation of the constant formula and obtain explicitly

\[
\alpha_0 = -S^*_\varepsilon \int_0^1 (S^*_\varepsilon)^{-2} s^{-1} \int_0^s t S^*_\varepsilon g_0 dt ds. \tag{4.33}
\]

Now, the result of Corollary 3.2 implies that, there exists a constant \( c > 0 \) such that

\[
\frac{1}{c} \left( \frac{\varepsilon}{r} \right)^{1/2} e^{\sqrt{2r}/\varepsilon} \leq S^*_\varepsilon \leq c \left( \frac{\varepsilon}{r} \right)^{1/2} e^{\sqrt{2r}/\varepsilon},
\]

for all \( r \geq \varepsilon \) and

\[
\frac{1}{c} \frac{r}{\varepsilon} \leq S^*_\varepsilon \leq c \frac{r}{\varepsilon},
\]

for all \( r \in (0, \varepsilon] \). Using these estimates, it is again an easy exercise to show that there exists \( c > 0 \) such that

\[
|\alpha_0| \leq c \|g_0\|_{E_0, \alpha}^{\mu - 2} r^{\mu - 2},
\]

for all \( r \geq \varepsilon \). And also that there exists \( \delta_0 \in \mathbb{R} \) such that

\[
|\alpha_0 - \varepsilon^{\mu} \delta_0 S^*_\varepsilon| \leq c \|g_0\|_{E_0, \alpha}^{\mu - 2} r^\mu,
\]

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for all $r \leq \varepsilon$, with

$$|\delta_0| \leq c \|g_0\|_{E^\alpha_{\mu-2}}.$$

Once we have derived these first estimates, the estimates for the derivatives follow at once. This ends the proof of the Lemma.

**Eigenspace corresponding to $n = \pm 1$.** Next, we study the equation

$$\begin{align*}
\mathcal{L}_\varepsilon w_1 &= f_1 \quad \text{in} \quad B_1 \setminus \{0\} \\
\mathcal{R} w_1 &= 0 \quad \text{on} \quad \partial B_1,
\end{align*}$$

(4.34)

when $w_1, f_1 \in \text{Span}\{h_{\pm 1} e^{\pm i\theta}\}$. This time, things are slightly more involved since we are dealing with a coupled system of ordinary differential equations. We shall take advantage from the fact that we already know four independent solutions of the homogeneous related problem, all of which are bounded near the origin. These solutions are given by

$$\begin{align*}
\Phi^+_{\varepsilon} &= \alpha_1 (r/\varepsilon)e^{i\theta} + \beta_1 (r/\varepsilon)e^{-i\theta}, \\
\Phi^-_{\varepsilon} &= i(\alpha_{-1} (r/\varepsilon)e^{i\theta} + \beta_{-1} (r/\varepsilon)e^{-i\theta}),
\end{align*}$$

and

$$\begin{align*}
\Psi^+_{\varepsilon} &= \alpha^*_1 (r/\varepsilon)e^{i\theta} + \beta^*_1 (r/\varepsilon)e^{-i\theta}, \\
\Psi^-_{\varepsilon} &= i(\alpha^*_{-1} (r/\varepsilon)e^{i\theta} + \beta^*_{-1} (r/\varepsilon)e^{-i\theta}).
\end{align*}$$

Let us recall that both $\Phi^\pm_{\varepsilon}$ decay at $\infty$ while both $\Psi^\pm_{\varepsilon}$ blow up exponentially at $\infty$. We prove the :

**Lemma 4.6** Assume that $\mu \in (1, 2)$. There exist some constants $\varepsilon_0 > 0$ and $c > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $f_1 \in E^\alpha_{\mu-2}(\overline{B_1} \setminus \{0\}) \cap \text{Span}\{h_{\pm 1} e^{\pm i\theta}\}$, there exists a unique $w_1$ solution of (4.34) which can be decomposed as

$$w_1 = v_1 + \varepsilon^{-1} (d_{-1} \Phi^-_{\varepsilon} + d_{+1} \Phi^+_{\varepsilon}),$$

where $v_1 \in E^2_\mu(\overline{B_1} \setminus \{0\})$ and $d_{\pm 1} \in \mathbb{R}$. In addition

$$|d_{-1}| + |d_{+1}| + \|v_1\|_{E^2_\mu} \leq c \|f_1\|_{E^\alpha_{\mu-2}}.$$

**Proof :** The proof of the Lemma goes as follows, first we construct a solution which has the correct behavior near the origin but is not equal to 0 on $\partial B_1$. Then, we use the two Jacobi fields $\Psi^\pm_{\varepsilon}$ to correct the boundary data of this solution in such a way that its real part is equal to 0 on $\partial B_1$. We obtain some uniform bound for this special solution and finally using all the Jacobi fields $\Phi^\pm_{\varepsilon}$ and $\Psi^\pm_{\varepsilon}$, we obtain the desired solution.

**Step 1.** We first prove that we can find a solution of

$$\begin{align*}
\mathcal{L}_\varepsilon w &= f_1 \quad \text{in} \quad B_1 \setminus \{0\} \\
\mathcal{R} w &= 0 \quad \text{on} \quad \partial B_1,
\end{align*}$$

(4.35)
which can be decomposed as

\[ w = v + \delta_{-1} \Psi_{\varepsilon}^{-1} + \delta_{+1} \Psi_{\varepsilon}^{+1}, \]

where the function \( v \) is bounded by a constant times \( r^\mu \) at the origin.

If we set \( v := a e^{i\theta} + b e^{-i\theta} \) and \( f_1 = \alpha e^{i\theta} + \beta e^{-i\theta} \), the existence of \( v \) follows from the existence of a solution for the following system of ordinary differential equations

\[
\begin{align*}
\frac{d^2a}{dr^2} + \frac{1}{r} \frac{da}{dr} - \frac{4}{r^2} a & = -\frac{1}{\varepsilon^2} (1 - 2S_\varepsilon^2) a + \frac{S_\varepsilon^2}{\varepsilon^2} b + \alpha \\
\frac{d^2b}{dr^2} + \frac{1}{r} \frac{db}{dr} & = -\frac{1}{\varepsilon^2} (1 - 2S_\varepsilon^2) b + \frac{S_\varepsilon^2}{\varepsilon^2} \pi + \beta,
\end{align*}
\]

having the correct behavior at 0. For \( r \) close to 0, the existence of such a solution follows as usual from a fixed point Theorem, then, the solution can be extended for all \( r > 0 \) without difficulties. Indeed, we can rewrite the above system as

\[
\begin{align*}
a &= -r^2 \int_r^{e^{i\varepsilon}} s^{-5} \int_0^s t^3 \left( \frac{S_\varepsilon^2}{\varepsilon^2} b - \frac{(1 - S_\varepsilon^2)}{\varepsilon^2} a + \alpha \right) dt ds \\
b &= \int_0^r s^{-1} \int_0^s t \left( \frac{S_\varepsilon^2}{\varepsilon^2} \pi - \frac{(1 - S_\varepsilon^2)}{\varepsilon^2} b + \beta \right) dt ds
\end{align*}
\]

where the constant \( \zeta > 0 \) is chosen small enough independently of \( \varepsilon \). The existence of a solution of this system for \( r \leq \zeta \varepsilon \) follows without any difficulty using the fact that \( S_\varepsilon \) is bounded by 1. We now choose \( \lambda \) as in Proposition 4.2. Since \( v \) is a solution of some ordinary differential equation, it is easy to see that, for all \( \zeta > \lambda \), there exists a constant \( c > 0 \) (depending on \( \zeta \) but independent of \( \varepsilon \in (0, \frac{1}{2\lambda}) \)) such that

\[
\sup_{B_{c+1}\setminus\{0\}} r^{-\mu} (|v| + r |\nabla v|) \leq c_\zeta \sup_{B_1\setminus\{0\}} r^{2-\mu} |f_1|. \tag{4.38}
\]

Now, we are going to chose \( \delta_{\pm 1} \in \mathbb{R} \) in such a way that \( \Re w = 0 \) on \( \partial B_1 \). To see that such a choice is possible, we observe that we have

\[
\Re (\delta_{+1} \Psi_{\varepsilon}^{+1}(e^{i\theta}) + \delta_{-1} \Psi_{\varepsilon}^{-1}(e^{i\theta})) = \delta_{+1} (\alpha^*_1(1/\varepsilon) + \beta^*_1(1/\varepsilon)) \cos \theta - \delta_{-1} (\alpha^*_{-1}(1/\varepsilon) - \beta^*_{-1}(1/\varepsilon)) \sin \theta.
\]

Moreover on \( \partial B_1 \), we also have \( \Re v \in \text{Span}\{\cos \theta, \sin \theta\} \). Since the functions \( \alpha^*_1 \) and \( \beta^*_1 \) have the same sign and since the functions \( \alpha^*_{-1} \) and \( \beta^*_{-1} \) have opposite signs we can conclude that it is always possible to choose \( \delta_{+1}, \delta_{-1} \in \mathbb{R} \) in such a way that

\[
\Re (\delta_{+1} \Psi_{\varepsilon}^{+1}(e^{i\theta}) + \delta_{-1} \Psi_{\varepsilon}^{-1}(e^{i\theta}) - v(e^{i\theta})) \neq 0.
\]

This ends the proof of the existence of \( w \).

**Step 2.** For the time being, we do not have any estimate of the norm of \( w \). This is the content of what follows. Let us recall that \( \lambda \) is chosen as in
Proposition 4.2 or Corollary 4.2. We are going to show that there exists a constant \( c > 0 \) independent of \( f_1 \) and \( \varepsilon \) such that the solution \( w \) constructed above satisfies

\[
\sup_{B_1 \setminus B_\lambda} r^{-\mu} \left( \frac{r^2}{\varepsilon^2} |w_r| + |w_i| \right) + \varepsilon^{-\mu} |\delta_{\pm 1}| \leq c \sup_{B_1 \setminus \{0\}} r^{2-\mu} |f_1|, 
\]

(4.39)

where, as usual we have decomposed \( w := w_r + i w_i \). We do not have a direct argument to derive this estimate, this is the reason why the proof of this result is by contradiction. If the above estimate were not true, there would exist sequences \( \varepsilon_j, f_j \) and \( w_j \) a sequence of solutions of

\[
\begin{cases}
\mathcal{L}_{\varepsilon_j} w_j = f_j & \text{in } B_1 \setminus \{0\} \\
\Re(w_j) = 0 & \text{on } \partial B_1,
\end{cases}
\]

which are constructed above, for which

\[
\sup_{B_1 \setminus \{0\}} r^{2-\mu} |f_j| = 1.
\]

and

\[
A_j := \sup_{B_1 \setminus B_{\lambda \varepsilon_j}} \left( r^{-\mu} \left( \frac{r^2}{\varepsilon_j^2} |w_{j,r}| + |w_{j,i}| \right) + \varepsilon_j^{-\mu} |\delta_{\pm 1,j}| \right)
\]

(4.40)

tends to \(+\infty\). We claim that

\[
C_j := \varepsilon_j^{-\mu} |\delta_{\pm 1,j}|,
\]

tends to \(+\infty\). To see this, let us decompose \( w_j := v_j + \delta_{\pm 1,j} \Psi^\pm + \delta_{-1,j} \Psi^{-1} \). Thanks to (4.38), we already know that the corresponding sequence of functions \( v_j \) is bounded by a constant times \( r^\mu \) and also that \( \nabla v_j \) is bounded by a constant times \( r^{\mu-1} \) in \( B_{\lambda \varepsilon_j} \setminus \{0\} \), all bounds being uniform with respect to \( j \). Hence, applying the result of Corollary 4.2, we see that there exists \( c > 0 \), which does not depend on \( j \), such that

\[
A_j \leq c(1 + C_j).
\]

(4.41)

Observe that we always have \( C_j \leq A_j \). This former inequality clearly shows that \( C_j \) also tends to \(+\infty\).

We define the sequences of rescaled functions

\[
\tilde{w}_j := C_j^{-1} \varepsilon_j^{-\mu} w_j(\varepsilon_j \cdot) \quad \text{and} \quad \tilde{f}_j := C_j^{-1} \varepsilon_j^{2-\mu} f_j(\varepsilon_j \cdot).
\]

Then we have \( \mathcal{L}_1 \tilde{w}_j = \tilde{f}_j \). Moreover \( \tilde{w}_j \) is bounded by a constant times \( r \) in \( B_\lambda \setminus \{0\} \) and, thanks to (4.41), we see that, for all \( \zeta > \lambda \), \( \tilde{w}_j \) is bounded by a constant times \( r^\mu \) in \( B_\zeta \setminus B_\lambda \), all bounds being uniform with respect to \( j \). Finally \( \tilde{f}_j \) tends to 0 on every compact subset of \( \mathbb{C} \setminus \{0\} \).
By construction, we have the decomposition of \( w_j \)

\[
    w_j := v_j + \delta_{\pm, j} \Psi_{\pm, j}^{1}. 
\]

And we have the corresponding decomposition of \( \tilde{w}_j \)

\[
    \tilde{w}_j := \tilde{v}_j + C_{j}^{-1} \varepsilon_{j}^{-\mu} \delta_{\pm, j} \Psi_{\pm, j}^{1}. 
\]

By virtue of Corollary 4.2, \( v_j \) is uniformly bounded by a constant times \( r^\mu \) on every \( B_{1/\varepsilon_j} \setminus B_{\lambda} \), and using (4.38) it is easy to see that \( \tilde{v}_j \) tends to 0 on every compact subset of \( \mathbb{C} \setminus \{0\} \). Hence, up to a subsequence, we may assume that \( \tilde{w}_j \) converges to a nontrivial linear combination of \( \Psi_{\pm, 1}^{1} \) which, by Corollary 4.2 is bounded by a constant times \( r^\mu \) in \( \mathbb{C} \setminus B_{\lambda} \). But all nonzero linear combinations of \( \Psi_{\pm, 1}^{1} \) blow up exponentially at \( \infty \). This is the desired contradiction, so the proof of (4.39) is complete.

**Step 3.** Up to now, we have proven the existence of \( w \) solution of (4.35), which can be decomposed as

\[
    w = v + \delta_{+1} \Psi_{\varepsilon}^{1} + \delta_{-1} \Psi_{\varepsilon}^{-1},
\]

in \( B_{\lambda/\varepsilon} \setminus \{0\} \). Moreover, we know that

\[
    \sup_{B_{\lambda/\varepsilon} \setminus \{0\}} r^{-\mu} |v| + \sup_{B_{1/\varepsilon} \setminus B_{\lambda}} r^{-\mu} \left( \frac{\varepsilon^2}{\varepsilon^2} |w_r | + |w_i| \right) + \varepsilon^{-\mu} |\delta_{\pm, 1}| \leq c \sup_{B_{1/\varepsilon} \setminus \{0\}} \varepsilon^{2-\mu} |f_1|,
\]

where as usual \( w := w_r + i w_i \).

We now add to this solution a suitable linear combination of \( \Phi_{\varepsilon}^{\pm, 1} \) and \( \Psi_{\varepsilon}^{\pm, 1} \) in order to ensure that we obtain a solution with 0 boundary data on \( \partial B_{1} \). So, we are looking for \( \hat{d}_{\pm, 1}, \hat{\delta}_{\pm, 1} \in \mathbb{R} \) such that

\[
    w(e^{i\theta}) = \hat{d}_{\pm, 1} \Phi_{\varepsilon}^{\pm, 1}(e^{i\theta}) + \hat{\delta}_{\pm, 1} \Psi_{\varepsilon}^{\pm, 1}(e^{i\theta}).
\]

The existence of these parameters follows easily from Lemma 3.6, which yields

\[
    \Phi_{\varepsilon}^{+} = -i \varepsilon \sin \theta + O(\varepsilon^3), \quad \Phi_{\varepsilon}^{-} = i \varepsilon \cos \theta + O(\varepsilon^3),
\]

and also

\[
    \Psi_{\varepsilon}^{+} = c_{+1} J_{\varepsilon}^{+}(1/\varepsilon) (\cos \theta + O(\varepsilon^2)), \quad \Psi_{\varepsilon}^{-} = c_{-1} J_{\varepsilon}^{+}(1/\varepsilon) (\sin \theta + O(\varepsilon^2)),
\]

on \( \partial B_{1} \). Furthermore, we also get the existence of a constant \( c > 0 \) such that

\[
    \varepsilon^{-2} J_{\varepsilon}^{+}(1/\varepsilon) |\hat{\delta}_{\pm, 1}| + \varepsilon |\hat{d}_{\pm, 1}| \leq c \sup_{\partial B_{1}} |w|.
\]

Observe in the first part of this estimate, the factor \( \varepsilon^{-2} \) which comes from the fact that, on the one hand we already have \( \Re w = 0 \) on \( \partial B_{1} \) and on the other hand we have \( \Re(\Phi_{\varepsilon}^{\pm, 1}) = O(\varepsilon^3) \) on \( \partial B_{1} \).
We set
\[ w_1 := w - \hat{\delta}_{\pm 1} \Psi_{\epsilon}^{\pm 1} - \hat{d}_{\pm 1} \Phi_{\epsilon}^{\pm 1}, \]
which is the solution to our problem. To complete the proof, let us check that the relevant estimates hold for \( \hat{w} \). We set
\[ d_{\pm 1} := \epsilon(\delta_{\pm 1} - \hat{\delta}_{\pm 1} - \hat{d}_{\pm 1}). \]
Obviously, (4.42) together with (4.43) yields
\[ |d_{\pm 1}| \leq c \sup_{B_1 \setminus \{0\}} r^{2-\mu} |f_1|. \tag{4.44} \]
Now, we may decompose \( \hat{w} \) in \( B_1 \setminus \{0\} \) as
\[ w_1 := v_1 + \frac{1}{\epsilon} d_{\pm 1} \Phi_{\epsilon}^{\pm 1}. \]
Using once more (4.43), we get
\[ \left| \hat{\delta}_{\pm 1} \Psi_{\epsilon}^{\pm 1} \right| \leq c \epsilon r^2 \frac{J^+}{J^0} (1/\epsilon) \leq c \epsilon r^{\mu - 2}, \]
in \( B_1 \setminus B_{\lambda \epsilon} \). Moreover, using (4.42) as well as Lemma 3.6, we also get
\[ \left| \delta_{\pm 1} \Phi_{\epsilon}^{\pm 1} \right| \leq c \epsilon^{\mu+1} r^{-1} \leq c \epsilon^2 r^{\mu - 2}, \]
in \( B_1 \setminus B_{\lambda \epsilon} \). Similarly, using (4.43) we obtain
\[ \left| \hat{\delta}_{\pm 1} \Phi_{\epsilon}^{\pm 1} \right| \leq c \epsilon r^{\mu - 2}, \]
Collecting the last estimates, (4.42) together with the identity
\[ v_1 = w - \hat{\delta}_{\pm 1} \Psi_{\epsilon}^{\pm 1} - (\delta_{\pm 1} - \hat{\delta}_{\pm 1}) \Phi_{\epsilon}^{\pm 1}, \]
we conclude that
\[ \sup_{B_1 \setminus B_{\lambda \epsilon}} r^{-\mu} \left( \frac{r^2}{\epsilon^2} |v_{1,r}| + |v_{1,i}| \right) \leq c \sup_{B_1 \setminus \{0\}} r^{2-\mu} |f_1|. \tag{4.45} \]
Now, we notice that, by construction, we have
\[ |\Psi_{\epsilon}^{\pm 1} - \Phi_{\epsilon}^{\pm 1}| \leq c \frac{r^2}{\epsilon^2}, \]
in \( B_{\lambda \epsilon} \setminus \{0\} \), for some constant independent of \( \epsilon \). Using, one last time, (4.42) and (4.43) together with the identity
\[ w_1 = v + (\delta_{\pm 1} - \hat{\delta}_{\pm 1}) (\Psi_{\epsilon}^{\pm 1} - \Phi_{\epsilon}^{\pm 1}) + \frac{1}{\epsilon} d_{\pm 1} \Phi_{\epsilon}^{\pm 1}, \]
we get
\[ \sup_{B_{\lambda} \setminus \{0\}} r^{-\mu} |v_1| \leq c \sup_{B_1 \setminus \{0\}} r^{2-\mu} |f_1|. \] (4.46)

This ensures the existence of \( w_1 \) as well as the desired decomposition. Furthermore the relevant estimates hold thanks to (4.44)-(4.46), since all the estimates for the derivatives follow from rescaled Schauder’s estimates.

Eigenspace corresponding to \( n = \pm 2 \). In this final step, we want to solve
\[ \begin{cases} \mathcal{L}_\varepsilon w_2 = f_2 & \text{in } B_1 \setminus \{0\} \\ w_2 = 0 & \text{on } \partial B_1, \end{cases} \] (4.47)
when \( w_2, f_2 \in \text{Span}\{h_{\pm 2} e^{\pm 2i\theta}\} \). For fix \( \varepsilon \), the existence of a solution is clear and, as the proof of Step 1, follows from Corollary 3.3, which states that, when restricted to the space of functions spanned by \( e^{\pm 2i\theta} \), the operator \( \mathcal{L}_\varepsilon \) is injective. We will therefore devote ourselves to the derivation of the correct estimates.

Lemma 4.7 Assume that \( \mu \in (1, 2) \). There exist some constants \( \varepsilon_0 > 0 \) and \( c > 0 \) such that, for all \( f_2 \in E^{\mu, \alpha}_{0, \varepsilon_0} \cap \text{Span}\{h_{\pm 2} e^{\pm 2i\theta}\} \) and for all \( \varepsilon \in (0, \varepsilon_0) \), there exists a unique \( w_2 \) solution of (4.47) which can be decomposed as
\[ w_2 = v_2 + \varepsilon \mu \eta(\varepsilon/\varepsilon) (\delta_{-2} \Psi_{\varepsilon-2} + \delta_{+2} \Psi_{\varepsilon+2}), \]
where \( v_2 \in E^{2, \alpha}_{\mu, \varepsilon}(B_1 \setminus \{0\}) \) and \( \delta_{\pm 2} \in \mathbb{R} \). In addition
\[ |\delta_{-2}| + |\delta_{+2}| + ||v_2||_{E^{2, \alpha}_{\mu, \varepsilon}} \leq c ||f_2||_{E^{\mu, \alpha}_{0, \varepsilon_0}}. \]

Proof: We decompose \( w_2 := w_r + i w_i \). For the time being, let us show that there exists a constant \( c > 0 \) independent of \( f_2 \) and \( \varepsilon \) such that
\[ \sup_{B_1 \setminus \{0\}} r^{2-\mu} \left( \varepsilon^{-2} |w_r| + \inf(\varepsilon^{-2}, \varepsilon^{-2}) |w_i| \right) \leq c \sup_{B_1 \setminus \{0\}} r^{2-\mu} |g_2|. \] (4.48)

Observe that this result is weaker than the result we would like to prove. The proof of these estimates is again by contradiction. Let us assume that the result is not true. Since these estimates are easily seen to be true if \( \varepsilon \) stays bounded away from 0, there would exist a sequence of solutions of \( \mathcal{L}_\varepsilon w_j = g_j \), with \( \varepsilon_j \to 0 \), such that
\[ A_j := \sup_{B_1 \setminus \{0\}} r^{2-\mu} \left( \varepsilon_j^{-2} |w_{j,r}| + \inf(\varepsilon_j^{-2}, \varepsilon_j^{-2}) |w_{j,i}| \right), \]
tends to +\( \infty \), while
\[ \sup_{B_1 \setminus \{0\}} r^{2-\mu} |g_j| = 1. \]

Furthermore, \( w_j, g_j \in \text{Span}\{h_{\pm 2} e^{\pm 2i\theta}\} \).

We define
\[ C_j := \sup_{B_{\lambda_j} \setminus \{0\}} \varepsilon_j^{-2} r^{2-\mu} |w_j| = +\infty, \]
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and we choose $\lambda$ as in Proposition 4.1. Thanks to Corollary 4.1, we know that

$$C_j \leq A_j \leq c (1 + C_j),$$

where the constant $c$ does not depend on $j$. In particular, this implies that $C_j$ tends to $+\infty$. Let $z_j \in \mathbb{B}_{\lambda \varepsilon_j} \setminus \{0\}$ be a point where the above supremum is achieved (observe that the supremum is always achieved since $w_j$ is bounded at the origin and $r^{\mu-2}$ blows up at 0). We define

$$\tilde{w}_j := A_j^{-1} |z_j|^{2-\mu} \varepsilon_j^{-2} w_j(|z_j|),$$

and

$$\tilde{g}_j := A_j^{-1} |z_j|^{4-\mu} \varepsilon_j^{-2} g_j(|z_j|).$$

We also set $r_j := \lambda \varepsilon_j / |z_j|$ and $\tilde{r}_j := 1 / |z_j|$. Then $L_{\varepsilon_j / |z_j|} \tilde{w}_j = \tilde{g}_j$. Moreover $\tilde{w}_j$ is bounded by $r^{\mu-2}$ in $B_{r_j} \setminus \{0\}$ and, thanks to Corollary 4.1, we also know that $\tilde{w}_j$ is bounded by a constant times $r^{\mu}$ in $B_{\tilde{r}_j} \setminus B_{r_j}$. Finally $\tilde{g}_j$ tends to 0 on every compact subset of $\mathbb{C} \setminus \{0\}$.

Up to a subsequence we may assume that $\varepsilon_j / |z_j|$ converges to $\varepsilon_{\infty} \in [1/\lambda, +\infty]$. We now distinguish two cases according to the value of $\varepsilon_{\infty}$.

1 - If $\varepsilon_{\infty} = +\infty$, then after the extraction of subsequences (if this is necessary) we obtain a nontrivial solution of

$$\Delta(e^{2i\theta} \tilde{w}) = 0 \quad \text{in} \quad \mathbb{C} \setminus \{0\},$$

which is bounded by a constant times $r^{\mu-2}$ and which belongs to $\text{Span}\{h_{\pm 2} e^{\pm 2i\theta}\}$. We decompose

$$\tilde{w} = \tilde{a} e^{2i\theta} + \tilde{b} e^{-2i\theta}.$$

Obviously, $\tilde{a}$ is a linear combination of $r^{3}$ and $r^{-3}$ while $\tilde{b}$ is a linear combination of $r$ and $r^{-1}$. Inspection of the behavior at both the origin and $\infty$ shows that $\tilde{w}$ cannot be bounded by a constant times $r^{\mu-2}$ since $\mu - 2 \in (-1, 1)$.

2 - If $\varepsilon_{\infty} < +\infty$, then after the extraction of subsequences, if this is necessary, we obtain a nontrivial solution of

$$L_{\varepsilon_{\infty}} \tilde{w} = 0 \quad \text{in} \quad \mathbb{C} \setminus \{0\},$$

which is bounded by a constant times $r^{\mu-2}$ if $r \leq \varepsilon_{\infty}$ and by a constant times $r^\mu$ if $r \geq \varepsilon_{\infty}$. Again, we decompose

$$\tilde{w} = \tilde{a} e^{2i\theta} + \tilde{b} e^{-2i\theta}.$$

Using the result of Proposition 3.1 and the fact that $\mu - 2 \in (-1, 1)$, we see that $\tilde{w}$ is bounded by a constant times $r$ near 0. Finally using the result of Proposition 3.2 together with the fact that $\mu < 2$, we see that $\tilde{w}$ decays at least like $r^{-2}$ near $\infty$. Therefore $\tilde{w}$ is a nontrivial bounded solution of $L_{\varepsilon_{\infty}} \tilde{w} = 0$ in $\mathbb{C}$. But this would contradict the result of Theorem 3.1.
Since we have reached a contradiction in all the cases, the proof of \((4.48)\) is complete. The next step consists in proving that one can choose \(\delta_{\pm 2} \in \mathbb{R}\) in such a way that

\[
\sup_{B_1 \setminus \{0\}} r^{-\mu} \left| w - \varepsilon^{\mu} \delta_{\pm 2} \Psi_{\varepsilon}^{\pm 2} \right| \leq c \sup_{B_1 \setminus \{0\}} r^{2-\mu} |f_2|, \tag{4.49}
\]

for some constant \(c > 0\) independent of \(\varepsilon\) and \(f_2\). In order to prove such a result, we write \(w = a e^{2i\theta} + b e^{-2i\theta}\) and \(f_2 = \alpha e^{2i\theta} + \beta e^{-2i\theta}\). We have already seen that, there exists \(a_0, b_0 \in \mathbb{C}\) such that the functions \(a\) and \(b\) are solutions of

\[
\begin{cases}
  a &= a_0 r^3 - r^3 \int_\varepsilon^r s^{-7} \int_0^s t^4 \left( S_{\varepsilon}^2 \left( \frac{S_{\varepsilon}^2}{\varepsilon^2} \right) a + \alpha \right) \, dt \, ds \\
  b &= b_0 r + r \int_0^r s^{-3} \int_0^s t^2 \left( S_{\varepsilon}^2 \left( \frac{S_{\varepsilon}^2}{\varepsilon^2} \right) b + \beta \right) \, dt \, ds.
\end{cases}
\]

Going back to the definition of \(\Psi_{\varepsilon}^{\pm 2}\), we see that it suffices to chose

\[\delta_{+2} + i \delta_{-2} = \varepsilon^{1-\mu} b_0.\]

Since we already know that

\[
\sup_{B_{1\lambda} \setminus \{0\}} \varepsilon^{-2} r^{2-\mu} \left| w \right| \leq c \sup_{B_1 \setminus \{0\}} r^{2-\mu} |f_2|,
\]

we easily find that

\[
\varepsilon^{3-\mu} |a_0| + \varepsilon^{1-\mu} |b_0| \leq c \sup_{B_1 \setminus \{0\}} r^{2-\mu} |f_2|.
\]

This proves the existence of \(\delta_{\pm 2}\) such that \((4.49)\) holds. Moreover, we also have

\[
|\delta_{\pm 2}| \leq c \sup_{B_1 \setminus \{0\}} r^{2-\mu} |f_2|.
\]

In order to complete the proof of the Lemma, it remains to apply rescaled Schauder’s estimates. \(\square\)

The proof of the Proposition is immediate once we have collected the results of all the different Lemmas. \(\square\)
Chapter 5

Families of Approximate Solutions with Prescribed Zero set

In this Chapter, we construct finite dimensional families of approximate solutions of the Ginzburg-Landau equation, which have prescribed zero set. In order for our approximate solutions to be close enough to true solutions of (1.1), we crucially will use the fact that the points $a_1, \ldots, a_N$ corresponding to the zero set of our solutions have to be critical points of the renormalized energy $W_g$ which is defined in (1.10). Finally, we derive some estimates which, in some sense, measure how close our approximate solution is from a true solution.

5.1 The approximate solution $\tilde{u}$

To begin with, we give a list of notation we will keep in the next Chapters. Then, we define the approximate solutions near its zero set. We end up this section by the definition of the approximate solution away from the zero set.

5.1.1 Notations

Assume we are given $g : \partial \Omega \to S^1$, a $C^{2,\alpha}$ function. We denote by $d(g)$ the topological degree of $g$. Notice that we do not necessarily assume that $\Omega$ is simply connected. In the case where $\partial \Omega$ has many connected components, the degree of $g$ is defined by taking into account the orientation of the different pieces of $\partial \Omega$.

Assume that we are given $N$ points $a_1, \ldots, a_N \in \Omega$. Further assume that,
for all $j = 1, \ldots, N$, we are given $d_j \in \{ -1, 1 \}$, such that
\[
\sum_{j=1}^{N} d_j = d(g). \tag{5.1}
\]

Finally, let us assume that $(a_j)_{j=1,\ldots,N}$ is a nondegenerate critical point of the renormalized energy $W_g$ which has been defined in (1.10), namely we want $\nabla^2 W_g$ to be a non degenerate bilinear form in $\mathbb{C}^n$. Recall that $W_g$ implicitly depends on the choice of $(d_j)_{j=1,\ldots,N}$.

Near $a_j$, we may write the limit solution $u_\ast$, which is defined in (1.6) as
\[
u_\ast := \left( \frac{z - a_j}{|z - a_j|} \right)^{d_j} e^{iH_j}, \tag{5.2}
\]
where the function $H_j$ is harmonic in some neighborhood of $a_j$. As already mentioned in Chapter 1, it follows from the work of F. Bethuel, H. Brezis and F. Hélein [11] that, since we have chosen $(a_j)_{j=1,\ldots,N}$ to be a critical point of $W_g$, we have
\[
\nabla H_j(a_j) = 0.
\]
For all $j = 1, \ldots, N$, it will be convenient to define
\[
\tau_j^0 := H_j(a_j). \tag{5.3}
\]
In all the subsequent chapters, we set
\[
\delta := \varepsilon^{1/2}. \tag{5.4}
\]

It will be convenient to adopt polar coordinates about each $a_j$, to this aim, we define $r_j$ and $\theta_j$ by
\[
r_j := |z - a_j| \quad \text{and} \quad e^{i\theta_j} := \frac{z - a_j}{|z - a_j|} \tag{5.5}
\]
Finally, we will denote
\[
S_j := S_\varepsilon(|z - a_j|). \tag{5.6}
\]

For all $j = 1, \ldots, N$, $L_{\varepsilon,j}$ will denote the conjugate of the linearized operator about $S_j e^{i(d_j \theta_j + \tau_j^0)}$. Let us notice that this operator does not depend on $\tau_j^0$ but does depend on $d_j$. Thanks to the results of the previous Chapter, the mapping properties of $L_{\varepsilon,j}$ are well known when $d_j = +1$. Observe that, when $d_j = -1$, it suffices to change $\theta$ into $-\theta$ to recover all results of Chapter 4 for the corresponding operator.

**Remark 5.1** In order to simplify the notations, we will assume from now on that all
\[
d_j = +1,
\]
and we will do it without further comment. The general case, when $d_j \in \{-1, +1\}$, requires very few modifications though the notations are more involved since, at each stage, we have to keep track of the sign of $d_j$.  

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5.1.2 The approximate solution near the zeros

Near each $a_j$, the approximate solution is defined to be a perturbation of the solution

$$S_j e^{i(\theta_j + \tau_{0,j})},$$

since we have assumed that all $d_j = +1$. The perturbation $w_j$ will be easy to define. However, good estimates of this function are quite difficult to derive. This will be the main content of this paragraph.

On each ball $B_{4\delta}(a_j)$, we define $w_j$ to be the solution of

$$\begin{cases}
\mathcal{L}_{\varepsilon,j} w_j = 0 & \text{in } B_{4\delta}(a_j) \setminus \{a_j\} \\
w_j = i(H_j - \tau_{0,j}) S_j & \text{on } \partial B_{4\delta}(a_j).
\end{cases} \tag{5.7}$$

The existence of $w_j$ follows at once from the result of Theorem 4.1. We even have some estimates for $w_j$, which are provided by the same Theorem. However, these estimates will not be precise enough for our purposes. Here, we are going to take advantage from the fact that the function $H_j$ is not arbitrary. Indeed, this function is harmonic, moreover we also know that, by assumption $\nabla H_j(a_j) = 0$. Hence we can state that

$$H_j - \tau_{0,j} \in \text{Span}\{r^n e^{\pm i n \theta_j} : n \geq 2\}.$$ 

In particular there exists some constant $c > 0$ such that

$$|H_j - \tau_{0,j}| \leq c r_j^2,$$

in $B_{2\delta}(a_j)$. This information will help us to obtain sharp estimates for $w_j$. In particular, we would like to prove that $w_j$ is close to $i(H_j - \tau_{0,j}) S_j$. More precisely, we have the :

**Proposition 5.1** There exists some constant $c > 0$ such that

$$|w_{j,r}| + |w_{j,i} - (H_j - \tau_{0,j}) S_j| \leq c \begin{cases}
\varepsilon r & \text{if } r \leq \varepsilon \\
\varepsilon^2 & \text{if } \varepsilon \leq r \leq 4\delta.
\end{cases} \tag{5.8}$$

In addition, for all $k \in \mathbb{N}$, there exists a constant $c_k > 0$ such that

$$|\nabla^k w_{j,r}| + |\nabla^k (w_{j,i} - (H_j - \tau_{0,j}) S_j)| \leq c_k \varepsilon^2 r^{-k} \quad \text{if } \varepsilon \leq r \leq 2\delta. \tag{5.9}$$

**Proof :** For the sake of simplicity in the notations, we will assume that $a_j = 0$ and drop the index $j$. To begin with, we perform a dilation of magnitude $1/\delta$. We set

$$w_\delta := w(\delta \cdot) \quad \text{and} \quad H_\delta := H(\delta \cdot).$$
Let us recall that, by definition \( \delta^2 = \varepsilon \) so we have
\[
S_{\varepsilon/\delta} = S_{\delta} := S(\cdot/\delta).
\]

Now, we have to solve
\[
\begin{cases}
\mathcal{L}_\delta w_\delta = 0 & \text{in } B_4 \setminus \{0\} \\
w_\delta = i (H_\delta - \tau^0) S_\delta & \text{on } \partial B_4.
\end{cases}
\]  
(5.10)

As we have claimed, the existence of \( w_\delta \) (and hence the existence of \( w \) itself) follows without difficulty from the result of Theorem 4.1. We therefore concentrate on the derivation of the relevant estimates.

**Step 1.** Here we construct an auxiliary function whose behavior is well controlled and which is close to \( w_\delta \). We define the imaginary part of the auxiliary function by
\[
\hat{w}_i := (H_\delta - \tau^0) S_\delta.
\]  
(5.11)

Since \( S \) tends to 1 at \( \infty \) we may always assume that \( \lambda > 0 \) is chosen sufficiently large so that
\[
\frac{1}{\delta^2} (1 - 3S^2_\delta) - \frac{1}{r^2} \leq -\frac{1}{\delta^2},
\]
in \( \mathbb{C} \setminus B_{\lambda \delta} \). In particular, this choice implies that the operator
\[
\Lambda := \Delta + \frac{1}{\delta^2} (1 - 3S^2_\delta) - \frac{1}{r^2},
\]
satisfies the maximum principle in \( B_4 \setminus B_{\lambda \delta} \). Hence, we are able to solve
\[
\begin{cases}
\Lambda \hat{w}_r = \frac{2}{r^2} \partial_\theta \hat{w}_i & \text{in } B_4 \setminus B_{\lambda \delta} \\
\hat{w}_r = 0 & \text{on } \partial B_4 \cup \partial B_{\lambda \delta}.
\end{cases}
\]  
(5.12)

Since the right hand side of (5.12) is bounded by a constant times \( \delta^2 \), we find that the constant function \( c \delta^4 \) can be used as a barrier function, provided the constant \( c > 0 \) is chosen large enough. Thus, we conclude that
\[
\sup_{B_4 \setminus B_{\lambda \delta}} |\hat{w}_r| \leq c \delta^4.
\]  
(5.13)

Now that we have proven this first estimate, it follows from (5.12) and the use of rescaled Schauder’s estimates that
\[
\sup_{B_4 \setminus B_{\lambda \delta}} |\nabla \hat{w}_r| \leq c \delta^3.
\]  
(5.14)

Now, observe that the function \( \partial_\theta \hat{w}_r \) satisfies
\[
\begin{cases}
\Lambda (\partial_\theta \hat{w}_r) = \frac{2}{r^2} \partial_\theta^2 \hat{w}_i & \text{in } B_4 \setminus B_{\lambda \delta} \\
\partial_\theta \hat{w}_r = 0 & \text{on } \partial B_4 \cup \partial B_{\lambda \delta}.
\end{cases}
\]
Again, the right hand side is bounded by a constant times $\delta^2$ and we conclude, as above, that

$$\sup_{B_4 \setminus B_{\lambda\delta}} |\partial_\theta \tilde{w}_r| \leq c \delta^4. \quad (5.15)$$

Using once more rescaled Schauder’s estimates, we also get

$$\sup_{B_4 \setminus B_{\lambda\delta}} |\nabla (\partial_\theta \tilde{w}_r)| \leq c \delta^3. \quad (5.16)$$

We want to improve (5.14). To this aim, we first notice that $\nabla \tilde{w}_r$ satisfies

$$\Lambda (\nabla \tilde{w}_r) = \nabla \left( \frac{2}{r^2} \partial_\theta \tilde{w}_r \right) + \nabla \left( \frac{3}{\delta^2} \delta^2 + \frac{1}{r^2} \right) \tilde{w}_r,$$

in $B_4 \setminus B_{\lambda\delta}$. Thanks to (5.13), we see that the second term on the right hand side of this equation is bounded by a constant times $\delta^4 r^{-3}$ and the first term on the right hand side of the same equation in bounded by $\delta^2/r$. In addition, thanks to (5.14), we know that $\nabla \tilde{w}_r$ is bounded by a constant times $\delta^3$ on $\partial B_4 \cup \partial B_{\lambda\delta}$. Increasing $\lambda$ if necessary, it is now easy to show that, provided the constant $c > 0$ is chosen large enough, the function

$$r \longrightarrow c (\delta^3 r^{-1} + \delta^3 e^{(r-4)/\delta}),$$

can be used as a barrier function. Therefore, we conclude that, for $\varepsilon$ small enough

$$\sup_{B_3 \setminus B_{\lambda\delta}} r |\nabla \tilde{w}_r| \leq c \delta^4.$$

Using similar arguments, we prove by induction that, for all $k \geq 0$

$$\sup_{B_2 \setminus B_{\lambda\delta}} r^k |\nabla^k \tilde{w}_r| \leq c \delta^4, \quad (5.17)$$

where the constant $c$ depends on $k$. Let us notice that the last two estimates do not hold up to $\partial B_4$.

We can now define the real part of auxiliary function by

$$\hat{w}_r := (1 - \eta_{\lambda\delta}) \tilde{w}_r \quad (5.18)$$

where, $\eta$ is a cutoff function identically equal to 1 in $B_1$ and equal to 0 outside $B_2$ and where $\eta_t := \eta(\cdot/t)$. Finally, the auxiliary function itself is given by

$$\hat{w} := \hat{w}_r + i \hat{w}_i.$$

**Step 2.** We now estimate $L_\delta \hat{w}$ in the space $E_{l-2}^{0,\alpha}$, for $l \in (1,2)$. Naturally, in the definition of $L_{E_{l-2}^{0,\alpha}}$ the parameter $\varepsilon$, which was used in Definition 4.4 has to be replaced by $\delta$. 

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We have
\[ F_r := \Delta \hat{w}_r + \frac{1}{\delta^2} \left( 1 - 3 S_\delta^2 \right) \hat{w}_r - \frac{1}{r^2} \hat{w}_r - \frac{2}{r^2} \partial_\theta \hat{w}_r \]
\[ = -\hat{w}_r \Delta \eta - 2 \nabla \hat{w}_r \nabla \eta - \eta \frac{2}{r^2} \partial_\theta \hat{w}_r. \]

Obviously, \( F_r \) is supported in \( B_{2\lambda} \) and, using (5.17) as well as the definition of \( \hat{w}_i \), we easily get
\[ \sup_{B_{2\lambda}} r^{-1} (|F_r| + r |\nabla F_r|) \leq c \delta. \]

Similarly, using the fact that \( H_\delta \) is harmonic and also that \( S \) is a solution of (2.2) we compute
\[ F_i := \Delta \hat{w}_i + \frac{1}{\delta^2} \left( 1 - S_\delta^2 \right) \hat{w}_i - \frac{1}{r^2} \hat{w}_i + \frac{2}{r^2} \partial_\theta \hat{w}_r \]
\[ = 2 \nabla S_\delta \nabla H_\delta + (1 - \eta) \frac{2}{r^2} \partial_\theta \hat{w}_r. \]

This time, using (5.15) and (5.16), we get the bound
\[ \sup_{B_{2\lambda}} r^{-1} (|F_i| + r |\nabla F_i|) \leq c \delta \]
\[ \sup_{B_\lambda \setminus B_{2\lambda}} r^2 (|F_i| + r |\nabla F_i|) \leq c \delta^4 \]
\[ \sup_{B_\lambda \setminus B_2} (|F_i| + r |\nabla F_i|) \leq c \delta^4. \]

It is now a simple exercise to check that
\[ \| L_\delta \hat{w} \|_{L^{\infty}_{1,2}} \leq c \delta^{4-l}. \]

Hence we also have \( \| L_\delta (w_\delta - \hat{w}) \|_{L^{\infty}_{1,2}} \leq c \delta^{4-l} \). In addition, we know that \( w_\delta - \hat{w} = 0 \) on \( \partial B_1 \). At this point, we apply the result of Theorem 4.1 with \( \varepsilon \) replaced by \( \delta \). Using the fact that both \( w_\delta \) and \( \hat{w} \) do not have any component corresponding to the eigenfrequencies \( n = -1, 0, +1 \), we see that the coefficients of \( \Phi_j^{\delta} \), for \( j = -1, 0, +1 \) and of \( \Psi_0^{\delta} \) are all equal to 0. Thus, we can already conclude that
\[ \sup_{B_\lambda \setminus \{0\}} (\delta r)^{-1} \left( \sum_{k=0}^{2} r^k |\nabla^k (w_\delta - \hat{w})| \right) \leq c \delta^2, \]
while
\[ \sup_{B_\lambda \setminus B_0} (\delta r)^{2-l} \left( \sum_{k=0}^{2} \delta^k |\nabla^k (w_\delta, r - \hat{w}_r)| \right) \leq c \delta^{8-2l} \]
\[ \sup_{B_\lambda \setminus B_0} (\delta r)^{-l} \left( \sum_{k=0}^{2} \delta^k |\nabla^k (w_\delta, i - \hat{w}_i)| \right) \leq c \delta^{4-2l}. \]
Performing the scaling backward, and taking into account that \( \varepsilon = \delta^2 \), we obtain

\[
\sup_{B_{r} \setminus \{0\}} r^{-1} \left( \sum_{k=0}^{2} r^k |\nabla^k (w - \hat{w}(-\cdot/\delta))| \right) \leq c \varepsilon, \tag{5.19}
\]

while

\[
\sup_{B_{4\delta} \setminus B_{r}} r^{2-t} \left( \sum_{k=0}^{2} \varepsilon^k |\nabla^k (w_r - \hat{w}_r(-\cdot/\delta))| \right) \leq c \varepsilon^{4-t}, \tag{5.20}
\]

\[
\sup_{B_{4\lambda\delta} \setminus B_{r}} r^{-1} \left( \sum_{k=0}^{2} r^k |\nabla^k (w_i - \hat{w}_i(-\cdot/\delta))| \right) \leq c \varepsilon^{2-t}.
\]

Observe that the proof of (5.8) for \( w_r \) is already complete.

**Step 3.** In this last step we derive (5.8) for \( w_i \) as well as (5.9) when \( k = 1 \).

It follows from (5.20), that

\[
\sup_{B_{4\delta} \setminus B_{r}} |\nabla^2 w_i| \leq c. \tag{5.21}
\]

Now, observe, as we have already done to derive (5.15) that \( \partial_\theta w_r \) satisfies

\[
\Delta (\partial_\theta w_r) + \left( \frac{1 - 3S^2}{\varepsilon^2} - \frac{1}{r^2} \right) \partial_\theta w_r = \frac{2}{r^2} \partial_\theta^2 w_i. \tag{5.22}
\]

Moreover, \( \partial_\theta w_r = 0 \) on \( \partial B_1 \) and, using (5.20) together with (5.15), we also have \( |\partial_\theta w_r| \leq c \varepsilon^2 \) on \( \partial B_{\lambda\varepsilon} \). Hence, we conclude that

\[
\sup_{B_{4\delta} \setminus B_{\lambda\varepsilon}} |\partial_\theta w_r| \leq c \varepsilon^2. \tag{5.23}
\]

Moreover, using rescaled Schauder’s estimates together with (5.20), we also get

\[
\sup_{B_{4\delta} \setminus B_{\lambda\varepsilon}} |\nabla (\partial_\theta w_r)| \leq c \varepsilon. \tag{5.24}
\]

We are now going to improve (5.20). To this aim, we define

\[
W_i := w_i - \hat{w}_i(-\cdot/\delta).
\]

A simple computation yields

\[
\Delta W_i + \left( \frac{1 - S^2}{\varepsilon^2} - \frac{1}{r^2} \right) W_i = -\frac{2}{r^2} \partial_\theta w_r - 2\nabla S \cdot \nabla H, \tag{5.25}
\]

and

\[
\Delta w_r + \left( \frac{1 - 3S^2}{\varepsilon^2} - \frac{1}{r^2} \right) w_r = \frac{2}{r^2} \partial_\theta W_i - \frac{2}{r^2} \partial_\theta \hat{w}_i. \tag{5.26}
\]

Both functions \( w_r \) and \( W_i \) vanish on \( \partial B_1 \). Furthermore, (5.20) implies that \( |w_r| + |W_i| \leq c \varepsilon^2 \) on \( \partial B_{\lambda\varepsilon} \).
Granted (5.23), we see that the right hand side of (5.25) is bounded by a constant times \( \varepsilon^2/r^2 \) and furthermore belongs to 

\[ \text{Span}\{h_{\pm n} e^{\pm in\theta} : n \geq 2\}. \]

Now, the asymptotics of \( S \) as \( r \) tends to +\( \infty \), show that 

\[
\left| (1 - S^2) \frac{\varepsilon^2}{r^2} - \frac{1}{r^2} \right| \leq c \frac{\varepsilon^2}{r^4} \leq c \lambda^{-2} r^{-2},
\]

in \( B_{4\delta} \setminus B_{\lambda \varepsilon} \). Applying the result of Lemma 5.4 which is given at the end of this Chapter (with \( \mu = 0 \) for example), we see that

\[
\sup_{B_{4\delta} \setminus B_{\lambda \varepsilon}} |W_i| \leq c \varepsilon^2,
\]

provided we have chosen \( \lambda \) large enough. In particular this ends the proof of (5.8). Rescaled Schauder’s estimates also yield

\[
\sup_{B_{4\delta} \setminus B_{\lambda \varepsilon}} r |\nabla W_i| \leq c \varepsilon^2,
\]

which proves (5.9) for \( w_i \), when \( k = 1 \). Finally, making use of (5.24) together with rescaled Schauder’s estimates, we also get

\[
\sup_{B_{4\delta} \setminus B_{\lambda \varepsilon}} r |\nabla^2 W_i| \leq c \varepsilon,
\]

Using once more (5.26), we have

\[
\Delta (\nabla w_r) + \left( \frac{1 - 3S^2}{\varepsilon^2} - \frac{1}{r^2} \right) \nabla w_r = \nabla \left( \frac{2}{r^2} \partial \theta W_i \right) - \nabla \left( \frac{2}{r^2} \partial \theta \hat{w}_i \right) - \nabla \left( \frac{(1 - 3S^2)}{\varepsilon^2} - \frac{1}{r^2} \right) w_r.
\]

Moreover, \( |\nabla w_r| \leq c \varepsilon \) on \( \partial B_{\lambda \varepsilon} \) and \( |\nabla w_r| \leq c \varepsilon^{3-l} \) on \( \partial B_{4\delta} \). This time, the right hand side of (5.27) is bounded by a constant times \( r^{-1} \). As in the proof of (5.17) we see that

\[
r \rightarrow c \left( \varepsilon^2 r^{-1} + \varepsilon^{3-l} e^{(r-4\delta)/\varepsilon} \right),
\]

can be used as a barrier function, provided the constant \( c \) is chosen large enough. Therefore, we conclude that, for \( \varepsilon \) small enough

\[
\sup_{B_{3\delta} \setminus B_{\lambda \varepsilon}} r |\nabla w_r| \leq c \varepsilon^2.
\]

Which ends the proof of (5.9) for \( w_r \), when \( k = 1 \).

The proof of (5.9) for general \( k \) can be done by induction using the same strategy. \( \square \)
5.1.3 The approximate solution away from the zeros

We now devote ourselves to the construction of our approximate solution away from the $a_j$. To this aim, we will solve the semilinear (scalar) equation

$$
\begin{cases}
\Delta \xi - |\nabla u_*|^2 \xi + \frac{(1 - \xi^2)}{\varepsilon^2} \xi = 0 & \text{in } \Omega_{\delta/2}, \\
\xi = S_\varepsilon + w_{j,r} & \text{on } \partial B_{\delta/2}(a_j) \\
\xi = 1 & \text{on } \partial \Omega.
\end{cases}
$$

(5.28)

Though, $\xi$ obviously depends on $\varepsilon$, we will not make this dependence explicit in the notation.

The existence of $\xi$ is fairly easy. However, information about the behavior of $\xi$ as $\varepsilon$ tends to 0 are harder to derive. This is the content of the following Lemma in which we prove that, as $\varepsilon$ tends to 0, the behavior of the function $\xi$ is qualitatively the same as the behavior of $S_\varepsilon$ away from 0.

**Lemma 5.1** There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, the above equation has a solution which satisfies for all $k \geq 1$

$$
\begin{cases}
1 - c \varepsilon^2 & \leq \xi \leq 1 & \text{in } \Omega_{\sigma} \\
1 - c \varepsilon^2 \frac{r_j^2}{r_j^2} & \leq \xi \leq 1 & \text{in } B_\sigma(a_j) \setminus B_{\delta/2}(a_j) \\
|\nabla^k \xi| & \leq c_k \varepsilon^2 r_j^{-2-k} & \text{in } B_{2\varepsilon}(a_j) \setminus B_{\delta}(a_j).
\end{cases}
$$

(5.29)

**Proof**: We set $\xi := 1 + w$. Then, we have to look for $w$ solution of

$$
\begin{cases}
\Delta w = |\nabla u_*|^2 (1 + w) + \frac{w(w + 1)(w + 2)}{\varepsilon^2} & \text{in } \Omega_{\delta/2} \\
w = S_\varepsilon + w_{j,r} - 1 & \text{on } \partial B_{\delta/2}(a_j) \\
w = 0 & \text{on } \partial \Omega.
\end{cases}
$$

(5.30)

Given the asymptotics of $S$ at $\infty$, we see that

$$-c \varepsilon \leq w \leq 0,$$

on each $\partial B_{\delta/2}(a_j)$.

To begin with, let us observe that $|\nabla u_*|^2 \leq c \varepsilon^{-1}$ in $\Omega_{\delta/2}$. Hence, the function

$$f(t) := |\nabla u_*|^2 (1 + t) + \frac{t(t + 1)(t + 2)}{\varepsilon^2},$$

is increasing in $[-1/4, +\infty)$, provided $\varepsilon$ is chosen small enough. We modify this function into a function $F$ which is increasing in all $\mathbb{R}$ and is equal to $f$ on $[-1/4, +\infty)$. We set

$$\tilde{F}(t) := \int_0^t \tilde{f} \, ds.$$
Let us define \( \tilde{w} \) to be the solution of
\[
\begin{cases}
\Delta \tilde{w} = \tilde{f}(\tilde{w}) & \text{on } \Omega_{3/2} \\
\tilde{w} = S_\varepsilon + w_{j,r} - 1 & \text{on } \partial B_{3/2}(a_j) \\
\tilde{w} = 0 & \text{on } \partial \Omega,
\end{cases}
\]
which is obtained by minimizing the functional
\[
E(\xi) := \int_{\Omega_{3/2}} |\nabla \xi|^2 + \int_{\Omega_{3/2}} \tilde{F}(\xi),
\]
among all functions \( \xi \in H^1(\Omega_{3/2}) \) satisfying the correct boundary conditions.

The maximum principle, together with the fact that \( \tilde{f} > 0 \) in \([0, +\infty)\), imply that \( \tilde{w} \) cannot achieve a positive maximum. Hence we already have
\[
\tilde{w} \leq 0.
\]
Now, provided the constant \( c > 0 \) is chosen large enough
\[
w_0 := -c \varepsilon^2 \sum_{j=1}^N r_j^{-2},
\]
satisfies
\[
\begin{cases}
\Delta w_0 \geq \tilde{f}(w_0) & \text{in } \Omega_{3/2} \\
w_0 \leq \tilde{w} & \text{on } \partial \Omega_{3/2}.
\end{cases}
\]
Hence \( \Delta(\tilde{w} - w_0) \leq \tilde{f}(\tilde{w}) - \tilde{f}(w_0) \). The function \( \tilde{f} \) being increasing, the maximum principle implies that \( \tilde{w} - w_0 \) cannot achieve a negative minimum in \( \Omega_{3/2} \).

Hence
\[
w_0 \leq \tilde{w} \leq 0,
\]
in \( \Omega_{3/2} \). In particular \( \tilde{w} \geq -1/4 \), for \( \varepsilon \) small enough, and therefore \( \tilde{w} \) is a solution of (5.30) which satisfies the first two inequalities of (5.29).

It remains to prove the estimates concerning the gradient of \( w \) in each \( B_{2a}(a_j) \setminus B_{3}(a_j) \). The proof of these is similar to what has already been done in the proof of Proposition 5.1. To simplify the notations, let us assume that \( a_j = 0 \). Notice that \( w \) satisfies
\[
\Delta w - 2 \frac{w}{\varepsilon^2} = |\nabla u_*|^2(1 + w) + \frac{(3w^2 + w^3)}{\varepsilon^2}.
\]
Hence, it follows from rescaled Schauder’s estimates that \( |\nabla w| \) is bounded by a constant times \( \varepsilon r^{-2} \) in \( B_{4a} \setminus B_{3/2} \). Now, we may write
\[
\Delta(\nabla w) - 2 \frac{\nabla w}{\varepsilon^2} = \nabla \left( |\nabla u_*|^2(1 + w) + \frac{(3w^2 + w^3)}{\varepsilon^2} \right).
\]
Using the estimate we have already proved, it is easy to see that the right hand side of this last equation is bounded by a constant times \(r^{-3}\) in \(B_{3\sigma} \setminus B_{3\delta/2}\). It is then a simple exercise to prove that, provided \(\varepsilon\) is chosen small enough, the function
\[
r \mapsto c\left(\varepsilon^2 r^{-3} + \varepsilon^{(\delta/2-r)/\varepsilon} + \varepsilon e^{(r-4\sigma)/\varepsilon}\right),
\]
can be used as a barrier function to prove that \(\nabla w\) is bounded by \(\varepsilon^2 r^{-3}\) in \(B_{3\sigma} \setminus B_{2\delta/3}\). The estimates for higher derivatives are then obtained by induction, following the same strategy.

In addition to the above result, we will also need the following Lemma which provides an estimate which is not as good as the one we have just proved but which turns out to be valid up to the boundary of \(\Omega\).

**Lemma 5.2** Let \(\xi\) be defined as above, then, for all \(k \geq 1\), there exists a constant \(c_k > 0\), independent of \(\varepsilon\), such that
\[
\sup_{\Omega_{\varepsilon}} |\nabla^k \xi| \leq c_k \varepsilon^{2-k}.
\]

**Proof**: The result has been obtained in the course of the proof of the previous Lemma.

In the next Proposition, we compare the function \(\xi\) with \(S_{\varepsilon}\).

**Proposition 5.2** For all \(k \geq 0\), there exists \(c_k > 0\), independent of \(\varepsilon\), such that
\[
|\nabla^k (\xi - S_{\varepsilon})| \leq c_k \varepsilon^2 r_j^{-k},
\]
in \(B_{2\sigma}(a_j) \setminus B_{\delta}(a_j)\).

**Proof**: As usual, we assume that \(a_j = 0\) and drop the indices \(j\), just to simplify the notations. Now, in \(B_{3\sigma} \setminus B_{3\delta/2}\), we have
\[
|\nabla u_\varepsilon| = |\nabla(\theta + H)|.
\]
Hence, the function \(\xi\) satisfies
\[
\Delta \xi - |\nabla(\theta + H)|^2 \xi + \frac{\xi}{\varepsilon^2} (1 - \xi^2) = 0,
\]
while the function \(S_\varepsilon\) solves
\[
\Delta S_\varepsilon - |\nabla \theta|^2 S_\varepsilon + \frac{S_\varepsilon}{\varepsilon^2} (1 - S_\varepsilon^2) = 0.
\]
Therefore, we obtain for the difference \(D := \xi - S_\varepsilon\)
\[
\Delta D - \left(\frac{\xi}{\varepsilon^2}(S_\varepsilon + \xi) - \frac{(1 - S_\varepsilon^2)}{\varepsilon^2} + |\nabla \theta|^2\right) D = \xi \nabla(2\theta + H) \cdot \nabla H.
\]
By definition of $\xi$, we have from (5.8)

$$|D| \leq c\varepsilon^2 \quad \text{on} \quad \partial B_{3/2}. $$

Moreover, it follows from (5.29) that

$$|D| \leq c\varepsilon^2 \quad \text{on} \quad \partial B_{3\sigma}. $$

Finally, let us remark that the right hand side of (5.34) is bounded by some constant $c > 0$ independent of $\varepsilon$ and that the potential is bounded from below by $-1/\varepsilon^2$, provided $\varepsilon$ is small enough. Then, the constant function $c\varepsilon^2$ can be used as a barrier function in $B_{3\sigma} \setminus B_{3/2}$, provided the constant $c$ is chosen large enough. This proves (5.33) for $k = 0$. Using rescaled Schauder’s estimates we also find that

$$\nabla |D| \leq c\varepsilon \quad \text{on} \quad \partial B_{3\sigma/4} \cup \partial B_{3\sigma}. $$

Now, $\nabla D$ satisfies an equation similar to (5.34) where, this time, the right hand side is bounded by a constant times $r^{-1}$. It is then easy to see that, for $\varepsilon$ small enough, the function

$$r \rightarrow c \left( \frac{\varepsilon^2}{r} + \varepsilon e^{(r-3\sigma)/\varepsilon} + \varepsilon e^{(3\sigma - 4r)/(4\varepsilon)} \right),$$

can be used as a barrier function to prove that

$$\sup_{B_{3\sigma/2} \setminus B_{3\sigma/5}} r |\nabla D| \leq c\varepsilon^2. $$

Hence, the proof of (5.33) is complete in the case where $k = 1$. The general case can then be obtained by induction. 

In our last Lemma, we want to show that the function

$$\tilde{w}_j := (\xi - S_j) + i (H_j - \tau_0^j) S_j,$$

is almost a solution of the homogeneous problem $L_{\varepsilon} w = 0$ in the region where it is defined. More precisely we have

**Lemma 5.3** For all $k \geq 0$, there exists some constant $c_k > 0$ such that

$$|\nabla^k (L_{\varepsilon} \tilde{w}_j)| \leq c_k \varepsilon^{2-k},$$

in $B_{2\sigma}(a_j) \setminus B_{\delta}(a_j)$. Where, by definition

$$\tilde{w}_j := ((\xi - S_j) + i (H_j - \tau_0^j) S_j).$$

**Proof :** As usual, let us assume that $a_j = 0$ and drop the index $j$. We set

$$F := L_{\varepsilon}((\xi - S_\varepsilon) + i (H - \tau^0) S_\varepsilon).$$

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Going back to the definition of $L_\varepsilon$ and using the equations satisfied by $\xi$ and $S_\varepsilon$, we find that

\begin{align*}
F &= |\nabla (\theta + H)|^2 \xi - \frac{\xi}{\varepsilon^2} (1 - \xi^2) - |\nabla \theta|^2 S_\varepsilon + \frac{S_\varepsilon}{\varepsilon^2} (1 - S_\varepsilon^2) \\
&+ \frac{(1 - 3S_\varepsilon^2)}{\varepsilon^2} (\xi - S_\varepsilon) - \frac{1}{r^2} (\xi - S_\varepsilon) + i \frac{2}{r^2} \partial \theta (\xi - S_\varepsilon) \\
&+ i \left( \Delta((H - \tau^0)S_\varepsilon) + \frac{S_\varepsilon}{\varepsilon^2} \left( 1 - S_\varepsilon^2 - \frac{\varepsilon^2}{r^2} \right) (H - \tau^0) \right) - \frac{2}{r^2} S_\varepsilon \partial \theta H.
\end{align*}

Using the fact that $H$ is harmonic, and hence $\Delta H = 0$, we can rearrange this expression into

\begin{align*}
F &= |\nabla H|^2 \xi + \frac{1}{\varepsilon^2} (2S_\varepsilon + \xi) (\xi - S_\varepsilon)^2 + 2 \frac{1}{r^2} (\xi - S_\varepsilon) \partial \theta H \\
&+ i \left( 2\nabla H \nabla S_\varepsilon + (H - \tau^0) \Delta S_\varepsilon + \frac{2}{r^2} \partial \theta (\xi - S_\varepsilon) \right) \\
&+ i \frac{S_\varepsilon}{\varepsilon^2} \left( 1 - S_\varepsilon^2 - \frac{\varepsilon^2}{r^2} \right) (H - \tau^0).
\end{align*}

Now, the estimates follow at once from the estimates of Lemma 5.33 and from the expansions of Theorem 3.1. \qed

### 5.2 A $3N$ dimensional family of approximate solutions

Thanks to the above work, we are now in a position to define our $3N$ dimensional family approximate solutions. Moreover, we will derive estimates which measure how far $\tilde{u}$ is from an exact solution.

#### 5.2.1 Definition of the family of approximate solutions

As usual, $\eta$ denotes a regular cutoff function such that $\eta = 1$ in $B_1$ and $\eta = 0$ in $\mathbb{C} \setminus B_2$. For all $t > 0$, we set

\[ \eta_t := \eta(\cdot / t). \]

and

\[ \eta_{t,j} := \eta((\cdot - a_j) / t). \]

First we define $\tilde{u}$ in each $B_{2r}(a_j)$. To simplify the notations, we will assume that $a_j = 0$ and drop the index $j$.

1 - In $B_{2r}$, we set

\[ \tilde{u} := \left( \eta_t (S_\varepsilon + w) + (1 - \eta_t) (S_\varepsilon + w_\varepsilon) e^{\frac{\varepsilon}{\varepsilon^2} \theta^\varepsilon} \right) e^{i(\theta + \tau^\varepsilon)}, \quad (5.35) \]
where the function $w$ is the one defined in (5.7).

2 - Next, the function $\tilde{u}$ is defined in $B_{2\sigma} \setminus B_{2\varepsilon}$ by the formula

$$\tilde{u} := (\eta_\delta (S_\sigma + w_\tau) + (1 - \eta_\delta)) e^{i (\theta + n_\delta (\frac{w_\tau}{\sigma} + \varepsilon) + (1 - \eta_\delta) H)}.$$ (5.36)

Finally, in $\Omega_{2\sigma}$, we simply define

$$\tilde{u} := \xi u_\ast.$$ (5.37)

Let us recall that, by definition, $u_\ast := e^{i(\theta + H)}$ in $B_{2\sigma}$.

Frequently, it will be convenient to write

$$\tilde{u} := |\tilde{u}| e^{i \tilde{\phi}},$$

where the real valued function $\tilde{\phi}$ is locally well defined away from the points $a_j$.

Once $\tilde{u}$ is defined, we have to define a $3N$ dimensional family of approximate solutions, which are obtained from $\tilde{u}$ by changing the position of the zeros of $\tilde{u}$ or slightly changing the phase of $\tilde{u}$ near each $a_j$. More precisely, given $d_{\pm 1} \in \mathbb{R}^N$, we define a family of diffeomorphism

$$\varphi_{d_{\pm 1}} : \Omega \rightarrow \Omega,$$

by

$$\varphi_{d_{\pm 1}} := z + \sum_{j=1}^{N} \eta_{\sigma, j} (d_{+1,j} + i d_{-1,j}).$$ (5.38)

Similarly, given $d_0 \in \mathbb{R}^N$, we define a regular family of real valued functions $\psi_{d_0}$ by

$$\psi_{d_0} := \sum_{j=1}^{N} \eta_{\sigma, j} d_{0,j}.$$ (5.39)

The $3N$ dimensional family of approximate solutions is then given by

$$\tilde{u}_{d_0, d_{\pm 1}} := e^{i \psi_{d_0}} \tilde{u} \circ \varphi_{d_{\pm 1}}.$$ (5.40)

When $d_0 = 0$ and $d_{\pm 1} = 0$, we will simply write $\tilde{u}$ instead of $\tilde{u}_{0,0}$.

Observe that, by construction, the zeros of $\tilde{u}_{d_0, d_{\pm 1}}$ are precisely the points $a_j - (d_{+1,j} + i d_{-1,j})$.

5.3 Estimates

We have already defined

$$N_\varepsilon (u) := \Delta u + \frac{u}{\varepsilon^2} (1 - |u|^2).$$

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Here our aim will be to derive a precise estimates of $N_\varepsilon(\tilde{u})$ which, in some sense, measures how far $\tilde{u}$ is from an exact solution of our problem.

Let $\tilde{L}_\varepsilon$ denote the linearized Ginzburg-Landau operator about $\tilde{u}$, namely

$$\tilde{L}_\varepsilon w := \Delta w + \frac{w}{\varepsilon^2} (1 - |\tilde{u}|^2) - 2\tilde{u} (w \cdot \tilde{u}).$$

We will also provide a precise estimate of the norm of $\tilde{L}_\varepsilon(D_{d_1} \tilde{u})$ and of $\tilde{L}_\varepsilon(D_{d_0} \tilde{u})$. We will need these estimates because, in the next Chapters, both $D_{d_0} \tilde{u}$ and $D_{d_1} \tilde{u}$ will play, for the operator $\tilde{L}_\varepsilon$, the rôle played by the functions $\Phi_j^\varepsilon$ for the operator $\Phi_\varepsilon$.

**Proposition 5.3** Assume that $\mu \in (1, 2)$. There exists some constant $c > 0$ independent of $\varepsilon$ such that

$$\|e^{-i\tilde{\phi}} N_\varepsilon(\tilde{u})\|_{E^{\mu, \alpha}_{\mu-2}} \leq c \delta^{2-2\alpha}. \quad (5.41)$$

Moreover, when the norms are restricted to $B_{2\varepsilon}(a_j)$, we also have

$$\|e^{-i\tilde{\phi}} N_\varepsilon(\tilde{u})\|_{E^{\mu, \alpha}_{\mu-2}} + \delta \|\nabla(e^{-i\tilde{\phi}} N_\varepsilon(\tilde{u}))\|_{E^{\mu, \alpha}_{\mu-2}} \leq c \delta^{4-\mu}. \quad (5.42)$$

and

$$\delta \|e^{-i\tilde{\phi}} \tilde{L}_\varepsilon(D_{d_1} \tilde{u})\|_{E^{\mu, \alpha}_{\mu-2}} + \|e^{-i\tilde{\phi}} \tilde{L}_\varepsilon(D_{d_0} \tilde{u})\|_{E^{\mu, \alpha}_{\mu-2}} \leq c \delta^{4-\mu}. \quad (5.43)$$

**Proof:** We devote ourselves to the derivation of (5.41) the proof of the other estimates being essentially the same.

**Step 1.** Derivation of the estimate in $B_{2\varepsilon}(a_j)$. For the sake of simplicity we assume that $a_j = 0$ and drop the indices $j$ in the notations. We have defined

$$\tilde{u} = \left(\eta_\varepsilon (S_\varepsilon + w) + (1 - \eta_\varepsilon) (S_\varepsilon + w_r) e^{i\theta} e^{\frac{i\varepsilon}{\varepsilon_0}}\right) e^{i(\theta + r\varepsilon)},$$

We claim that we can also write this identity as

$$\tilde{u} = (S_\varepsilon + \tilde{w}) e^{i(\theta + r\varepsilon)},$$

in all $B_{2\varepsilon}$, where the difference $w - \tilde{w}$ is supported in $B_{2\varepsilon} \setminus B_\varepsilon$ and satisfies, for all $k \geq 1$

$$|\nabla^k(w - \tilde{w})| \leq c_k \varepsilon^{4-k}.$$ 

Indeed, to obtain this result is suffices to expend in $B_{2\varepsilon} \setminus B_\varepsilon$

$$(S_\varepsilon + w_r) e^{\frac{i\varepsilon}{\varepsilon_0}} = S_\varepsilon + w_r + i w_1 + O(|w|^2),$$

and the claim follows from (5.8) and (5.9).
With this notation, we find that
\[ e^{-i(\theta + \tau^0)} N_\varepsilon(\tilde{u}) = L_\varepsilon \tilde{w} - \frac{\tilde{w} + 2S_\varepsilon}{\varepsilon^2} \nabla |\tilde{w}|^2 - \frac{S_\varepsilon}{\varepsilon^2} \tilde{w}^2, \]
in $B_{2\varepsilon}$. Here, we have used the fact $S_\varepsilon$ is a solution of (2.2). Observe that, since $w$ solves (5.7) we find, $L_\varepsilon \tilde{w} = 0$ in $B_\varepsilon$, while, in $B_{2\varepsilon} \setminus B_\varepsilon$, we have
\[ L_\varepsilon \tilde{w} = L_\varepsilon (\tilde{w} - w), \]
which can be estimated by a constant times $\varepsilon^2$ and whose $k$-th derivative is bounded by a constant times $\varepsilon^{2-k}$.

Using once more (5.8) and (5.9), we conclude
\[ \sup_{B_{2\varepsilon} \setminus \{0\}} r^{-3} (|N_\varepsilon(\tilde{u})| + r |\nabla N_\varepsilon(\tilde{u})|) \leq c\varepsilon^{-1}. \quad (5.44) \]

**Step 2.** We now turn to the evaluation of $N_\varepsilon(\tilde{u})$ in $B_{2\delta} \setminus B_{2\varepsilon}$. To this aim it will be convenient to write $\tilde{u}$ in the form
\[ \tilde{u} = (S_\varepsilon + \tilde{w}_r) e^{i(\theta + r^0 + \frac{\tilde{w}_r}{S_\varepsilon})}, \]
where
\[ \tilde{w}_r := \eta_\delta w_r + (1 - \eta_\delta) (\xi - S_\varepsilon), \]
and where
\[ \frac{\tilde{w}_r}{S_\varepsilon} := \eta_\delta \frac{w_r}{S_\varepsilon} + (1 - \eta_\delta) (H - \tau^0). \]
Observe that the function $\tilde{w} - w$ has compact support in $B_{2\delta} \setminus B_\delta$.

An easy computation leads to
\[
e^{-i\tilde{\theta}} N_\varepsilon(\tilde{u}) = \Delta \tilde{w}_r + \frac{\tilde{w}_r}{\varepsilon^2} (1 - 3S_\varepsilon^2) - \frac{1}{r^2} \tilde{w}_r - \frac{2}{r^2} \partial_\theta \tilde{w}_i + i \left( S_\varepsilon \Delta \left( \frac{\tilde{w}_r}{S_\varepsilon} \right) + 2 \nabla \left( \frac{\tilde{w}_r}{S_\varepsilon} \right) \nabla S_\varepsilon + \frac{2}{r^2} \partial_\theta \tilde{w}_r \right) + 2i \nabla \tilde{w}_r \nabla \left( \frac{\tilde{w}_r}{S_\varepsilon} \right) + i \left( \frac{\tilde{w}_r}{S_\varepsilon} \right) \tilde{w}_r - \frac{2}{r^2} \partial_\theta \tilde{w}_i - \left| \nabla \left( \frac{\tilde{w}_r}{S_\varepsilon} \right) \right|^2 (\tilde{w}_r + S_\varepsilon) - \frac{\tilde{w}_r^2}{\varepsilon^2} (3S_\varepsilon + \tilde{w}_r). \]
The second line can be simplified since
\[ S_\varepsilon \Delta \left( \frac{\tilde{w}_r}{S_\varepsilon} \right) + 2 \nabla \left( \frac{\tilde{w}_r}{S_\varepsilon} \right) \nabla S_\varepsilon = \Delta \tilde{w}_r - \frac{\Delta S_\varepsilon}{S_\varepsilon} \tilde{w}_i. \]
Finally, using the fact that $S$ solves (3.3), we conclude that
\[
e^{-i\tilde{\theta}} N_\varepsilon(\tilde{u}) = L_\varepsilon \tilde{w} + 2i \nabla \tilde{w}_r \nabla \left( \frac{\tilde{w}_r}{S_\varepsilon} \right) + i \Delta \left( \frac{\tilde{w}_r}{S_\varepsilon} \right) \tilde{w}_r - \frac{2}{r^2} \partial_\theta \tilde{w}_i - \left| \nabla \left( \frac{\tilde{w}_r}{S_\varepsilon} \right) \right|^2 (\tilde{w}_r + S_\varepsilon) - \frac{\tilde{w}_r^2}{\varepsilon^2} (3S_\varepsilon + \tilde{w}_r). \]
By construction

\[ L \tilde{w} = 0, \]

in \( B_{4\delta} \) and \( \tilde{w} = w \) in \( B_{\delta} \setminus B_{2\epsilon} \). Hence we also have \( L \tilde{w} = 0 \) in this set. Using (5.8), (5.9) as well as (5.33), we can evaluate the nonlinear terms in the previous computation and we conclude that

\[
\sup_{B_{\delta} \setminus B_{2\epsilon}} r^{-2} (|N_{\epsilon}(\tilde{u})| + r |\nabla(N_{\epsilon}(\tilde{u}))|) \leq c.
\]

We set

\[ \tilde{w} := (\xi - S_{\epsilon}) + i(H - \tau^0) S_{\epsilon}. \]

Since \( \tilde{w} = \eta w + (1 - \eta) \tilde{w} \), we may now write

\[ L_{\epsilon} \tilde{w} = (w - \hat{w}) \Delta \eta + 2\nabla(w - \hat{w}) \nabla \eta + (1 - \eta) L_{\epsilon}(w - \hat{w}). \]

Moreover, we have, for all \( k \geq 0 \), the estimate

\[ |\nabla^k (w - \hat{w})| \leq c_k \delta^{4-k}, \]

in \( B_{2\delta} \setminus B_{\delta} \), which follows from (5.8), (5.9) and from (5.33). This, together with the result of Lemma 5.3, proves that

\[
\sup_{B_{2\delta} \setminus B_{\delta}} \delta^{-2} (|L_{\epsilon} \tilde{w}| + \delta |\nabla(L_{\epsilon} \tilde{w})|) \leq c.
\]

Hence, we have

\[
\sup_{B_{2\delta} \setminus B_{2\epsilon}} r^{-2} (|N_{\epsilon}(\tilde{u})| + r |\nabla(N_{\epsilon}(\tilde{u}))|) \leq c. \tag{5.45}
\]

Step 3. In \( \Omega_{2\delta} \), we have defined

\[ \tilde{u} := \xi u_{*}. \]

Since \( \xi \) is a solution of

\[ \Delta \xi - |\nabla u_{*}|^2 \xi + \frac{\xi}{\epsilon^2} (1 - \xi^2) = 0, \]

and, in addition, since we also know that

\[ \Delta u_{*} = -|\nabla u_{*}|^2 u_{*}, \]

we obtain

\[ N_{\epsilon}(\tilde{u}) = 2\nabla u_{*} \nabla \xi. \tag{5.46} \]

Observe that we have the identity

\[ \nabla u_{*} = i u_{*} \nabla(\theta + H). \]
We now take advantage from the fact that
\[ \nabla \theta \cdot \nabla S_\varepsilon = 0, \]
in \( B_{2\varepsilon} \setminus \{0\} \). Hence, we may write
\[ e^{-i\tilde{\phi}} \mathcal{N}_\varepsilon(\tilde{u}) = 2i \nabla (\theta + H) \nabla \xi = i(\nabla \theta \nabla (\xi - S_\varepsilon) + \nabla H \nabla \xi). \]
Making use of (5.29) and (5.33), we can estimate
\[ \sup_{B_{2\varepsilon} \setminus B_{\varepsilon}} r^2 (|\mathcal{N}_\varepsilon(\tilde{u})| + r |\nabla (\mathcal{N}_\varepsilon(\tilde{u}))|) \leq c \varepsilon^2. \]
And also from Lemma 5.2
\[ \sup_{\Omega_{2\varepsilon}} (|\mathcal{N}_\varepsilon(\tilde{u})| + \varepsilon |\nabla (\mathcal{N}_\varepsilon(\tilde{u}))|) \leq c \varepsilon. \]
Collecting the last two estimates together with (5.44) and (5.45), it follows at once that
\[ |\mathcal{N}_\varepsilon(\tilde{u})|_{C^{0,\alpha}} \leq c \delta^{2-2\alpha}. \]
This ends the proof of the estimate. \( \square \)

### 5.4 Appendix

Here, we prove the Lemma which is needed in the third Step of the proof of Proposition 5.1.

**Lemma 5.4** Assume that \( \mu \in (0, 2) \). There exists a constant \( c > 0 \) such that, for all \( r_1 > r_0 > 0 \) and for all
\[ w, g \in \text{Span}\{h_{\pm n} e^{\pm i n \theta} : n \geq 2\}, \]
satisfying
\[ \begin{cases} \Delta w + V w &= g & \text{in } B_{r_1} \setminus \overline{B_{r_0}}, \\ w &= 0 & \text{on } \partial B_{r_1}, \end{cases} \]
we have
\[ \sup_{B_{r_1} \setminus B_{r_0}} r^{-\mu} |w| \leq c \left( \sup_{\partial B_{r_0}} r^{-\mu} |w| + \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\mu} |g| \right), \]
provided the potential \( V \in \mathbb{R} \) only depends on \( r \) and satisfies
\[ \sup_{[r_0, r_1]} r^2 |V| \leq 4 - \mu^2. \]
Proof: As usual we define \( \tilde{w} \) to be the solution of

\[
\begin{cases}
\Delta \tilde{w} = 0 & \text{in } B_{r_1} \setminus \overline{B_{r_0}} \\
\tilde{w} = 0 & \text{on } \partial B_{r_1} \\
\tilde{w} = w & \text{on } \partial B_{r_0},
\end{cases}
\]

The maximum principle provides the estimate

\[
\sup_{B_{r_1} \setminus B_{r_0}} |\tilde{w}| \leq \sup_{\partial B_{r_0}} |w|,
\]

from which it follows that

\[
\sup_{B_{r_1} \setminus B_{r_0}} r^{-\mu} |\tilde{w}| \leq \sup_{\partial B_{r_0}} r^{-\mu} |w|,
\]

since we have assumed \( \mu \geq 0 \).

Now, we decompose \( w - \tilde{w} \) and \( g - V \tilde{w} \) into Fourier series

\[
g - V \tilde{w} = \sum_{|n| \geq 2} h_n e^{in\theta}, \quad \text{and} \quad w - \tilde{w} = \sum_{|n| \geq 2} w_n e^{in\theta}.
\]

Since

\[
\Delta (w - \tilde{w}) + V (w - \tilde{w}) = g - V \tilde{w},
\]

and \( w - \tilde{w} = 0 \) on \( \partial B_{r_1} \cup \partial B_{r_0} \), we see that, for all \( |n| \geq 2 \), the function \( w_n \) is a solution of

\[
\frac{d^2 w_n}{dr^2} + \frac{1}{r} \frac{dw_n}{dr} - \frac{n^2}{r^2} w_n + V w_n = h_n,
\]

with \( w_n(r_1) = w_n(r_0) = 0 \). Observe that by assumption \( r^2 |V| \leq 4 \) and hence, the maximum principle holds for the operator

\[
\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} - \frac{n^2}{r^2} + V.
\]

Furthermore

\[
|h_n| \leq Mr^{\mu - 2},
\]

where we have set

\[
M := \sup_{B_{r_1} \setminus B_{r_0}} r^{2-\mu} (|g| + 4r^{-2} |\tilde{w}|).
\]

Notice that we have used once more the fact that \( r^2 |V| \leq 4 \). It is now easy to see that the function

\[
r \rightarrow \frac{M}{n^2 - \mu^2 - \sup (r^2 |V|)} r^\mu,
\]

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can be used as a barrier function to prove that
\[ |w_n| \leq \frac{M}{n^2 - \mu^2 - \sup (r^2 |V|)} r^\mu. \]

Finally, summation over \( n \) yields
\[ |w - \tilde{w}| \leq cr^\mu. \]

And the result follows at once. \qed
Chapter 6

Existence of Ginzburg-Landau Vortices

In this Chapter, we collect the result of the previous Chapter to derive the existence of Ginzburg-Landau vortices. More precisely, given a nondegenerate critical point of the renormalized energy, we prove the existence of a branch of solutions of the Ginzburg-Landau equation which is parameterized by $\varepsilon$ and which is close to the approximate solution $\tilde{u}$ which has been constructed in Chapter 5. To obtain this branch of solutions, we rephrase our problem as a fixed point problem. As a simple Corollary of our construction, we will obtain a local uniqueness result for the solution of the Ginzburg-Landau equation.

6.1 Statement of the result

The aim of this Chapter is to show that, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, we can prove the existence of a unique solution of

$$N_\varepsilon(u) = 0,$$

which is close to the approximate solution $\tilde{u}$, we have constructed in Chapter 5.

We choose $\mu \in (1, 2)$. It will be convenient to define the space

$$E := E^2_{\mu}((\Omega \setminus \Sigma) \oplus \bigoplus_{j=1}^N \text{Span} \{ \varepsilon^\mu \eta_{\varepsilon,j} \Psi^0_{\varepsilon,j}, \varepsilon^\mu \eta_{\varepsilon,j} \Psi^{\pm 2}_{\varepsilon,j} \} ),$$

where as usual $\eta$ is a cutoff function equal to 1 in $B_1$ and equal to 0 outside $B_2$, and where we have set

$$\eta_{\varepsilon,j} := \eta((\cdot - a_j)/\varepsilon),$$

and

$$\Psi^k_{\varepsilon,j} := \Psi^k_{\varepsilon}((\cdot - a_j).$$
Naturally, the space $\mathcal{E}$ is endowed with the norm
\[
\|v\|_\mathcal{E} := \|v\|_{E^{2,\alpha}_\mu} + |\delta_0| + |\delta_{\pm 2}|.
\]
We also rename
\[
\mathcal{F} := E^{0,\alpha}_{\mu-2}(\Omega \setminus \Sigma).
\]
We define
\[
\chi_\varepsilon := \sum_{j=1}^N \eta_{c,j}.
\]
Recall that we have decomposed the approximate solution $\tilde{u}$ as
\[
\tilde{u} := e^{i\tilde{\phi}} |\tilde{u}|.
\]
Now, for all $d_0 \in \mathbb{R}^N$, $d_{\pm 1} \in \mathbb{R}^N$ and $w \in \mathcal{E}$, we define
\[
u(d_0, d_{\pm 1}, w) := e^{i\tilde{\phi}} \left( e^{i\tilde{\phi}} \left( |\tilde{u}| + w_r \right) e^{i(1 - \chi_\varepsilon)\frac{w}{\varepsilon^2}} + i \chi_\varepsilon w_i \right) \circ \varphi_{d_{\pm 1}}. \tag{6.1}
\]
Where we recall that we have defined in (5.38) and in (5.39)
\[
\psi_{d_0} := \sum_{j=1}^N \eta_{\sigma,j} d_{0,j}.
\]
and
\[
\varphi_{d_{\pm 1}} := z + \sum_{j=1}^N \eta_{\sigma,j} (d_{+1} + i d_{-1}).
\]
Observe that, when $w = 0$, we simply have
\[
u(d_0, d_{\pm 1}, 0) = e^{i\tilde{\phi}} \left( \tilde{u} \circ \varphi_{d_{\pm 1}} \right) = \tilde{u}_{d_0,d_{\pm 1}}.
\]
Let us briefly comment on the structure of $\nu(d_0, d_{\pm 1}, w)$. Notice that, to first order, that is for $d_0, d_{\pm 1}$ and $w$ small, we have
\[
u(d_0, d_{\pm 1}, w) \sim \tilde{u} + e^{i\tilde{\phi}} \left( w + \sum_{j} \left( d_0 \Phi_{0,j}^0 + \frac{1}{\varepsilon} d_{\pm 1,j} \Phi_{0,j}^{\pm 1} \right) \right). \tag{6.2}
\]
The presence of $e^{i\tilde{\phi}}$ is justified by the fact that, when linearizing the nonlinear Ginzburg-Landau operator about $\tilde{u}$, we want to find an operator close to $\tilde{L}_\varepsilon$, which is the conjugate linearized operator. Now, the analysis of the nonlinear terms which appear in $\mathcal{N}_\varepsilon$, shows that the expression of $\nu(d_0, d_{\pm 1}, w)$ given by (6.1) is more adapted than the simpler expression given by (6.2). To be more precise, the structure of $\nu(d_0, d_{\pm 1}, w)$ is intimately linked to or analysis of the linearized operator as well as the definition of the spaces we are working with.
Finally we define the nonlinear mapping
\[ M(d_0, d_{\pm 1}, w) := e^{-i\tilde{\phi}} e^{-i(1-\chi_\epsilon) \frac{\tilde{\psi}}{\tilde{w}}} \left( e^{-i\tilde{\psi}d_0} N_\epsilon (u(d_0, d_{\pm 1}, w)) \right) \circ \varphi^{-1}_{d_{\pm 1}}. \]

Again, let us briefly try to justify the rather complicated expression of this nonlinear mapping. As above, the presence of \( e^{-i\tilde{\phi}} \) is justified by the fact that, when linearizing the nonlinear Ginzburg-Landau operator about \( \tilde{u} \), we want to find an operator close to \( \tilde{L}_\epsilon \), which is the conjugate linearized operator. Finally, it turns out that the presence of \( e^{-i(1-\chi_\epsilon) \frac{\tilde{\psi}}{\tilde{w}}} \) is needed to improve the estimates of some nonlinear terms.

For the time being, let us observe that \( M \) is well defined, and even \( C^1 \), from a neighborhood of \((0, 0, 0)\) in \( \mathbb{R}^N \times (\mathbb{R}^N \times \mathbb{R}^N) \times E \) into \( F \).

**Theorem 6.1** Assume that \( \mu \in (1, 2) \). There exist \( R > 0 \) and \( \epsilon_0 > 0 \), such that, for all \( \epsilon \in (0, \epsilon_0) \), there exists a unique \((d_0, d_{\pm 1}, w)\) solution of
\[ M(d_0, d_{\pm 1}, w) = 0, \]
in the ball of radius \( R \) of \( \mathbb{R}^N \times (\mathbb{R}^N \times \mathbb{R}^N) \times E \). Moreover, there exists \( c > 0 \), such that for all \( \epsilon \in (0, \epsilon_0) \), the unique solution \((d_0, d_{\pm 1}, w)\) satisfies
\[ |d_0|\mathbb{R}^N + |d_{\pm 1}|\mathbb{R}^N \times \mathbb{R}^N + \|w\|_E \leq c\epsilon^{1-\alpha}. \]

Before we proceed to the proof of the Theorem, let us rephrase the result as :

**Theorem 6.2** Assume that \( d_j = \pm 1 \), for all \( j = 1, \ldots, N \). Let \((a_i)_{i=1,\ldots,N}\) be a nondegenerate critical point of \( W_q \). Then, there exists \( \epsilon_0 > 0 \) and for all \( \epsilon \in (0, \epsilon_0) \) there exists a solution \( u_\epsilon \) of (1.5) with exactly \( N \) isolated zeros \( a_1(\epsilon), \ldots, a_N(\epsilon) \) such that :
\[ \lim_{\epsilon \to 0} a_j(\epsilon) = a_j, \]
and
\[ \lim_{\epsilon \to 0} u_\epsilon = u_* \quad \text{in} \quad C^{2,\alpha}_\loc(\Omega \setminus \Sigma). \]

Where \( \Sigma := \{a_1, \ldots, a_N\} \) and where \( u_* \) is given by (1.6).

Hence, the proof of Theorem 1.9 is complete. As a byproduct of our construction, we obtain some information about the functions \( u_\epsilon \). For example, we have the :

**Proposition 6.1** For all \( \alpha' > 0 \) there exists a constant \( c_{\alpha'} > 0 \) independent of \( \epsilon \in (0, \epsilon_0) \) such that
\[ |a_j(\epsilon) - a_j| \leq c_{\alpha'} \epsilon^{\alpha'}. \]
Further properties of $u_\varepsilon$ will be proven in the last section of this Chapter.

For the time being let us write Taylor’s formula. It turns out to be easier for the estimates we will have to perform, not to use the usual formula but rather

$$M(d_0, d_{\pm 1}, w) = M(0, 0, 0) + D M_{(0,0,0)}(d_0, d_{\pm 1}, w)$$

$$+ \int_0^1 (D M_{(td_0, td_{\pm 1}, 0)} - D M_{(0,0,0)}) (d_0, d_{\pm 1}, 0) \, dt$$

$$+ (D M_{(d_0, d_{\pm 1}, 0)} - D M_{(0,0,0)}) (0, 0, w)$$

$$+ \int_0^1 (D M_{(d_0, d_{\pm 1}, tw)} - D M_{(d_0, d_{\pm 1}, 0)}) (0, 0, w) \, dt.$$

Observe that

$$M(0, 0, 0) = e^{i \tilde{\phi} N_\varepsilon(u)}.$$

For the sake of simplicity, we will denote the nonlinear terms by

$$Q_1(d_0, d_{\pm 1}) := \int_0^1 (D M_{(td_0, td_{\pm 1}, 0)} - D M_{(0,0,0)}) (d_0, d_{\pm 1}, 0) \, dt,$$

$$Q_2(d_0, d_{\pm 1}, w) := (D M_{(d_0, d_{\pm 1}, 0)} - D M_{(0,0,0)}) (0, 0, w),$$

and

$$Q_3(d_0, d_{\pm 1}, w) := \int_0^1 (D M_{(d_0, d_{\pm 1}, tw)} - D M_{(d_0, d_{\pm 1}, 0)}) (0, 0, w) \, dt.$$

The reason why we have performed this rather unusual Taylor’s expansion is that we want to avoid mixed second order differential of the form $D_{d_0} D_w M$ or $D_{d_{\pm 1}} D_w M$ evaluated at some point $(d_0, d_{\pm 1}, w)$ with $w \neq 0$, since these are not easy to estimate.

Clearly, in order to prove Theorem 6.1, it is sufficient to find a fixed point for the mapping

$$(d_0, d_{\pm 1}, w) \quad \rightarrow \quad - (D M_{(0,0,0)})^{-1} (M(0, 0, 0) + Q_1(d_0, d_{\pm 1})$$

$$+ Q_2(d_0, d_{\pm 1}, w) + Q_3(d_0, d_{\pm 1}, w)).$$

Now, the strategy of the proof of the different results of this Chapter is as follows: First, we show that $D M_{(0,0,0)}$ is close to $\tilde{L}_\varepsilon$ and thus, it is an isomorphism from $\mathbb{R}^N \times (\mathbb{R}^N \times \mathbb{R}^N) \times \mathcal{E}$ into $\mathcal{F}$. Next we prove that the three nonlinear mappings $Q_k$ are contraction mappings when restricted to a suitable neighborhood of $(0, 0, 0)$ in $\mathbb{R}^N \times (\mathbb{R}^N \times \mathbb{R}^N) \times \mathcal{E}$. 
6.2 The linear mapping $DM_{(0,0,0)}$

We have an explicite expression for $DM_{(0,0,0)}$, namely

$$
DM_{(0,0,0)}(d_0, d_{±1}, w) = \tilde{\mathcal{L}}_\varepsilon \left( w + \sum_{j=1}^{N} (d_{0,j} \Phi_{\varepsilon,j}^0 + d_{±1,j} \Phi_{\varepsilon,j}^{±1}) \right) 
- e^{-i\phi} \nabla N_\varepsilon(\tilde{u}) \nabla (\varphi_{d_{±1}} - z) 
- i e^{-i\phi} N_\varepsilon(\tilde{u}) \left( \psi_{d_{0}} + (1 - \chi_\varepsilon) \frac{w_j}{|w|} \right). 
$$

(6.4)

We can now prove the :

**Proposition 6.2** There exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, linear operator

$$
DM_{(0,0,0)}(\beta, b, w) : \mathbb{R}^N \times (\mathbb{R}^N \times \mathbb{R}^N) \times \mathcal{E} \rightarrow \mathcal{F},
$$

is an isomorphism whose inverse is bounded independently of $\varepsilon \in (0, \varepsilon_0)$.

**Proof :** First, we use (5.41), in Proposition 5.3, to get

$$
\| e^{-i\phi} N_\varepsilon(\tilde{u}) (1 - \chi_\varepsilon) \frac{w_j}{|w|} \|_F \leq c \delta^{2-2\alpha} \| w \|_\mathcal{E}.
$$

While, using (5.42), we obtain

$$
\| i e^{-i\phi} N_\varepsilon(\tilde{u}) \psi_{d_{0}} \|_F \leq c \delta^{4-\mu} \| d_{0} \|_{\mathbb{R}^N},
$$

and

$$
\| e^{-i\phi} \nabla N_\varepsilon(\tilde{u}) \nabla (\varphi_{d_{±1}} - z) \|_F \leq c \delta^{3-\mu} \| d_{±1} \|_{\mathbb{R}^N \times \mathbb{R}^N}.
$$

Then, the result follows from a perturbation argument, using the result of Theorem ??.

Finally, let us notice that (5.41) also yields the estimate

$$
\| M(0, 0, 0) \|_F \leq c \delta^{2-2\alpha}. 
$$

(6.5)

6.3 Estimates of the nonlinear terms

Here, we estimate in turn the nonlinear terms $Q_k$. We will frequently use the following basic computation for some real or complex valued functions $f, h$ and some complex valued function $g$, defined in some domains of $\mathbb{C}$

$$
(e^{-ih} \Delta \left( (f \circ g) e^{ih} \right)) = f \circ g \left( i \Delta h - |\nabla h|^2 \right) + 2i Df_g(\nabla g \nabla h) + D^2f_g(\nabla g, \nabla h) + Df_g(\Delta g).
$$

Where

$$
Df_g(\nabla g \nabla h) := Df_g(\partial_x g \partial_x h + \partial_y g \partial_y h),
$$

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and where

\[ D^2 f_g(\nabla g, \nabla g) := D^2 f_g(\partial_x g, \partial_x g) + D^2 f_g(\partial_y g, \partial_y g). \]

We apply this computation to \( g := \varphi_{d\pm 1} \) and \( h := \psi_{d_0} \). Recall that, in \( B_{\sigma}(a_j) \), we simply have

\[
\varphi_{d\pm 1} = z + (d_{+1,j} + i d_{-1,j}) \eta_{\sigma,j} \quad \text{and} \quad h := \psi_{d_0} = d_{0,j} \eta_{\sigma,j}.
\]

For the sake of simplicity in the notations, let us assume that \( a_j = 0 \), drop all indices and write \( d_1 \) instead of \( d_{+1} + i d_{-1} \). We obtain

\[
(e^{-i\psi} \Delta (e^{i\psi} f \circ \varphi)) \circ \varphi^{-1} = \Delta f + |\nabla \eta|^2 \circ \varphi^{-1} D^2 f (d_1, d_1) + 2 D^2 f (d_1, \nabla \eta \circ \varphi^{-1}) + (\Delta \eta + 2 i d_0 |\nabla \eta|^2) \circ \varphi^{-1} D f (d_1) + (i d_0 \Delta \eta - d_0^2 |\nabla \eta|^2) \circ \varphi^{-1} f.
\]

### 6.3.1 Estimates of \( Q_1 \)

The nonlinear term \( Q_1 \) is probably the easiest to estimate. This is the content of the following:

**Lemma 6.1** There exists \( c > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_0) \), we have

\[
\|Q_1(d_0, d_{\pm 1}) - Q_1(\tilde{d}_0, \tilde{d}_{\pm 1})\|_{\mathcal{F}} \leq c \, l_1 \left( \|d_0 - \tilde{d}_0\|_{\mathbb{R}^N} + \|d_{\pm 1} - \tilde{d}_{\pm 1}\|_{\mathbb{R}^N \times \mathbb{R}^N} \right),
\]

provided \( l_1 \leq 1 \). Here, we have defined

\[
l_1 := \|d_0\|_{\mathbb{R}^N} + \|d_{\pm 1}\|_{\mathbb{R}^N \times \mathbb{R}^N} + \|	ilde{d}_0\|_{\mathbb{R}^N} + \|	ilde{d}_{\pm 1}\|_{\mathbb{R}^N \times \mathbb{R}^N}.
\]

**Proof**: First of all let us notice that \( \mathcal{M}(d_0, d_{\pm 1}, 0) \) does not depend on \( d_0 \) nor on \( d_{\pm 1} \) in \( \Omega_{2\sigma} \cup \bigcup_{j=1}^{N} B_{\sigma}(a_j) \). Therefore, we have

\[
Q_1(d_0, d_{\pm 1}) - Q_1(\tilde{d}_0, \tilde{d}_{\pm 1}) = 0,
\]

there. Finally, in \( B_{2\sigma}(a_j) \setminus B_{\sigma}(a_j) \), we have

\[
\mathcal{M}(d_0, d_{\pm 1}, 0) = e^{-i\phi_0} \left( e^{i\psi_{d_0}} \Delta (e^{i\psi_{d_0}} \tilde{u} \circ \varphi_{d_{\pm 1}}) \right) \circ \varphi_{d_{\pm 1}}^{-1} + \frac{\|\tilde{u}\|}{\varepsilon^2} (1 - |\tilde{u}|^2).
\]

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Recall that, in \( B_{2\varepsilon}(a_j) \setminus B_{\varepsilon}(a_j) \), we have
\[
\tilde{u} := \xi u_a.
\]
Then, using (6.6), it is easy to see that all partial differential (to any order) of \( \mathcal{M}(d_0, d_{\pm 1}, 0) \), with respect to \( d_0 \) and \( d_{\pm 1} \) are bounded independently of \( \varepsilon \). The result then follows at once. \( \square \)

### 6.3.2 Estimates of \( Q_2 \)

We now turn to the estimate of \( Q_2 \). This is slightly more complicated.

**Lemma 6.2** There exists \( c > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_0) \), we have
\[
\| Q_2(d_0, d_{\pm 1}, w) - Q_2(d_0, \tilde{d}_{\pm 1}, \tilde{w}) \| \leq c l_1 \| w - \tilde{w} \| \varepsilon
\]
provided \( l_1 \leq 1 \). Here, we have defined
\[
l_2 := \| w \|_{L^2_{\alpha, \beta}(|\mathbb{P}_d \setminus \Omega_{2\varepsilon})} + \| \tilde{w} \|_{L^2_{\alpha, \beta}(|\mathbb{P}_d \setminus \Omega_{2\varepsilon})}.
\]

**Proof:** Again, we observe that \( \mathcal{M}(d_0, d_{\pm 1}, w) \) does not depend on \( d_0 \) nor on \( d_{\pm 1} \) in \( \Omega_{2\varepsilon} \cup_2 B_{\varepsilon}(a_j) \). Therefore, \( Q_2 \) is identically equal to 0 in this set. It remains to evaluate \( Q_2 \) in \( B_{2\varepsilon}(a_j) \setminus B_{\varepsilon}(a_j) \). But, in this set, we have
\[
\mathcal{M}(d_0, d_{\pm 1}, w) = e^{-i\phi} e^{-\frac{w}{2\varepsilon}} \left( e^{-i\psi_{d_0}} \Delta \left( e^{i\psi_{d_0}} (\xi \varphi_{d_{\pm 1}}) \right) \varphi_{d_{\pm 1}}^{-1} \right.
\]
\[
\left. + \frac{|u| + w_r}{\varepsilon^2} (1 - (|u| + w_r)^2), \right.
\]
where we have defined
\[
f := e^{i(\tilde{\phi} + \frac{w}{2\varepsilon})} (|u| + w_r).
\]
In particular, the partial differential of \( \mathcal{M} \) with respect to \( w \), computed at \( (d_0, d_{\pm 1}, 0) \) reads
\[
D_w \mathcal{M}(d_0, d_{\pm 1}, 0)(0, 0, w) = e^{-i\phi} \left( e^{-i\psi_{d_0}} \Delta \left( e^{i\psi_{d_0}} (\xi \varphi_{d_{\pm 1}}) \right) \varphi_{d_{\pm 1}}^{-1} \right.
\]
\[
\left. - i \frac{w_i}{|u|} e^{-i\phi} \left( e^{-i\psi_{d_0}} \Delta \left( e^{i\psi_{d_0}} \tilde{u} \circ \varphi_{d_{\pm 1}} \right) \right) \right. \varphi_{d_{\pm 1}}^{-1}
\]
\[
\left. + \frac{w_r}{\varepsilon^2} (1 - 3|u|^2), \right.
\]
Thus, we obtain
\[
Q_2(d_0, d_{\pm 1}, w) = e^{-i\phi} \left( e^{-i\psi_{d_0}} \Delta \left( e^{i\psi_{d_0}} (\xi \varphi_{d_{\pm 1}}) \right) \varphi_{d_{\pm 1}}^{-1} \right.
\]
\[
\left. - e^{-i\phi} \Delta (e^{i\phi} w) \right.
\]
\[
\left. - i \frac{w_i}{|u|} e^{-i\phi} \left( e^{-i\psi_{d_0}} \Delta \left( e^{i\psi_{d_0}} \tilde{u} \circ \varphi_{d_{\pm 1}} \right) \right) \right. \varphi_{d_{\pm 1}}^{-1}
\]
\[
\left. - i \frac{w_i}{|u|} e^{-i\phi} \Delta \tilde{u}, \right.
\]
\[
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\]
The result is now an easy consequence of (6.6), if we decompose
\[ Q_2(d_0, d_{\pm 1}, w) - Q_2(\tilde{d}_0, \tilde{d}_{\pm 1}, \tilde{w}) = Q_2(d_0, d_{\pm 1}, w) - Q_2(d_0, d_{\pm 1}, \tilde{w}) + Q_2(d_0, d_{\pm 1}, \tilde{w}) - Q_2(\tilde{d}_0, \tilde{d}_{\pm 1}, \tilde{w}). \]

At first glance, one would expect a term \( \varepsilon^{-\alpha} \) in front of \( \|w\|_\varepsilon \), in the statement of the result, since we only have
\[ \|\nabla^2 w_r\|_{c_{\varepsilon, \alpha}} \leq c \varepsilon^{-\alpha} \|w_r\|_{c_{\varepsilon, \alpha}}. \] (6.7)

But this is not the case. The key observation is that, in the expression of \( Q_2 \), all terms involving second order partial derivatives of \( w \) are of the form
\[ |\nabla \eta|^2 \circ \varphi^{-1} D^2 w(d_1, d_1) \] and \( D^2 w(d_1, \nabla \eta \circ \varphi^{-1}) \).

Hence, the imaginary part of \( Q_2 \) only involves second partial derivatives of \( w_i \) and does not involve any second order partial derivatives of \( w_r \). In particular we do not have to use the estimate (6.7).

### 6.3.3 Estimates of \( Q_3 \)

The derivation of the estimates for \( Q_3 \) are far more involved than the estimates of the previous nonlinear quantities. We prove here the Lemma:

**Lemma 6.3** There exists \( c > 0 \) such that, for all \( \varepsilon \in (0, \varepsilon_0) \), we have
\[ \|Q_3(d_0, d_{\pm 1}, w) - Q_3(\tilde{d}_0, \tilde{d}_{\pm 1}, \tilde{w})\|_\varepsilon \leq c l_3 \|w - \tilde{w}\|_\varepsilon \]
\[ + \ c l_2 \left( \|d_0 - \tilde{d}_0\|_{\mathbb{R}^N} \ + \ \|d_{\pm 1} - \tilde{d}_{\pm 1}\|_{\mathbb{R}^N \times \mathbb{R}^N} \right), \] (6.8)
provided \( l_1 \leq 1 \), \( l_2 \leq 1 \) and \( l_3 \leq 1 \). Where, by definition
\[ l_3 := (\|w_i\|_{c_{\varepsilon, \alpha}} + \|w_r\|_{c_{\varepsilon, \alpha}}) + (\|\tilde{w}_i\|_{c_{\varepsilon, \alpha}} + \|\tilde{w}_r\|_{c_{\varepsilon, \alpha}}). \]

**Proof:** The proof of the estimate being rather technical, we have divided it into several parts.

**Step 1.** Preliminary computation. In order to simplify the notations, we will write for short
\[ u(w) := (|\tilde{u}| + w_r) \ + \ i \chi_{\varepsilon} w_i e^{-i(1 - \chi_{\varepsilon}) \tilde{w}}. \]
It will also be convenient to decompose \( \mathcal{M} \) into
\[ \mathcal{M}(d_0, d_{\pm 1}, w) := \mathcal{M}_I(d_0, d_{\pm 1}, w) + \mathcal{M}_II(w), \]
where we have defined
\[ \mathcal{M}_I(d_0, d_{\pm 1}, w) := e^{-i\phi} e^{-i(1 - \chi_{\varepsilon}) \tilde{w}} \]
\[ \times \left( e^{-i\psi_{\text{str}}} \Delta \left( e^{i\psi_{\text{str}}} e^{i(1 - \chi_{\varepsilon}) \tilde{w}} u(w) \circ \varphi_{d_{\pm 1}} \right) \right) \circ \varphi_{d_{\pm 1}}^{-1}. \] (6.9)
So, we can write
\[ \mathcal{M}_{14}(w) := \frac{u(w)}{\varepsilon^2} \left(1 - |u(w)|^2\right). \] (6.10)

Already observe that \( \mathcal{M}_{14} \) does not depend on \( d_0 \) nor on \( d_{\pm 1} \).

**Step 2.** Study of \( \mathcal{M}_{14} \). To begin with, let us simplify the expression of \( \mathcal{M}_{14} \) in \( B_2(a_j) \). In this set \( \chi \equiv 1 \), hence, the above expressions yield
\[ \mathcal{M}_{14}(w) = \frac{|\tilde{u}| + w}{\varepsilon^2} \left(1 - (|\tilde{u}| + w)^2\right). \]

So, we can write
\[ \mathcal{M}_{14}(w) = \text{terms independent of } w + \text{terms linear in } w \]
\[ - \frac{1}{\varepsilon^2} \left(|\tilde{u}|^2 + 2|\tilde{u}|w + w^2\right). \]

Since the terms which do not depend on \( w \) and the terms which are linear in \( w \) are irrelevant for the computation of \( (DM_{14, tw} - DM_{14, 0})(w) \), we find easily
\[ (DM_{14, tw} - DM_{14, 0})(w) = - \frac{1}{\varepsilon^2} \left(2|\tilde{u}|^2 + 4|\tilde{u}|w + 3w^2\right). \] (6.11)

In \( B_{2\varepsilon}(a_j) \setminus B_\varepsilon(a_j) \), the expression for \( \mathcal{M}_{14} \) is much more involved but does not introduce any additional difficulty. More precisely, we have
\[ \mathcal{M}_{14}(w) = \text{terms independent of } w + \text{terms linear in } w \]
\[ - \frac{|\tilde{u}|}{\varepsilon^2} \left(w_x^2 + \chi_x^2 w_t^2 + 2(|\tilde{u}| + w_r) \chi_x w_i \sin \left((1 - \chi_x) \frac{w_i}{|\tilde{u}|}\right)\right) \]
\[ - \frac{1}{\varepsilon^2} \left(w_r + i\chi_x w_i e^{-i(1 - \chi_x) \frac{w_i}{|\tilde{u}|}}\right) \]
\[ \times \left(2|\tilde{u}| w_r + w_t^2 + \chi_x^2 w_i^2 + 2(|\tilde{u}| + w_r) \chi_x w_i \sin \left((1 - \chi_x) \frac{w_i}{|\tilde{u}|}\right)\right) \]
\[ + \frac{i}{\varepsilon^2} \chi_{xx} w_i (1 - |\tilde{u}|^2) \left(e^{-i(1 - \chi_x) \frac{w_i}{|\tilde{u}|}} - 1\right). \]

Since the terms which do not depend on \( w \) and the terms which are linear in \( w \) are irrelevant for the computation of \( (DM_{14, tw} - DM_{14, 0})(w) \), we find easily that
\[ (DM_{14, tw} - DM_{14, 0})(w) = \frac{t}{\varepsilon^2} Q_{tw}(w_r, w_i), \]

where \( Q_{tw}(\cdot, \cdot) \) is a quadratic form whose coefficients are functions bounded in \( C_{0, 0}^1(B_{2\varepsilon} \setminus B_\varepsilon) \), provided the norm of \( w \) is bounded, say by 1, in \( C_{0, 0}^1(B_{2\varepsilon} \setminus B_\varepsilon) \).

We finally turn to the expression of \( \mathcal{M}_{14} \) in \( \Omega_{2\varepsilon} \). This time \( \chi \equiv 0 \) and we find
\[ \mathcal{M}_{14}(w) := \frac{|\tilde{u}| + w_r}{\varepsilon^2} \left(1 - (|\tilde{u}| + w_r)^2\right). \]
So, we can write

\[ M_{II}(w) = \text{terms independent of } w + \text{terms linear in } w \]

\[ - \frac{1}{\varepsilon^2} \left( 3 |\tilde{u}| w_r^2 + w_r^3 \right). \]

Since the terms which do not depend on \( w \) and the terms which are linear in \( w \) are irrelevant for the computation of \((DM_{II,w} - DM_{II,0})(w)\), we find easily

\[ (DM_{II,w} - DM_{II,0})(w) = -\frac{1}{\varepsilon^2} \left( 6 t |\tilde{u}| w_r^2 + 3 t^2 w_r^3 \right). \] (6.12)

At this stage, the key observation is that this quantity only involves \( w_r \) and does not involve any \( w_i \). This is important since, in \( \Omega_\sigma \), we can bound \(|w_r| \leq c \varepsilon^2 \|w\|E\), while we only have \(|w_i| \leq c \|w\|E\). Hence this expression remains bounded uniformly in \( \varepsilon \) if \( \|w\|E \) does.

**Step 3. Study of \( M_I \).** Once again we use the fact that in \( \Omega_{2\sigma} \cup_j B_\sigma(u_j) \), the mapping \( M_I \) does not depend on \( d_0 \) nor on \( d_{\pm 1} \). Hence, in this set, (6.9) reduces to

\[ M_I(d_0, d_{\pm 1}, w) = e^{-i\tilde{\phi}}e^{-i(1-\chi_{\varepsilon})\frac{w_i}{|\tilde{u}|}}\Delta \left( e^{i\phi}\left( (\tilde{u} + w_r)e^{i(1-\chi_{\varepsilon})\frac{w_i}{|\tilde{u}|}} + i\chi_{\varepsilon}w_i \right) \right), \]

Hence, we have

\[ M_I(d_0, d_{\pm 1}, w) = u(w) \left( i\Delta(\tilde{\phi} + (1 - \chi_{\varepsilon})\frac{w_i}{|\tilde{u}|}) - |\nabla(\tilde{\phi} + (1 - \chi_{\varepsilon})\frac{w_i}{|\tilde{u}|})|^2 \right) + 2i \nabla u(w) \nabla(\tilde{\phi} + (1 - \chi_{\varepsilon})\frac{w_i}{|\tilde{u}|}) + \Delta u(w). \]

With little work, we find

\[ M_I(d_0, d_{\pm 1}, w) = \text{terms independent of } w + \text{terms linear in } w \]

\[ + 2 \left( i \nabla w_r - w_r \nabla \tilde{\phi} \right) \nabla((1 - \chi_{\varepsilon})\frac{w_i}{|\tilde{u}|}) \]

\[ + i w_r \Delta((1 - \chi_{\varepsilon})\frac{w_i}{|\tilde{u}|}) - (|\tilde{u}| + w_r) \nabla((1 - \chi_{\varepsilon})\frac{w_i}{|\tilde{u}|})^2 \]

\[ + i e^{-i\tilde{\phi}} \Delta(e^{i\phi} \chi_{\varepsilon} w_i)(e^{-i(1-\chi_{\varepsilon})\frac{w_i}{|\tilde{u}|}} - 1). \]
Hence
\[(DM_{I,(d_0,d_{±1},tw)} - DM_{I,(d_0,d_{±1},0)})(0,0,w)\]
\[= 2t \left(i \nabla w_r - w_r \nabla \phi \right) \nabla ((1 - \chi_\varepsilon) \frac{w_1}{|w|})
+ tw_r \left(i \Delta((1 - \chi_\varepsilon) \frac{w_1}{|w|}) - \nabla((1 - \chi_\varepsilon) \frac{w_1}{|w|})^2 \right)
- t(|\tilde{u}| + tw_r)|\nabla((1 - \chi_\varepsilon) \frac{w_1}{|w|})|^2
+ i e^{-i\tilde{\phi}} \Delta(e^{i\tilde{\phi}} \chi_\varepsilon w_i) (e^{-i(1-\chi_\varepsilon) t} \tilde{\varepsilon} - 1)
+ t(1 - \chi_\varepsilon) \frac{w_1}{|u|} e^{-i\tilde{\phi}} \Delta(e^{i\tilde{\phi}} \chi_\varepsilon w_i) e^{-i(1-\chi_\varepsilon) t} \frac{w_1}{|u|}.
\] (6.13)

At this stage, the key observation is that this quantity does not involve any second order partial derivatives of \(w_r\). This is important since, in \(\Omega_\sigma\), we can bound \(\nabla^2 w_i|_{\partial^\alpha} \leq c\|w\|_{\mathcal{E}}\), while we only have \(\nabla^2 w_r|_{\partial^\alpha} \leq c \varepsilon^{-\alpha} \|w\|_{\mathcal{E}}\). Hence this expressions remains bounded uniformly in \(\varepsilon\) if \(\|w\|_{\mathcal{E}}\) does.

The expression of \(\mathcal{M}_I\) in \(B_{2\varepsilon}(a_j) \setminus B_\varepsilon(a_j)\) is much more involved since this time it depends on \(d_0\) and \(d_{±1}\) and we have to use extensively (6.6). However, it does not introduce any new difficulty in the estimate. In particular, the key point is that \((DM_{I,(d_0,d_{±1},tw)} - DM_{I,(d_0,d_{±1},0)})(0,0,w)\) does not contain any second order partial derivatives of \(w_r\).

**Step 4.** The result then follows at once from the expressions given in (6.11), (6.12) and (6.13). \(\square\)

### 6.4 The fixed point argument

Clearly, collecting the results of Proposition 6.2, Lemma 6.1, Lemma 6.2 and Lemma 6.3, one can show that there exists \(R > 0\) such that for all \(\varepsilon \in (0,\varepsilon_0)\), the operator
\[(d_0, d_{±1}, w) \rightarrow - (DM_{I,(0,0,0)})^{-1} (\mathcal{M}(0,0,0) + Q_1(d_0, d_{±1})
+ Q_2(d_0, d_{±1}, w) + Q_3(d_0, d_{±1}, w)).\]

is a contraction from the ball of radius \(R\) in \(\mathbb{R}^N \times (\mathbb{R}^N \times \mathbb{R}^N) \times \mathcal{E}\) into itself. Application of a standard fixed point Theorem for contraction mapping already gives the local uniqueness result of Theorem 6.1. Using in addition (6.5), we conclude that, the unique fixed point \((d_0, d_{±1}, w)\) satisfies the estimate
\[|d_0|_{\mathbb{R}^N} + |d_{±1}|_{\mathbb{R}^N \times \mathbb{R}^N} + \|w\|_{\mathcal{E}} \leq c\varepsilon^{1-\alpha},\]

where the constant \(c\) is independent of \(\varepsilon \in (0,\varepsilon_0)\).
6.5 Further information about the branch of solutions

We finally derive some informations about the branch of solutions we have constructed.

To begin with, let us define what we call the density of area of a vector valued function $u$ defined in some domain in $\mathbb{R}^2$.

**Definition 6.5** Assume that $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$. We set

$$\text{Area}(u) := \det(Du).$$

(6.14)

We now give some alternate definitions. For example, if we identify $\mathbb{R}^2$ with $\mathbb{C}$, the definition of the area becomes

$$\text{Area}(u) := -\frac{i}{2} (\partial_x u \partial_y \bar{u} - \partial_x \bar{u} \partial_y u).$$

(6.15)

Using polar coordinates, we obtain

$$\text{Area}(u) := -\frac{i}{2r} (\partial_r u \partial_\theta \bar{u} - \partial_r \bar{u} \partial_\theta u).$$

(6.16)

As a first example, we may compute the density of area corresponding to the radially symmetric solution of the Ginzburg-Landau equation

$$u_\varepsilon := S_\varepsilon e^{i \theta},$$

which is a model for any solution of the equation near a zero (of degree one). Using (6.16), we find

$$\text{Area}(u_\varepsilon) = \frac{1}{r} S_\varepsilon \partial_r S_\varepsilon.$$  

Since $\partial_r S_\varepsilon > 0$, we conclude that

$$\text{Area}(u_\varepsilon) > 0,$$

in all $\mathbb{C}$. What is more, given the asymptotic behavior of $S_\varepsilon$ at 0 and at $\infty$, we see that there exists a constant $c > 0$ such that

$$|\text{Area}(u)| \geq c \frac{\varepsilon^2}{\sup(\varepsilon^4, r^4)},$$

(6.17)

in $\mathbb{C}$.

Our purpose is now to study to what extent the previous inequality holds for the solutions we have constructed. More precisely, we prove the
Proposition 6.3 There exist $c > 0$, $\sigma_0 > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, if $u_\varepsilon$ denotes the solution of Theorem 6.2, then

$$|\text{Area}(u_\varepsilon)| \geq c \frac{\varepsilon^2}{\sup(\varepsilon^4, r^4)},$$

in each $B_{\sigma_0}(a_j)$, where

$$r := |z - a_j(\varepsilon)|,$$

is the distance from the zero of $u_\varepsilon$ in $B_{\sigma}(a_j)$.

Proof: For the sake of simplicity, let us assume that $a_j(\varepsilon) = 0$, $d_j = +1$ and let us drop the $j$ indices.

Up to a translation, the solution $u_\varepsilon$ is a perturbation of $\tilde{u}$ in $B_{2\sigma}$. In particular, we have

$$u_\varepsilon = e^{id_0} \nabla \tilde{u} + \mathcal{O}(\varepsilon^{1-\alpha} r^{\mu-1}).$$

(6.18)

We may always assume that $d_0 = 0$ since we have the identity

$$\text{Area}(e^{id_0} u) = \text{Area}(u).$$

Now, observe that, not only do we have (6.17) but, decreasing the constant $c$ if necessary, we also have

$$c \frac{\varepsilon^2}{\sup(\varepsilon^4, r^4)} \leq \text{Area}(S_\varepsilon e^{i\theta}) \leq c^{-1} \frac{\varepsilon^2}{\sup(\varepsilon^4, r^4)}.$$  

(6.19)

Step 1. In $B_{2\varepsilon}$, we have using (5.8) and (5.9),

$$\nabla \tilde{u} = \nabla(S_\varepsilon e^{i(\theta + r^\alpha)}) + \mathcal{O}(\varepsilon).$$

Combining this estimate together with (6.18), we find that

$$\text{Area}(u_\varepsilon) = \text{Area}(S_\varepsilon e^{i\theta}) + \mathcal{O}(1 + \varepsilon^{-\alpha} r^{\mu-1})$$

$$\geq c \varepsilon^{-2}.$$

Step 2. In $B_{2\delta} \setminus B_{2\varepsilon}$, we have using (5.8), (5.9) and (5.33),

$$\nabla \tilde{u} = \nabla(S_\varepsilon e^{i(\theta + H)}) + \mathcal{O}(\varepsilon^2 r^{-1}).$$

Combining this estimate together with (6.18), we find that

$$\text{Area}(u_\varepsilon) = \text{Area}(S_\varepsilon e^{i(\theta + H)}) + \mathcal{O}(\varepsilon^2 r^{-2} + \varepsilon^{1-\alpha} r^{\mu-2}).$$

Moreover, using (6.16), we can compute explicitly

$$\text{Area}(S_\varepsilon e^{i(\theta + H)}) = \frac{1}{r} (1 + \partial_\theta H) S_\varepsilon \partial_\varepsilon S_\varepsilon.$$
Hence, we conclude using (6.19) that
\[
\text{Area } (u_\varepsilon) = \text{Area } (S_\varepsilon e^{i\theta}) + O(\varepsilon^2 r^{-2} + \varepsilon^{1-\alpha} r^{\mu-2}) \\
\geq \varepsilon^2 r^{-4}.
\]
Observe that this estimate still holds in $B_{1.5} \setminus B_{2.5}$.

**Step 3.** In $B_{\sigma} \setminus B_{2.5}$, we have explicitly
\[
\tilde{u} = \xi u = \xi e^{i(\theta + H)},
\]
and
\[
u_\varepsilon = (\xi + w_r) e^{i(\theta + H + \frac{w}{\xi})},
\]
where $w$ is the solution given by the fixed point argument. Hence, we find using (6.16)
\[
\text{Area } (u_\varepsilon) = \frac{1}{r} (1 + \partial_\theta H + \xi^{-1} \partial_\theta w_i) (\xi + w_r) \partial_r (\xi + w_r).
\]
For the time being, we only know that
\[
|w_r| + \varepsilon |\partial_r w_r| \leq c \varepsilon^{1-\alpha} r^{\mu},
\]
in $B_{2.5} \setminus B_{2.5}$. And this estimate is not sufficient for our purpose. However, we can improve this estimate using the strategy we have already used in the proof of Proposition 5.2. More precisely, the function $u_\varepsilon$ is a solution of
\[
\Delta u_\varepsilon + \frac{u_\varepsilon}{\varepsilon^2} (1 - |u_\varepsilon|^2) = 0.
\]
Taking the real part of this identity, we obtain
\[
\Delta w_r - |\nabla (\theta + H + \frac{w_i}{\xi})|^2 (\xi + w_r) + \frac{w_r + \xi}{\varepsilon^2} (1 - |\xi + w_r|^2) = 0.
\]
Making use of the fact that $\xi$ is a solution of (5.28), we conclude that
\[
\Delta w_r = \left(|\nabla (\theta + H + \frac{w_i}{\xi})|^2 + \frac{1}{\varepsilon^2} (3\xi^2 + 3\xi w_r + w_r^2 - 1)\right) w_r
\]
\[
= \left(|\nabla (\theta + H + \frac{w_i}{\xi})|^2 - |\nabla \theta|^2\right) \xi.
\]
The right hand side of this equation is easily seen to be bounded by a constant, independently of $\varepsilon$. Moreover $|w_r| \leq c \varepsilon^{1-\alpha} \delta^\mu$ on $\partial B_{2.5}$ and $|w_r| \leq c \varepsilon^{1-\alpha}$ on $\partial B_{2.5}$.

If $\varepsilon$ is small enough, we can assume that the potential in the above equation satisfies
\[
\left(|\nabla (\theta + H + \frac{w_i}{\xi})|^2 + \frac{1}{\varepsilon^2} (3\xi^2 + 3\xi w_r + w_r^2 - 1)\right) \geq \frac{1}{\varepsilon^2}.
\]
The maximum principle shows that the function
\[ r \rightarrow c \left( \varepsilon^2 + \varepsilon^{1-\alpha} \delta^{\mu} e^{(2\delta - r)/\varepsilon} + \varepsilon^{1-\alpha} e^{(r-2\sigma)/\varepsilon} \right), \]
can be used as a barrier function to prove that \( w_r \) is bounded by a constant times \( \varepsilon^2 \) in \( B_{3\sigma/2} \setminus B_{3\delta} \), provided \( \varepsilon \) is chosen small enough. Similarly, we prove that \( \partial_r w_r \) is bounded by a constant times \( \varepsilon^2 r^{-1} \) in \( B_{\sigma} \setminus B_{4\delta} \), provided \( \varepsilon \) is chosen small enough.

Using (5.33) together with the fact that
\[ |w_r| + r \left| \partial w_r \right| \leq c \varepsilon^2 \quad \text{and} \quad \left| \partial_\theta w_r \right| \leq c \varepsilon^{1-\alpha} r^\mu, \]
we get
\[
\text{Area}(u_\varepsilon) = \frac{1}{r} (1 + \partial_\theta H + O(\varepsilon^{1-\alpha} r^\mu))(S_\varepsilon + O(\varepsilon^2))(\partial_r S_\varepsilon + O(\varepsilon^2 r^{-1}))
\]
\[ = \text{Area} (S_\varepsilon e^{i\theta}) + O(\varepsilon^2 r^{-2} + \varepsilon^{3-\alpha} r^{\mu-4}). \]

Therefore, using (6.19), we conclude that \( \text{Area} (u_\varepsilon) \geq c \varepsilon^2 r^{-4} \) in \( B_{\sigma_0} \setminus B_{4\delta} \) provided \( \sigma_0 \) is chosen small enough (but fixed). This completes the proof of the Proposition. \( \square \)
Chapter 7

Elliptic Operators in Weighted Sobolev Spaces.

In this Chapter, we prove some estimates for solutions of elliptic problems in weighted Sobolev spaces. Again, the aim of this Chapter is not to provide a thorough description of the theory of elliptic operators in weighted Sobolev spaces but rather to provide simple proofs of some results which are needed in the subsequent Chapters. Further results can be found in the references already given in Chapter 2.

7.1 General overview

To begin with, let us explain what are the weighted Sobolev spaces we have in mind. Let \( B_1 \subset \mathbb{R}^n \) be the unit ball of \( \mathbb{R}^n \), \( n \geq 2 \). We define the weighted Lebesgue spaces as follows:

Definition 7.6 Given \( \nu \in \mathbb{R} \), the space \( L^2_\nu(B_1 \setminus \{0\}) \) is defined to be the set of functions \( w \in L^2_{\text{loc}}(B_1 \setminus \{0\}) \) for which the following norm is finite

\[
\|w\|_{L^2_\nu} := \left( \int_{B_1} |w|^2 r^{-2\nu-2} \, dx \right)^{1/2}.
\] (7.1)

In other words, we simply have

\[
L^2_\nu(B_1 \setminus \{0\}) := r^\nu L^2(B_1, r^{-2} \, dx).
\]

Granted this definition, we may now define the weighted Sobolev spaces by:

Definition 7.7 Given \( k \in \mathbb{N} \) and given \( \nu \in \mathbb{R} \), we the space \( H^k_\nu(B_1 \setminus \{0\}) \) is defined to be the set of functions \( w \in L^2_{\text{loc}}(B_1 \setminus \{0\}) \) for which the following
norm is finite
\[ \|w\|_{H^k}^2 := \sum_{j=0}^{k} \|\nabla^j w\|_{L^2_{\nu-j}}. \] (7.2)

The relation between weighted Sobolev spaces and weighted Hölder spaces is given by the following simple observation:
\[ C_0^{\alpha, \mu}(B_1 \setminus \{0\}) \hookrightarrow w \in L^2_{\nu}(B_1 \setminus \{0\}), \]
for all \( \nu < \mu + \frac{n-2}{2} \).

Obviously, for all \( k \geq 2 \)
\[ \Delta : H^k_{\nu}(B_1 \setminus \{0\}) \twoheadrightarrow H^{k-2}_{\nu-2}(B_1 \setminus \{0\}), \]
is well defined and bounded and the same is true for any operator of the form \( \Delta + c r^{-2} \).

Paralleling what we have done for weighted Hölder spaces, we can also investigate the mapping properties of the Laplacian when defined between weighted Sobolev spaces. The results are essentially the same and show that, again, the indicial roots play a crucial rôle in understanding the mapping properties of the Laplacian. In particular, if we set
\[ \delta_j^\pm = \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + \lambda_j}, \] (7.3)
we can prove the :

**Proposition 7.1** Assume that \( \nu > \frac{2-n}{2} \) and that \( \nu \notin \{\delta_j^\pm : j \in \mathbb{N}\} \). Let \( j_0 \in \mathbb{N} \) be the least index for which \( \nu < \delta_j^{+} \). Then
\[ \Delta : H^2_{\nu,D}(B_1 \setminus \{0\}) \twoheadrightarrow L^2_{\nu-2}(B_1 \setminus \{0\}), \]
injective and the dimension of its cokernel is given by \( j_0 \). Here the subscript \( D \) refers to the fact that the functions are equal to 0 on \( \partial B_1 \).

**Proof** : The proof is essentially identical to the proof of the corresponding result (see Proposition 2.4) in weighted Hölder spaces.

We can also prove the :

**Proposition 7.2** Assume that \( \nu < \frac{n-2}{2} \) and that \( \nu \notin \{\delta_j^\pm : j \in \mathbb{N}\} \). Let \( j_0 \in \mathbb{N} \) be the least index for which \( \nu > \delta_j^{-} \). Then,
\[ \Delta : H^2_{\nu,D}(B_1 \setminus \{0\}) \twoheadrightarrow L^2_{\nu-2}(B_1 \setminus \{0\}), \]
is surjective and the dimension of its kernel is given by \( j_0 \).
Proof : This time the proof is close to the proof of Proposition 2.6. □

We will not develop any further the theory of elliptic operators in weighted Sobolev spaces but we will rather concentrate our attention on some specific estimates which will be needed in the subsequent Chapters.

7.2 Estimates for the Laplacian

Our first result is related to the study of the Laplacian in the unit punctured ball of \( \mathbb{R}^2 \), when the weight parameter \( \nu \) is in the surjectivity range.

Lemma 7.1 Let \( B_1 \) be the unit ball of \( \mathbb{R}^2 \) and assume that \( \nu \in (-1,0) \). If \( f \in L^2_{\nu-2}(B_1 \setminus \{0\}) \) and if \( w \in H^1_{\nu}(B_1 \setminus \{0\}) \) is a weak solution of

\[
\begin{align*}
\Delta w &= f \quad \text{in} \quad B_1 \setminus \{0\} \\
w &= 0 \quad \text{on} \quad \partial B_1.
\end{align*}
\]

(7.4)

Then, there exists \( c_0 \in \mathbb{R} \) such that \( \Delta w = f + c_0 \delta_0 \) in \( B_1 \) and

\[
\|\nabla w\|^2_{L^2_{\nu-1}} \leq \frac{2}{\nu^2(1-\nu)^2} \|f\|^2_{L^2_{\nu-2}} + \frac{1}{2\pi \nu^2} \left( c_0 + \int_{B_1} f \right)^2,
\]

(7.5)

provided all integrals are well defined.

Proof : We begin by the following simple observation. \textit{A priori} \( w \) does not extend to a weak solution of \( \Delta w = f \) in \( B_1 \). However, since we have assumed that

\[
\int_{B_1} r^{-2-2\nu}|w|^2,
\]

is finite with \( \nu \in (-1,0) \), necessarily there exists \( c_0 \in \mathbb{R} \) such that

\[
\Delta w = f + c_0 \delta_0,
\]

in \( B_1 \). In addition, without loss of generality, we may assume that \( f \) is supported away from the origin since the general case can be obtained using a simple density argument.

We now, decompose both \( w \) and \( f \) as

\[
w := v_0 + v \quad \text{and} \quad f := g_0 + g,
\]

where both \( v \) and \( g \) belong to \( \text{Span}\{h_n e^{in\theta} : n \neq 0\} \), while \( v_0 \) and \( g_0 \) only depend on \( r \).

Step 1. Notice that \( v \) is a weak solution of

\[
\begin{align*}
\Delta v &= g \quad \text{in} \quad B_1 \\
v &= 0 \quad \text{on} \quad \partial B_1.
\end{align*}
\]
Further observe that $v(r, \cdot) \in \text{Span}\{e^{in\theta} : n \neq 0\}$, hence, we always have
\[
\int_{\partial B_r} |v|^2 d\theta \leq \int_{\partial B_r} |\partial_\theta v|^2 d\theta.
\]
Multiplying this inequality by $r^{-1-2\nu}$ and integrating over $(0, 1)$, we obtain
\[
\int_{B_1} r^{-2\nu-2} |v|^2 \leq \int_{B_1} r^{-2\nu-2} |\partial_\theta v|^2.
\] (7.6)

Now, we multiply the equation by $-r^{-2\nu} v$ and integrate by parts. We obtain
\[
-\int_{B_1} r^{-2\nu} v g = \int_{B_1} r^{-2\nu} |\nabla v|^2 + \frac{1}{2} \int_{B_1} r^{-2\nu} \nabla |v|^2.
\]
Since
\[
\frac{1}{2} \int_{B_1} r^{-2\nu} \nabla |v|^2 = -2\nu^2 \int_{B_1} r^{-2\nu-2} |v|^2,
\]
we get
\[
-\int_{B_1} r^{-2\nu} v g = \int_{B_1} r^{-2\nu} |\nabla v|^2 - 2\nu^2 \int_{B_1} r^{-2\nu} |v|^2.
\]
Which after the use of Cauchy-Schwarz inequality becomes
\[
\int_{B_1} r^{-2\nu} |\nabla v|^2 - 2\nu^2 \int_{B_1} r^{-2\nu-2} |v|^2
\leq \left( \int_{B_1} r^{2-2\nu} |g|^2 \right)^{1/2} \left( \int_{B_1} r^{-2\nu-2} |v|^2 \right)^{1/2}.
\]
Finally, we use (7.6) to conclude that
\[
\int_{B_1} r^{-2\nu} |\nabla v|^2 - 2\nu^2 \int_{B_1} r^{-2\nu-2} |\partial_\theta v|^2
\leq \left( \int_{B_1} r^{2-2\nu} |g|^2 \right)^{1/2} \left( \int_{B_1} r^{-2\nu-2} |\partial_\theta v|^2 \right)^{1/2}.
\] (7.7)

Notice that all integrations are valid since $g$ is supported away from the origin and hence $v$ is bounded by a constant times $r$ near 0.

Now, we multiply the equation by $r^{1-2\nu} \partial_r v$ and integrate by parts. This time, we get
\[
\int_{B_1} r^{1-2\nu} g \partial_r v = \nu \int_{B_1} r^{-2\nu} (|\partial_r v|^2 - r^{-2} |\partial_\theta v|^2) + \frac{1}{2} \int_{\partial B_1} |\partial_r v|^2.
\] (7.8)

In particular, we have the inequality
\[
\int_{B_1} r^{1-2\nu} g \partial_r v \geq \nu \int_{B_1} r^{-2\nu} |\partial_r v|^2 - \nu \int_{B_1} r^{-2-2\nu} |\partial_\theta v|^2.
\]
which becomes, after the application of Cauchy-Schwarz inequality

\[
\nu \int_{B_1} r^{-2\nu} |\partial_r v|^2 - \nu \int_{B_1} r^{-2-2\nu} |\partial_\theta v|^2 \\
\leq \left( \int_{B_1} r^{2-2\nu} |g|^2 \right)^{1/2} \left( \int_{B_1} r^{-2\nu} |\partial_r v|^2 \right)^{1/2}.
\]  

(7.9)

In order to simplify the discussion, we set

\[ X^2 := \int_{B_1} r^{-2\nu} |\partial_r v|^2 \quad Y^2 := \int_{B_1} r^{-2-2\nu} |\partial_\theta v|^2 \]

and

\[ G^2 := \int_{B_1} r^{2-2\nu} |g|^2. \]

The inequalities (7.7) and (7.9) are then translated into

\[
\nu (X^2 - Y^2) \leq G X^2 + (1 - 2\nu^2) Y^2 \leq G Y^2.
\]

We can multiply the first inequality by \(-\nu\) and add it to the second inequality to get

\[
(1 - \nu^2) (X^2 + Y^2) \leq G (Y - \nu X).
\]

And, since \(\nu \in (-1, 0)\), we conclude

\[
X^2 + Y^2 \leq \frac{2}{(1 - \nu^2)^2} G^2.
\]

That is

\[
\int_{B_1} r^{-2\nu} |\nabla v|^2 \leq \frac{2}{(1 - \nu^2)^2} \int_{B_1} r^{2-2\nu} |g|^2.
\]  

(7.10)

**Step 2.** It remains to prove a similar estimate for \(v_0\). We use once more (7.8) which, since all functions are now radial functions, reads

\[-\nu \int_{B_1} r^{-2\nu} |\partial_r v_0|^2 = - \int_{B_1} r^{1-2\nu} g_0 \partial_r v_0 + \pi |\partial_r v_0(1)|^2.\]

We use Cauchy-Schwarz inequality to get

\[
-\nu \int_{B_1} r^{-2\nu} |\partial_r v_0|^2 \leq \left( \int_{B_1} r^{2-2\nu} |g_0|^2 \right)^{1/2} \left( \int_{B_1} r^{-2\nu} |\partial_r v_0|^2 \right)^{1/2} + \pi |\partial_r v_0(1)|^2.
\]

It is then an easy exercise to see that

\[
\nu^2 \int_{B_1} r^{-2\nu} |\partial_r v_0|^2 \leq \int_{B_1} r^{2-2\nu} |g_0|^2 + 2 \pi |\partial_r v_0(1)|^2.
\]

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Now, \( \Delta v_0 = g_0 + c_0 \delta_0 \), which after integration over \( B_1 \) yields
\[
2 \pi \partial_r v_0(1) = \int_{B_1} g_0 + c_0.
\]
Hence
\[
\nu^2 \int_{B_1} r^{-2\nu} |\partial_r v_0|^2 \leq \int_{B_1} r^{-2\nu} |g_0|^2 + \frac{1}{2\pi} \left( \int_{B_1} g + c_0 \right)^2.
\]
(7.11)

The result of the Lemma is obtained by summing (7.10) with (7.11). \( \square \)

We now derive a similar result in more general domains. As usual, \( \Omega \) is a regular open subset of \( \mathbb{R}^2 \) and \( \Sigma := \{a_1, \ldots, a_N\} \) is a finite set of points in \( \Omega \).

We chose \( \sigma > 0 \) in such a way that, if \( a_i \neq a_j \in \Sigma \), then \( B_{2\sigma}(a_i) \) and \( B_{2\sigma}(a_j) \) are disjoint and both included in \( \Omega \). We define weighted Lebesgue spaces on \( \Omega \setminus \Sigma \) by:

**Definition 7.8** Given \( \nu \in \mathbb{R} \), the space \( L^2_{\nu}(\Omega \setminus \Sigma) \) is defined to be the set of functions \( w \in L^2_{\text{loc}}(\Omega \setminus \Sigma) \) for which the following norm is finite
\[
\|w\|_{L^2_{\nu}} := \left( \int_{\Omega} |w|^2 \text{dist}(x, \Sigma)^{-2\nu-2} \, dx \right)^{1/2}.
\]
(7.12)

Our main result reads:

**Proposition 7.3** Assume that \( \nu \in (-1, 0) \). There exists a constant \( c > 0 \) and, for all \( w \in L^2_{\nu}(\Omega \setminus \Sigma) \) weak solution of
\[
\begin{align*}
\Delta w = f & \quad \text{in} \quad \Omega \setminus \Sigma \\
 w = 0 & \quad \text{on} \quad \partial \Omega,
\end{align*}
\]
(7.13)

there exists \( c_1, \ldots, c_N \in \mathbb{R} \) such that
\[
\Delta w = f + \sum_{j=1}^{N} c_j \delta_{a_j},
\]
in \( \Omega \) and
\[
\|\nabla w\|_{L^2_{\nu-1}}^2 \leq c \left( \|f\|_{L^2_{\nu}}^2 + \sum_{j=1}^{N} \left( c_j + \int_{B_{2\sigma}(a_j)} f \right)^2 \right),
\]
(7.14)
provided all integrals are well defined.
Proof: The proof of the result relies on Lemma 7.1 and on the domain decomposition method we have already used in Chapter 2 and in Chapter 6.

Applying Lemma 7.1 to each \( B_\sigma(a_j) \), we already see that there exists \( c_j \in \mathbb{R} \) such that
\[
\Delta w = f + c_j \delta_{a_j},
\]
in \( B_\sigma(a_j) \).

Step 1. Given \( \phi := (\phi_1, \ldots, \phi_N) \in \Pi_j H^{3/2}(\partial B_\sigma(a_j)) \), we define \( v_{ext} \) to be the harmonic extension of the \( \phi_j \) in \( \Omega_\sigma \) which takes the value 0 on \( \partial \Omega \). Namely
\[
\begin{align*}
\Delta v_{ext} &= 0 \quad \text{in} \quad \Omega_\sigma \\
v_{ext} &= \phi_j \quad \text{on} \quad \partial B_\sigma(a_j) \\
v_{ext} &= 0 \quad \text{on} \quad \partial \Omega,
\end{align*}
\]
and we may also define \( v_{int} \) to be the harmonic extension of the \( \phi_j \) in each \( B_\sigma(a_j) \). Namely
\[
\begin{align*}
\Delta v_{int} &= 0 \quad \text{in each} \quad B_\sigma(a_j) \\
v_{int} &= \phi_j \quad \text{on each} \quad \partial B_\sigma(a_j).
\end{align*}
\]

As we have already done in Chapter 2 and in Chapter 6, we define the Dirichlet to Neumann mapping by
\[
DN(\phi) := \left( (\partial_1 v_{ext} - \partial_1 v_{int})(\partial B_\sigma(a_1)), \ldots, (\partial_N v_{ext} - \partial_N v_{int})(\partial B_\sigma(a_N)) \right).
\]
It is clear that
\[
DN : \Pi_j H^{3/2}(\partial B_\sigma(a_j)) \rightarrow \Pi_j H^{1/2}(\partial B_\sigma(a_j)),
\]
is a well defined bounded linear operator. Moreover, \( DN \) is a first order elliptic operator whose principal symbol is \(-2|\xi|\). Hence, in order to prove that \( DN \) is an isomorphism, it is enough to show that \( DN \) is injective.

Assume that \( DN(\phi) = 0 \). Then the function \( v \) which is equal to \( v_{ext} \) in \( \Omega_\sigma \) and equal to \( v_{int} \) in \( \cup_j B_\sigma(a_j) \) is harmonic in all \( \Omega \) and has 0 boundary values on \( \partial \Omega \). Thus \( v = 0 \) and hence \( \phi = 0 \). This proves the injectivity of \( DN \).

Step 2. Now, we define \( w_{int} \) and \( w_{ext} \) to be the solutions of the following equations
\[
\begin{align*}
\Delta w_{int} &= f + c_j \delta_{a_j} \quad \text{in each} \quad B_\sigma(a_j) \\
w_{int} &= 0 \quad \text{on each} \quad \partial B_\sigma(a_j),
\end{align*}
\]
and
\[
\begin{align*}
\Delta w_{ext} &= f \quad \text{in} \quad \Omega_\sigma \\
w_{ext} &= 0 \quad \text{on} \quad \partial \Omega_\sigma.
\end{align*}
\]
Finally, we set
\[ \phi := DN^{-1} (\partial_{r_1} w_{\text{ext}} - \partial_{r_1} w_{\text{int}})_{|\partial B_\sigma(a_1)}, \ldots, (\partial_{r_N} w_{\text{ext}} - \partial_{r_N} w_{\text{int}})_{|\partial B_\sigma(a_N)} \].

(7.17)

We will denote \( \phi := (\phi_1, \ldots, \phi_N) \). We claim that there exists \( c > 0 \) such that, for all \( j = 1, \ldots, N \), we have
\[ \| \phi_j \|_{H^{3/2}(\partial B_\sigma(a_j))} \leq c \left( \| f \|_{L^2(\Omega \setminus \Sigma)}^2 + \left( c_j + \int_{B_\sigma(a_j)} f \right)^2 \right). \]

(7.18)

For the sake of simplicity in the notations, let us assume that \( a_j = 0 \) and let us drop the \( j \) indices. First, we use standard trace embedding results to get
\[ \| \partial_r w_{\text{int}} \|_{H^{1/2}(\partial B_\sigma)} \leq c \| \nabla w_{\text{int}} \|_{H^1(B_\sigma \setminus B_{\sigma/2})}. \]

(7.19)

Now, using (7.15), we can state that there exists \( c > 0 \) such that
\[ \| \nabla w_{\text{int}} \|_{H^1(B_\sigma \setminus B_{\sigma/3})} \leq c \left( \| \nabla w_{\text{int}} \|_{L^2(B_\sigma \setminus B_{\sigma/3})} + \| f \|_{L^2(B_\sigma \setminus B_{\sigma/3})} \right). \]

(7.19)

Using once more the fact that \( w_{\text{int}} \) satisfies (7.15) together with the result of Lemma 7.1, we conclude that
\[ \| \nabla w_{\text{int}} \|_{L^2(B_\sigma \setminus B_{\sigma/3})} \leq c \left( \| f \|_{L^2(\Omega \setminus \Sigma)}^2 + \left( c_j + \int_{B_\sigma} f \right)^2 \right). \]

Therefore
\[ \| \partial_r w_{\text{int}} \|_{H^{1/2}(\partial B_\sigma)} \leq c \left( \| f \|_{L^2(\Omega \setminus \Sigma)}^2 + \left( c_j + \int_{B_\sigma} f \right)^2 \right). \]

(7.20)

Similarly, we get
\[ \| \partial_r w_{\text{ext}} \|_{H^{1/2}(\partial B_\sigma)} \leq c \| f \|_{L^2(\Omega \setminus \Sigma)}^2. \]

(7.21)

The claim follows at once from (7.20) and (7.21).

**Step 3.** Let \( \phi \) be given by (7.17). We define \( v_{\text{ext}} \) to be the harmonic extension of the \( \phi_j \) in \( \Omega_\sigma \) which takes the value 0 on \( \partial \Omega \). And we define \( v_{\text{int}} \) to be the harmonic extension of the \( \phi_j \) in each \( B_\sigma(a_j) \). By construction of \( \phi \), it is clear that the following identity holds
\[ \left\{ \begin{array}{ll}
    w = w_{\text{int}} - v_{\text{int}} & \text{in } \bigcup_{j=1}^N B_\sigma(a_j) \\
    w = w_{\text{ext}} - v_{\text{ext}} & \text{in } \Omega_\sigma.
\end{array} \right. \]

(7.22)
In $\Omega_{\sigma}$, we make use of the following standard estimates
\[
\|v_{\text{ext}}\|_{H^1(\Omega_{\sigma})} \leq c \sum_{j=1}^{N} \|\phi_j\|_{H^{3/2}(\partial B_{\sigma}(a_j))},
\] (7.23)
and
\[
\|w_{\text{ext}}\|_{H^1(\Omega_{\sigma})} \leq c \|f\|_{L^2(\Omega_{\sigma})}.
\] (7.24)

In each $B_{\sigma}(a_j)$, standard regularity results yield
\[
\|\nabla v_{\text{int}}\|_{L^2(B_{\sigma}(a_j))} \leq c \|\phi_j\|_{H^{3/2}(\partial B_{\sigma}(a_j))}.\]
(7.25)
Moreover, since $w_{\text{int}}$ satisfies (7.15), Lemma 7.1 yields
\[
\|\nabla w_{\text{int}}\|_{L^2(B_{\sigma}(a_j))}^2 \leq c \left( \|f\|_{L^2(B_{\sigma}(a_j))}^2 + \left( c_j + \int_{B_{\sigma}(a_j)} f \right)^2 \right).\]
(7.26)

Combining (7.23)-(7.26) with (7.18) leads to the desired estimate. The proof of the Proposition is therefore complete.

Following the strategy we have used to prove Lemma 7.1, we can prove the:

**Proposition 7.4** Assume that $\nu \in (0, 1)$. There exists $c > 0$ such that, for all $f \in L^2_{\nu - 2}(\mathbb{R}^2 \setminus \{0\})$, if
\[
u := \frac{1}{2\pi} \log r \ast f,
\]
then
\[
\|u - u(0)\|_{H^2} \leq c \|f\|_{L^2_{\nu - 2}}.
\]

**Proof:** We define
\[
w := u - u(0),
\]
and we decompose both $w$ and $f$ as
\[
w := v_0 + v \quad \text{and} \quad f := g_0 + g,
\]
where both $v$ and $g$ belong to Span$\{h_n e^{in\theta} : n \neq 0\}$, while $v_0$ and $g_0$ only depend on $r$.

**Step 1.** To begin with, let us assume that $g$ has compact support in $\mathbb{R}^2 \setminus \{0\}$. There is no loss of generality in doing so since, the general case can be obtained by a classical density argument. Under this assumption, since $v$ is a weak solution of
\[
\Delta v = g,
\]
then $v$ is smooth at the origin and $v(0) = 0$. Hence, there exists a constant $c > 0$ (depending on $g$) such that
\[
|v| + r |\nabla v| \leq cr,
\]

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in $B_1$. Furthermore, it is classical to see that there also exists a constant $c > 0$ (depending on $g$) such that
\[ |v| \leq c r^{-1} \quad \text{and} \quad |\nabla v| \leq c r^{-2}, \tag{7.27} \]
in $\mathbb{R}^2 \setminus B_1$.

Further observe that $v(r, \cdot) \in \text{Span}\{e^{in\theta} : n \neq 0\}$, hence
\[ \int_{\partial B_r} |v|^2 \, d\theta \leq \int_{\partial B_r} |\partial_\theta v|^2 \, d\theta. \]
Multiplying this inequality by $r^{-1-2\nu}$ and integrating over $(0, +\infty)$, we obtain
\[ \int_{\mathbb{R}^2} r^{-2\nu-2} |v|^2 \leq \int_{\mathbb{R}^2} r^{-2\nu-2} |\partial_\theta v|^2. \tag{7.28} \]
Now, we multiply the equation $\Delta v = g$ by $r^{-2\nu} v$ and integrate by parts over $B_R$. We obtain
\[
\int_{B_R} r^{-2\nu} v g = -\int_{B_R} r^{-2\nu} |\nabla v|^2 + 2 \nu^2 \int_{B_R} r^{-2-2\nu} |v|^2 \\
+ \int_{\partial B_R} r^{-2\nu} v \partial_r v + \nu \int_{\partial B_R} r^{-1-2\nu} |v|^2.
\]
We let $R$ tend to $+\infty$ and obtain thanks to (7.27)
\[ \int_{\mathbb{R}^2} r^{-2\nu} v g = -\int_{\mathbb{R}^2} r^{-2\nu} |\nabla v|^2 + 2 \nu^2 \int_{\mathbb{R}^2} r^{-2-2\nu} |v|^2. \tag{7.29} \]
Multiplying the equation by $r^{1-2
u} \partial_r v$ and integrating by parts over $B_R$, we also get
\[
\int_{B_R} r^{1-2\nu} g \partial_r v = \nu \int_{B_R} r^{-2\nu} (|\partial_r v|^2 - r^{-2} |\partial_\theta v|^2) \\
+ \frac{1}{2} \int_{\partial B_R} r^{-1-2\nu} (|\partial_r v|^2 - r^{-2} |\partial_\theta v|^2).
\]
Again, we let $R$ tend to $+\infty$ and obtain
\[ \int_{\mathbb{R}^2} r^{1-2\nu} g \partial_r v = \nu \int_{\mathbb{R}^2} r^{-2\nu} (|\partial_r v|^2 - r^{-2} |\partial_\theta v|^2). \tag{7.30} \]
Using this identity together with (7.29) yields
\[
(1 - \nu^2) \int_{\mathbb{R}^2} r^{-2\nu} |\partial_r v|^2 + (1 + \nu^2) \int_{\mathbb{R}^2} r^{-2-2\nu} |\partial_\theta v|^2 \\
-2 \nu^2 \int_{\mathbb{R}^2} r^{-2-2\nu} |v|^2 = -\int_{\mathbb{R}^2} r^{-2\nu} v g - \nu \int_{\mathbb{R}^2} r^{1-2\nu} g \partial_r v.
\]
Making use of (7.28), we get
\[ (1 - \nu^2) \int_{\mathbb{R}^2} r^{-2\nu} |\nabla v|^2 \leq -\int_{\mathbb{R}^2} r^{-2\nu} v g - \nu \int_{\mathbb{R}^2} r^{1-2\nu} g \partial_r v. \]
Now, we make use of Cauchy-Schwarz inequality and once more of (7.28), to conclude that

\[(1 - \nu^2)^2 \int_{\mathbb{R}^2} r^{-2\nu} |\nabla v|^2 \leq \int_{\mathbb{R}^2} r^{2-2\nu} |g|^2\]

It still follows from (7.28) that we also get

\[(1 - \nu^2)^2 \int_{\mathbb{R}^2} r^{-2-2\nu} |v|^2 \leq \int_{\mathbb{R}^2} r^{2-2\nu} |g|^2\]

Then, using classical regularity results for solutions of elliptic equations we get

\[\int_{\mathbb{R}^2} r^{2-2\nu} |\nabla^2 v|^2 \leq c \int_{\mathbb{R}^2} r^{2-2\nu} |g|^2\]

for some constant \(c\) which does not depend on \(g\).

\[\text{Step 2.}\]

It remains to prove the relevant estimate for \(v_0\). Observe that (7.30) still holds and gives us

\[\int_{\mathbb{R}^2} r^{1-2\nu} g_0 \partial_r v_0 = \nu \int_{\mathbb{R}^2} r^{-2\nu} |\partial_r v_0|^2.\]

and hence

\[\nu^2 \int_{\mathbb{R}^2} r^{-2\nu} |\partial_r v_0|^2 \leq \int_{\mathbb{R}^2} r^{2-2\nu} |g_0|^2. \quad (7.31)\]

Finally, notice that, since \(\nu > 0\), we have (provided all integrals are well defined)

\[\int_0^\infty r^{-1-2\nu} h^2 = -\frac{1}{2\nu} \int_0^\infty \partial_r (r^{-2\nu}) h^2 = \frac{1}{\nu} \int_0^\infty r^{-2\nu} h \partial_r h \leq \frac{1}{\nu} \left( \int_0^\infty r^{-1-2\nu} h^2 \right)^{1/2} \left( \int_0^\infty r^{-1-2\nu} |\partial_r h|^2 \right)^{1/2}.\]

Hence

\[\int_0^\infty r^{-1-2\nu} h^2 \leq \frac{1}{\nu^2} \int_0^\infty r^{-1-2\nu} |\partial_r h|^2.\]

Applying this inequality to \(v_0\) and using (7.31) we conclude that

\[\int_{\mathbb{R}^2} r^{-2\nu} |\partial_r v_0|^2 + \int_{\mathbb{R}^2} r^{-2-2\nu} |v_0|^2 \leq \left( \frac{1}{\nu^2} + \frac{1}{\nu^2} \right) \int_{\mathbb{R}^2} r^{2-2\nu} |g_0|^2.\]

This ends the proof of the Proposition. \(\square\)
7.3 Estimates for some elliptic operator in divergence form

We now derive, for the operator

\[ w \rightarrow \text{div}(\rho^{-2} \nabla w), \]

results which are similar in their spirit to those we have obtained in the previous section for the Laplacian. Since, the assumptions we will make on the function \( \rho \) are very weak, we cannot obtain results as sharp as the ones we have obtained for the Laplacian. However, the results we obtain will be sufficient for our purposes.

The counterpart of Lemma 7.1 is given by the following Lemma in which the function \( \rho \) is assumed to satisfy the following properties:

1. The function \( \rho \) is smooth in \( B_1 \setminus \{0\} \).
2. The function \( \rho \) only depends on \( r \) and is increasing.
3. There exists a constant \( c_0 > 0 \) such that

\[ \rho \geq c_0 r \quad \text{and} \quad r \frac{\partial_r \rho}{\rho} \leq \frac{1}{c_0} \quad \text{in} \quad B_1. \]

Under these assumptions, we have:

**Lemma 7.2** Assume that \( \nu \in (1 - 2^{-1/2}, 1) \). Let \( w \) be a weak solution of

\[
\begin{aligned}
-\text{div}(\rho^{-2} \nabla w) &= \rho^{-2} f + \text{div} g \quad \text{in} \quad B_1 \setminus \{0\} \\
w &= 0 \quad \text{on} \quad \partial B_1,
\end{aligned}
\]

and further assume that

\[
\int_{B_1} r^{2 - 2\nu} \rho^{-2} |\nabla w|^2 < +\infty.
\]

Then

\[
\begin{aligned}
\int_{B_1} r^{2 - 2\nu} \rho^{-2} |\nabla w|^2 &\leq c \left( \int_{B_1} r^{4 - 2\nu} \rho^{-2} |f|^2 + \int_{B_1} r^{2 - 2\nu} \rho^2 |g|^2 \\
&\quad + \left( \int_{\partial B_1} \rho^{-2} \partial_r w \right)^2 + \left( \int_{\partial B_1} g \cdot x \right)^2 \right),
\end{aligned}
\]

where the constant \( c \) only depends on \( \nu \) and \( c_0 \).
Before we proceed to the proof of this result, let us briefly comment on the reason why some information on \( w \) is needed on the right hand side of the estimate. To simplify the discussion, let us assume that \( \rho \) is given by

\[
\rho := r^\alpha,
\]

where \( \alpha \in (0, 1) \). In this particular case, the function \( w_0 = 1 - r^{2\alpha} \) is a solution to the homogeneous problem

\[
-\text{div} (\rho^{-2} \nabla w_0) = 0,
\]
in \( B_1 \setminus \{0\} \). Clearly, because of this solution of the homogeneous problem exists, one cannot hope to estimate \( w \), the solution of (7.33) just in term of \( f \) and \( g \).

Yet another way to understand this extra term is to notice that \( a \text{ priori} \) \( w \) satisfies (7.33) away from the origin but, just as in Lemma 7.1, the function \( w \) is not a weak solution of the equation in all \( B_1 \). Indeed, one can see that there exists \( c_0 \in \mathbb{R} \) such that

\[
-\text{div} (\rho^{-2} \nabla w_0) = \rho^{-2} f + \text{div} g + c_0 \delta_0,
\]
in \( B_1 \). Somehow, the extra term is just here to evaluate the constant \( c_0 \).

**Proof:** As usual, we decompose \( w \), \( f \) and \( g \) as

\[
w := w_0 + w_1, \quad \text{and} \quad f := f_0 + f_1,
\]
where \( w_1 \) and \( f_1 \) belong to \( \text{Span}\{h_n e^{in\theta} : n \neq 0\} \), while \( w_0 \) and \( f_0 \) only depend on \( r \).

Moreover, we decompose \( g := g_0 + g_1 \) in such a way that \( \text{div} g_1 \) belongs to \( \text{Span}\{h_n e^{in\theta} : n \neq 0\} \), while \( \text{div} g_0 \) only depend on \( r \). If we write

\[
g := (g^x, g^y),
\]
we have

\[
\text{div} g = (\cos \theta \partial_r g^x - \frac{1}{r} \sin \theta \partial_\theta g^x) + (\sin \theta \partial_r g^y + \frac{1}{r} \cos \theta \partial_\theta g^y).
\]

And, if we define the function

\[
h_0 := \frac{1}{2\pi} \int_0^{2\pi} (\cos \theta g^x + \sin \theta g^y) \, d\theta,
\]
we get explicitely

\[
\text{div} g_0 = \frac{1}{2\pi r} \int_{\partial B_r} \text{div} g = \frac{1}{r} h_0 + \partial_r h_0.
\]

**Step 1.** First, let us observe that, since \( w_1 \in \text{Span}\{h_n e^{in\theta} : n \neq 0\} \), we have

\[
\int_{\partial B_r} |w_1|^2 \leq \int_{\partial B_r} |\partial_\theta w_1|^2,
\]

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which, once multiplied by $r^{-2\nu} \rho^{-2}$ and integrated over $(0, 1)$, yields

$$
\int_{B_1} r^{-2\nu} \rho^{-2} |w_1|^2 \leq \int_{B_1} r^{-2\nu} \rho^{-2} |\partial_\nu w_1|^2. \tag{7.34}
$$

This inequality is in particular useful to justify all the integrations by parts.

**Step 2.** Multiplying the equation satisfied by $w_1$ by $r^{2-2\nu} w_1$ and integrating over $B_1$, we get

$$
\int_{B_1} r^{2-2\nu} w_1 (\rho^{-2} f_1 + \text{div} g_1) = \int_{B_1} r^{2-2\nu} \frac{\|\nabla w_1\|^2}{\rho^2} + \frac{1}{2} \int_{B_1} \rho^{-2} \nabla r^{2-2\nu} \nabla |w_1|^2.
$$

Moreover, we have

$$
\frac{1}{2} \int_{B_1} \rho^{-2} \nabla r^{2-2\nu} \nabla |w_1|^2 = -2 (1 - \nu)^2 \int_{B_1} r^{-2\nu} \rho^{-2} |w_1|^2 + 2 (1 - \nu) \int_{B_1} r^{1-2\nu} \partial_\rho \frac{|w_1|^2}{\rho^2}.
$$

The function $\rho$ being increasing and $\nu$ being less than 1, we conclude that

$$
\int_{B_1} r^{2-2\nu} \rho^{-2} |\nabla w_1|^2 - 2 (1 - \nu)^2 \int_{B_1} r^{-2\nu} \rho^{-2} |w_1|^2 \\
\leq \int_{B_1} r^{2-2\nu} \rho^{-2} f_1 - \int_{B_1} g_1 \nabla (r^{2-2\nu} w_1).
$$

Using (7.34) together with Cauchy-Schwarz inequality, we conclude that

$$
\int_{B_1} r^{2-2\nu} \rho^{-2} |\partial_\nu w_1|^2 + (1 - 2 (1 - \nu)^2) \int_{B_1} r^{-2\nu} \rho^{-2} |\partial_\nu w_1|^2 \\
\leq \left( \int_{B_1} r^{4-2\nu} \rho^{-2} |f_1|^2 \right)^{1/2} \left( \int_{B_1} r^{-2\nu} \rho^{-2} |\partial_\nu w_1|^2 \right)^{1/2} \\
+ c_\nu \left( \int_{B_1} r^{2-2\nu} \rho^2 |g_1|^2 \right)^{1/2} \left( \int_{B_1} r^{2-2\nu} \rho^{-2} |\nabla w_1|^2 \right)^{1/2},
$$

for some constant $c_\nu$ only depending on $\nu$. Since we have assumed that $\nu > 1 - 2^{-1/2}$, this implies that

$$
\int_{B_1} r^{2-2\nu} \rho^{-2} |\nabla w_1|^2 \leq c \left( \int_{B_1} r^{4-2\nu} \rho^{-2} |f_1|^2 + \int_{B_1} r^{2-2\nu} \rho^2 |g_1|^2 \right). \tag{7.35}
$$

**Step 3.** We now consider the proof of the estimate for $w_0$. The equation satisfied by $w_0$ reads

$$
- \partial_r (\rho^{-2} r \partial_r w_0) = \rho^{-2} r f_0 + h_0 + r \partial_r h_0. \tag{7.36}
$$
Multiplying this equation by $r^{3-2\nu} \partial_r w_0$ and integrating between 0 and 1, we get

\[
(1 - \nu) \int_{B_1} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2 + \int_{B_1} r^{3-2\nu} \partial_r \rho \rho^{-3} |\partial_r w_0|^2 = \pi \rho^{-2}(1) |\partial_r w_0|^2(1) + \int_{B_1} r^{3-2\nu} (\rho^{-2} f_0 + r^{-1} h_0 + \partial_r h_0) \partial_r w_0.
\]

We focus on the term

\[
\int_{B_1} r^{3-2\nu} \partial_r h_0 \partial_r w_0,
\]

which appears in the last integral on the right hand side of (7.37). We use (7.36) together with an integration by parts to obtain

\[
\int_{B_1} r^{3-2\nu} \partial_r h_0 \partial_r w_0 = \int_{B_1} r^{3-2\nu} h_0 (f_0 + r^{-1} \rho^2 h_0 + \rho^2 \partial_r h_0) + 2 \pi h_0(1) \partial_r w_0(1) - (3 - 2\nu) \int_{B_1} r^{2-2\nu} \partial_r w_0 h_0 - 2 \int_{B_1} r^{3-2\nu} \partial_r \rho \rho^{-1} \partial_r w_0 h_0.
\]

(7.38)

Finally, we evaluate the term

\[
\int_{B_1} r^{3-2\nu} \rho^2 h_0 \partial_r h_0,
\]

which appears in the first integral on the right hand side of (7.38). We write

\[
\int_{B_1} r^{3-2\nu} \rho^2 h_0 \partial_r h_0 = \frac{1}{2} (\rho^2 |h_0|^2)(1) - \int_{B_1} r^{2-2\nu} (2 - \nu + r^{-1} \rho \partial_r \rho) \rho^2 |h_0|^2.
\]

(7.39)

Observe that for all $\kappa > 0$, we have

\[
ab \leq \kappa a^2 + \frac{1}{4\kappa} b^2
\]

(7.40)

Hence, using Cauchy-Schwarz inequality, we obtain

\[
\int_{B_1} r^{3-2\nu} \rho^{-2} f_0 \partial_r w_0 \leq \kappa \int_{B_1} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2 + \frac{1}{4\kappa} \int_{B_1} r^{4-2\nu} \rho^{-2} |f_0|^2.
\]

Similarly we estimate

\[
\int_{B_1} r^{2-2\nu} h_0 \partial_r w_0 \leq \kappa \int_{B_1} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2 + \frac{1}{4\kappa} \int_{B_1} r^{2-2\nu} \rho^2 |h_0|^2.
\]
Finally, we simply bound

\[ h_0(1) \partial_r w_0(1) \leq \frac{1}{2} (|h_0|^2(1) + |\partial_r w_0|^2(1)), \]

and

\[ \int_{B_1} r^{3-2\nu} h_0 f_0 \leq \frac{1}{2} \int_{B_1} r^{4-2\nu} \rho^{-2} |f_0|^2 + \frac{1}{2} \int_{B_1} r^{2-2\nu} \rho^2 |h_0|^2. \]

Combining (7.37), (7.38) and (7.39), with the above inequalities, we conclude that

\[ \int_{B_1} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2 \leq c (\rho^2(1) |h_0|^2(1) + \rho^{-2}(1) |\partial_r w_0|^2(1)) \]

\[ + c \left( \int_{B_1} r^{4-2\nu} \rho^{-2} |f_0|^2 + \int_{B_1} r^{2-2\nu} \rho^2 |h_0|^2 \right), \]

where the constant \( c \) only depends on \( \nu \) and on \( c_0 \). To obtain the result in the statement of the Lemma, it is enough to notice that

\[ \partial_r w_0(1) = \frac{1}{2\pi} \int_{\partial B_1} \partial_r w, \]

and, by definition

\[ h_0(1) = \frac{1}{2\pi} \int_{\partial B_1} g x. \]

The result of the Lemma is then a consequence of (7.35) and (7.41).

We now derive a similar result in more general domains. We keep the notations of the previous section and assume that \( \rho \) satisfies the following properties:

1. The function \( \rho \) is smooth in \( \Omega \setminus \Sigma \).
2. In each \( B_\sigma(a_j) \), the function \( \rho \) only depends on \( r_j := |x - a_j| \) and is increasing.
3. There exists a constant \( c_0 > 0 \) such that

\[ \rho \geq c_0 r_j \quad \text{and} \quad r_j \frac{\partial_r \rho}{\rho} \leq \frac{1}{c_0}, \]

in each \( B_\sigma(a_j) \)

\[ \rho \geq c_0, \]

in \( \Omega_\sigma \).

Under these assumptions, we have the:
Proposition 7.5 Assume that $\nu \in (1 - 2^{-1/2}, 1)$. For all $w$ weak solution of
\[
\begin{cases}
-\text{div}(\rho^{-2} \nabla w) = \rho^{-2} f + \text{div } g & \text{in } \Omega \setminus \Sigma \\
\partial_\nu w = 0 & \text{on } \partial \Omega,
\end{cases}
\tag{7.42}
\]
if we further assume that
\[
\int_\Omega r^{2-2\nu} \rho^{-2} |\nabla w|^2 < +\infty.
\]
Then
\[
\int_\Omega r^{2-2\nu} \rho^{-2} |\nabla w|^2 \leq c \left( \int_\Omega r^{4-2\nu} \rho^{-2} |f|^2 + \int_\Omega r^{2-2\nu} \rho^2 |g|^2 \right) + c \sum_{j=1}^N \left( \int_{\partial B_\sigma(a_j)} \rho^{-2} \partial_r w \right)^2 + c \sum_{j=1}^N \left( \int_{\partial B_\sigma(a_j)} g \cdot (x - a_j) \right)^2,
\tag{7.43}
\]
where $r := \text{dist}(x, \Sigma)$ and where the constant $c > 0$ only depends on $\nu$ and on the constant $c_0$.

Proof: We proceed exactly like in the proof of Proposition 7.1 using Lemma 7.2 instead of Lemma 7.1. In particular, the operator
\[
\overline{\text{DN}} : \Pi_j H^{3/2}(\partial B_\rho(a_j)) \longrightarrow H^{1/2}(\partial B_\sigma(a_j)),
\]
can be defined as before to be the difference of the two Dirichlet to Neumann operators in $\Omega_\sigma$ and in $\cup_j B_\sigma(a_j)$, corresponding to the operator $\Lambda : w \longrightarrow -\text{div}(\rho^{-2} \nabla w)$ with 0 Neumann data on $\partial \Omega$. However, this time $\Lambda$ has one dimensional Kernel and one dimensional Cokernel. This induces a one dimensional Kernel and a one dimensional Cokernel for $\overline{\text{DN}}$. More precisely, we have
\[
\text{Ker} \overline{\text{DN}} = \text{Span}\{(1, \ldots, 1)\},
\]
and
\[
\text{Range} \overline{\text{DN}} = \{(\phi_1, \ldots, \phi_N) \in \Pi_j H^{1/2}(\partial B_\sigma(a_j)) | \sum_j \int_{\partial B_\sigma(a_j)} \phi_j = 0\}.
\]
Beside this remark, the proof follows the proof of Proposition 7.1, so we omit the details. \(\square\)

We end this Chapter with a result which is close to the result of Proposition 7.5. The result involves a function $\rho$ which is assumed to satisfy the following set of assumptions:

1. The function $\rho$ is smooth in $\mathbb{R}^2 \setminus \{0\}$, only depends on $r$ and is increasing.
2. There exists a constant $c_0 > 0$ such that
\[ \rho \geq c_0 r, \]
in each $B_1 \setminus \{0\}$,
\[ c_0 \leq \rho \leq \frac{1}{c_0}, \]
in $\mathbb{R} \setminus B_1$ and
\[ r \frac{\partial_r \rho}{\rho} \leq \frac{1}{c_0}, \]
in $\mathbb{R}^2 \setminus \{0\}$.

Under these assumptions, our result reads:

**Proposition 7.6** Assume that $\nu \in (1 - 2^{-1/2}, 1)$. There exists $c > 0$ and $\varsigma \in (0, \sigma)$, such that, for all $w$ weak solution of
\[ -\text{div}(\rho^{-2} \nabla w) + w = \rho^{-2} f + \text{div} g, \]
in $\mathbb{R}^2 \setminus \{0\}$, if we assume that
\[ \int_{\mathbb{R}^2} r^{2-2\nu} \rho^{-2} |\nabla w|^2 < +\infty, \]
and if $\lim_{\infty} |w| = 0$, then
\[ \int_{\mathbb{R}^2} r^{2-2\nu} \rho^{-2} |\nabla w|^2 \leq c \left( \int_{\mathbb{R}^2} r^{4-2\nu} \rho^{-2} |f|^2 + \int_{\mathbb{R}^2} r^{2-2\nu} \rho^2 |g|^2 \right) + \left( \int_{\partial B_\varsigma} h_0 \partial_r w - \int_{B_\varsigma} w \right)^2 + \left( \int_{\partial B_\varsigma} g \cdot x \right)^2, \]
provided $\rho(\varsigma) > 1/2$.

**Proof:** As in the proof of Lemma 7.2, we decompose $w$, $f$ and $g$ as
\[ w := w_0 + w_1, \quad \text{and} \quad f := f_0 + f_1, \]
where $w_1$ and $f_1$ belong to $\text{Span}\{h_n e^{in\theta} : n \neq 0\}$, while $w_0$ and $f_0$ only depend on $r$. As in the proof of Lemma 7.2, we will write
\[ \text{div} g_0 = \frac{1}{r} h_0 + \partial_r h_0. \]

Moreover, we decompose $g := g_0 + g_1$ in such a way that $\text{div} g_1$ belongs to $\text{Span}\{h_n e^{in\theta} : n \neq 0\}$, while $\text{div} g_0$ only depend on $r$.

**Step 1.** Following Step 1 and Step 2 in the proof of Lemma 7.2, we already obtain
\[ \int_{\mathbb{R}^2} r^{2-2\nu} \rho^{-2} |\nabla w_1|^2 \leq c \left( \int_{\mathbb{R}^2} r^{4-2\nu} \rho^{-2} |f_1|^2 + \int_{\mathbb{R}^2} r^{2-2\nu} \rho^2 |g_1|^2 \right). \]
Step 2. We now would like to derive the relevant estimate for $w_0$. Keeping
the notations of Step 3 in the proof of Lemma 7.2, we see that $w_0$ is a solution of
\[ -\partial_r \left( \rho^{-2} r \partial_r w_0 \right) + r w_0 = \rho^{-2} r f_0 + h_0 + r \partial_r h_0. \] (7.46)

1 - Estimate on $\partial B_\varsigma$. By definition of $h_0$, we have
\[ h_0(\varsigma) = \frac{1}{2\pi} \varsigma^{-2} \int_{\partial B_\varsigma} g \cdot x. \] (7.47)

Now, since we have assumed that $\lim_{\varsigma \to \infty} w_0 = 0$, we can write
\[ w_0 = -\int_r^{+\infty} \partial_r w_0 \, dr. \]

Hence, applying Cauchy-Schwarz inequality, we get the pointwise bound
\[ r^{2-2\nu} |w_0|^2 \leq c \int \rho^{2-2\nu} |\partial_r w_0|^2 \leq c \int \rho^{2-2\nu} |\partial_r w_0|^2, \] (7.48)
for some constant $c$ depending on $\nu$ and $c_0$. Observe that we have used the assumption $\rho \geq c_0$ to obtain the last inequality. Integration of this inequality over $B_\varsigma$ yields
\[ \int_{B_\varsigma} r^{2-2\nu} |w_0|^2 \leq c \varsigma^2 \int \rho^{2-2\nu} |\partial_r w_0|^2. \] (7.49)

Observe that
\[ \int_{B_\varsigma} w = \int_{B_\varsigma} w_0 \quad \text{and} \quad \int_{\partial B_\varsigma} \partial_r w = \int_{\partial B_\varsigma} \partial_r w_0. \]

Hence, we can estimate $|\partial_r w_0(\varsigma)|^2$ in the following way
\[ \frac{\varsigma^2}{\rho^2(\varsigma)} |\partial_r w_0(\varsigma)|^2 = \left( \frac{1}{2\pi} \int_{\partial B_\varsigma} \frac{1}{\rho^2} \partial_r w_0 \right)^2 \leq c \left( \int_{\partial B_\varsigma} \frac{1}{\rho^2} \partial_r w - \int_{B_\varsigma} w \right)^2 + c \varsigma^{2\nu} \int_{B_\varsigma} r^{2-2\nu} |w_0|^2 \]
\[ \leq c \left( \int_{\partial B_\varsigma} \frac{1}{\rho^2} \partial_r w - \int_{B_\varsigma} w \right)^2 + c \varsigma^{2\nu+2} \int_{B_\varsigma} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2. \] (7.50)

We have used Cauchy-Schwarz inequality to obtain the second estimate and
(7.49) to obtain the last inequality.
2 - Estimate over $B_\varsigma$. Multiplying (7.46) by $r^{3-2\nu} \partial_r w_0$ and integrating between 0 and $\varsigma$, we get

$$
\int_{B_\varsigma} r^{3-2\nu} (\rho^{-2} f_0 + r^{-1} h_0 + \partial_r h_0) \partial_r w_0 = (1 - \nu) \int_{B_\varsigma} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2 \\
+ \int_{B_\varsigma} r^{3-2\nu} \rho^{-3} \partial_r \rho |\partial_r w_0|^2 \\
- (2 - \nu) \int_{B_\varsigma} r^{2-2\nu} |w_0|^2 \\
- \pi \varsigma^{4-2\nu} (\rho^{-2} |\partial_r w_0|^2 - |w_0|^2)(\varsigma).
$$

The function $\rho$ being increasing, we obtain the inequality

$$
(1 - \nu) \int_{B_\varsigma} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2 \leq \int_{B_\varsigma} r^{3-2\nu} (\rho^{-2} f_0 + r^{-1} h_0 + \partial_r h_0) \partial_r w_0 \\
+ (2 - \nu) \int_{B_\varsigma} r^{2-2\nu} |w_0|^2 \\
+ \pi \varsigma^{4-2\nu} (\rho^{-2} |\partial_r w_0|^2 - |w_0|^2)(\varsigma).
$$

Arguing as in Step 3 of the proof of Lemma 7.2, we conclude that

$$
\int_{B_\varsigma} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2 \leq c \varsigma^{4-2\nu} (\rho^2(\varsigma) |h_0|^2(\varsigma) + \rho^{-2}(\varsigma) |\partial_r w_0|^2(\varsigma)) \\
+ c \left( \int_{B_\varsigma} r^{4-2\nu} \rho^{-2} |f_0|^2 + \int_{B_\varsigma} r^{2-2\nu} \rho^2 |h_0|^2 \right) \\
+ c \int_{B_\varsigma} r^{2-2\nu} |w_0|^2,
$$

where the constant $c$ only depends on $\nu$ and on $c_0$.

Using (7.50) and (7.51), we conclude that

$$
\int_{B_\varsigma} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2 \leq c \left( \int_{\mathbb{R}^2} r^{4-2\nu} \rho^{-2} |f|^2 + \int_{\mathbb{R}^2} r^{2-2\nu} \rho^2 |g|^2 \right) \\
+ \frac{1}{\rho^2} \left( \int_{\partial B_\varsigma} w - \int_{B_\varsigma} w \right)^2 + \left( \int_{\partial B_\varsigma} g \cdot x \right)^2) \\
+ c (\varsigma^2 + \varsigma^4 \rho^2(\varsigma)) \int_{\mathbb{R}^2} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2,
$$

for some constant $c$ only depending on $\nu$ and $c_0$. The key point here is that the constant $c$ does not depend on $\varsigma$.

3 - Estimate on $\mathbb{R}^2 \setminus B_\varsigma$. On $\mathbb{R}^2 \setminus B_\varsigma$, we multiply (7.46) by $r^{2-2\nu} w_0$ and
integrate by parts. We obtain

\[
\begin{align*}
\int_{\mathbb{R}^2 \setminus B_s} r^2 \rho^{-2} |\partial_r w_0|^2 + \int_{\mathbb{R}^2 \setminus B_s} r^2 \nu |w_0|^2 & = \int_{\mathbb{R}^2 \setminus B_s} r^2 \rho^{-2} f_0 w_0 \\
- \int_{\mathbb{R}^2 \setminus B_s} r^2 \nu h_0 \partial_r w_0 & - 2 (1 - \nu) \int_{\mathbb{R}^2 \setminus B_s} r^2 \rho^{-2} w_0 \partial_r w_0 \\
- 2 (1 - \nu) \int_{\mathbb{R}^2 \setminus B_s} r^2 \nu \partial_r w_0 & - 2 \pi \varsigma^{3-2\nu} (\rho^{-2} w_0 \partial_r w_0 + h_0 w_0) (\varsigma).
\end{align*}
\]  

(7.53)

We now evaluate each term which appears on the right hand side of this inequality. Let \( \tau_1, \tau_2 \) two positive parameters, to be fixed later on. We use the inequality

\[a b \leq \kappa a^2 + \frac{1}{4\kappa} b^2,\]  

(7.54)

to bound

\[- \int_{\mathbb{R}^2 \setminus B_s} r^2 \nu h_0 \partial_r w_0 \leq \tau_1 \int_{\mathbb{R}^2} r^2 \nu \rho^{-2} |\partial_r w_0|^2 + \frac{1}{4\tau_1} \int_{\mathbb{R}^2} r^2 \nu \rho^2 |h_0|^2.\]

Similarly, we have

\[
\int_{\mathbb{R}^2 \setminus B_s} r^2 \nu f_0 w_0 \leq \tau_1 \int_{\mathbb{R}^2} r^2 \nu \rho^{-2} |w_0|^2 + \frac{1}{4\tau_2} \int_{\mathbb{R}^2} r^2 \nu \rho^2 |f_0|^2.
\]

and

\[- (1 - \nu) \int_{\mathbb{R}^2 \setminus B_s} r^2 \nu h_0 w_0 \leq \tau_2 \int_{\mathbb{R}^2 \setminus B_s} r^2 \nu \rho^{-2} |w_0|^2 + \frac{1 - \nu}{4\tau_2} \int_{\mathbb{R}^2} r^2 \nu \rho^2 |h_0|^2.\]

We use (7.47) and (7.48) to get the bounds

\[
2 \pi \varsigma^{3-2\nu} (h_0 w_0)(\varsigma) \leq \tau_1 \int_{\mathbb{R}^2 \setminus B_s} r^2 \nu \rho^{-2} |\partial_r w_0|^2 + \frac{c}{\tau_1} \varsigma^{4-2\nu} \left( \int_{\partial B_s} g \cdot x \right)^2.
\]
Similarly, we use (7.48) and (7.50) to get
\[
2\pi \varsigma^{3-2\nu} (\rho^{-2} w_0 \partial_r w_0)(\varsigma) \leq 2\pi \left( \tau_1 (\varsigma^{1-\nu} w_0^2) + \frac{1}{4\tau_1} (\varsigma^{2-\nu} \rho^{-2} \partial_r w_0)^2 \right) \\
\leq \frac{c}{\tau_1} \varsigma^{2-2\nu} \left( \int_{\partial B_1} \frac{1}{\rho^2} \partial_r w - \int_{B_1} w \right)^2 \\
+ c \left( \tau_1 + \frac{1}{\tau_1} \right) \int_{\mathbb{R}^2} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2.
\]

It remains to estimate
\[
-2 (1 - \nu) \int_{\mathbb{R}^2 \setminus B_1} r^{1-2\nu} \rho^{-2} w_0 \partial_r w_0.
\]

We decompose
\[
2 \int_{\mathbb{R}^2 \setminus B_1} r^{1-2\nu} \rho^{-2} w_0 \partial_r w_0 = 2 \int_{\mathbb{R}^2 \setminus B_1} r^{1-2\nu} \rho^{-2} w_0 \partial_r w_0 \\
+ 2 \int_{B_1 \setminus B_1} r^{1-2\nu} \rho^{-2} w_0 \partial_r w_0.
\]

Using once more (7.54), we see that the first integral is bounded by
\[
2 \int_{\mathbb{R}^2 \setminus B_1} r^{1-2\nu} \rho^{-2} w_0 \partial_r w_0 \leq \int_{\mathbb{R}^2} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2 \\
+ \int_{\mathbb{R}^2 \setminus B_1} r^{2-2\nu} |w_0|^2.
\]

Since \( \rho \) is increasing and since we integrate over \( r \geq 1 \), we get
\[
2 \int_{\mathbb{R}^2 \setminus B_1} r^{1-2\nu} \rho^{-2} w_0 \partial_r w_0 \leq \int_{\mathbb{R}^2} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2 \\
+ \rho^{-2} (1) \int_{\mathbb{R}^2 \setminus B_1} r^{2-2\nu} |w_0|^2.
\]

Finally, for the second integral, we can write
\[
2 \int_{B_1 \setminus B_1} r^{1-2\nu} \rho^{-2} w_0 \partial_r w_0 \leq \int_{B_1 \setminus B_1} \rho^{-2} |\partial_r w_0|^2 \\
+ \int_{B_1 \setminus B_1} r^{2-4\nu} \rho^{-2} |w_0|^2. \tag{7.55}
\]

We use on last time (7.46), multiply it by \( w_0 \) and integrate to get
\[
\int_{\mathbb{R}^2 \setminus B_1} \rho^{-2} |\partial_r w_0|^2 + \int_{\mathbb{R}^2 \setminus B_1} |w_0|^2 = \int_{\mathbb{R}^2 \setminus B_1} \rho^{-2} f_0 w_0 \\
+ \int_{\mathbb{R}^2 \setminus B_1} h_0 \partial_r w_0 - 2\pi \varsigma (h_0 - \rho^{-2} \partial_r w_0)(\varsigma) w_0(\varsigma).
\]

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Using (7.54), we can state

\[ \int_{\mathbb{R}^2 \setminus B_{c}} \rho^{-2} |\partial_r w_0|^2 + \int_{\mathbb{R}^2 \setminus B_{c}} |w_0|^2 \leq c \int_{\mathbb{R}^2 \setminus B_{c}} \rho^{-4} |f_0|^2 \]

\[ + \int_{\mathbb{R}^2 \setminus B_{c}} \rho^2 |h_0|^2 - 4\pi \zeta (h_0 - \rho^{-2} \partial_r w_0)(\zeta) w_0(\zeta). \]

This, together with (7.55) implies that

\[ 2 \int_{B_1 \setminus B_{c}} r^{1-2\nu} \rho^{-2} w_0 \partial_r w_0 \leq c_0 \int_{\mathbb{R}^2} r^{4-2\nu} \rho^{-2} |f_0|^2 + c_0 \int_{\mathbb{R}^2} r^{2-2\nu} \rho^2 |h_0|^2 \]

\[ + 4\pi \zeta^3 \nu^4 \left( \rho^{-4} (|h_0| + \rho^{-2} |\partial_r w_0|)|w_0| \right)(\zeta), \]

where the constant \( c_0 \) depends on \( \zeta \). Notice that we have implicitly used the fact that \( 1 - 2\nu < 0 \). Now we use (7.47), (7.48) and (7.50) to get

\[ 2 \int_{B_1 \setminus B_{c}} r^{1-2\nu} \rho^{-2} w_0 \partial_r w_0 \leq c_0 \int_{\mathbb{R}^2} r^{4-2\nu} \rho^{-2} |f_0|^2 \]

\[ + c_0 \int_{\mathbb{R}^2} r^{2-2\nu} \rho^2 |h_0|^2 \]

\[ + c_0 \left( \int_{\partial B_c} g \cdot x \right)^2 \]

\[ + c_0 \left( \int_{\partial B_c} \frac{1}{\rho^2} \partial_r w - \int_{B_c} w \right)^2 \]

\[ + (\tau_1 + \frac{c}{\tau_1} \nu^4 \rho^{-4}(\zeta)) \int_{\mathbb{R}^2} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2. \]

4 - Final argument. Collecting all above estimates in (7.53) together with (7.52), we obtain

\[ (\nu - c\tau_1 - \frac{c}{\tau_1} \nu^4 \rho^{-4}(\zeta)) \int_{\mathbb{R}^2} r^{2-2\nu} \rho^{-2} |\partial_r w_0|^2 \]

\[ + (1 - (1 - \nu) \rho^{-2}(1 - 2\tau_2)) \int_{\mathbb{R}^2 \setminus B_{c}} r^{2-2\nu} |w_0|^2 \]

\[ \leq c_0 \int_{\mathbb{R}^2} r^{4-2\nu} \rho^{-2} |f_0|^2 + c_0 \int_{\mathbb{R}^2} r^{2-2\nu} \rho^2 |h_0|^2 \]

\[ + c_0 \left( \int_{\partial B_c} g \cdot x \right)^2 + c_0 \left( \int_{\partial B_c} \frac{1}{\rho^2} \partial_r w - \int_{B_c} w \right)^2. \]

Now, let us assume that

\[ \rho^4(\zeta) \geq 1/2. \]

In particular, this implies that

\[ 1 - (1 - \nu) \rho^{-2}(1) > 0, \]

since the function \( \rho \) is increasing and since we have assumed that \( 1 - 2^{-1/2} < \nu \). The result then follows at once by first choosing \( \tau_1 \) and \( \tau_2 \) to be small enough but fixed and then letting \( \zeta \) tend to 0. \( \square \)

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Chapter 8

Generalized Pohozaev Formula for $\rho$–Conformal Fields

In this Chapter, we derive a Pohozaev type formula in order to compare two solutions of the Ginzburg-Landau equation. This formula is a modification of the standard Pohozaev formula which is commonly used in the study of semilinear elliptic equations.

In the beginning of the Chapter we briefly recall the use of Pohozaev identity in the classical framework. Then, we derive a Pohozaev type identity for the Ginzburg-Landau equation, first in the particular case of radially symmetric solutions in the unit ball and then in the general case.

8.1 The Pohozaev formula in the classical framework

To shed light on the developments of this Chapter, we slightly digress and briefly recall the classical use of Pohozaev identity and its variational interpretation. We refer the reader to the work of R. Schoen [87] for geometric consequences of this identity.

Let $\Omega$ be an open subset of $\mathbb{R}^n$ and $p > 1$, we are interested in positive solutions of the equation

$$\begin{cases} \Delta u + u^p = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

(8.1)

Recall that a conformal Killing vector field $X := (X^1, \ldots, X^n)$ in $\mathbb{R}^n$ is a vector field which corresponds to the infinitesimal action of a conformal transformation.
It satisfies
\[ \partial_{x_i} X^j + \partial_{x_j} X^i = c \delta_{i,j}, \]
where the function \( x \in \mathbb{R}^n \rightarrow c(x) \in \mathbb{R} \) depends on \( X \) and where \( \delta_{i,j} \) is the usual Kronecker’s symbol. In \( \mathbb{R}^n, n \geq 3 \) the space of conformal Killing vector fields has dimension \( \frac{(n+1)(n+2)}{2} \) and is generated by
\[
X_1 = b, \quad \forall b \in \mathbb{R}^n, \\
X_2 = x, \\
X_3 = (b \cdot x) c - (c \cdot x) b, \quad \forall b, c \in \mathbb{R}^n, \\
X_4 = (b \cdot x) x - \frac{1}{2} |x|^2 b, \quad \forall b, c \in \mathbb{R}^n.
\]
Notice that, in dimension \( n = 2 \), the conformal Killing vector fields are generated by holomorphic function and thus the space if in this case infinite dimensional.

Now, assume that \( u \) is a solution of \( \Delta u + u^p = 0. \) Multiplying this equation by \( \partial_{X} u := \sum_{i=1}^{n} X^i \partial_{x_i} u, \)
it is an easy exercise to see that
\[
\text{div} \left( (\partial_{X} u) \nabla u - \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{p+1} u^{p+1} \right) X \right) + \frac{c}{2} \left( \frac{n-2}{2} |\nabla u|^2 - \frac{n}{p+1} u^{p+1} \right) = 0, \tag{8.2}
\]
where \( c(x) \) is the function which appears in the definition of the conformal Killing vector field.

Multiplying (8.1) by \( u \) we also find
\[
\text{div} (u \nabla u) = |\nabla u|^2 - u^{p+1}.
\]
Hence we conclude that
\[
\text{div} \left( \partial_{X} u \nabla u - \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} u^{p+1} \right) X + \frac{n-2}{8} c \nabla u^2 - u^2 \nabla c \right) = \frac{c}{2} \left( \frac{n}{p+1} - \frac{n-2}{2} \right) u^{p+1}, \tag{8.3}
\]
The main feature of this kind of identity is that it involves on the left hand side a quantity which is in divergence form and on the right hand side, a quantity whose sign is controlled.

Assuming that \( u \) is regular enough (say in \( H^1(\Omega) \cap L^{p+1}(\Omega) \)), we see that \( u \) solution of \( \Delta u + u^p = 0 \) is a critical point of the functional
\[
E(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \frac{1}{p+1} \int_{\Omega} |u|^{p+1}.
\]
Namely
\[
\frac{dE}{dt} (u + tw)|_{t=0} = 0,
\]
for all \(w \in H^1(\Omega) \cap L^{p+1}(\Omega)\) with compact support in \(\Omega\).

It is interesting to observe that the Pohozaev identity may also be derived from the assumption that \(u\) is a critical point of \(E\) with respect to variations on the parameterization of the domain (these critical points are usually called stationary critical point). Indeed, let
\[
\phi : \Omega \longrightarrow \mathbb{R}^n,
\]
be a vector field in \(\Omega\), with compact support in \(\Omega\). We define, for \(t\) small enough
\[
u_t := u(\cdot + t\phi).
\]
By definition, \(u\) is said to be a critical point of \(E\) with respect to variations on the parameterization of the domain, if
\[
\frac{dE}{dt}(u_t)|_{t=0} = 0, \quad (8.4)
\]
for all sufficiently smooth vector field \(\phi\) having compact support in \(\Omega\). Obviously, assuming that \(u\) is regular enough, we have the expansion
\[
u_t = u + t \partial X u + O(t^2),
\]
and, using this, we find explicitly
\[
\frac{dE}{dt}(u_t)|_{t=0} = \int_\Omega \nabla u \nabla (\partial_x u) - \int_\Omega u^p \partial_x u.
\]
Integrating by parts, we see that (8.4) becomes
\[
\sum_{i,j} \int_\Omega \partial_x u \partial_x u (\partial_x \phi^i + \partial_x \phi^j) - \int_\Omega \left(\frac{1}{2} |\nabla u|^2 - \frac{1}{p + 1} u^{p+1}\right) \div \phi = 0. \quad (8.5)
\]
The Pohozaev identity then follows from this identity using some appropriate vector fields. More precisely, we define for all \(\varepsilon > 0\)
\[
\chi_\varepsilon := 1 - \eta \left(\frac{\dist(x, \partial \Omega)}{\varepsilon}\right),
\]
where \(\eta\) is a cutoff function identically equal to 1 in \([0,1]\) and equal to 0 in \([2, +\infty)\). Given any conformal Killing vector field \(X\), we set
\[
\phi_\varepsilon := \chi_\varepsilon X,
\]

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and we apply (8.5) to $\phi_\varepsilon$. We find

$$
\int_\Omega (\partial_X u) (\nabla \chi_\varepsilon \nabla u) - \int_\Omega \left( \frac{1}{2} |\nabla u|^2 + \frac{1}{p+1} u^{p+1} \right) \partial_X \chi_\varepsilon + \int_\Omega \frac{c}{2} \chi_\varepsilon \left( \frac{n-2}{2} |\nabla u|^2 - \frac{1}{p+1} u^{p+1} \right) = 0.
$$

The Pohozaev formula (8.2), integrated over $\Omega$, follows at once by letting $\varepsilon$ tend to 0. We refer for example to [24] for the details.

A classical application of this Pohozaev identity is the following:

**Theorem 8.1** Assume that $\Omega$ is a starshaped domain. Then, there are no nonzero positive smooth solutions of (8.1), if $p \geq \frac{n+2}{n-2}$.

**Proof:** Assume that $\Omega$ is starshaped with respect to the origin and apply (8.3) to the vector field $X = x$. We find

$$
\text{div} \left( \partial_X u \nabla u + \left( \frac{1}{2} |\nabla u|^2 - \frac{1}{p+1} u^{p+1} \right) X + \frac{n-2}{4} \nabla u^2 \right) = \left( \frac{n}{p+1} - \frac{n-2}{2} \right) u^{p+1}.
$$

Integrating this identity over $\Omega$ and using the fact that $u = 0$ on $\partial \Omega$, we conclude that

$$
\int_{\partial \Omega} \partial_X u \partial_\nu u - \frac{1}{2} |\nabla u|^2 (x \cdot \nu) = \left( \frac{n}{p+1} - \frac{n-2}{2} \right) \int_\Omega u^{p+1},
$$

where $\nu$ denotes the outward pointing normal to $\partial \Omega$. Hence

$$
\frac{1}{2} \int_{\partial \Omega} |\nabla u|^2 (x \cdot \nu) = \left( \frac{n}{p+1} - \frac{n-2}{2} \right) \int_\Omega u^{p+1},
$$

since $\Omega$ is starshaped we have $(x \cdot \nu) > 0$ on $\partial \Omega$. This implies that $\nabla u = 0$ on $\partial \Omega$ and we conclude that $u$ is identically equal to 0 too, using for example Hopf Lemma [26].

### 8.2 Comparing Ginzburg-Landau solutions through Pohozaev’s argument

Before, we begin our study of the Pohozaev identity in the framework of the Ginzburg-Landau equation, let us list the notations we will use in the subsequent Chapters.

#### 8.2.1 Notations

Let us recall that we have already defined the scalar product in $\mathbb{C}$ by

$$
z \cdot z' = xx' + yy'.
$$
We will adopt the same notation for the scalar product in $\mathbb{R}^2$.

In the subsequent Chapters, it will be convenient to use the following notation

$$ z \wedge z' := (iz) \cdot z' := xy' - yx'. $$

Observe that the following identities hold

$$ z \wedge z = 0, \quad \text{and} \quad z \wedge z' = -z' \wedge z. $$

Given a vector field $X = (X^x, X^y)$, we will adopt the notation

$$ \partial X := X^x \partial_x + X^y \partial_y. $$

In the particular case where the vector field $\nu := (\nu_x, \nu_y)$ is the outward normal vector field on the boundary of a subset $\omega$, we will write

$$ \partial \nu := \nu_x \partial_x + \nu_y \partial_y, $$

just to be more consistent with usual notations.

Finally, given a vector field $X = (X^x, X^y)$, we set

$$ X^\perp := (X^y, X^x), $$

which corresponds to a rotation of angle $-\pi/2$. Similarly, we define

$$ \nabla^\perp := (\partial_y, -\partial_x). $$

In the particular case where the vector field $\nu := (\nu_x, \nu_y)$ is the outward normal vector field on the boundary of a subset $\omega$, we will set

$$ \tau := \nu^\perp, $$

and

$$ \partial \tau := \nu_y \partial_x - \nu_x \partial_y. $$

Observe that, with all these definitions, we have

$$ \nabla^\perp w \cdot \nu = -\partial z w. $$

### 8.2.2 The comparison argument in the case of radially symmetric data

We show how to adapt the Pohozaev identity in order to compare two solutions of the Ginzburg-Landau equation in a simple setting. To begin with, let us consider the Ginzburg-Landau equation in the unit ball, with boundary data simply given by $e^{i\theta}$. Namely

$$ \begin{cases} 
\Delta u + \frac{u}{\varepsilon^2} (1 - |u|^2) = 0 & \text{in} \quad B_1 \\
u = e^{i\theta} & \text{on} \quad \partial B_1.
\end{cases} \quad (8.6) $$

We recall the following simple result:
Theorem 8.2 [29], [34] There exists a unique, non constant solution of (8.6) which is of the form
\[ u_\varepsilon := \tilde{S}_\varepsilon e^{i\theta}. \]

This solution \( \tilde{S}_\varepsilon \) satisfies
\[ \frac{d\tilde{S}_\varepsilon}{dr} > 0, \quad \text{for all} \quad r \geq 0, \]
and \( 0 < \tilde{S}_\varepsilon < 1 \), for all \( r > 0 \).

Now, let \( u \) be any other solution of (8.6) which satisfies \( u(0) = 0 \).

Following P. Mironescu [65], we set
\[ w := \frac{u}{u_\varepsilon}. \]

It is an easy exercise to check that \( w \) is a solution of
\[ \Delta w + 2 \frac{\partial_r \tilde{S}_\varepsilon}{\tilde{S}_\varepsilon} \partial_r w + 2 i \frac{1}{r^2} \partial_\theta w + w \tilde{S}_\varepsilon^2 \frac{(1 - |w|^2)}{\varepsilon^2} = 0, \quad \text{(8.7)} \]
in \( B_1 \setminus \{0\} \). Furthermore, \( w \) is a critical point of the functional
\[ E(w) := \int_{B_1} \tilde{S}_\varepsilon^2 |\nabla w|^2 + 2 \int_{B_1} \tilde{S}_\varepsilon^2 w \wedge \partial_\theta w + \frac{1}{2\varepsilon^2} \int_{B_1} \tilde{S}_\varepsilon^4 (1 - |w|^2)^2. \quad \text{(8.8)} \]

In order to get the Pohozaev identity we are interest in, we consider the vector field \( X := (x, y) \), and we take the scalar product of equation (8.7) with
\[ \partial_X w := x \partial_x w + y \partial_y w = r \partial_r w. \]

It is an easy exercise to prove that
\[ \partial_X w \cdot \Delta w = \partial_x \left( \partial_X w \cdot \partial_x w - \frac{x}{2} |\nabla w|^2 \right) + \partial_y \left( \partial_X w \cdot \partial_y w - \frac{y}{2} |\nabla w|^2 \right). \]

We will write for short
\[ \partial_X w \cdot \Delta w = \text{div} \left( \partial_X w \cdot \nabla w - \frac{X}{2} |\nabla w|^2 \right). \quad \text{(8.9)} \]

Similarly, we have
\[ 2 \partial_X w \cdot \left( \frac{\partial_r \tilde{S}_\varepsilon}{\tilde{S}_\varepsilon} \partial_r w \right) = \frac{2}{r} \frac{\partial_r \tilde{S}_\varepsilon}{\tilde{S}_\varepsilon} |\partial_X w|^2. \quad \text{(8.10)} \]
We also obtain
\[ 2 \partial_X w \cdot \left( \frac{i}{r^2} \partial_\theta w \right) = - \partial_x (w \wedge \partial_y w) + \partial_y (w \wedge \partial_x w). \]

We recall that we have defined
\[ \nabla \perp X := (X^y, -X^x). \]

Hence, the previous formula can be written for short
\[ 2 \partial_X w \cdot \left( \frac{i}{r^2} \partial_\theta w \right) = - \text{div} (w \wedge \nabla \perp w). \tag{8.11} \]

Finally, we have
\[ \partial_X w \cdot \left( \frac{S^2_x}{\varepsilon^2} w (1 - |w|^2) \right) = \frac{1}{4 \varepsilon^2} \text{div} \left( \frac{S^2_x}{\varepsilon^2} (1 - |w|^2)^2 X \right) \]
\[ + \frac{1}{4 \varepsilon^2} (1 - |w|^2)^2 \text{div} (S^2_x X). \tag{8.12} \]

Combining (8.7)-(8.12), together with the fact that \( w \) is a solution of (8.6), we conclude that
\[ \text{div} \left( \partial_X w \cdot \nabla w - \frac{1}{2} |
abla w|^2 X - w \wedge \nabla \perp w - \frac{S^2_x}{4 \varepsilon^2} (1 - |w|^2)^2 X \right) = - \frac{2}{r} \frac{\partial_r S_x}{S_x} |\partial_X w|^2 - \frac{1}{4 \varepsilon^2} (1 - |w|^2)^2 \text{div} (S^2_x X). \tag{8.13} \]

This is, in a particular case, the Pohozaev identity for Ginzburg-Landau equation we are interested in. For the time being, let us observe that on the left hand side we have a quantity which is in divergence form and on the right hand side, we have a negative quantity.

We can also give another form of the Pohozaev identity by integrating the previous identity over any \( \omega \) which has compact support in \( B_1 \setminus \{0\} \). We find
\[ \int_{\partial \omega} \left( \partial_X w \cdot \partial_\nu w - \frac{1}{2} |
abla w|^2 X \cdot \nu + w \wedge \partial_r w - \frac{S^2_x}{4 \varepsilon^2} (1 - |w|^2)^2 X \cdot \nu \right) = - \int_{\omega} \frac{2}{r} \frac{\partial_r S_x}{S_x} |\partial_X w|^2 - \frac{1}{4 \varepsilon^2} \int_{\omega} (1 - |w|^2)^2 \text{div} (S^2_x X), \tag{8.14} \]

where we recall that
\[ \partial_\nu := \nu_x \partial_x + \nu_y \partial_y \quad \text{and} \quad \partial_r := \nu_y \partial_x - \nu_x \partial_y, \]
and where \( \nu := (\nu_x, \nu_y) \) is the outward unit normal to \( \partial \omega \).

In the subsequent sections of this Chapter, we will explain how this identity can be used to prove that \( w \) is identically equal to 1 and hence that \( u = S_x e^{i \theta} \).
But, before we do so, we explain how to generalize this identity.
8.3 $\rho-$conformal vector fields

In order to be able to derive a Pohozaev formula in order to compare two solutions of the Ginzburg-Landau equation in the general case and not only in the special case where $u_\varepsilon$ is the axially symmetric solution, we have to introduce the notion of $\rho-$conformal vector fields.

These $\rho-$conformal vector fields are a generalization of the well known conformal vector fields. As will be explained later, this notion is only relevant in two dimensions. We define:

**Definition 8.9** Let $\rho$ be a function defined in a 2 dimensional domain $\omega$. We will say that a vector-field $X$ is $\rho-$conformal in $\omega$ if there exists a vector field $\tilde{X}$ (which may eventually be singular) solution of

$$\begin{cases}
\text{div}(\rho^2 \tilde{X}) = 0 & \text{in } \mathcal{D}'(\omega) \\
\rho^2 \text{curl} \tilde{X} = 0 & \text{in } \mathcal{D}'(\omega),
\end{cases}$$

such that $X$ is given by

$$X = \frac{\tilde{X}^\perp}{|X|^2}.$$  \hspace{1cm} (8.15)

We recall that, for any vector $X = (X^x, X^y)$, we have defined

$$X^\perp := (X^y, -X^x).$$

Observe that, when the function $\rho$ is taken to be a constant, say $\rho = 1$, then $\rho-$conformal vector fields are just conformal vector fields (i.e. the infinitesimal action of a conformal transformation). Indeed, in this case, the vector field $X$ is 1-conformal if and only if the vector field $\tilde{X}$ verifies both

$$\text{div} \tilde{X} = 0 \quad \text{and} \quad \text{curl} \tilde{X} = 0.$$  \hspace{1cm} (8.16)

It is then an easy exercise (see Lemma 8.1 below) to show that these conditions are satisfied if and only if the complex valued function $f := X^x + iX^y$ is holomorphic, which in turn is equivalent to the fact that

$$|\partial_x f|^2 = |\partial_y f|^2 \quad \text{and} \quad \partial_x f \cdot \partial_y f = 0,$$

which means that $X$ is conformal.

As another example of a $\rho$-conformal vector field, let us simply consider

$$X = (x, y),$$

which is a conformal vector field. Let us show that this is also a $\rho$-conformal vector field for any function $\rho$ only depending on $r$. More precisely, let us
assume that $\rho$ is a smooth function in $\mathbb{C} \setminus \{0\}$, only depending on $r$, which satisfies $\rho(0) = 0$. Then, we define
\[ \tilde{X} = \frac{1}{(x^2 + y^2)^{1/2}}(y, -x). \]

Clearly, we have
\[ \begin{cases} 
\text{div}(\rho^2 \tilde{X}) = 0 & \text{in } \mathcal{D}'(\mathbb{C}) \\
\rho^2 \text{curl} \tilde{X} = 0 & \text{in } \mathcal{D}'(\mathbb{C}). 
\end{cases} \tag{8.17} \]

In particular $X = (x, y)$ is a $\rho$-conformal vector field.

**Remark 8.1** Assume $\rho$ does not vanish on a simply connected domain $\omega$, then on $\omega$ we can write $\tilde{X} := \nabla \phi$. Clearly, $\text{div}(\rho^2 \nabla \phi) = 0$ and hence $\phi$ is a critical point of the functional
\[ \phi \rightarrow \int_{\Omega} \rho^2 |d\phi|^2 \, dx. \]

Observe that, if we were in dimension $n \geq 3$, then a conformal change of metric would make $\phi$ an harmonic function for that new metric. However, in dimension 2, such change of metric does not exists, this is why $\rho$-conformal vector fields are not simply conformal fields in dimension 2.

### 8.4 Conservation laws

Let us start with the following basic computation:

**Lemma 8.1** Let $\tilde{X}$ be any smooth field which does not vanish and let $X$ be the field given by
\[ X = \frac{\tilde{X}^\perp}{|\tilde{X}|^2} \]

Then, we have
\[ (\partial_x X^x - \partial_y X^y) = 2 X^x X^y \text{div} \tilde{X} + ((X^y)^2 - (X^x)^2) \text{curl} \tilde{X}, \]
and
\[ (\partial_y X^x + \partial_x X^y) = ((X^y)^2 - (X^x)^2) \text{div} \tilde{X} - 2 X^x X^y \text{curl} \tilde{X}. \]

**Proof:** Observe that the following identities hold
\[ \frac{\partial}{\partial z} (X^x - i X^y) = \frac{1}{2} (\partial_x X^x - \partial_y X^y) - \frac{i}{2} (\partial_y X^x + \partial_x X^y), \tag{8.18} \]
and
\[ \frac{\partial}{\partial z} (\tilde{X}^x - i \tilde{X}^y) = \frac{1}{2} (\partial_x \tilde{X}^x - \partial_y \tilde{X}^y) - \frac{i}{2} (\partial_y \tilde{X}^x + \partial_x \tilde{X}^y). \tag{8.19} \]
Moreover, given the definition of the vector fields $X$ and $\tilde{X}$ we have

$$X^x - i X^y = \frac{\tilde{X}^y + i \tilde{X}^x}{(X^x)^2 + (X^y)^2} = \frac{i}{X^x + i X^y}.$$  

Therefore,

$$i \frac{\partial}{\partial z} \left( \frac{1}{X^x + i X^y} \right) = \frac{i}{(X^x + i X^y)^2} \frac{\partial}{\partial z} \left( \tilde{X}^x + i \tilde{X}^y \right).$$

Inserting (8.18) and (8.19) into this last identity leads to the desired result. □

Identity (8.9) is a particular case of a more general identity we now state.

**Lemma 8.2** Let $\omega \subset \mathbb{C}$ and let $w$ be a complex valued function defined in $\omega$. Then, for all vector field $X := (X^x, X^y)$ defined in $\omega$, we have

$$\partial X^w \cdot \Delta w = \text{div} \left( \partial X^w \cdot \nabla w - \frac{1}{2} |\nabla w|^2 X \right)$$

$$= \frac{1}{2} (|\partial_x w|^2 - |\partial_y w|^2) (\partial_x X^y - \partial_y X^x)$$

$$- \partial_x w \cdot \partial_y w (\partial_x X^y + \partial_y X^x). \quad (8.20)$$

**Proof:** The proof is straightforward and left to the reader. □

As a first application, we obtain the following *conservation law* for conformal vector fields in dimension 2:

**Corollary 8.1** Let $X$ be a conformal vector field, then

$$\partial X^w \cdot \Delta w = \text{div} \left( \partial X^w \cdot \nabla w - \frac{1}{2} |\nabla w|^2 X \right) \quad (8.21)$$

**Proof:** Simply use Lemma 8.1 and Lemma 8.2. □

The denomination *conservation law* is justified by the following identity:

$$\frac{d}{dt} \int_\omega |\nabla w \circ (x + tX(x))|^2 = -2 \int_\omega \partial X^w \cdot \Delta w + 2 \int_{\partial \omega} \partial X^w \cdot \partial_n w, \quad (8.22)$$

which holds provided $w$ is smooth enough.

Combining Lemma 8.1 and Lemma 8.2, we obtain:

**Lemma 8.3** Let $\omega \subset \mathbb{C}$ and let $w$ be a complex valued function defined in $\omega$. Let $X := (X^x, X^y)$ be a vector field defined in $\omega$, we set

$$\tilde{X} := -\frac{X^\perp}{|X|^2}.$$  

Then, provided curl $\tilde{X} = 0$, we have

$$\partial X^w \cdot \Delta w = \text{div} \left( \partial X^w \cdot \nabla w - \frac{1}{2} |\nabla w|^2 X \right) - \partial X^w \cdot \partial X^w \text{ div } \tilde{X}. \quad (8.23)$$
Proof: Observe that

\[
\partial_X w \cdot \partial_{X^\perp} w = (X_x \partial_x w + X^y \partial_y w) (X^y \partial_x w - X^x \partial_y w)
\]

\[
= X^x X^y (|\partial_x w|^2 - |\partial_y w|^2) \\
+ ((X^y)^2 - (X^x)^2) \partial_x w \cdot \partial_y w.
\]

The result is then a simple Corollary of Lemma 8.2 and Lemma 8.1. \(\square\)

For further reference, we state:

**Lemma 8.4** Assume that \(X\) is a vector field and that the function \(\rho\) does not vanish. Define

\[
\tilde{X} := -\frac{X^\perp |X|^2}{\rho^2},
\]

Then

\[
\text{div} \tilde{X} = \frac{2}{|X|^2} \frac{\partial X^\perp \rho}{\rho} + \frac{1}{\rho^2} \text{div}(\rho^2 \tilde{X}).
\]

Moreover,

\[
\text{div} \tilde{X}^\perp = 0,
\]

provided \(\text{curl} \tilde{X} = 0\).

**Proof:** For the first formula, we simply compute

\[
\text{div} \tilde{X} = \text{div}(\rho^{-2} \rho^2 \tilde{X}) = -2 \frac{\partial X^\perp \rho}{\rho} + \frac{1}{\rho^2} \text{div}(\rho^2 \tilde{X}).
\]

The second formula follows from the identity

\[
\text{div} \tilde{X}^\perp = \text{curl} \tilde{X},
\]

which holds for any vector field. \(\square\)

Thanks to all the above preliminary computations, we can derive the following conservation law:

**Proposition 8.1** Let \(X\) be a vector field and let \(\rho\) be a function which does not vanish. We set

\[
\tilde{X} := -\frac{X^\perp |X|^2}{\rho^2}.
\]

Provided \(\text{curl} \tilde{X} = 0\), we have

\[
\partial_X w \cdot \left( \Delta w + \frac{2}{\rho} \nabla \rho \nabla w \right) = \text{div}(\partial_X w \cdot \nabla w - \frac{1}{2} |\nabla w|^2 X) \\
+ \frac{2}{|X|^2} \frac{\partial X^\perp \rho}{\rho} |\partial_X w|^2 \\
- \frac{1}{\rho^2} \partial_X w \cdot \partial_X^\perp \rho \text{div}(\rho^2 \tilde{X}). \quad (8.24)
\]

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Proof: Observe that the following identity holds
\[ \partial_{X^\perp} \partial_X \varphi + \partial_X \rho \partial_X w = |X|^2 \nabla \rho \nabla w. \]
The result is then a simple consequence of Lemma 8.3 and Lemma 8.4. □

Observe that, when \( X \) is a \( \rho \)-conformal vector field, identity (8.24) can be simplified and we obtain the:

**Corollary 8.2** Let \( X \) be a \( \rho \)-conformal vector field, then

\[
\rho^{-2} \partial_X w \cdot \text{div} (\rho^2 \nabla w) = \text{div} (\partial_X w \cdot \nabla w - \frac{1}{2} |\nabla w|^2 X) + \frac{2}{|X|^2} \frac{\partial_X \rho}{\rho} |\partial_X w|^2.
\]

(8.25)

It is interesting to compare this formula with the corresponding formula (8.21) we have obtained in Corollary 8.1 when \( X \) is a conformal vector field. Clearly, the (8.25) involves the extra term

\[
\frac{2}{|X|^2} \frac{\partial_X \rho}{\rho} |\partial_X w|^2.
\]

For later applications, the sign of this term will be of crucial importance. This will be further commented in the last section of this Chapter in which we will also give a geometric meaning to this quantity.

The denomination *conservation law* is justified by the following identity:

\[
\frac{d}{dt} \int_\omega \rho^2 \left| \nabla w \circ (x + t \frac{X}{\rho^2}(x)) \right|^2 = 2 \int_\omega \partial_X w \cdot \frac{\text{div} (\rho^2 \nabla w)}{\rho^2} + 2 \int_{\partial \omega} \partial_X w \cdot \frac{\partial w}{\partial \nu},
\]

(8.26)

which holds provided \( w \) is smooth enough.

### 8.4.1 Comparing solutions through a Pohozaev type formula: the general case

We now apply all the previous computation to the Ginzburg-Landau equation.

### 8.4.2 Conservation laws for Ginzburg-Landau equation

Assume that \( v \) is a solution of the Ginzburg-Landau equation

\[
\Delta v + \frac{\rho}{\varepsilon^2} (1 - |v|^2) v = 0.
\]

(8.27)

We set

\[
\tilde{X} := \frac{1}{|v|^2} v \wedge \nabla v,
\]

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These two functions are well defined away from the points where $v = 0$ or $v \wedge \nabla v = 0$. We have the following simple but fundamental result:

**Proposition 8.2** Assume that $v$ is a solution of (8.27). Then the vector field $X$ defined above is a $|v|$-conformal vector field.

To check this, it is easier to write

$$v := |v| e^{i\phi}.$$  

This is always possible, at least locally, away from the zero set of $v$. The fact that $v$ is a solution of the Ginzburg-Landau equation implies that

$$\text{div}(|v|^2 \phi) = 0.$$  

However, with this definitions, we have the alternate definition of $X$ which is given by

$$X = \frac{\nabla \phi}{|\nabla \phi|^2}, \quad \text{and} \quad \bar{X} = \nabla \phi.$$  

Hence, $X$ is a $|v|$-conformal vector field.

Applying the result of Proposition 8.1, we obtain the following

**Corollary 8.3** Let $w$ be any regular map and $v$ a solution of (8.27) in $\omega$. Further assume that

$$v \neq 0 \quad \text{and} \quad v \wedge \nabla v \neq 0,$$  

in $\omega$. Then the following identity holds

$$\partial_X w \cdot \left(\Delta w + \frac{2}{|v|} \nabla |v| \nabla w\right) = \text{div} \left(\partial_X w \cdot \nabla w - \frac{1}{2} |\nabla w|^2 X\right) + \frac{2}{|X|^2} \frac{\partial_X |v|}{|v|} |\partial_X w|^2,$$

where $X$ is the vector fields defined in (8.28).

It is interesting to investigate the behavior of the above defined vector field for Ginzburg-Landau equations, as the parameter $\varepsilon$ tends to 0. To this aim, let us assume that $v_\varepsilon$ is a sequence of solutions of (8.27) and further assume that, as $\varepsilon$ tends to 0, the sequence $v_\varepsilon$ converges to the harmonic $S^1$ valued function $v_\ast$, given by

$$v_\ast = \prod_{j=1}^{n} \left(\frac{z - a_j}{|z - a_j|}\right)^{d_j} e^{i\phi},$$

where the function $\phi$ is harmonic. For example, this is the case if the $v_\varepsilon$ are minimizers of the Ginzburg-Landau functional. Naturally, the convergence of
the sequence is understood away from the points $a_j$. It is of interest to compute, using (8.28), the vector field $X_\ast$ corresponding to $v_\ast$. Since $v_\ast$ is $S^1$ valued, we find

$$X_\ast = \frac{v_\ast \wedge \nabla^\bot v_\ast}{|v_\ast \wedge \nabla^\bot v_\ast|^2}.$$ 

Away from the points $a_j$ we can write, at least locally

$$v_\ast := e^{i\phi_\ast}.$$ 

Hence we find the alternate formula for $X_\ast$

$$X_\ast = \frac{\nabla^\bot \phi_\ast}{|\nabla \phi_\ast|^2}.$$ 

Observe that, near $a_j$ (which is taken to be equal to 0 in order to simplify the notations), we have

$$\phi_\ast = d\theta + H$$

where $d$ is the degree of $v_\ast$ around 0 and $H$ is a harmonic function which has a critical point at the origin. It is then an easy exercise to see that

$$\nabla \phi_\ast = d \left(-\frac{y}{r^2}, \frac{x}{r^2}\right) + \mathcal{O}(r).$$

And hence

$$X_\ast = \pm(x, y) + \mathcal{O}(r^3).$$

In particular, near each $a_j$, the vector field $X_\ast$ is close to the “usual” conformal vector field which is used in Pohozaev formula.

### 8.4.3 The Pohozaev formula

In this section, we generalize (8.13) to the case where $\Omega$ is an arbitrary bounded domain and where $v$ is not necessary the axially symmetric solution of the Ginzburg-Landau equation.

Let us give a definition which will be useful throughout this Chapter.

**Definition 8.10** We will say that a $C^1$ complex valued function, defined in some domain $\omega$, is non degenerate if

$$v \neq 0 \quad \text{and} \quad v \wedge \nabla v \neq 0,$$

in all $\omega$.

Observe that, if $v \neq 0$, we can write, at least locally,

$$v := |v| e^{i\phi},$$

and we find

$$v \wedge \nabla v = \nabla \phi.$$
Hence, the second condition in the Definition simply becomes
\[ \nabla \phi \neq 0. \]

Assume that \( u \) is a solution of the Ginzburg-Landau equation and assume that \( v \) is an arbitrary complex valued function. If we define the function
\[ w := \frac{u}{v}, \]
we have

Lemma 8.5 \textit{The function } \( w \) \textit{is a solution of}
\[ \Delta w + \frac{2}{v} \nabla w \nabla v + w|v|^2 \frac{(1 - |w|^2)}{\varepsilon^2} = -\frac{w}{v} \left( \Delta v + \frac{v(1 - |v|^2)}{\varepsilon^2} \right). \tag{8.30} \]

\textbf{Proof} : The proof follows from the identity
\[ \Delta \left( \frac{u}{v} \right) + \frac{2}{v} \nabla v \nabla w = \frac{1}{v^2} (v \Delta u - u \Delta v). \]
The result of the Lemma follows at once. \( \square \)

If in addition we assume that the function \( v \) is also a solution of the Ginzburg-Landau equation, the above formula simplifies and we obtain
\[ \Delta w + \frac{2}{v} \nabla w \nabla v + w|v|^2 \frac{(1 - |w|^2)}{\varepsilon^2} = 0. \tag{8.31} \]
Using this, we get the

Proposition 8.3 \textit{Assume that } \( u \) \textit{and } \( v \) \textit{are two solutions of the Ginzburg-Landau equation, which are defined in } \( \omega \). \textit{Further assume that } \( v \) \textit{is nondegenerate in } \( \omega \). \textit{Define}
\[ X = |v|^2 \frac{v \wedge \nabla^2 v}{|v \wedge \nabla^2 v|^2} \quad \text{and} \quad \tilde{X} := \frac{1}{|v|^2} v \wedge \nabla v. \]
Then the following identity holds :
\[ \text{div} \left( \partial_X w \cdot \nabla w - \frac{1}{2} |\nabla w|^2 X - w \wedge \nabla^2 w - \frac{1}{4\varepsilon^2} |v|^2 (1 - |w|^2)^2 X \right) \]
\[ = -\frac{1}{4\varepsilon^2} (1 - |w|^2)^2 \text{div}(|v|^2 X) - \frac{2}{|X|^2} \frac{\partial_X |v|}{|v|} |\partial_X w|^2. \tag{8.32} \]

\textbf{Proof} : Observe that
\[ \frac{2}{|v|} \nabla w \nabla |v| = \frac{2}{v} \nabla w \nabla v - 2i \partial \tilde{X} w, \]

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and also
\[ \partial_X w \cdot (2i \partial_X w) = \frac{2}{|X|^2} \partial_X w \wedge \partial_X w \]
\[ = -2 \partial_x w \wedge \partial_y w \]
\[ = -\operatorname{div}(w \wedge \nabla^\perp w). \]

Using these two identities together with Corollary 8.3, we obtain
\[ \partial_X w \cdot \left( \Delta w + \frac{2}{v} \nabla v \nabla w \right) = \operatorname{div} \left( \partial_X w \cdot \nabla w - \frac{1}{2} |\nabla w|^2 X - w \wedge \nabla w \right) \]
\[ + \frac{2}{|X|^2} \frac{\partial_X |v|}{|v|} |\partial_X w|^2. \]

Now, using the equation satisfied by \( w \) we find
\[ \frac{|v|^2}{\varepsilon^2} \partial_X w \cdot w (1 - |w|^2) = -\operatorname{div} \left( \partial_X w \cdot \nabla w - \frac{1}{2} |\nabla w|^2 X - w \wedge \nabla w \right) \]
\[ - \frac{2}{|X|^2} \frac{\partial_X |v|}{|v|} |\partial_X w|^2. \]

Finally, we have
\[ \partial_X w \cdot w |v|^2 (1 - |w|^2) = -\frac{1}{4} \operatorname{div} \left( |v|^2 (1 - |w|^2)^2 X \right) \]
\[ + \frac{1}{4} (1 - |w|^2)^2 \operatorname{div} (|v|^2 X). \]

which together with the previous identity completes the proof. \( \square \)

The identity (8.32) is the exact generalization of the one established in (8.13) in the axially symmetric case. Again, this is a conservation law type identity which could have been obtained starting from the fact that \( w \) is a critical point with respect to the variations of the domain \( \omega \), of the functional
\[ E(w) := \int_{\omega} |v|^2 |\nabla w|^2 + 2 \int_{\omega} (v \wedge \nabla v) \cdot (w \wedge \nabla w) + \frac{1}{2\epsilon^2} \int_{\omega} |v|^4 (1 - |w|^2)^2. \quad (8.33) \]

When we do not assume that \( v \) is a solution of the Ginzburg-Landau equation anymore, we have:

**Proposition 8.4** Assume that \( u \) is a solution of the Ginzburg-Landau equation, which is defined in \( \omega \). Further assume that \( v \) is a complex valued function which is nondegenerate in \( \omega \). Define

\[ X = |v|^2 \frac{v \wedge \nabla^\perp v}{|v \wedge \nabla^\perp v|^2} \quad \text{and} \quad \tilde{X} := \frac{1}{|v|^2} v \wedge \nabla v. \]

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If we assume that \(\text{curl}\vec{X} = 0\) then the following identity holds:

\[
\text{div} \left( \partial_X w \cdot \nabla w - \frac{1}{2} |\nabla w|^2 X - w \wedge \nabla^\perp w - \frac{1}{4\epsilon^2} |v|^2 (1 - |w|^2)^2 X \right)
\]

\[
= -\frac{1}{4\epsilon^2} (1 - |w|^2)^2 \text{div}(|w|^2 X) - \frac{2}{|X|^2} \frac{\partial_X |v|}{|v|} |\partial_X w|^2
\]

\[
- \frac{1}{|v|^2} \partial_X w \cdot \partial_X^\perp w \text{div}(v \wedge \nabla v)
\]

\[
+ \frac{1}{v} \partial_X w \cdot w \left( \Delta v + \frac{v}{\epsilon^2} (1 - |v|^2) \right).
\]

(8.34)

**Proof**: The proof is similar to the proof of Proposition 8.3. However, since we do not assume that \(v\) is a solution of the Ginzburg-Landau equation, the vector field \(X\) is not a \(|v|\)-conformal vector field anymore. Hence \(\text{div}(|v|^2 \vec{X}) \neq 0\). The reason why the quantity

\[
-\frac{1}{|v|^2} \partial_X w \cdot \partial_X^\perp w \text{div}(\vec{X}|v|^2),
\]

which comes from Proposition 8.1, has to appear in the right hand side of the identity. Finally, observe that

\[
\vec{X}|v|^2 = v \wedge \nabla v,
\]

in order to get the desired formula. \(\square\)

### 8.4.4 Integration of the Pohozaev formula

In proposition 8.3, we have seen that if \(u\) and \(v\) are two solutions of the Ginzburg-Landau equation, which are defined in \(\Omega\). If we assume that \(v\) is nondegenerate in a subset \(\omega \subset \Omega\) and if we define

\[
X = |v|^2 \frac{v \wedge \nabla^\perp v}{|v \wedge \nabla^\perp v|^2} \quad \text{and} \quad \vec{X} := \frac{1}{|v|^2} v \wedge \nabla v.
\]

Then the following identity holds:

\[
\int_{\partial\omega} \left( \partial_X w \cdot \partial_\nu w - \frac{1}{2} |\nabla w|^2 X \cdot \nu + w \wedge \partial_\nu w - \frac{1}{4\epsilon^2} |v|^2 (1 - |w|^2)^2 X \cdot \nu \right)
\]

\[
= -\frac{1}{4\epsilon^2} \int_\omega (1 - |w|^2)^2 \text{div}(|w|^2 X) - \int_\omega \frac{2}{|X|^2} \frac{\partial_X |v|}{|v|} |\partial_X w|^2,
\]

where \(\nu\) denotes the outward unit normal to \(\partial\omega\) and \(\tau = \nu^\perp\).

Now, we would like to understand what happens when \(\omega\) tends to \(\Omega\). In addition to all the above assumptions, let us assume that the following properties are satisfies:
1. The functions \( u \) and \( v \) have the same boundary data on \( \partial \Omega \).

2. All the zeros of \( v \) are isolated and 0 is a regular value of \( v \). We will denote by \( \Sigma \) the zero set of \( v \).

3. Away from the zero set of \( v \)
   \[
   v \wedge \nabla v \neq 0.
   \]
   For all \( s > 0 \) small enough, we define
   \[
   \omega_s = \{ z \in \Omega : \text{dist}(z, \Sigma) > s \}.
   \]

Granted all the above definitions, we prove

**Lemma 8.6** Under the above assumptions, we have
\[
\lim_{s \to 0} \int_{\partial \omega} \left( \partial_X w \cdot \partial_\nu w - \left( \frac{1}{2} |\nabla w|^2 + \frac{|v|^2}{4 \varepsilon^2} (1 - |w|^2)^2 \right) X \cdot \nu + w \wedge \partial_\tau w \right) = \frac{1}{2} \int_{\partial \Omega} |\nabla w|^2 X \cdot \nu.
\]

**Proof:** The proof of this Lemma follows at once since, on \( \partial \Omega \), we have \( |w| = 1 \), \( \partial_\tau w = 0 \) and
\[
\partial_\nu w \cdot \partial_X w = |\partial_\nu w|^2 X \cdot \nu = |\nabla w|^2 X \cdot \nu.
\]
The details are left to the reader. \( \square \)

Let \( a_j \in \Sigma \) be a zero of \( v \). For the sake of simplicity, let us assume that \( a_j = 0 \). Then we have

**Lemma 8.7** Under the above assumptions, we have
\[
\lim_{s \to 0} \int_{\partial B_s} \left( \partial_X w \cdot \partial_\nu w - \left( \frac{1}{2} |\nabla w|^2 + \frac{|v|^2}{4 \varepsilon^2} (1 - |w|^2)^2 \right) X \cdot \nu + w \wedge \partial_\tau w \right) = 0,
\]
where \( \nu := -\frac{1}{r}(x, y) \) and \( \tau = \nu^\perp \).

**Proof:** In order to simplify the discussion, we will write
\[
v = f(\theta) r + O(r^2) \quad \text{and} \quad w = h(\theta) + O(r).
\]
These expansions do hold since we have assumed that 0 is a regular value of \( v \).

Near 0 we can also write
\[
v = a x + b y + O(r^2),
\]

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where $a, b \in \mathbb{C}$. Going back to the definition of $X$ and $\tilde{X}$, we find

$$X = |f|^2 (a \wedge b)^{-1} (x, y) + O(r^2),$$

and

$$\tilde{X} = |f|^{-2} (a \wedge b) (-y, x) + O(1).$$

In particular, since the leading term in $w$ only depends on $\theta$, we see that

$$\partial_X w = O(r) \quad \text{and} \quad \partial_\nu w = O(1),$$

and finally

$$X \cdot \nu = O(1).$$

Thanks to these expansions, we already obtain

$$\lim_{s \to 0} \int_{\partial B_s} \partial_X w \cdot \partial_\nu w = 0.$$

With the above notations, we have

$$\lim_{s \to 0} s \left( \frac{1}{2} (X \cdot \nu) |\nabla w|^2 + w \wedge (\nabla \| \wedge w \cdot \nu) \right)_{\partial B_s} = - \frac{1}{2} (a \wedge b)^{-1} |f|^2 |\partial_\theta h|^2 - (ih) \cdot (\partial_\theta h).$$

The function $w$ is a solution of (8.31), moreover, we have already seen the identity

$$\frac{1}{\nu} \nabla w = \frac{1}{\|w\|} \nabla |w| \cdot \nabla w + 2i \partial_\nu w.$$

Hence, we get

$$r^2 \Delta w + \frac{r^2}{|w|} \nabla |w| \cdot \nabla w + 2i r^2 \partial_\nu w + r^2 \omega |w|^2 \frac{1 - |w|^2}{\varepsilon^2} = 0.$$

Granted the above notations, we may let $r$ tends to 0 in the above identity and we find

$$\partial_\theta \left( |f|^2 \partial_\theta h \right) + 2i (a \wedge b) \partial_\theta h = 0.$$

Taking the scalar product with $|g|^2 h$ and integrating over $\partial B_s$ we conclude that

$$\lim_{s \to 0} \int_{\partial B_s} \left( \frac{1}{2} |\nabla w|^2 X \cdot \nu - w \wedge \partial_\tau w \right) = 0,$$

as claimed. \hfill \Box

Collecting the results of the last two Lemmas, we find:

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Theorem 8.3 Assume that $u$ and $v$ are two solutions of the Ginzburg-Landau equation, which are defined in $\Omega$ and have the same boundary data on $\partial \Omega$. Further assume that $0$ is a regular value of $v$ and, away from the zero set of $v$

$v \wedge \nabla v \neq 0$.

Then the following identity holds:

\[ 2 \int_{\partial \Omega} \partial_X w \cdot \partial_\nu w - \int_{\partial \Omega} |\nabla w|^2 X \cdot \nu = \frac{1}{2\pi^2} \int_\Omega (1 - |w|^2)^2 \text{div}(|v|^2 X) \]

\[ - \int_\Omega \frac{4}{|X|^2} \frac{\partial_X |v|}{|v|} \partial_X w^2 = 0, \]

where as usual $\nu$ denotes the outward unit normal to $\partial \Omega$ and where

\[ X = |v|^2 \frac{v \wedge \nabla \perp v}{|v \wedge \nabla \perp v|^2}. \]

8.5 Uniqueness Results

8.5.1 A few uniqueness results

We can already obtain some uniqueness results, thanks to the Pohozaev formula given in Theorem 8.3. These uniqueness results were originally established by P. Mironescu. The arguments used by P. Mironescu are exactly those which are developed in the second section of this Chapter. Let us mention that the idea of using Pohozaev formula to study uniqueness questions for Ginzburg-Landau equations goes back to the work of S. Chanillo and M. Kiessling in [20].

For example, as a beautiful application of Theorem 8.3, we have the following result due to P. Mironescu:

Theorem 8.4 [64] Assume that $u$ is a solution of the Ginzburg-Landau equation (8.6). Further assume that $u(0) = 0$ then $u$ is radially symmetric, namely

$u = \tilde{S}_x e^{i\theta}$.

Proof: We set $v := \tilde{S}_x e^{i\theta}$ and we find explicitely

$X = (x, y)$.

We set

$w := \frac{u}{v}$.

Observe that, since $w \equiv 1$ on $\partial B_1$, we can write $|\nabla w| = |\partial_\nu w|$ on $\partial B_1$. Hence

\[ 2 \int_{\partial B_1} \partial_X w \cdot \partial_\nu w - \int_{\partial B_1} |\nabla w|^2 = \int_{\partial B_1} |\nabla w|^2. \]
The assumptions of Theorem 8.3 hold and thus, we obtain

\[ \int_{\partial B_1} |\nabla w|^2 + \frac{1}{\varepsilon^2} \int_{B_1} (1 - |w|^2)^2 (\tilde{S}_\varepsilon^2 + \tilde{S}_\varepsilon \partial_r \tilde{S}_\varepsilon) + 4 \int_{B_1} r \frac{\partial_r \tilde{S}_\varepsilon}{\tilde{S}_\varepsilon} |\partial_r w|^2 = 0. \]

The function \( \tilde{S}_\varepsilon \) being increasing, we conclude that \( \partial_r w \equiv 0 \). Since \( w = 1 \) on \( \partial B_1 \), we get \( u = v \). \( \square \)

We also obtain the:

**Theorem 8.5** [64] Assume that \( u \) is a solution of

\[ \Delta u + u (1 - |u|^2) = 0, \]

in \( \mathbb{C} \). Further assume that

\[ \int_\mathbb{C} (1 - |u|^2)^2 < +\infty, \quad \text{and} \quad \deg \left( \frac{u}{|u|}, \infty \right) = 1. \]

Then, there exists \( \tau_0 \in \mathbb{R} \) and \( a \in \mathbb{C} \) such that

\[ e^{-i\tau_0} u(\cdot - a) = S e^{i\theta}, \]

where \( S \) is defined in Theorem 3.1.

**Proof:** Since the degree of \( u \) at \( \infty \) is nonzero, the function \( u \) has to vanish in \( \mathbb{C} \). Without loss of generality, we may assume that \( u(0) = 0 \).

Now, we use the radially symmetric solution defined in Theorem 3.1. We set \( v := S e^{i\theta} \).

In this case we have

\[ X = (x, y). \]

Integrating (8.34) over \( B_R \setminus B_s \) and letting \( s \) tend to 0, we obtain using Lemma 8.6

\[ \int_{\partial B_R} \left( r |\partial_r w|^2 - r^{-1} |\partial_\theta w|^2 + 2 w \wedge \partial_\theta w - \frac{r}{2\varepsilon^2} S^2 (1 - |w|^2)^2 \right) \]

\[ = -\frac{1}{\varepsilon^2} \int_{B_R} (1 - |w|^2)^2 (S^2 + S \partial_r S) - 4 \int_{B_R} r \frac{\partial_r S}{S} |\partial_r w|^2. \] (8.35)

Finally, we let \( R \) tend to \( +\infty \). Using the assumptions together with informations about the behavior at infinity of \( u \) which are provided by [15] and [90], one proves that the left hand side of (8.35) tends to zero. This implies that \( |w| \equiv 1 \) and \( \partial_r w \equiv 0 \). Going back to the equation satisfied by \( w \) we now find

\[ \partial_{\theta^2} w + 2 i \partial_\theta w = 0. \]

Since \( |w| \equiv 1 \) we find that, either

\[ w = e^{i\tau_0} \]

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for some $\tau_0 \in \mathbb{R}$, or
\[
    w = e^{i(\tau_0 - 2\theta)},
\]
for some $\tau_0 \in \mathbb{R}$. However, the later case is not possible otherwise the degree of
$u$ at infinity would be $-1$, which contradicts our assumption. Hence $w \equiv e^{i\tau_0}$ as claimed.

\[\square\]

**Remark 8.2** The uniqueness question for degrees different from $\pm 1$ is largely open and could happen not to be true as it is conjectured in [69].

### 8.5.2 Uniqueness results for semilinear elliptic problems

To our knowledge, the first time where a uniqueness result was established for
semilinear elliptic problems using the quotient of two solutions $u$ and $v$ arose in
a work by H. Brezis and L. Oswald in [16], where they gave a different proof of
the following result of A. M. Krasnosel’skii:

**Theorem 8.6** Let $\Omega$ be a bounded regular domain of $\mathbb{R}^n$ and $f$ be a smooth function defined in $[0, +\infty)$ such that $f(t)/t$ is strictly decreasing. Assume $u_1$ and $u_2$ are positive solutions of
\[
    \begin{cases}
    -\Delta u = f(u) & \text{in } \Omega \\
    u = 0 & \text{on } \partial \Omega.
    \end{cases}
\]  
(8.36)

Then $u_1 = u_2$.

**Proof**: We define as usual
\[
w := \frac{u_2}{u_1},
\]
which is a solution of
\[
div(u_1^2 \nabla w) = u_2 f(u_1) - u_1 f(u_2).
\]
Observe that, thanks to Hopf Lemma, $\partial \nu u_1 \neq 0$ on $\partial \Omega$ and hence we see that
\[
w = \frac{\partial \nu u_2}{\partial \nu u_1},
\]
on $\partial \Omega$.

Now, we multiply the equation satisfied by $w$ by $w$ itself and integrate by
parts over $\Omega$. We obtain
\[
\int_{\Omega} u_1^2 |\nabla w|^2 + \int_{\Omega} f(u_1) (u_1 - u_2) u_2 = 0.
\]
Since we have assumed that $f(t)/t$ is strictly decreasing we conclude that
necessarily $u_2 = u_1$ in $\Omega$. \[\square\]

For Ginzburg-Landau equations, a similar argument arose first in the work of
L. Lassoued and P. Mironescu [41]. The comparison argument through division
is also very efficient, as it was observed in [21], to prove the following uniqueness
result of D. Ye and F. Zhou [100].
Proposition 8.5 Let $\Omega$ be a simply connected domain. Assume that $u$ and $v$ are two solutions of the Ginzburg-Landau equation
\[
\Delta u + \frac{u}{\varepsilon^2} (1 - |u|^2) = 0,
\]
in $\Omega$ such that $u = v$ on $\partial\Omega$ and which do not vanish in $\Omega$. Further assume that
\[
|u|^2 + |v|^2 - \varepsilon^2 (|\nabla v|^2 + |\nabla u|^2) > 0,
\]
in $\Omega$. Then $u = v$.

Proof: We set
\[
w := \frac{u}{v},
\]
and
\[
\tilde{X} := \frac{v \wedge \nabla v}{|v|^2}.
\]
We have already seen that $w$ satisfies
\[
\Delta w + \frac{2}{v} \nabla v \nabla w + w |v|^2 \left( \frac{1 - |w|^2}{\varepsilon^2} \right) = 0.
\]
Furthermore, we have also seen that
\[
\frac{2}{v} \nabla w \nabla v = \frac{2}{|v|} \nabla w |v| + 2i \partial_{\tilde{X}} w.
\]
Hence
\[
div(|v|^2 \nabla w) + 2i |v|^2 \partial_{\tilde{X}} w + w |v|^2 \left( \frac{1 - |w|^2}{\varepsilon^2} \right) = 0.
\]
Taking the scalar product with $w$ and integrating by parts over $\Omega$, we find
\[
\int_{\partial\Omega} |v|^2 w \cdot \partial_{\nu} w - \int_{\Omega} |v|^2 |\nabla w|^2 + 2 \int_{\Omega} |v|^2 w \cdot (i \partial_{\tilde{X}} w)
\]
\[
+ \frac{1}{\varepsilon^2} \int_{\Omega} |u|^2 (1 - |w|^2) = 0.
\]
Now, since $u = v$ on $\partial\Omega$ we can write
\[
\int_{\partial\Omega} |v|^2 w \cdot \partial_{\nu} w = \int_{\partial\Omega} \left( |u|^2 \Re(u^{-1} \partial_{\nu} u) - |v|^2 \Re(v^{-1} \partial_{\nu} v) \right).
\]
In addition, observe that
\[
|v|^2 w \cdot (i \partial_{\tilde{X}} w) = (v \wedge \nabla v) (\nabla w \wedge w).
\]
Hence we get
\[
\int_{\partial\Omega} \left( |u|^2 \Re(u^{-1} \partial_{\nu} u) - |v|^2 \Re(v^{-1} \partial_{\nu} v) \right) - \int_{\Omega} |v|^2 |\nabla w|^2
\]
\[
+ 2 \int_{\Omega} (v \wedge \nabla v) (\nabla w \wedge w) + \frac{1}{\varepsilon^2} \int_{\Omega} |u|^2 (1 - |w|^2) = 0.
\]
We claim that
\[ \int_{\Omega} \frac{1}{|w|^2} (v \wedge \nabla v) (\nabla w \wedge w) = 0. \]
Indeed, \( \Omega \) being simply connected, we can write
\[ w := |w| e^{i\psi} \quad \text{and} \quad v := |v| e^{i\phi}. \]
Using these notations, we obtain
\[ \frac{1}{|w|^2} (v \wedge \nabla v) (\nabla w \wedge w) = |v|^2 \nabla \phi \nabla \psi. \]
Therefore
\[ \int_{\Omega} \frac{1}{|w|^2} (v \wedge \nabla v) (\nabla w \wedge w) = \int_{\partial \Omega} |v|^2 \psi \partial_{\nu} \Phi - \int_{\Omega} \text{div}(|v|^2 \nabla \phi) \psi. \]
Now, \( w \equiv 1 \) on \( \partial \Omega \) and hence \( \psi \equiv 0 \) on \( \partial \Omega \). Furthermore, \( v \) is a solution of the Ginzburg-Landau equation and thus\[ \text{div}(|v|^2 \nabla \phi) = 0. \]
This ends the proof of the claim.

Going back to (8.38), we obtain
\[ \int_{\partial \Omega} \left( |u|^2 \Re(u^{-1} \partial_{\nu} u) - |v|^2 \Re(v^{-1} \partial_{\nu} v) \right) - \int_{\Omega} |v|^2 |
abla w|^2 + 2 \int_{\Omega} \frac{|v|^2 - |u|^2}{|u|^2} (v \wedge \nabla v) (\nabla w \wedge w) + \frac{1}{\varepsilon^2} \int_{\Omega} |u|^2 (1 - |u|^2) = 0. \]

We make use of the inequality \( 2ab \leq a^2 + b^2 \), to get
\[ 2 \int_{\Omega} \frac{|v|^2 - |u|^2}{|u|^2} (v \wedge \nabla v) (\nabla w \wedge w) \leq \int_{\Omega} |v|^2 |
abla w|^2 + \int_{\Omega} \frac{|
abla v|^2}{|u|^2 |v|^2} (|u|^2 - |v|^2)^2, \]
which together with the previous identity yields
\[ \int_{\partial \Omega} \left( |u|^2 \Re(u^{-1} \partial_{\nu} u) - |v|^2 \Re(v^{-1} \partial_{\nu} v) \right) + \frac{1}{\varepsilon^2} \int_{\Omega} \frac{|u|^2}{|v|^2} (|v|^2 - |u|^2) \geq \int_{\Omega} \frac{|
abla v|^2}{|u|^2 |v|^2} (|u|^2 - |v|^2)^2. \]
Interchanging \( u \) and \( v \), we also obtain
\[ \int_{\partial \Omega} \left( |u|^2 \Re(u^{-1} \partial_{\nu} u) - |v|^2 \Re(v^{-1} \partial_{\nu} v) \right) + \frac{1}{\varepsilon^2} \int_{\Omega} \frac{|v|^2}{|u|^2} (|u|^2 - |v|^2) \geq \int_{\Omega} \frac{|
abla u|^2}{|u|^2 |v|^2} (|u|^2 - |v|^2)^2. \]
Making the sum of the two inequalities we get
\[ \frac{1}{\varepsilon^2} \int_{\Omega} |u|^2 + |v|^2 \geq \int_{\Omega} \frac{\nabla u}{|u|^2} (|v|^2 - |u|^2)^2. \]
Using the assumption (8.37), we conclude that $|u| = |v|$ in $\Omega$.

We now obtain from (8.39)
\[ \int_{\partial \Omega} (|u|^2 \Re(u^{-1} \partial_n u) - |v|^2 \Re(v^{-1} \partial_n v)) - \int_{\Omega} |v|^2 |\nabla w|^2 = 0. \]
Interchanging $u$ and $v$ and summing the two identities we conclude that
\[ \int_{\Omega} |v|^2 |\nabla \left( \frac{u}{v} \right) |^2 + \int_{\Omega} |u|^2 |\nabla \left( \frac{v}{u} \right) |^2 = 0. \]
Hence, $w$ is constant equal to 1 in $\Omega$.

As a first corollary of the previous Proposition, we have the :

**Theorem 8.7** Assume that $\Omega$ is a simply connected bounded domain of $\mathbb{C}$ and assume that $g : \partial \Omega \rightarrow S^1$ is a smooth map of degree 0. Then there exists $\varepsilon_0$ (only depending on $\Omega$ and $g$) such that for all $\varepsilon \in (0, \varepsilon_0)$, the functional
\[ E_{\varepsilon}(u) := \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_{\Omega} (1 - |u|^2)^2, \]
admits a unique minimizer among maps in $H^1(\Omega)$ which satisfy $u = g$ on $\partial \Omega$.

**Proof** : Since the boundary data $g$ has degree 0, it is easy to see that there exists a constant $c > 0$ only depending on $\Omega$ and $g$ such that
\[ \frac{1}{2} \int_{\Omega} |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_{\Omega} (1 - |u|^2)^2 \leq c, \]
for any minimizer. Arguing as in [10], one proves that all minimizers satisfy the lower bound
\[ |u| \geq 1/2, \]
for all $\varepsilon$ small enough, say $\varepsilon \in (0, \varepsilon_0)$. Still using the analysis of [10], we can prove that there exists a constant $c > 0$ such that $|\nabla u| \leq c$ provided $u$ is a minimizer of $E_{\varepsilon}$ and $\varepsilon \in (0, \varepsilon_0)$. Obviously, for $\varepsilon$ small enough the assumptions of Proposition 8.5 are fulfilled and the result follows.

Proposition 8.5 also yields the following result of M. Comte and P. Mironescu [21].

**Theorem 8.8** Assume that $\Omega$ be a bounded simply connected domain of $\mathbb{C}$ and assume that $g : \partial \Omega \rightarrow S^1$ is a smooth map of degree 0. For all $\alpha \in (0, 1)$, there exists $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$ the solution of
\[
\begin{cases}
\Delta u + u \frac{(1 - |u|^2)}{\varepsilon^2} = 0 & \text{in } \Omega \\
u = g & \text{on } \partial \Omega,
\end{cases}
\]
satisfying
\[ |u| \geq \alpha > 0 \quad \text{in} \quad \Omega, \]
is unique and minimizes \( E_\varepsilon \) among maps in \( H^1(\Omega) \) whose boundary data is given by \( g \).

**Proof:** It is proven in [10] that there exists \( c > 0 \) such that for all \( u \) is a critical point of \( E_\varepsilon \) and if \( |u| \geq 1/2 \), then \( |\nabla u| \leq c \). The result is again a consequence of Proposition 8.5.

### 8.6 Dealing with general nonlinearities

We explain how the previous analysis can be carried out for a large class of equations. In doing so, we show that our analysis is not completely specific to the Ginzburg-Landau equation.

#### 8.6.1 A pohozaev formula for general nonlinearities

In this section, we extend the Pohozaev identity we have obtained for solutions of the Ginzburg-Landau equation to critical points of any functional of the form

\[
E(u) := \int_\Omega |\nabla u|^2 - \int_\Omega F(|u|^2),
\]

(8.40)

where the function \( F \) is assumed to be smooth in \([0, \infty)\) and \( u \) is a complex valued function. Critical points of this functional are solution of the following Euler-Lagrange equation

\[
\Delta u + u f(|u|^2) = 0,
\]

(8.41)

where \( f \) denotes the derivative of \( F \). Observe that the Ginzburg-Landau equation corresponds to the choice

\[
F(t) = -\frac{1}{2\varepsilon^2} (1 - t)^2.
\]

Following closely the strategy of the proof of Theorem 8.3, we establish the following result:

**Theorem 8.9** Assume that \( u \) and \( v \) are two solutions of (8.41), which are defined in \( \Omega \) and have the same boundary data on \( \partial \Omega \). Further assume that 0 is a regular value of \( v \) and, away from the zero set of \( v \)

\[
v \wedge \nabla v \neq 0.
\]
Then the following identity holds:

\[
\int_\Omega \nabla \cdot X |v|^2 (F(|v|^2) - F(|u|^2) - f(|v|^2) (|v|^2 - |u|^2))
+ \int_\Omega \partial_X |v|^2 \frac{\partial_X |v|^2}{|v|^2} (f(|v|^2) - f(|u|^2) - f'(|v|^2) (|v|^2 - |u|^2))
+ \int_\Omega \frac{\partial_X |v|^2}{|v|^2} \left( F(|u|^2) - F(|v|^2) - f(|u|^2) (|u|^2 - |v|^2) \right)
+ 4 \int_\omega \frac{\partial_X |v|}{|v|} |\partial_X w|^2 = 2 \int_{\partial \Omega} \partial_X w \cdot \partial w - \int_{\partial \Omega} |\nabla w|^2 X \cdot \nu,
\]

(8.42)

where, by definition

\[
X := \frac{|v \wedge \nabla v|}{|v \wedge \nabla v|^2}.
\]

**Proof:** To begin with, observe that the vector field \(X\) defined in the statement of the result is a \(|v|-\text{conformal}\) vector field. In particular the result of Proposition 8.1 still holds.

The function

\[
w := \frac{u}{v},
\]

verifies the equation

\[
\Delta w + 2\nabla w \frac{\nabla w}{w} + w \left( f(|u|^2) - f(|v|^2) \right) = 0.
\]

Hence, Proposition 8.1 yields

\[
\partial_X w \cdot w \left( f(|v|^2) - f(|u|^2) \right) = \nabla \cdot (\partial_X w \cdot \nabla w - \frac{1}{2} |\nabla w|^2 X)
+ 2 \frac{\partial_X |v|}{|v|} |\partial_X w|^2.
\]

If we write

\[
\partial_X w \cdot w = \frac{1}{2} \partial_X |w|^2,
\]

it remains to compute

\[
\frac{1}{2} \partial_X \left( \frac{|u|^2}{|v|^2} \right) \left( f(|v|^2) - f(|u|^2) \right).
\]

For the sake of simplicity in the notations, we set

\[
U := |u|^2 \quad \text{and} \quad V := |v|^2.
\]
It is a simple exercise to check that the following identity holds

\[
\partial X \left( \frac{U}{V} \right) (f(U) - f(V)) = \text{div} \left( (F(U) - F(V)) - f(V)(U - V) \right) \frac{X}{V}
\]

\[+ \frac{1}{V} (F(U) - F(V) - f(V)(U - V)) \text{div} X\]

\[- \frac{1}{V} \partial_X V (f(U) - f(V) - f'(V)(U - V))\]

\[- \frac{1}{V^2} \partial_X V (F(V) - F(U) - f(U)(V - U)).\]

Hence, we conclude that

\[
\text{div} \left( 2 \partial_X w \cdot \nabla w - |\nabla w|^2 X \right) + \frac{4}{|X|^2} \frac{\partial_X |v|}{|v|} |\partial_X w|^2
\]

\[+ \text{div} \left( (F(|u|^2) - F(|v|^2)) - f(|v|^2)(|u|^2 - |v|^2) \right) \frac{X}{|v|^2} \]

\[- \frac{1}{|v|^2} \left( \text{div} X \right) (F(|u|^2) - F(|v|^2) - f(|v|^2)(|u|^2 - |v|^2))\]

\[- \frac{1}{|v|^2} \partial_X |v|^2 \left( f(|u|^2) - f(|v|^2) - f'(|v|^2)(|u|^2 - |v|^2)\right)\]

\[- \frac{1}{|v|^4} \partial_X |v|^2 \left( F(|v|^2) - F(|u|^2) - f(|u|^2)(|v|^2 - |u|^2)\right) = 0.\]

We now integrate this identity over \( \omega \subset \Omega \), where \( \omega \) does not contain any zero of \( v \) and let \( \omega \) tend to \( \Omega \) to find the desired formula. The details been identical to what we have already done in the special case of the Ginzburg-Landau equation, we omit them.

\[\square\]

**Remark 8.3** Observe that if we assume that both \( F \) and its derivative \( f \) are concave then

\[F(a) - F(b) - f(a)(a - b) \geq 0,\]

and

\[f(a) - f(b) - f'(a)(a - b) \geq 0.\]

**8.6.2 Uniqueness results for general nonlinearities**

The uniqueness results we have obtained in Theorem 8.4 and Theorem 8.5 for Ginzburg-Landau equation hold for more general nonlinearities. Recall that the Ginzburg-Landau equation corresponds to the choice

\[F(t) = \frac{1}{2e^2} (1 - t)^2 \quad \text{and} \quad f(t) = \frac{1}{e^2} (1 - t).\]

Using Theorem 8.9 and following the same strategy of proof, we obtain the counterpart of Theorem 8.4 for a wide class of nonlinearities.
Theorem 8.10 Assume that both $F$ and $f$ are concave, $f(0) > 0$ and $f(1) = 0$. Let $u$ be a solution of
\[ \Delta u + u f(|u|) = 0 \quad \text{in} \quad B_1 \]
\[ u = e^{i\theta} \quad \text{on} \quad \partial B_1. \]
Further assume that $u(0) = 0$ then $u$ is radially symmetric, namely that $u$ can be written as
\[ u = \rho e^{i\theta}, \]
where $\rho$ only depends on $r$.

Similarly we also obtain the counterpart of Theorem 8.5:

Theorem 8.11 Assume that both $F$ and $f$ are concave, $f(0) > 0$ and $f(1) = 0$. Let $u$ be a solution of
\[ \Delta u + u f(|u|^2) = 0, \]
in $\mathbb{C}$. Further assume that
\[ \int_{\mathbb{C}} F(|u|^2) < +\infty, \quad \text{and} \quad \deg \left( \frac{u}{|u|}, \infty \right) = 1. \]
Then, there exists $\tau_0 \in \mathbb{R}$ and $a \in \mathbb{C}$ such that
\[ e^{-i(\theta + \tau_0)} u(\cdot - a) \]
is a radial function.

8.6.3 More about the quantities involved in the Pohozaev identity
We conclude this Chapter by some remark concerning some quantities which appear in the Pohozaev formula.

Proposition 8.6 Let $v$ be a nondegenerate complex valued function defined in some open subset of $\mathbb{C}$. Assume that the vector field $X$ is defined by
\[ X := |v|^2 \frac{v \wedge \nabla^\perp v}{|v \wedge \nabla^\perp v|^2}. \]
Then
\[ \frac{\partial |v|}{|v|} = \frac{|X|^2}{|v|^2} \text{Area}(v). \tag{8.43} \]
Where the Area has been defined at the end of Chapter 7.

If we further assume that $\text{curl} \, X = 0$, then we have
\[ \text{div} \, X = 2 \frac{|\tilde{X}|}{|v|} k, \tag{8.44} \]
where $k$ is the curvature of the flow-lines of $\tilde{X}$, namely the solutions of
\[ \dot{\gamma}(t) = \tilde{X}(\gamma(t)). \tag{8.45} \]

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Proof: Locally, we may assume that
\[ v := |v| e^{i\phi}. \]
Then \( X = |\nabla \phi|^{-2} \nabla \perp \phi \) and hence
\[
\frac{\partial_X |v|}{|v|} = \frac{1}{|\nabla \phi|^2 |v|^2} (|v| \nabla |v| \nabla \perp \phi) = \frac{|X|^2}{|v|^2} \det(Dv),
\]
and this yields (8.43).

Assume that \( \gamma \) is a solution of
\[
\dot{\gamma} = \tilde{X}(\gamma).
\]
We identify \( \gamma \) with a point of \( \mathbb{R}^2 \times \{0\} \). In which case it is well known that the curvature \( k \) of \( \gamma \) is given by the third component of the vector
\[
\frac{\dot{\gamma} \times \ddot{\gamma}}{|\dot{\gamma}|^3},
\]
where \( \times \) denotes the cross product in \( \mathbb{R}^3 \). With little work, we obtain explicitly
\[
|\tilde{X}|^3 k = \tilde{X}_x \tilde{X}_y (\partial_y \tilde{X}^y - \partial_x \tilde{X}^x) + (\tilde{X}_x)^2 \partial_x \tilde{X}^y - (\tilde{X}_y)^2 \partial_y \tilde{X}^x.
\]
Since we have assumed that \( \text{curl} \tilde{X} = 0 \) we have \( \partial_x \tilde{X}^y = \partial_y \tilde{X}^x \). Hence we also have
\[
2 |\tilde{X}|^3 k = 2 \tilde{X}_x \tilde{X}_y (\partial_y \tilde{X}^y - \partial_x \tilde{X}^x) + ((\tilde{X}_x)^2 - (\tilde{X}_y)^2) (\partial_x \tilde{X}^y + \partial_y \tilde{X}^x).
\]
Finally, we use the fact that
\[
\tilde{X} = -\frac{X \perp}{|X|^2},
\]
which, together with Lemma 8.1, implies that
\[
\left( \partial_x \tilde{X}^x - \partial_y \tilde{X}^y \right) = -2 \tilde{X}_x \tilde{X}_y \text{div} X - ((\tilde{X}_y)^2 - (\tilde{X}_x)^2) \text{curl} X,
\]
and
\[
\left( \partial_y \tilde{X}^x + \partial_x \tilde{X}^y \right) = -\left( (\tilde{X}_y)^2 - (\tilde{X}_x)^2 \right) \text{div} X + 2 \tilde{X}_x \tilde{X}_y \text{curl} X.
\]
We conclude that
\[
2 |\tilde{X}|^3 k = |\tilde{X}|^4 \text{div} X,
\]
which is the desired result.

It is of interest to put in perspective the result of this last Proposition and the Pohozaev formula for general nonlinearities such which has been derived in Theorem 8.9, with the uniqueness results of Theorem 8.10 and Theorem 8.11. Indeed, it should be clear that global uniqueness results such as these deeply
rly on the fact that for the radially symmetric solution the Area density is positive and also that the curvature of the flow lines associated to the vector field $\tilde{X}$ (which is nothing but the gradient of the phase of the radially symmetric solution) is also positive.

Another key feature of these uniqueness theorems is that we do not assume that the other solution which is compared to the radially symmetric solution has only one zero at the origin but simply that the set of zeros of $u$ contains the origin.

Hence, we can more generally state the:

**Theorem 8.12** Assume that both $F$ and $f$ are concave. Let $u$ and $v$ be two solutions of
\[
\begin{align*}
\Delta u + u f(|u|^2) &= 0 \quad \text{in} \quad \Omega \\
u &= g \quad \text{on} \quad \partial \Omega.
\end{align*}
\]
Assume that $v \wedge \nabla v \neq 0$ away from the zero set of $v$, that the density area of $v$ is positive in $\Omega$ and that the curvature of the flow lines of the gradient of the phase of $v$ is positive. If in addition
\[v^{-1}(\{0\}) \subset u^{-1}(\{0\}),\]
then $u = v$. 

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Chapter 9

The rôle of the zeros in the uniqueness question

Using the results of Chapter 8 and Chapter 9, we prove that, under some natural upper bound on the energy, two solutions of the Ginzburg-Landau equation which have the same boundary data and the same zeros are necessarily identical for small values of the parameter $\varepsilon$.

9.1 The zero set of solutions of Ginzburg-Landau equations

In this part we study the zero set of a solution of

$$
\begin{aligned}
\Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) &= 0 \quad \text{in} \quad \Omega \\
\frac{u}{\varepsilon} &= g \quad \text{on} \quad \partial \Omega,
\end{aligned}
$$

(9.1)

where $g$ takes its values into $S^1$. More precisely, we prove that, provided the energy corresponding to $u$ is appropriately bounded, then $u$ has a finite number of isolated zeros.

**Proposition 9.1** Let $c_0 > 0$ be given. Assume that, for all $\varepsilon \in (0, \varepsilon_0)$, the function $u_\varepsilon$ is a solution of (9.1) which satisfies

$$
E_\varepsilon(u) := \frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_\Omega (1 - |u|^2)^2 \leq c_0 \log \frac{1}{\varepsilon}.
$$

(9.2)

Further assume that there exist $\Sigma := \{a_1, \ldots, a_N\} \subset \Omega$ such that, as $\varepsilon$ tends to 0, the sequence $u_\varepsilon$ converges in $C^{2,\alpha}_{\text{loc}}(\Omega \setminus \Sigma)$ to the $S^1$ valued harmonic map

$$
u_* := \prod_{j=1}^N \left( \frac{z - a_j}{|z - a_j|} \right)^{d_j} e^{i\phi},
$$

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where the function $\phi$ is harmonic in $\Omega$ and where $d_j = \pm 1$. Then, for $\varepsilon$ small enough, there exists $N$ distinct points $a_1(\varepsilon), \ldots, a_N(\varepsilon)$ such that:

1. The zero set of $u_\varepsilon$ is given by \{${a_1(\varepsilon), \ldots, a_N(\varepsilon)}$\}.
2. For all $j = 1, \ldots, N$, $\lim_{\varepsilon \to 0} |a_j(\varepsilon) - a_j| = 0$.
3. For all $j = 1, \ldots, N$, the degree of $u_\varepsilon$ at $a_j(\varepsilon)$ is equal to $d_j$.

In addition, for all $j = 1, \ldots, N$, if we define the sequence of rescaled functions

$$\tilde{u}_\varepsilon(z) := u_\varepsilon(\varepsilon z + a_j(\varepsilon)).$$

Then, there exists $\tau_j \in \mathbb{R}$ such that, up to a subsequence, for all $\gamma > 0$ and for all $k \in \mathbb{N}$, the sequence of rescaled functions $\tilde{u}_\varepsilon$ converges as $\varepsilon$ tends to $0$ to $Se^{i(\theta_j + \tau_j)}$ in $C^k_{\text{loc}}(B_\gamma)$. Where $Se^{i\theta}$ is the radially symmetric solution of the Ginzburg-Landau equation, which has been defined in Theorem 3.1.

The proof of this result is decomposed in several steps. To begin with let us recall the following Pohozaev formula which was derived in [11].

**Lemma 9.1** Assume that $u$ is a solution of

$$\Delta u + \frac{u}{\varepsilon^2} (1 - |u|^2) = 0,$$

and define $X := (x, y)$. Then

$$\text{div} \left( \partial_X u \cdot \nabla u - \frac{1}{2} |\nabla u|^2 X - \frac{1}{4\varepsilon^2} (1 - |u|^2)^2 X \right) + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 = 0.$$

**Proof:** Simply take the scalar product of the equation satisfied by $u$ with $\partial_X u$ and proceed exactly as in the previous Chapter. $\square$

Now, we prove the following $\eta$-compactness Lemma:

**Lemma 9.2** [\eta-compactness Lemma] Let $c_1 > 1$ be given. There exists $\eta > 0$ such that if $u$ is a solution of

$$\Delta u + \frac{u}{\varepsilon^2} (1 - |u|^2) = 0,$$

which satisfies

$$||\nabla u||_{L^\infty} \leq \frac{c_1}{4\varepsilon}, \quad (9.3)$$

and if there exists $z \in \mathbb{C}$ and $R > 2\varepsilon$ for which

$$\int_{B_R(z)} |\nabla u|^2 + \frac{1}{2\varepsilon^2} \int_{B_R(z)} (1 - |u|^2)^2 \leq \eta \log \frac{R}{2\varepsilon}, \quad (9.4)$$

then

$$|u| \geq \frac{1}{2} \quad \text{in} \quad B_{R/2}(z).$$
Proof: We choose $\eta = \frac{\pi}{16 c_1^2}$ and argue by contradiction. Assume that, for some $z' \in B_{R/2}(z)$, we have $|u|(z') < 1/2$. For the sake of simplicity in the notations, let us assume that $z' = 0$. Thanks to (9.3) we see that

$$|u| < \frac{3}{4}, \quad \text{in} \quad B_{\varepsilon/c_1}.$$ 

Hence, we have

$$\frac{1}{\varepsilon^2} \int_{B_{\varepsilon/c_1}} (1 - |u|^2)^2 > \frac{\pi}{16 c_1^2}. \quad (9.5)$$

Now, we integrate the Pohozaev formula which was derived in the previous Lemma, over the ball of radius $r$, we find

$$\frac{1}{\varepsilon^2} \int_{B_r} (1 - |u|^2)^2 = r \int_{\partial B_r} \left( \frac{1}{r^2} |\partial_r u|^2 - |\partial_r u|^2 + \frac{1}{2r^2} (1 - |u|^2)^2 \right) \leq r \int_{\partial B_r} |\nabla u|^2 + \frac{r}{2\varepsilon^2} \int_{\partial B_r} (1 - |u|^2)^2.$$ 

Integrating this inequality over $r \in (\varepsilon, R/2)$, we find, using (9.4),

$$\int_{r=\varepsilon}^{R/2} \frac{1}{r} \left( \int_{B_r} (1 - |u|^2)^2 \right) dr \leq \eta \log \left( \frac{R}{2\varepsilon} \right).$$

Thanks to the mean value formula, we can conclude that there exists $r_0 \in [\varepsilon, R/2]$ such that

$$\frac{1}{\varepsilon^2} \int_{B_{r_0}} (1 - |u|^2)^2 \leq \eta,$$

which is not possible thanks to our choice of $\eta$ and (9.5). Hence $|u|(z') \geq 1/2$. $\square$

Remark 9.1 There exists a corresponding version of the $\eta$-compactness Lemma for the Ginzburg-Landau equation in higher dimensions. However the proof is not as straightforward as the one in dimension 2. We refer to [82], [46] and [47] for further details. Notice also that there is a corresponding version of the $\eta$-compactness Lemma up to the boundary $\partial \Omega$ for any smooth $S^1$ valued Dirichlet boundary condition independent of $\varepsilon$. We refer to [46] for a proof of such a result for minimizers in dimension greater or equal to 3.

Using the previous Lemma, we obtain the following zero-set covering Lemma. Observe that, at this stage, the results do not depend on the fact that the degrees of the limiting vortices are $\pm 1$.

Lemma 9.3 Under the assumptions of Proposition 9.1, there exist $\tilde{\varepsilon}_0 > 0$, $n \in \mathbb{N}$ and $\lambda \geq 1$ only depending on $\Omega$, $g$ and $c_0$ and, for all $\varepsilon \in (0, \tilde{\varepsilon}_0)$, there exist $n_\varepsilon$ points $z_1, \ldots, z_{n_\varepsilon} \in \Omega$ such that:

1. $\{ z \in \Omega : |u| < 1/2 \} \cap B_{\lambda \varepsilon}(z_j) \neq \emptyset.$
2. $\{z \in \Omega : |u| < 1/2\} \subset \bigcup_{j=1}^{n_{\varepsilon}} B_{\lambda\varepsilon}(z_j)$.

3. For all $\varepsilon \in (0, \varepsilon_0)$, $n_{\varepsilon} \leq n$.

4. For all $k \neq l$, $B_{2\lambda\varepsilon}(z_k) \cap B_{2\lambda\varepsilon}(z_l) = \emptyset$.

**Proof:** The proof of this Lemma combines arguments from [10], [11], [12], [96] and [21]. In order to simplify the presentation we work away from the boundary of $\Omega$. However, similar arguments can be carried out up to $\partial\Omega$.

**Step 1.** First of all, it follows from [10] that
\[ \|\nabla u\|_{L^{\infty}} \leq \frac{c_1}{4\varepsilon}, \] (9.6)
where the constant $c_1$ only depends on $\Omega$, $g$ and $c_0$.

**Step 2.** We have already established the following Pohozaev identity in Lemma 9.1
\[ \frac{1}{\varepsilon^2} \int_{B_{\varepsilon}(z)} (1 - |u|^2)^2 = r \int_{\partial B_{r}(z)} \left( \frac{1}{r^2} |\partial_{\nu}u|^2 - |\partial_r u|^2 \right) + \frac{r}{2\varepsilon^2} \int_{\partial B_{r}(z)} (1 - |u|^2)^2. \] (9.7)

The key point, which was introduced in [12] and independently in [96], is to apply the above Pohozaev formula on ball of size $\varepsilon^\alpha$, for some $\alpha \in (0, 1)$.

Assume that $\alpha \in (0, 1)$ and $\beta \in (0, \alpha)$ are fixed. As we have already done in the proof of Lemma 9.2, we use the bound of energy (9.2) together with the above Pohozaev identity (9.7), to obtain
\[ \frac{1}{\varepsilon^2} \int_{\varepsilon^\alpha}^{r} \frac{1}{r} \left( \int_{B_r(z)} (1 - |u|^2)^2 \right) dr \leq c_0 \log \frac{1}{\varepsilon}, \]
and using the mean value formula we deduce the existence of $r \in (\varepsilon^\alpha, \varepsilon^\beta)$ such that
\[ \frac{1}{\varepsilon^2} \int_{B_r(z)} (1 - |u|^2)^2 \leq \frac{c_0}{\alpha - \beta}. \] (9.8)

**Step 3.** We claim that there exists a positive integer $n_\alpha$ which does not depend on $\varepsilon$ such that the set
\[ \mathcal{Z} := \{z' \in B_{\varepsilon^\alpha}(z) : |u(z')| < 1/2\}, \]
is included in at most $n_\alpha$ balls of radius $\varepsilon$. The proof of this claim is close to what we have already done in the proof of Lemma 9.2. Let $z' \in B_{\varepsilon^\alpha}(z)$ such that $|u(z')| < 1/2$, then using (9.6), we see that $|u| \leq 3/4$ in the ball of center $z$ and radius $\varepsilon/c_1$. Hence we have the lower bound
\[ \frac{1}{\varepsilon^2} \int_{B_{\varepsilon/c_1}(z')} (1 - |u|^2)^2 \geq \frac{\pi}{16 c_1^2}. \] (9.9)
However, we have proved that
\[
\frac{1}{\varepsilon^2} \int_{B_{\varepsilon^{\alpha}}} (1 - |u|^2)^2 \leq \frac{c_0}{\alpha - \beta}.
\]
An easy covering argument yields that there are at most \( n_{\alpha} := \frac{4^3 \varepsilon^2 c_0}{(\alpha - \beta) \pi} \) balls of radius \( \varepsilon \) which are needed to cover \( \mathcal{Z} \).

Granted the \( \eta \)-compactness Lemma and the global bound of the energy (9.2), we conclude that, for any \( \alpha \in (0, 1) \), the set of points where \( |u| < 1/2 \) can be covered by at most \( N_{\alpha} \) balls of radius \( \varepsilon^{\alpha} \), where \( N_{\alpha} \) only depend on \( \Omega, g \) and \( c_0 \) but does not depend on \( \varepsilon \).

Step 4. It is not difficult to see that we can choose \( \lambda \geq 1 \) in such a way that, if we use balls of radius \( \lambda \varepsilon \) instead of balls of radius \( \varepsilon \) to cover \( \mathcal{Z} \),
\[
\forall l \neq k \quad B_{2\lambda^{\varepsilon}}(z_l) \cap B_{2\lambda^{\varepsilon}}(z_k) = \emptyset. \tag{9.11}
\]
Indeed if (9.11) is not satisfied for \( \lambda = 1 \), there exists \( z_l \neq z_k \) such that
\[
B_{2\varepsilon}(z_l) \cap B_{2\varepsilon}(z_k) \neq \emptyset.
\]
In which case we take \( \lambda = 4 \) and eliminate \( z_k \) from the list of points. Then we argue by induction until (9.11) is satisfied. Since the number of points is bounded by \( n_{\alpha} \), the process end up with some \( \lambda \leq 4^{n_{\alpha}} \).

To conclude, we have obtained the existence of \( n \in \mathbb{N} \) independent of \( \varepsilon \) and a set of points \( z_1, \ldots, z_n \) in \( \Omega \), such that \( n_{\varepsilon} \leq n \) and
\[
\{ z \in \Omega : |u|(z) < 1/2 \} \subset \bigcup_{l=1}^{n\varepsilon} B_{\lambda^{\varepsilon}}(z_l).
\]
Which together with (9.11) completes the proof of the Lemma.

**Remark 9.2** For further use, it is useful to observe that, increasing \( \lambda \) if necessary, we may assume that in addition to the properties of Lemma 9.3, we can assume that
\[
S(r) \geq 3/4,
\]
for all \( r \geq \lambda \), where \( S \) is the function which has been defined in Theorem 3.1.

Before we proceed to the proof of Proposition 9.1, we need a last Lemma

**Lemma 9.4** Under the hypothesis of Proposition 9.1, if \( \sigma > 0 \) is fixed such that, for all \( j = 1, \ldots, N \), \( B_{2\sigma}(a_j) \subset \Omega \) and \( B_{2\sigma}(a_j) \cap B_{2\sigma}(a_k) = \emptyset \), then
\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{B_{\varepsilon}(a_j)} (1 - |u|^2)^2 \leq 2\pi + O(\sigma^2). \tag{9.12}
\]
Proof: It follows from Theorem X.3 in [11] that, for all $\sigma > 0$,

$$||u| - 1| \leq c_\sigma \varepsilon^2,$$

(9.13)
on $\partial B_\sigma(a_j)$. Moreover, we have already seen that the fact that $(a_j)_{j=1,\ldots,N}$ is a critical point of the renormalized energy $W_g$ can be characterized by the fact that the $S^1$ valued harmonic map

$$u_* := \prod_{j=1}^{N} \left( \frac{z - a_j}{|z - a_j|} \right)^{d_j} e^{i\phi},$$
can be written in the neighborhood of each $a_j$ in the following way

$$u_* = \left( \frac{z - a_j}{|z - a_j|} \right)^{d_j} e^{iH_j},$$
with

$$\nabla H_j(a_j) = 0.$$  
(9.14)
Here $H_j$ is a harmonic function in a neighborhood of $a_j$ (see [11] Corollary VIII.1).

Now, Pohozaev formula (9.7), yields

$$\frac{1}{\varepsilon^2} \int_{B_\sigma(z_l)} (1 - |u|^2)^2 = r \int_{\partial B_\sigma(z_l)} \left( |\partial_r u|^2 - \frac{1}{r^2} |\partial_\theta u|^2 \right)$$
$$+ \frac{r}{2\varepsilon^2} \int_{\partial B_\sigma(z_l)} (1 - |u|^2)^2.$$

Using (9.13) together with (9.14), we may pass to the limit as $\varepsilon$ tends to 0 in this identity and obtain

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{B_\sigma(z_l)} (1 - |u|^2)^2 \leq 2\pi d_j^2 + O(\sigma^2).$$
(9.15)
Finally, we use the fact that $|d_j| = 1$ to conclude. $\square$

We are now in a position to prove Proposition 9.1.

Proof of proposition 9.1 : We keep the notations of the proof of Lemma 9.3. A priori many $z_l$ may converge, as $\varepsilon$ tends to 0 to the same vortex $a_j$. Let us denote by $Z_j$ the set of points of $\{z_1, \ldots, z_{n_\varepsilon}\}$ which converge to $a_j$. We shall prove that, provided $\varepsilon$ is chosen small enough, the set $Z_j$ contains exactly one point, hence we will have $n_\varepsilon = N$. The other statements of the Proposition will then follow immediately.

Step 1. We claim that

$$\sup_{j=1,\ldots,n_\varepsilon} \left( \sup_{z_{j,k},z_{j,l} \in Z_j} \frac{|z_{j,k} - z_{j,l}|}{\varepsilon} \right),$$
(9.16)
stays bounded independently of $\varepsilon$. To prove this, we argue by contradiction and assume that the claim is not true. Then, we may assume that (up to a subsequence) there exists $j, k$ and $l$ such that

$$\lim_{\varepsilon \to 0} \frac{|z_{j,k} - z_{j,l}|}{\varepsilon} = +\infty.$$ 

We set

$$\tilde{u}_\varepsilon := u_\varepsilon(\varepsilon z + z_{j,k}), \quad \text{and} \quad r_\varepsilon := \frac{|z_{j,k} - z_{j,l}|}{2\varepsilon}.$$ 

Thanks to (9.8), we see that

$$\Delta \tilde{u}_\varepsilon + \tilde{u}_\varepsilon (1 - |\tilde{u}_\varepsilon|^2) = 0,$$

in $B_{r_\varepsilon}$ and

$$\int_{B_{r_\varepsilon}} (1 - |\tilde{u}_\varepsilon|^2)^2 \leq \frac{c_0}{\alpha - \beta}.$$ 

Extracting some subsequence, if this is necessary, we may assume that the sequence $\tilde{u}_\varepsilon$ converges in $C^k$ topology, for any $k \in \mathbb{N}$ and on any compact subset of $\mathbb{C}$, to $\hat{u}$ solution of

$$\Delta \hat{u} + \hat{u} (1 - |\hat{u}|^2) = 0 \quad \text{ in } \mathbb{C},$$

$$\int_{\mathbb{C}} (1 - |\hat{u}|^2)^2 < +\infty.$$ 

Solutions of (9.17) are well understood thanks to the work of H. Brezis, F. Merle and T. Riviè re [15], and also to the work of I. Shafrir [90], where it is proven that $|\hat{u}|$ tends to 1 at $\infty$ and also that

$$\int_{\mathbb{C}} (1 - |\hat{u}|^2)^2 = 2 \pi \deg(\hat{u}, \infty)^2.$$ 

Observe that this identity implies that $\hat{u}$ is constant if its degree at infinity is 0. However, since $\inf_{B_{\lambda\varepsilon}} \tilde{u}_\varepsilon < 1/2$, this implies that $\inf_{B_{\lambda}} |\hat{u}| \leq 1/2$. This implies that $\hat{u}$ is not constant and hence, we have

$$\int_{\mathbb{C}} (1 - |\hat{u}|^2)^2 \geq 2\pi.$$ 

Therefore, we have

$$\frac{1}{\varepsilon^2} \int_{B_{r_\varepsilon}(z_{j,k})} (1 - |u_\varepsilon|^2)^2 \geq 2\pi + o_\varepsilon(1).$$

Similarly, we also have

$$\frac{1}{\varepsilon^2} \int_{B_{r_\varepsilon}(z_{j,l})} (1 - |u_\varepsilon|^2)^2 \geq 2\pi + o_\varepsilon(1).$$
Combining this two inequalities, we get that

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \int_{B_\varepsilon(a_j)} (1 - |u_\varepsilon|^2)^2 \geq 4\pi,
\]

which clearly contradicts the result of Lemma 9.4. This ends the proof of the claim.

**Step 2.** Thanks to the claim proved in Step 1, we can say that there exists a constant \( c > 0 \) such that for any two points \( z_{j,k}, z_{j,l} \in Z_j \), we have

\[|z_{j,k} - z_{j,l}| \leq c\varepsilon.\]

As already mentioned in Remark 9.2, the constant \( \lambda \) which appears in the statement of Lemma 9.3 can be chosen in such a way that

\[S(r) \geq 3/4,\]

for all \( r \geq \lambda \), where the function \( S \) has been defined in Theorem 3.1.

We claim that

\[|u_\varepsilon| \geq 1/2,\]

in \( B_{\varepsilon}(z_{j,k}) \setminus B_{\lambda\varepsilon}(z_{j,k}) \).

We argue by contradiction and assume that there exists a sequence \( u_\varepsilon \) satisfying all the above hypothesis for which

\[\inf_{B_{\varepsilon}(z_{j,k}) \setminus B_{\lambda\varepsilon}(z_{j,k})} |u_\varepsilon| < 1/2,\]

We set

\[\tilde{u}_\varepsilon := u_\varepsilon(\varepsilon z + z_{j,k}).\]

Arguing as in Step 1, we see that, up to a subsequence, the sequence \( \tilde{u}_\varepsilon \) converges to \( \hat{u} \) solution of

\[\Delta \hat{u} + \hat{u} (1 - |\hat{u}|^2) = 0,\]

in \( \mathbb{C} \), and thanks to (9.8), we also have

\[\int_{\mathbb{C}} (1 - |\hat{u}|^2)^2 \leq \frac{c_0}{\alpha - \beta^2}.
\]

Moreover, by construction \( \hat{u}(0) = 0 \) and

\[\deg(\hat{u}, \infty) = \lim_{\varepsilon \to 0} \deg \left( \frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_\varepsilon(a_j) \right) = d_j.\]

We can then use the uniqueness result we have already proved in Theorem 8.5 and deduce that \( \hat{u} \) is given by \( Se^{i(\theta + \tau_j)} \) for some \( \tau_j \in \mathbb{R} \), where \( Se^{i\theta} \) is the radially symmetric solution defined in Theorem 3.1. In particular, we have

\[|\hat{u}| \geq 3/4,\]

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in $\mathbb{C} \setminus B_\lambda$. However, by assumption
\[
\inf_{B_\varepsilon(z_j,k)\setminus B_\lambda(z_j,k)} |u_\varepsilon| < 1/2,
\]
which implies that
\[
\inf_{B_\varepsilon\setminus B_\lambda} |\hat{u}| \leq 1/2,
\]
which is the desired contradiction.

In particular, thanks to the above claim, we can state that the set $Z_j$ reduces to a point.

Step 3. It follows from Step 3 that $n_\varepsilon = N$ as claimed. Moreover, we have also proved that near $a_j$ the solution $u_\varepsilon$ has a unique zero $a_j(\varepsilon)$ and necessarily the degree of $u_\varepsilon/|u_\varepsilon|$ at $a_j(\varepsilon)$ is given by the degree of $u_\ast$ at $a_j$. Hence 1, 2 and 3 in Proposition 9.1 are proved. The last statement of the Proposition also follows from Step 2.

**9.2 A uniqueness result**

In all this section, we will assume that the following assumptions hold:

1. $(a_1, \ldots, a_N)$ is a critical point of the renormalized energy $W_g$, for $d_j = \pm 1$.

2. For all $\varepsilon \in (0, \varepsilon_0)$, $u_\varepsilon$ and $v_\varepsilon$ are solutions of the Ginzburg-Landau equation
\[
\begin{cases}
\Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2) &= 0 \quad \text{in } \Omega \\
u &= g \quad \text{on } \partial \Omega.
\end{cases}
\]

3. There exists $c_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$
\[
\frac{1}{2} \int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2} \int_\Omega (1 - |u|^2)^2 < c_0 \log \frac{1}{\varepsilon},
\]
holds for $u = u_\varepsilon$ and $u = v_\varepsilon$.

4. As $\varepsilon$ tends to 0, both sequences $u_\varepsilon$ and $v_\varepsilon$ converge to $u_\ast$, the $S^1$ valued harmonic map associated to $a_1, \ldots, a_N$, which is explicitly given by
\[
u_\ast := \prod_{j=1}^N \left(\frac{z - a_j}{|z - a_j|}\right)^{d_j} e^{i\phi}.
\]

Under these assumptions, we prove the following Theorem which is one of the main achievement of the two previous Chapters.
Theorem 9.1 Assume that the above assumptions (1)-(4) hold. Further assume that
\[
u^{-1}(\{0\}) = v^{-1}(\{0\}).
\]
Then, there exists \(\tilde{\varepsilon}_0 > 0\), such that, for any \(\varepsilon \in (0, \tilde{\varepsilon}_0)\), we have
\[
\nu = v.
\]

It will be convenient to denote by \(\{a_1(\varepsilon), \ldots, a_N(\varepsilon)\}\) the common zero set of \(\nu\) and \(v\). We will assume that the constant \(\sigma > 0\) is chosen in such a way that, for all \(\varepsilon \in (0, \varepsilon_0)\)
\[
B_{2\sigma}(a_j(\varepsilon)) \subset \Omega,
\]
and, for all \(j \neq k\)
\[
B_{2\sigma}(a_j(\varepsilon)) \cap B_{2\sigma}(a_k(\varepsilon)) = \emptyset.
\]
Finally, for all \(r \in (0, \sigma)\), we set
\[
\tilde{\Omega}_r := \Omega \cup \bigcup_{j=1}^{N} B_r(a_j(\varepsilon)).
\]

Before we proceed to the proof of Theorem 9.1, we will need pointwise estimates on quantities depending on \(\nu\) and \(v\). This is the content of the following section.

9.2.1 Preliminary results

This Lemma is concerned with universal estimates for \(\nu\) and \(v\).

Lemma 9.5 Assume that the above assumptions (1)-(4) hold. Then, for all \(k \in \mathbb{N}\), there exists \(c_k > 0\) such that for all \(k \geq 0\) and for all \(\varepsilon \in (0, \varepsilon_0)\)
\[
|\nabla^k(1 - |\nu|) + |\nabla^k(1 - |v|)| | \leq c_k \varepsilon^2 r^{-2-k}
\]
\[
|\nabla^k \nu| + |\nabla^k v| \leq c_k r^{-k},
\]
where \(r\) denotes the distance function to the zero set of \(v\).

Proof: The proof of (9.19) was first established for minimizers of the Ginzburg-Landau functional in [21] and was then extended to critical points in [22] (see Theorem 1). As explained in [22], the main step consists in proving that, for all \(j = 1, \ldots, N\) and for all \(\varepsilon \in (0, \varepsilon_0)\)
\[
\int_{B_{2\sigma}(a_j(\varepsilon)) \setminus B_{r}(a_j(\varepsilon))} |\nabla\psi_j|^2 \leq C,
\]
for some \(C\) independent of \(\varepsilon\), where \(\psi_j\) is the “excess of phase” defined by the identity
\[
e^{i\psi_j} := \left( \frac{z - a_j(\varepsilon)}{|z - a_j(\varepsilon)|} \right)^{-d_j} |v| \frac{v}{|v|},
\]
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in $B_{\sigma}(a_j(\varepsilon))$ and where $a_1(\varepsilon), \ldots, a_N(\varepsilon)$ are the zeros of $v$.

The proof of this later fact makes use of ideas from [15] where a similar estimate is proved for the “excess of phase” but on all of $\mathbb{C}$. The result for arbitrary $k$ follows by induction. We refer to the step B.6 of the proof of Theorem 1 in [10] for the details. □

In the next Lemma, we collect some information about the $|v_\varepsilon|$-conformal vector field related to $v_\varepsilon$.

**Lemma 9.6** Assume that the above assumptions (1)-(4) hold. Denote by $X$ the vector field

$$X := |v_\varepsilon|^2 \frac{v_\varepsilon \wedge \nabla^\perp v_\varepsilon}{|v_\varepsilon| \wedge \nabla^\perp v_\varepsilon|^2}.$$  

Then for all $\alpha > 0$, there exist $\bar{\varepsilon}_0 > 0$, $c > 0$ and $\sigma_0 > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}_0)$, we have

$$|X| \leq c r \quad \text{and} \quad d_j \text{div} X \geq 2 - \alpha, \quad (9.20)$$

in each $B_{\sigma_0}(a_j(\varepsilon))$, where $r$ denotes the distance to $\{a_1(\varepsilon), \ldots, a_N(\varepsilon)\}$, the zero set of $v_\varepsilon$ and

$$d_j \frac{\partial X|v_\varepsilon|}{|v_\varepsilon|} \geq \frac{1}{c}, \quad (9.21)$$

in each $B_{2\varepsilon}(a_j(\varepsilon))$.

Moreover, for all $j = 1, \ldots, N$ and for all $\gamma > 0$, there exists $c > 0$, independent of $\varepsilon \in (0, \bar{\varepsilon}_0)$, such that

$$d_j \frac{\partial X|v_\varepsilon|}{|v_\varepsilon|} \geq c, \quad (9.22)$$

in each $B_{\gamma\varepsilon}(a_j(\varepsilon))$.

**Proof:** Take a point $a_j(\varepsilon)$ and assume $d_j = +1$ (the case where $d_j = -1$ can be treated similarly). To keep the notations short, we will omit the $\varepsilon$ indices.

**Step 1.** From the arguments in the proof of Lemma 9.3, we know that $\tilde{v}_\varepsilon := v(\varepsilon z + a_j(\varepsilon))$ converges in $C^k$ norm, on any bounded set of $\mathbb{C}$, to the radially symmetric solution $S e^{i(\theta + \tau_j)}$ defined in Theorem 3.1.

Thus it is not difficult to see that, if we denote

$$\frac{\tilde{v}}{|\tilde{v}|} \wedge \nabla^\perp \frac{\tilde{v}}{|\tilde{v}|} = \nabla^\perp \theta + \nabla^\perp \omega, \quad (9.23)$$

we have, for any fixed $c > 0$

$$\lim_{\varepsilon \to 0} (||r^2 \nabla^2 \omega||_{L^\infty(B_{c\varepsilon})} + ||r \nabla \omega||_{L^\infty(B_{c\varepsilon})}) = 0. \quad (9.24)$$

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We set
\[ \tilde{X} := \frac{v \wedge \nabla \tilde{v}}{|v \wedge \nabla \tilde{v}|^2} = \frac{1}{\varepsilon} X (\varepsilon z + a_j(\varepsilon)). \]  
(9.25)

We have, on one hand,
\[ \left| \tilde{X} - \frac{\nabla \tilde{v}}{|\nabla \tilde{v}|^2} \right| \leq c |r^2 \nabla \omega|, \]  
(9.26)

and on the other hand
\[ \left| \nabla \left( \tilde{X} - \frac{\nabla \tilde{v}}{|\nabla \tilde{v}|^2} \right) \right| \leq c \left( |r^2 \nabla^2 \omega| + |r \nabla \omega| \right), \]  
(9.27)

where we have used the fact that \(|r^2 \nabla^2 \omega| + |r \nabla \omega|\) is uniformly bounded, independently of \(\varepsilon\). Observe that
\[ \partial_x \left( \frac{\partial_y \theta}{|\nabla \theta|^2} \right) - \partial_y \left( \frac{\partial_x \theta}{|\nabla \theta|^2} \right) = 2. \]  
(9.28)

Combining (9.23)-(9.28), we establish (9.20) and (9.22) in \(B_{c\varepsilon}(a_j(\varepsilon))\) for any fixed \(c\).

**Step 2.** We claim that the first inequality which appears in (9.20) still holds in \(B_{\sigma}(a_j(\varepsilon)) \setminus B_{c\varepsilon}(a_j(\varepsilon))\). We argue by contradiction and assume that the result is not true. There would exist a sequence \(z_{\varepsilon}\) such that \(|z_{\varepsilon}|^{-1} |X(z_{\varepsilon})|\) tends to \(+\infty\). Since we have already proved that (9.20) holds on any ball of size \(c \varepsilon\), we can assume that
\[ \lim_{\varepsilon \to 0} \frac{r_{\varepsilon}}{\varepsilon} = +\infty. \]

Furthermore, since \(v_{\varepsilon}\) converges to \(u_*\) away from the \(a_j\), this implies that \(\lim_{\varepsilon \to 0} |z_{\varepsilon}| = 0\). We set \(r_{\varepsilon} := |z_{\varepsilon}|\) and
\[ \tilde{v}_{\varepsilon} := v(r_{\varepsilon} z + a_j(\varepsilon)). \]

Obviously \(\tilde{v}_{\varepsilon}\) verifies
\[ \Delta \tilde{v}_{\varepsilon} + \left( \frac{r_{\varepsilon}}{\varepsilon} \right)^2 \tilde{v}_{\varepsilon} (1 - |\tilde{v}_{\varepsilon}|^2) = 0, \]  
(9.29)
in \(B_{\sigma/r_{\varepsilon}}\). In particular \(\Delta \tilde{v}_{\varepsilon}\) and \(\tilde{v}_{\varepsilon}\) are collinear. Moreover, we know from Lemma 9.5 that
\[ |\nabla^k \tilde{v}_{\varepsilon}| \leq c_k |z|^{-k}. \]

In particular
\[ |\tilde{v}_{\varepsilon}| (1 - |\tilde{v}_{\varepsilon}|^2) \leq c \left( \frac{\varepsilon}{r_{\varepsilon}} \right)^2 |z|^{-2}. \]  
(9.30)

Up to a subsequence, we may then pass to the limit as \(\varepsilon\) tends to 0 and obtain a map \(\tilde{v}\), defined in \(\mathbb{C} \setminus \{0\}\) such that \(\Delta \tilde{v}\) and \(\tilde{v}\) are collinear. Moreover, it follows from (9.30) that \(\tilde{v}\) is either identically 0 or \(S^1\) valued. However, observe that
|\tilde{v}_\varepsilon| is bounded from below by 1/2 outside $B_\lambda$. Hence $\tilde{v}$ cannot be identically equal to 0 and we can conclude that $\tilde{v}$ is a $S^1$ valued harmonic map defined in $\mathbb{C} \setminus \{0\}$.

We also have an additional information: since the degree of $v_\varepsilon$ at $a_j(\varepsilon)$ is $+1$, we can state that the degree of $\tilde{v}$ is also 1. Hence

$$\tilde{v} := e^{i(\theta + \tau)}.$$  

However,

$$\tilde{X}_\varepsilon := \frac{1}{r_\varepsilon} X(r_\varepsilon z + a_j(\varepsilon)),$$

also converges to

$$\tilde{X} := \frac{\nabla \perp \theta}{|\nabla \theta|^2},$$

which contradicts the fact that $|r_\varepsilon^{-1} X(z_\varepsilon)| \to +\infty$. This ends the proof of the first inequality in (9.20).

Step 3. We prove now that the second inequality in (9.20) also holds in $B_{\sigma_0}$ provided $\sigma_0$ is fixed small enough. To begin with, let us denote

$$X_0 := \frac{u_* \wedge \nabla \perp u_*}{|u_* \wedge \nabla \perp u_*|^2}.$$

Then, we may choose $\sigma_0 > 0$ in such way that

$$\text{div} X_0 > 2 - \alpha,$$

in each $B_{\sigma_0}(a_j)$. Indeed, if $(r, \theta)$ are radial coordinates about $a_j$, we may write $u_* := e^{i(\theta + H_j)}$ near $a_j$ where $H_j$ is harmonic and $\nabla H_j(a_j) = 0$. Hence

$$X_0 = \frac{\nabla \perp \theta}{|\nabla \theta|^2} + O(r^2),$$

near each $a_j$. The existence of $\sigma_0$ is then immediate.

Again, we argue by contradiction and assume that for some $\alpha > 0$, there exists a sequence $z_\varepsilon \in B_{\sigma_0}(a_j(\varepsilon))$ such that $\text{div} X(z_\varepsilon) < 2 - \alpha$. We set $r_\varepsilon := |z_\varepsilon|$. Since we have already proved that (9.20) holds on any ball of size $c\varepsilon$, we can assume that

$$\lim_{\varepsilon \to 0} \frac{r_\varepsilon}{\varepsilon} = +\infty.$$  

Furthermore, since $v_\varepsilon$ converges to $u_*$ away from the $a_j$, (9.31) implies that $\lim_{\varepsilon \to 0} |z_\varepsilon| = 0$. We set

$$\tilde{v}_\varepsilon := v(r_\varepsilon z + a_j(\varepsilon)).$$

The analysis which is detailed in Step 2 shows that

$$\tilde{X}_\varepsilon := \frac{1}{r_\varepsilon} X(r_\varepsilon z + a_j(\varepsilon)),$$
also converges to

\[ \hat{X} := \frac{\nabla \perp \theta}{|\nabla \theta|^2}. \]

Observe that \( \text{div} \hat{X} = 2 \).

By assumption \( \text{div} \hat{X}_\varepsilon(z_\varepsilon/r_\varepsilon) < 2 - \alpha \) and letting \( \varepsilon \) tends to 0 we conclude that there exists \( z \in \partial B_1 \) such that \( \text{div} \hat{X}(z) \leq 2 - \alpha \). This is the desired contradiction, hence the second inequality in (9.20) is proved. \( \square \)

Let us assume that \( u \) is a solution of the Ginzburg-Landau in \( \Omega \). Then we have

\[ \text{div} (u \wedge \nabla u) = 0. \]

Hence, there exist a function \( \psi_u \) defined in \( \Omega \) such that

\[ \nabla \perp \psi_u = u \wedge \nabla u. \]

Let us denote by \( \{ b_1, \ldots, b_N \} \) the zero set of \( u \), and let \( d_j \in \mathbb{Z} \) be the degree of \( \frac{u}{|u|} \) at \( b_j \). Away from the zeros of \( u \) we may write

\[ u := |u| e^{i\phi_u}. \]

In which case we find that

\[ |u|^{-2} \nabla \psi_u = -\nabla \perp \phi_u. \]

It is now a simple exercise to see that

\[ \text{div} \left( \frac{1}{|u|^2} \nabla \psi_u \right) = 2 \pi \sum_{j=1}^{N} d_j \delta_{b_j}. \]

Moreover, if \( u := g \) on \( \partial \Omega \), we have

\[
\begin{align*}
\partial_r \psi_u &= - (u \wedge \nabla \perp u) \cdot \nu \\
&= u \wedge \partial_r u \\
&= g \wedge \partial_r g.
\end{align*}
\]

To summarize, we have proved that \( \psi_u \) is a solution of

\[
\left\{ \begin{array}{ll}
\text{div} \left( \frac{1}{|u|^2} \nabla \psi_u \right) &= 2 \pi \sum_{j=1}^{N} d_j \delta_{b_j} & \text{in } \Omega \\
\partial_r \psi_u &= g \wedge \partial_r g & \text{on } \partial \Omega.
\end{array} \right. \tag{9.32}
\]

It will be important to observe that also we have, for all \( \sigma > 0 \) small enough

\[ \int_{\partial B_\varepsilon(b_j)} \frac{1}{|u|^2} \partial_r \psi_u = 2 \pi \sigma d_j. \tag{9.33} \]
where \( r = |z - b_j| \). This identity simply follows from the fact that
\[
\frac{1}{|u|^2} \partial_r \psi_u = \partial_r \phi_u,
\]
together with the fact that \( d_j \) is the degree of \( \frac{u}{|u|} \) on \( \partial B_{\sigma}(b_j) \).

In the following Lemma we obtain an estimate for the weighted \( L^2 \) norm of \( \nabla(\psi_u - \psi_v) \) in all \( \Omega \) in terms of the \( L^2 \) norm \( |u_\varepsilon| - |v_\varepsilon| \) in \( \Omega_\varepsilon \) and the weighted \( L^2 \) norm of \( \partial X(u/v) \) in \( \Omega \setminus \Omega_{2\varepsilon} \). Observe that, the use of weighted \( L^2 \) spaces is crucial since similar estimates would not hold in \( L^2 \). More precisely we have :

**Lemma 9.7** Assume that \( \nu \in (1 - 2^{-1/2}, 1) \) and further assume that the hypothesis of Theorem 9.1 hold. Then, there exists \( c > 0 \) and \( \tilde{\varepsilon}_0 > 0 \) such that, for all \( \varepsilon \in (0, \tilde{\varepsilon}_0) \), we have
\[
\int_\Omega r^{2 - 2\nu} |\nabla(\psi_u - \psi_v)|^2 \leq c \int_{\Omega_\varepsilon} r^{2 - 2\nu} (|u_\varepsilon| - |v_\varepsilon|)^2 + c \int_{\Omega \setminus \Omega_{2\varepsilon}} r^{2 - 2\nu} |\partial X(u/v)|^2 .
\]

(9.34)

Where \( r \) denotes the distance from \( z \) to the zero set of \( v_\varepsilon \) and where \( X \) is the vector field defined by
\[
X := |v_\varepsilon|^2 \frac{v_\varepsilon \wedge \nabla v_\varepsilon}{|v_\varepsilon|^2}.
\]

**Proof:** For the sake of simplicity in the notations, let us drop the \( \varepsilon \) indices.

**Step 1.** Since \( u \) and \( v \) have the same zero set and both converge to \( u_* \), we find, using (9.32) that
\[
\text{div} \left( \frac{1}{|u|^2} \nabla \psi_u \right) = \text{div} \left( \frac{1}{|v|^2} \nabla \psi_v \right).
\]

Hence, the function \( \xi := \psi_v - \psi_u \) satisfies
\[
\begin{aligned}
-\Delta \xi + \frac{2}{|u|} \nabla |u| \nabla \xi &= |u|^2 \text{div} \left( \frac{1}{|u|^2} - \frac{1}{|v|^2} \right) \nabla \psi_v \\
|\partial_\nu \xi &= 0 \quad \text{on} \quad \partial \Omega,
\end{aligned}
\]

(9.35)

Moreover, using (9.33) we see that
\[
\int_{\partial B_{\sigma}(a_j(\varepsilon))} \frac{1}{|u|^2} \partial_r \psi_u = \int_{\partial B_{\sigma}(a_j(\varepsilon))} \frac{1}{|v|^2} \partial_r \psi_v,
\]

(9.36)

where \( r := |z - a_j(\varepsilon)| \).

**Step 2.** Let us denote by
\[
S_j := S \left( \frac{z - a_j(\varepsilon)}{\varepsilon} \right),
\]

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where \( S \) has been defined in Theorem 3.1. We claim that
\[
\lim_{\varepsilon \to 0} \left( \sup_{B_\varepsilon(a_j(\varepsilon))} \left| \frac{|u|}{S_j} - 1 \right| \right) = 0. \tag{9.37}
\]
We first prove that, for all fixed \( \gamma > 1 \), (9.37) holds when \( B_\varepsilon(a_j(\varepsilon)) \) is replaced by \( B_{\gamma \varepsilon}(a_j(\varepsilon)) \). We argue by contradiction and assume that there exists a sequence \( u_\varepsilon \) for which
\[
\sup_{B_{\gamma \varepsilon}(a_j(\varepsilon))} \left| \frac{|u_\varepsilon|}{S_j} - 1 \right| \geq \alpha > 0.
\]
As usual, we set
\[
\tilde{u}_\varepsilon := u_\varepsilon(\varepsilon z + a_j(\varepsilon)).
\]
We have already seen that, up to a subsequence, \( \tilde{u}_\varepsilon \) converges in \( C^k \) topology to \( S e^{i(\theta + \tau_j)} \) on any compact of \( \mathbb{C} \). Using the properties of \( S \) together with the fact that both \( \tilde{u}_\varepsilon \) and \( S \) vanish at the origin, it is easy to see that
\[
\lim_{\varepsilon \to 0} \sup_{B_{\gamma \varepsilon}} \left( |\tilde{u}_\varepsilon| - S \right) = 0.
\]
Hence we find
\[
\lim_{\varepsilon \to 0} \sup_{B_\gamma} \left| \frac{|\tilde{u}_\varepsilon|}{S} - 1 \right| = 0,
\]
which clearly contradicts our assumption. This proves (9.37) on any ball of radius \( \gamma \varepsilon \).

Now that we have obtained (9.37) on any ball of radius \( \gamma \varepsilon \), we can argue as in Step 2 of the proof of Lemma 9.6 to obtain the corresponding result on \( B_\varepsilon(a_j(\varepsilon)) \).

Step 3. Let \( \eta \) be the cutoff function equal to 1 in \( B_1 \) and equal to 0 outside \( B_2 \). We set
\[
\eta_j := \eta \left( \frac{z - a_j(\varepsilon)}{\varepsilon} \right).
\]
Using these notations we decompose the right hand side of (9.35) in the following way
\[
\text{div} \left( \left( \frac{1}{|v|^2} - \frac{1}{|u|^2} \right) \nabla \psi_v \right) = \text{div} \left( 1 - \eta_j \right) \left( \frac{|u|^2 - |v|^2}{|u|^2 |v|^2} \nabla \psi_v \right) \]
\[
+ \frac{1}{|v|^2} \nabla \psi_v \nabla \left( \eta_j \frac{|u|^2 - |v|^2}{|u|^2} \right). \tag{9.38}
\]
If we set
\[
f = \frac{|u|^2}{|v|^2} \nabla \psi_v \nabla \left( \eta_j \frac{|u|^2 - |v|^2}{|u|^2} \right),
\]
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and
\[ g = (1 - \eta_j) \left( \frac{1}{|u|^2} - \frac{1}{|u|^2} \right) \nabla \psi_v. \]

We can write
\[ -\text{div}(|u|^{-2} \nabla \xi) = |u|^{-2}f + \text{div}g. \]

We would like to apply the result of Proposition 7.5. However, the function $|u|$ does not satisfy all the required hypothesis since this function does not depend only on $|z - a_j(\epsilon)|$ near each $a_j(\epsilon)$. This is the reason why we introduce the function
\[ \rho := N \sum_{j=1}^{N} \tilde{\eta}_j S_j + \left( 1 - \sum_{j=1}^{N} \tilde{\eta}_j \right) |u|, \]
where we have set
\[ \tilde{\eta}_j := \eta \left( \frac{z - a_j(\epsilon)}{\sigma} \right). \]

Obviously, this time, the function $\rho$ fulfills all the required assumptions. Granted this, we will write
\[ -\text{div}(\rho^{-2} \nabla \xi) = |u|^{-2}f + \text{div}g + \text{div}((|u|^{-2} - \rho^{-2}) \nabla \xi). \]

Hence, for any $1 - 2^{-1/2} < \nu < 1$, we obtain from Proposition 7.5, the estimate
\[
\int_{\Omega} r^{2-2\nu} \rho^{-2} |\nabla \xi|^2 \leq c \left( \int_{\Omega} r^{1-2\nu} \rho^{2} |u|^{-4} |f|^2 + \int_{\Omega} r^{2-2\nu} \rho^{2} |g|^2 \right) \\
+ c \sum_{j=1}^{N} \left( \int_{\partial B_{r_j}(a_j)} \rho^{-2} \partial_{r_j} \xi \right)^2 \\
+ c \sum_{j=1}^{N} \left( \int_{\partial B_{r_j}(a_j)} g \cdot (z - a_j) \right)^2,
\]
where $r$ is the distance from $z$ to $\{a_1(\epsilon), \ldots, a_N(\epsilon)\}$, $r_j := |z - a_j(\epsilon)|$ and where $\tilde{\sigma}$ is fixed in $[\sigma/2, \sigma]$. Observe that the constant $c > 0$ only depends on $\nu$ and does not depend on $\tilde{\sigma}$ provided $\tilde{\sigma}$ stays bounded from below and from above, say $\tilde{\sigma} \in [\sigma/2, \sigma]$.

**Step 4.** It follows from (9.37) that
\[
\lim_{\epsilon \to 0} \sup_{B_{r_j}(a_j(\epsilon))} \frac{|u|}{\rho} - 1 = 0.
\]

Therefore, for $\epsilon$ small enough
\[
c \int_{\Omega} r^{2-2\nu} \rho^{2} (|u|^{-2} - \rho^{-2})^2 |\nabla \xi|^2 \leq \frac{1}{4} \int_{\Omega} r^{2-2\nu} \rho^{-2} |\nabla \xi|^2.
\]
We now make use of (9.36) which implies that
\[
\int_{\partial B_{r}(a_{j}(\varepsilon))} \rho^{-2} \partial_{r_{j}} \xi = \int_{\partial B_{r}(a_{j}(\varepsilon))} (\rho^{-2} - |v|^{-2}) \partial_{r_{j}} \psi_{v},
\]
\[
- \int_{\partial B_{r}(a_{j}(\varepsilon))} (\rho^{-2} - |u|^{-2}) \partial_{r_{j}} \psi_{u}.
\]
Using (9.19) in Lemma 9.5, we see that \(\partial_{r_{j}} \psi_{u}\) and \(\partial_{r_{j}} \psi_{v}\) are bounded from above and \(|u|\) and \(|v|\) are bounded from below, independently of \(\varepsilon\). Thus, we have
\[
\left( \int_{\partial B_{r}(a_{j}(\varepsilon))} \rho^{-2} \partial_{r_{j}} \xi \right)^{2} \leq c \int_{\partial B_{r}(a_{j}(\varepsilon))} (|u| - |v|)^{2}. \quad (9.41)
\]
Still using (9.19) in Lemma 9.5, we obtain
\[
\left( \int_{\partial B_{r}(a_{j})} g \cdot (z - a_{j}) \right)^{2} \leq c \int_{\partial B_{r}(a_{j}(\varepsilon))} (|u| - |v|)^{2}. \quad (9.42)
\]
Collecting (9.40)-(9.42) into (9.39) we obtain
\[
\int_{\Omega} r^{2 - 2 \nu} \rho^{-2} |\nabla \xi|^{2} \leq c \left( \int_{\Omega} r^{4 - 2 \nu} \rho^{2} |f|^{2} + \int_{\Omega} r^{2 - 2 \nu} \rho^{2} |g|^{2} \right) + c \sum_{j=1}^{N} \int_{\partial B_{r}(a_{j}(\varepsilon))} (|u| - |v|)^{2}. \quad (9.43)
\]
Using (9.37) together with the fact that \(u\) and \(v\) converge to \(u_{*}\), we get
\[
|v|/2 \leq |u| \leq 2|v|, \quad (9.44)
\]
for all \(\varepsilon\) small enough. Moreover, since we have \(|\nabla \psi_{v}| \leq c |v| |\nabla v|\), (9.19) in Lemma 9.5 implies that
\[
|\nabla \psi_{u}| \leq c r^{-1} |v|.
\]
Hence (9.43) becomes
\[
\int_{\Omega} r^{2 - 2 \nu} |u|^{-2} |\nabla \xi|^{2} \leq c \sum_{j=1}^{N} \int_{B_{r}(a_{j}(\varepsilon))} r^{4 - 2 \nu} \frac{1}{|v|^{2}} \left| \nabla \psi_{v} \nabla \left( \frac{|v|}{|u|} \right) \right|^{2}
+ c \int_{\partial B_{r}(a_{j}(\varepsilon))} r^{-2 \nu} (|u| - |v|)^{2}
+ c \sum_{j=1}^{N} \int_{\partial B_{r}(a_{j}(\varepsilon))} (|u| - |v|)^{2}, \quad (9.45)
\]
In order to obtain this estimate, we have extensively used the fact that both \(|u|\) and \(|v|\) can be assumed to be bounded from below by 1/2 in \(\Omega \setminus \bigcup_{j} B_{\varepsilon}(a_{j}(\varepsilon))\) provided \(\varepsilon\) is chosen small enough.
Step 5. For all $j = 1, \ldots, N$, we can choose $\tilde{\sigma} \in [\sigma/2, \sigma]$ in such a way that
\[
\int_{\partial B_\sigma(a_j(\epsilon))} (|u| - |v|)^2 \leq \frac{2}{\sigma} \int_{B_\sigma(a_j(\epsilon)) \setminus B_{\sigma/2}(a_j(\epsilon))} (|u| - |v|)^2.
\]
The choice of $\tilde{\sigma}$ may depend on $u$ and $v$ but this is irrelevant since in (9.39), the constant $c$ does not depend on $\tilde{\sigma} \in [\sigma/2, \sigma]$. Thus, (9.45) yields
\[
\int_{\Omega} r^{2-2\nu} |u|^{-2} |\nabla \xi|^2 \leq c \sum_{j=1}^{N} \int_{B_{2\sigma}(a_j(\epsilon))} r^{4-2\nu} \frac{1}{|v|^2} \left| \nabla \psi_v \nabla \left( \frac{|v|}{|u|} \right) \right|^2 + c \int_{\tilde{\Omega}_\sigma} r^{-2\nu} (|u| - |v|)^2.
\]
This is not exactly the estimate which appears in the statement of the Lemma. To obtain the relevant estimate, we begin by the following simple observation
\[
\nabla \left( \frac{|v|}{|u|} \right) = -\frac{|v|^2}{|u|^2} \nabla \left( \frac{|u|}{|v|} \right),
\]
which together with (9.44) yields
\[
\left| \nabla \psi_v \nabla \left( \frac{|v|}{|u|} \right) \right|^2 \leq c \left| \nabla \psi_v \nabla \left( \frac{|u|}{|v|} \right) \right|^2.
\]
Now, we compute
\[
\nabla \left( \frac{u}{v} \right) = \left( \nabla \left( \frac{|u|}{|v|} \right) + i \frac{|u|}{|v|} \nabla (\phi_u - \phi_v) \right) e^{i(\phi_u - \phi_v)}.
\]
Hence
\[
\left| \nabla \psi_v \nabla \left( \frac{|v|}{|u|} \right) \right|^2 \leq \left| \nabla \psi_v \nabla \left( \frac{u}{v} \right) \right|^2.
\]
Therefore, we obtain
\[
\int_{\Omega} r^{2-2\nu} |u|^{-2} |\nabla \xi|^2 \leq c \sum_{j=1}^{N} \int_{B_{2\sigma}(a_j(\epsilon))} r^{4-2\nu} \frac{1}{|v|^2} \left| \nabla \psi_v \nabla \left( \frac{u}{v} \right) \right|^2 + c \int_{\tilde{\Omega}_\sigma} r^{-2\nu} (|u| - |v|)^2.
\]
We end up using the fact that
\[
X = |u|^2 \frac{\nabla \psi_v}{|\nabla \psi_v|^2},
\]
and hence
\[
\nabla \psi_v \left| \nabla \psi_v \right| = X \left| X \right|.
\]
Since $|\nabla \psi_v| \leq c r^{-1} |v|$, we get

$$
\frac{1}{|v|} |\nabla \psi_v \nabla \left( \frac{u}{v} \right)| = \frac{1}{|v|} |\nabla \psi_u| |\partial_X \left( \frac{u}{v} \right) | \leq c \frac{r}{|r|} |\partial_X \left( \frac{u}{v} \right) |.
$$

Using this, the estimate follows at once. □

In the next Lemma, we obtain an $L^2$ bound for $\varepsilon^{-1}(|u_\varepsilon| - |v_\varepsilon|)$ and also $\nabla(|u_\varepsilon| - |v_\varepsilon|)$ in $\tilde{\Omega}_R$ in terms of the $L^2$ norm of $|u_\varepsilon| - |v_\varepsilon|$ and $\nabla(\psi_u - \psi_v)$ in a slightly larger set. Let us recall that

$$
\tilde{\Omega} := \Omega \setminus \bigcup_j B_R(a_j(\varepsilon)).
$$

**Lemma 9.8** Assume that the assumptions of Theorem 9.1 hold. Then, there exists $c > 0$, $\gamma_0 > 0$ and $\tilde{\varepsilon}_0 > 0$ such that, for all $\varepsilon \in (0, \tilde{\varepsilon}_0)$ and for all $R \in [\gamma_0 \varepsilon, \sigma]$, we have

$$
\frac{1}{\varepsilon^2} \int_{\tilde{\Omega}_R} (|u_\varepsilon| - |v_\varepsilon|)^2 + \int_{\tilde{\Omega}_R} |\nabla(|u_\varepsilon| - |v_\varepsilon|)|^2
$$

$$
\leq c \frac{\varepsilon^2}{R^2} \left( \frac{1}{\varepsilon^2} \int_{\tilde{\Omega}_{R/2}} (|u_\varepsilon| - |v_\varepsilon|)^2 + \int_{\tilde{\Omega}_{R/2}} |\nabla(\psi_u - \psi_v)|^2 \right).
$$

(9.47)

**Proof :** As usual, we drop the $\varepsilon$ indices in order to simplify the notations. In $\Omega \setminus \{a_1(\varepsilon), \ldots, a_N(\varepsilon)\}$, the following equations hold

$$
\Delta |u| + |u| \left( \frac{1 - |u|^2}{\varepsilon^2} \right) - |u| |\nabla \phi_u|^2 = 0,
$$

and also

$$
\Delta |v| + |v| \left( \frac{1 - |v|^2}{\varepsilon^2} \right) - |v| |\nabla \phi_v|^2 = 0.
$$

Since $|\nabla \phi_u| = |u|^2 |\nabla \phi_u|$ and $|\nabla \phi_v| = |v|^2 |\nabla \phi_v|$, we see that the function $|u| - |v|$ satisfies

$$
\Delta(|u| - |v|) - V(|u| - |v|) = F, \quad (9.48)
$$

where we have set

$$
V := \frac{1}{\varepsilon^2} (|u|^2 + |u| |v| + |v|^2 - 1) + \frac{|\nabla \psi_u|^2}{|u|^4} - \frac{|\nabla \psi_v|^2}{|u|^4 |v|^4} (|u| + |v|),
$$

and

$$
F := \frac{|v|}{|u|^4} (|\nabla \psi_u|^2 - |\nabla \psi_v|^2).
$$

Obviously, if $\varepsilon$ is small enough and if $\gamma_0$ is sufficiently large, the potential $V$ satisfies

$$
V \geq \frac{1}{\varepsilon^2}, \quad (9.49)
$$

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in $\tilde{\Omega}_{\gamma_0 \varepsilon/2}$. If this is the case, we claim that
\[
\frac{1}{\varepsilon^2} \int_{\tilde{\Omega}_R} (|u| - |v|)^2 \leq c \left( \int_{\tilde{\Omega}_{R/2}} |\nabla (\psi_u - \psi_v)|^2 + \frac{1}{R^2} \int_{\tilde{\Omega}_{R/2}} (|u| - |v|)^2 \right),
\tag{9.50}
\]for some constant $c > 0$ independent of $R \in [\gamma_0 \varepsilon, \sigma]$.

We set
\[
\chi := 1 - \sum_{j=1}^N \eta \left( 2 \frac{z - a_j(\varepsilon)}{R} \right),
\]
where $\eta$ is some cutoff function identically equal to 1 in $B_1$ and equal to 0 outside $B_2$.

We set $D := |u| - |v|$ and compute
\[
\Delta (\chi^2 D) = V \chi^2 D + \chi^2 F + 2 \nabla (\chi^2) \nabla D + \Delta (\chi^2) D.
\]

Now, we multiply by $D$ and integrate by parts over $\Omega$ to obtain
\[
\int_{\Omega} V \chi^2 |D|^2 + \int_{\Omega} \chi^2 |\nabla D|^2 + 2 \int_{\Omega} \chi D \nabla \chi \nabla D + \int_{\Omega} \chi^2 F D = 0.
\]

Using the inequality $-\frac{A^2}{2} - 2 B^2 \leq 2 A B$, we conclude that
\[
\int_{\Omega} V \chi^2 |D|^2 + \frac{1}{2} \int_{\Omega} \chi^2 |\nabla D|^2 - 2 \int_{\Omega} |D|^2 |\nabla \chi|^2 + \int_{\Omega} \chi^2 F D \leq 0.
\]

Now
\[
a b \leq \frac{1}{2 \varepsilon^2} a^2 + \frac{\varepsilon^2}{2} b^2
\]
and hence, we get
\[
\left| \int_{\Omega} \chi^2 F D \right| \leq \frac{1}{2 \varepsilon^2} \int_{\Omega} \chi^2 |D|^2 + \frac{\varepsilon^2}{2} \int_{\Omega} \chi^2 |F|^2.
\]

Thanks to (9.49), we obtain
\[
\frac{1}{\varepsilon^2} \int_{\tilde{\Omega}_R} |D|^2 + \int_{\tilde{\Omega}_R} |\nabla D|^2 \leq c \left( \frac{1}{R^2} \int_{\tilde{\Omega}_{R/2}} |D|^2 + \varepsilon^2 \int_{\tilde{\Omega}_{R/2}} |F|^2 \right),
\]
as desired. The estimate is then an easy consequence of (9.19) which implies that $|\nabla \psi_v|$ is bounded by a constant times $r^{-1}$ and $|u|$ and $|v|$ are bounded from below by $1/2$ in $\tilde{\Omega}_{\gamma_0 \varepsilon}$ provided $\varepsilon$ is small enough and $\gamma_0$ is chosen large enough. $\square$

The main results of this section are stated in the next two Corollaries in which we control the $L^2$ norms of $\varepsilon^{-1} (|u_c| - |v_c|)$ and $\nabla (|u_c| - |v_c|)$ in the complement of small ball centered at the $a_j(\varepsilon)$ in terms of the $L^2$ norm of $|u_c| - |v_c|$ and $\nabla (\psi_u - \psi_v)$ in small balls centered at the $a_j(\varepsilon)$.
Corollary 9.1 Assume that the hypothesis of Theorem 9.1 hold. Then, there exists \( c > 0, \gamma_0 > 0 \) and \( \bar{\varepsilon}_0 > 0 \) such that, for all \( \varepsilon \in (0, \bar{\varepsilon}_0) \) and for all \( R \in [\gamma_0 \varepsilon, \sigma] \), we have

\[
\frac{1}{\varepsilon^2} \int_{\Omega_R} (|u_\varepsilon| - |v_\varepsilon|)^2 + \int_{\Omega_R} |\nabla((|u_\varepsilon| - |v_\varepsilon|))^2 \leq c \left( \frac{\varepsilon}{R} \right)^2 \left( \frac{1}{\varepsilon^2} \int_{\Omega_n \setminus \Omega_R} (|u_\varepsilon| - |v_\varepsilon|)^2 + \int_{\Omega \setminus \Omega_{2\varepsilon}} \frac{1}{|X|^2} \left| \partial X \left( \frac{u_\varepsilon}{v_\varepsilon} \right) \right|^2 \right). \tag{9.51}
\]

where \( X \) is the vector field defined by

\[
X := |v_\varepsilon|^2 \frac{v_\varepsilon \wedge \nabla^\perp v_\varepsilon}{|v_\varepsilon \wedge \nabla^\perp v_\varepsilon|^2}.
\]

Proof: It follows from Lemma 9.8 that

\[
\frac{1}{\varepsilon^2} \int_{\Omega_R} (|u_\varepsilon| - |v_\varepsilon|)^2 + \int_{\Omega_R} |\nabla((|u_\varepsilon| - |v_\varepsilon|)|2 \leq c \left( \frac{\varepsilon}{R} \right)^2 \left( \frac{1}{\varepsilon^2} \int_{\Omega_n \setminus \Omega_R} (|u_\varepsilon| - |v_\varepsilon|)^2 + \int_{\Omega \setminus \Omega_{2\varepsilon}} \frac{1}{|X|^2} \left| \partial X \left( \frac{u_\varepsilon}{v_\varepsilon} \right) \right|^2 \right). \tag{9.52}
\]

We make use of Lemma 9.7 to get

\[
\int_{\Omega_{R/2}} |\nabla(\psi u - \psi v)|^2 \leq c \left( \frac{\varepsilon}{R} \right)^{2-2\nu} \left( \frac{1}{\varepsilon^2} \int_{\Omega_n \setminus \Omega_R} (|u_\varepsilon| - |v_\varepsilon|)^2 + \int_{\Omega \setminus \Omega_{2\varepsilon}} \frac{1}{|X|^2} \left| \partial X \left( \frac{u_\varepsilon}{v_\varepsilon} \right) \right|^2 \right). \tag{9.53}
\]

which, together with (9.52) yields

\[
\frac{1}{\varepsilon^2} \int_{\Omega_R} (|u_\varepsilon| - |v_\varepsilon|)^2 + \int_{\Omega_R} |\nabla((|u_\varepsilon| - |v_\varepsilon|)|2 \leq c \left( \frac{\varepsilon}{R} \right)^2 \left( \frac{1}{\varepsilon^2} \int_{\Omega_n \setminus \Omega_R} (|u_\varepsilon| - |v_\varepsilon|)^2 + \int_{\Omega \setminus \Omega_{2\varepsilon}} \frac{1}{|X|^2} \left| \partial X \left( \frac{u_\varepsilon}{v_\varepsilon} \right) \right|^2 \right). \tag{9.54}
\]

Now, if \( \gamma_0 \) is chosen large enough, we conclude that

\[
\frac{c}{R^2} \int_{\Omega_R} (|u_\varepsilon| - |v_\varepsilon|)^2 + \frac{c \varepsilon^{-2\nu}}{R^{4-2\nu}} \int_{\Omega_n \setminus \Omega_R} (|u_\varepsilon| - |v_\varepsilon|)^2 \leq \frac{1}{2\varepsilon^2} \int_{\Omega_R} (|u_\varepsilon| - |v_\varepsilon|)^2,
\]

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for all $R \in [\gamma_0 \varepsilon, \sigma]$. Hence we have

$$
\frac{1}{\varepsilon^2} \int_{\tilde{\Omega}_R} (|u_\varepsilon| - |v_\varepsilon|)^2 + \int_{\tilde{\Omega}_R} |\nabla (|u_\varepsilon| - |v_\varepsilon|)|^2 \\
\leq \frac{c}{R^2} \int_{\tilde{\Omega}_R \setminus \tilde{\Omega}_{R/2}} (|u_\varepsilon| - |v_\varepsilon|)^2 \\
+ c \varepsilon^{2 - 2\nu} \int_{\tilde{\Omega}_R \setminus \Omega_\varepsilon} (|u_\varepsilon| - |v_\varepsilon|)^2 \\
+ c \left( \frac{\varepsilon}{R} \right)^{4 - 2\nu} \int_{\tilde{\Omega}_R \setminus \tilde{\Omega}_{2\varepsilon}} \frac{1}{|X|^2} \left| \partial_X \left( \frac{u_\varepsilon}{v_\varepsilon} \right) \right|^2.
$$

The proof of the result then follows at once since $\nu < 1$. \qed

Using similar arguments, we also have

**Corollary 9.2** Assume that $\nu \in (1 - 2^{-1/2}, 1)$. Then, under the assumptions of the previous Corollary, we have

$$
\int_{\tilde{\Omega}_R} |\nabla (\psi u_\varepsilon - \psi v_\varepsilon)|^2 \leq c \left( \frac{\varepsilon}{R} \right)^{2 - 2\nu} \left( \frac{1}{\varepsilon^2} \int_{\tilde{\Omega}_R \setminus \tilde{\Omega}_\varepsilon} (|u_\varepsilon| - |v_\varepsilon|)^2 \\
+ \int_{\tilde{\Omega}_R \setminus \tilde{\Omega}_{2\varepsilon}} \frac{1}{|X|^2} \left| \partial_X \left( \frac{u_\varepsilon}{v_\varepsilon} \right) \right|^2 \right).
$$

**Proof :** Indeed, we already have from Lemma 9.7

$$
\int_{\tilde{\Omega}_R} |\nabla (\psi u_\varepsilon - \psi v_\varepsilon)|^2 \leq + \frac{c}{R^2} \int_{\tilde{\Omega}_R} (|u_\varepsilon| - |v_\varepsilon|)^2 \\
+ c \left( \frac{\varepsilon}{R} \right)^{2 - 2\nu} \left( \frac{1}{\varepsilon^2} \int_{\tilde{\Omega}_R \setminus \tilde{\Omega}_\varepsilon} (|u_\varepsilon| - |v_\varepsilon|)^2 \\
+ \int_{\tilde{\Omega}_R \setminus \tilde{\Omega}_{2\varepsilon}} \frac{1}{|X|^2} \left| \partial_X \left( \frac{u_\varepsilon}{v_\varepsilon} \right) \right|^2 \right),
$$

the result is then a straightforward consequence of the previous Corollary which allows us to bound the first term on the right hand side of the inequality. \qed

**9.2.2 The proof of Theorem 9.1**

The proof of Theorem 9.1 relies on a combination of the Pohozaev identity we have obtained in Chapter 9 with the estimates in weighted Sobolev spaces which has been established in Lemma 9.7.

To begin with, for any $\tilde{\sigma} \in [\sigma, 2\sigma]$, we apply Proposition 8.1 to $w := \frac{u}{v}$, on any $B_{\tilde{\sigma}}(a_j(\varepsilon))$. To keep the notations short we will write $B_r$ instead of $B_r(a_j(\varepsilon))$. We get

$$
\frac{1}{4\varepsilon^2} \int_{B_r} \text{div}(|w|^2 X) \left( 1 - |w|^2 \right)^2 + 2 \int_{B_r} \frac{1}{|X|^2} \frac{\partial_X|v|}{|v|} |\partial_X w|^2 \\
= - \int_{\partial B_r} \partial_X w \cdot \partial_{\nu} w + \frac{1}{2} \int_{\partial B_r} |\nabla w|^2 X \cdot \nu \quad (9.53)
$$

$$
+ \frac{1}{4\varepsilon^2} \int_{\partial B_r} (1 - |w|^2)^2 |v|^2 X \cdot \nu - \int_{\partial B_r} w \wedge \partial_{\nu} w,
$$

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where
\[ X := |u|^2 \frac{v \wedge \nabla^\perp v}{|u \wedge \nabla^\perp v|^2}. \]

**Step 1.** Observe that
\[ \int_{\partial B_\sigma} 1 \wedge \partial_r w = 0, \]
since on \( \partial B_\sigma \) we have\[ \partial_r = -\frac{1}{\sigma} \partial_\theta. \]
Hence we can write \[ \int_{\partial B_\sigma} w \wedge \partial_r w = \int_{\partial B_\sigma} (w - 1) \wedge \partial_r w. \]
We may now multiply (9.53) by \( d_j \) (which is the degree of \( u_* \) at \( a_j \)) and obtain, using (9.19), (9.20) together with the above identity
\[
\begin{align*}
\frac{d_j}{4\epsilon^2} \int_{B_\sigma} \text{div}(|v|^2 X) (1 - |w|^2)^2 &+ 2d_j \int_{B_\sigma} \frac{1}{|X|^2} \frac{\partial_X |v|}{|v|} |\partial_X w|^2 \\
&\leq c \int_{\partial B_\sigma} (|\nabla (u - v)|^2 + |u - v|^2) \\
&\quad + \frac{c}{\epsilon^2} \int_{\partial B_\sigma} (|u| - |v|)^2,
\end{align*}
\]
for some constant \( c > 0 \) which does not depend on \( \tilde{\sigma} \in [\sigma, 2\sigma] \). Notice that, we can always choose \( \tilde{\sigma} \) in such a way that
\[ \int_{\partial B_\sigma} (|\nabla (u - v)|^2 + |u - v|^2) \leq \frac{4}{\sigma} \int_{B_{2\sigma} \setminus B_\sigma} (|\nabla (u - v)|^2 + |u - v|^2), \]
and also
\[ \int_{\partial B_\sigma} (|u| - |v|)^2 \leq \frac{4}{\sigma} \int_{B_{2\sigma} \setminus B_\sigma} (|u| - |v|)^2. \]
Hence, we conclude that
\[
\begin{align*}
\frac{d_j}{4\epsilon^2} \int_{B_\sigma} \text{div}(|v|^2 X) (1 - |w|^2)^2 &+ 2d_j \int_{B_\sigma} \frac{1}{|X|^2} \frac{\partial_X |v|}{|v|} |\partial_X w|^2 \\
&\leq c \int_{B_{2\sigma} \setminus B_\sigma} (|\nabla (u - v)|^2 + |u - v|^2) \\
&\quad + \frac{c}{\epsilon^2} \int_{B_{2\sigma} \setminus B_\sigma} (|u| - |v|)^2.
\end{align*}
\]

**Step 2.** Away from the zero set, we may write
\[ u := |u| e^{i\phi_u} \quad \text{and} \quad v := |v| e^{i\phi_v}. \]
Hence
\begin{align*}
|u - v|^2 + |\nabla (u - v)|^2 \leq c \left( (|u| - |v|)^2 + |\phi_u - \phi_v|^2 \right)
+ |\nabla (|u| - |v|)|^2 + |\nabla (\phi_u - \phi_v)|^2, 
\end{align*}

in \( \tilde{\Omega}_\sigma \).

Since \( \phi_u = \phi_v \) on \( \partial \Omega \), we can estimate the \( L^2 \) norm of \( \phi_u - \phi_v \) in \( \tilde{\Omega}_\sigma \), in terms of the \( L^2 \) norm of \( \nabla (\phi_u - \phi_v) \) in the same set. Using this together with
\[ \nabla \phi_u = \frac{\nabla^\perp \psi_u}{|u|^2} \quad \text{and} \quad \nabla \phi_v = \frac{\nabla^\perp \psi_v}{|v|^2}, \]

we easily obtain
\begin{align*}
\int_{B_{2\sigma} \setminus B_\sigma} (|\nabla (u - v)|^2 + |u - v|^2) \leq & \int_{\tilde{\Omega}_\sigma} \left( (|u| - |v|)^2 + |\nabla (|u| - |v|)|^2 \right) \\
&+ \int_{\tilde{\Omega}_\sigma} |\nabla (\psi_u - \psi_v)|^2.
\end{align*}

Where we have used the fact that \( |v| \) and \( |u| \) are uniformly bounded from below in \( \tilde{\Omega}_\sigma \). Therefore, (9.54) yields
\begin{align*}
\frac{d_j}{4 \varepsilon^2} \int_{B_\sigma} \text{div}(|v|^2 X) (1 - |w|^2)^2 &+ 2 d_j \int_{B_\sigma} \frac{1}{|X|^2} |\partial_X| |\partial_X w|^2 \\
&\leq \frac{c}{\varepsilon^2} \int_{\tilde{\Omega}_\sigma} (|u| - |v|)^2 \\
&+ c \int_{\tilde{\Omega}_\sigma} |\nabla (|u| - |v|)|^2 \\
&+ c \int_{\tilde{\Omega}_\sigma} |\nabla (\psi_u - \psi_v)|^2. 
\end{align*}

(9.55)

\textbf{Step 3.} Let \( \gamma \) be a positive number chosen larger than \( \gamma_0 \) which appears in Corollary 9.1 and Corollary 9.2. Using Lemma 9.5 and Lemma 9.6, we can bound
\[ |\partial_X| \leq c \varepsilon^2. \]

Moreover, we have
\[ \frac{1}{|X|^2} |\partial_X w| \leq |\nabla w|. \]

Hence we can estimate
\[ \left| \int_{B_{2\sigma} \setminus B_\sigma} \frac{1}{|X|^2} |\partial_X| |\partial_X w|^2 \right| \leq c \int_{B_{2\sigma} \setminus B_\sigma} \frac{\varepsilon^2}{|X|^2} |\nabla w|^2. \]

We have already seen that
\[ \nabla \left( \frac{u}{v} \right) = \left( \nabla \left( \frac{|u|}{|v|} \right) + i \frac{|u|}{|v|} \nabla (\phi_u - \phi_v) \right) e^{i(\phi_u - \phi_v)}. \]

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Therefore, we can estimate

\[ |\nabla w|^2 \leq c \left( \frac{1}{r^2} |u - v|^2 + |\nabla (|u - v|)|^2 + |\nabla (\psi_u - \psi_v)|^2 \right), \]

on \( \tilde{\Omega}_{\varepsilon} \), since we have from Lemma 9.5

\[ |\nabla |u|| + |\nabla |v|| \leq c \varepsilon^2 r^{-3} \leq c r^{-1}, \]

in \( \tilde{\Omega}_{\varepsilon} \). We conclude that

\[
\left| \int_{B_{\sigma \setminus B_{\varepsilon}}} \frac{1}{|X|^2} \frac{\partial_X|v|}{|v|} |\partial_X w|^2 \right| \leq \frac{c}{\gamma^2} \int_{B_{\sigma \setminus B_{\varepsilon}}} |\nabla (|u - v|)|^2 + \frac{c}{\gamma^2 \varepsilon^2} \int_{B_{\sigma \setminus B_{\varepsilon}}} (|u - v|)^2 + \frac{c}{\gamma^2} \int_{B_{\sigma \setminus B_{\varepsilon}}} |\nabla (\psi_u - \psi_v)|^2.
\] (9.56)

This inequality, together with (9.55) yields

\[
\frac{d_j}{4 \varepsilon^2} \int_{B_{\varepsilon}} \text{div}(|v|^2 X) (1 - |w|^2)^2 + 2 d_j \int_{B_{\varepsilon}} \frac{1}{|X|^2} \frac{\partial_X|v|}{|v|} |\partial_X w|^2
\leq \frac{c}{\varepsilon^2} \int_{\Omega_{\varepsilon}} (|u - v|)^2 + c \int_{\Omega_{\varepsilon}} |\nabla (|u - v|)|^2 + c \int_{\Omega_{\varepsilon}} |\nabla (\psi_u - \psi_v)|^2 + \frac{c}{\gamma^2 \varepsilon^2} \int_{B_{\sigma \setminus B_{\varepsilon}}} |\nabla (\psi_u - \psi_v)|^2.
\]

In order to conclude, we sum the previous inequality over \( j \) and use Corollary 9.1 and Corollary 9.2 to get

\[
\sum_{j=1}^{N} \frac{d_j}{4 \varepsilon^2} \int_{B_{\varepsilon}} \text{div}(|v|^2 X) (1 - |w|^2)^2 + 2 \sum_{j=1}^{N} d_j \int_{B_{\varepsilon}} \frac{1}{|X|^2} \frac{\partial_X|v|}{|v|} |\partial_X w|^2
\leq c \left( \varepsilon^{2-2\nu} + \gamma^{2\nu-4} \right) \left( \frac{1}{\varepsilon^2} \int_{\Omega_{\varepsilon} \setminus \tilde{\Omega}_{\varepsilon}} (|u - v|)^2 + \int_{\Omega_{\varepsilon} \setminus \tilde{\Omega}_{\varepsilon}} \frac{1}{|X|^2} |\partial_X w|^2 \right).
\] (9.57)

We have

\[ d_j \text{div}(|v|^2 X) = 2 d_j |v| \partial_X |v| + d_j |v|^2 \partial_X v. \]

Granted Lemma 9.6 we already have

\[ d_j \text{div}(|v|^2 X) \geq 0, \] (9.58)

in \( B_{\gamma \varepsilon} \), provided \( \varepsilon \) is small enough. Moreover, we also have

\[ |2 d_j |v| \partial_X |v|| \leq c \frac{\varepsilon^2}{r^2}, \]

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and
\[ d_j |v|^2 \text{div } X \geq \frac{1}{2}, \]
in \( B_\sigma \). Hence we find that
\[ d_j \text{div}(|v|^2 X) \geq \frac{1}{4}, \] (9.59)
in \( B_\sigma \setminus B_{\gamma \varepsilon} \) provided \( \gamma \) is chosen large enough.

Finally, we also have
\[ \frac{\partial X |v|}{|v|} \geq 0, \] (9.60)
in \( B_{\gamma \varepsilon} \) and
\[ \frac{\partial X |v|}{|v|} \geq \frac{1}{c}, \] (9.61)
in \( B_{2\varepsilon} \) provided \( \varepsilon \) is small enough.

Using (9.58)-(9.61) we see that, for all \( \varepsilon \) small enough, the right hand side of (9.57) is less than the left hand side of (9.57) provided the constant \( \gamma \) has been fixed large enough. In particular, \( w \equiv 0 \) and therefore \( u = v \) for all \( \varepsilon \) small enough. \( \Box \)
Chapter 10

Solving uniqueness questions

In this Chapter, we prove the main uniqueness result which is stated in Chapter 1. The proof of this result combines the construction of the solutions of the Ginzburg-Landau equation we have performed in Chapter 3 through Chapter 7 with the Pohozaev formula for the Ginzburg-Landau equation as established in Chapter 9. Indeed, using the machinery developed in Chapter 10, we are going to compare any sequence of solutions of the Ginzburg-Landau equation to the solutions we have constructed in Chapter 7. One of the major new difficulty we are confronted with is that we do not assume that the two solutions have the same zero set.

10.1 Statement of the uniqueness result

Let $(a_j)_{j=1,...,N}$ be a non degenerate critical point of the renormalized energy $W_g$, where we assume that all $d_j = \pm 1$. For all $\varepsilon \in (0, \varepsilon_0)$, we denote by $v_\varepsilon$ the solutions of the Ginzburg-Landau equation

$$
\begin{cases}
\Delta u + \frac{u}{\varepsilon^2} (1 - |u|^2) = 0 & \text{in } \Omega \\
u = g & \text{on } \partial \Omega,
\end{cases}
$$

which we have constructed in Chapter 7. The result we want to prove reads:

**Theorem 10.1** Let $\varepsilon_n$ be a sequence which tends to 0 and $u_{\varepsilon_n}$ a sequence of solutions of the Ginzburg-Landau equation (10.1) with $\varepsilon = \varepsilon_n$. Assume that, as $n$ tends to $+\infty$, the sequence $u_{\varepsilon_n}$ converges to the $S^1$ valued harmonic map $u_*$ which is associated to $(a_1, \ldots, a_N)$. Further assume that there exists $c_0 > 0$ such that

$$
\frac{1}{2} \int_{\Omega} |\nabla u_{\varepsilon_n}|^2 + \frac{1}{4 \varepsilon_n^2} \int_{\Omega} (1 - |u_{\varepsilon_n}|^2)^2 \leq c_0 \log 1/\varepsilon_n,
$$

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holds. Then, for \( n \) large enough, we have
\[
u_{\varepsilon_n} = v_{\varepsilon_n}.
\]
In the sequel, it will be convenient to skip the \( n \) indices and simply denote \( u_{\varepsilon} \) instead of \( u_{\varepsilon_n} \) and \( v_{\varepsilon} \) instead of \( v_{\varepsilon_n} \).

## 10.2 Proof of the uniqueness result

### 10.2.1 Geometric modification of the family \( v_{\varepsilon} \)

Thanks to the result of Proposition 9.1, we know that the zero set of \( u_{\varepsilon} \) is given by \( \{a_1(\varepsilon), \ldots, a_N(\varepsilon)\} \) where

\[
\lim_{\varepsilon \to 0} a_j(\varepsilon) = a_j.
\]
Moreover, by construction, the zero set of \( v_{\varepsilon} \) is given by \( \{b_1(\varepsilon), \ldots, b_N(\varepsilon)\} \) where

\[
\lim_{\varepsilon \to 0} b_j(\varepsilon) = a_j.
\]

There is \textit{a priori} no reason why \( a_j(\varepsilon) \) should be equal to \( b_j(\varepsilon) \) moreover, we know that both solutions \( u_{\varepsilon} \) and \( v_{\varepsilon} \) look like a rescaled version of \( S e^{i\theta} \), the solution on all \( \mathbb{C} \), up to some shift in the phase. Our aim is here to modify the solution \( v_{\varepsilon} \) in such a way that the zeros of the two functions do match and the phases at each \( a_j(\varepsilon) \) also match. We will denote by \( d \) the distance between the zeros of \( u_{\varepsilon} \) and the zeros of \( v_{\varepsilon} \). Namely

\[
d := \max_j |a_j(\varepsilon) - b_j(\varepsilon)|.
\]

It will be convenient to denote by

\[
r := \text{dist}(z; \{a_1(\varepsilon), \ldots, a_N(\varepsilon)\}).
\]

and, as in the previous Chapter, the parameter \( \sigma > 0 \) is assumed to be chosen in such a way that, for all \( j = 1, \ldots, N \), \( B_{4\sigma}(a_j(\varepsilon)) \subset \Omega \) and, if \( j \neq k \)

\[
B_{2\varepsilon}(a_j(\varepsilon)) \cap B_{2\varepsilon}(a_k(\varepsilon)) = \emptyset.
\]

We also set, for all \( R \in (0, \sigma) \)

\[
\tilde{\Omega}_R := \Omega \setminus \bigcup_{j=1}^N B_R(a_j(\varepsilon)).
\]

**Step 1.** Translation of the zeros of \( v_{\varepsilon} \). As in Chapter 6, we introduce for all \( c = (c_1, \ldots, c_N) \in \mathbb{C}^N \) small enough, the diffeomorphism

\[
\varphi_c := z + \sum_{j=1}^N \eta \left( \frac{2z - a_j}{\sigma} \right) c_j,
\]

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where $\eta$ is a cutoff function equal to 1 in $B_1$ and equal to 0 outside $B_2$.

We define

$$\hat{v}_\varepsilon := v_\varepsilon \circ \varphi(b(\varepsilon) - a(\varepsilon),$$

where we have set

$$a(\varepsilon) := (a_1(\varepsilon), \ldots, a_N(\varepsilon)) \quad \text{and} \quad b(\varepsilon) := (b_1(\varepsilon), \ldots, b_N(\varepsilon)).$$

**Step 2.** Phase shift near each $a_j(\varepsilon)$. Observe that, since $v_\varepsilon$ is a solution of the Ginzburg-Landau equation, we know that

$$\text{div}(v_\varepsilon \wedge \nabla v_\varepsilon) = 0.$$

Therefore

$$\text{div}(\hat{v}_\varepsilon \wedge \nabla \hat{v}_\varepsilon),$$

has compact support in $\bigcup_{j=1}^N B_{\varepsilon}(a_j(\varepsilon)) \setminus B_{\varepsilon/2}(a_j(\varepsilon))$ and we easily find that, for all $k \geq 0$, there exists $c_k > 0$ such that

$$\sup_\Omega |\nabla^k \text{div}(\hat{v}_\varepsilon \wedge \nabla \hat{v}_\varepsilon)| \leq c_k d. \quad (10.2)$$

Let us denote by $K$ the solution of

$$\begin{cases}
\Delta K &= \frac{1}{|\hat{v}_\varepsilon|^2} \text{div}(\hat{v}_\varepsilon \wedge \nabla \hat{v}_\varepsilon) \quad \text{in} \quad \Omega \\
K &= 0 \quad \text{on} \quad \partial \Omega.
\end{cases} \quad (10.3)$$

It follows at once from (10.2) that, for all $k \geq 0$ there exists $c_k > 0$ such that

$$\sup_\Omega |\nabla^k K| \leq c_k d. \quad (10.4)$$

We define

$$\tilde{K} := \left(1 - \sum_{j=1}^N \eta \left(\frac{z - a_j(\varepsilon)}{\varepsilon}\right) \right) K + \sum_{j=1}^N \eta \left(\frac{z - a_j(\varepsilon)}{\varepsilon}\right) K(a_j(\varepsilon)). \quad (10.5)$$

With this definition, $\tilde{K}$ is constant in each $B_\varepsilon(a_j(\varepsilon)) \setminus B_{\varepsilon/2}(a_j(\varepsilon))$ and, thanks to (10.4) we have, for all $k \geq 1$

$$|\nabla^k \tilde{K}| \leq c d \varepsilon^{k-1}, \quad (10.6)$$

in each $B_\varepsilon(a_j(\varepsilon)) \setminus B_{\varepsilon/2}(a_j(\varepsilon))$. Furthermore, (10.4) still holds in $\tilde{\Omega}_\varepsilon$. The second geometric modification of $v_\varepsilon$ is defined by

$$\tilde{v}_\varepsilon := \hat{v}_\varepsilon e^{-i\tilde{K}}.$$

Let us state some simple properties of $\tilde{v}_\varepsilon$. First of all we have

$$\Delta \tilde{v}_\varepsilon + \tilde{v}_\varepsilon \frac{\varepsilon}{\varepsilon^2} (1 - |\tilde{v}_\varepsilon|^2) = 0, \quad (10.7)$$

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in each $B_{\epsilon/2}(a_j(\epsilon))$. Away from the zero set of $\tilde{v}_\epsilon$ we may write (at least locally)

$$\tilde{v}_\epsilon := |\tilde{v}_\epsilon| e^{i\tilde{\phi}_\epsilon}.$$ 

Hence

$$\Delta \tilde{v}_\epsilon + \tilde{v}_\epsilon \frac{\epsilon}{\epsilon^2} (1 - |\tilde{v}_\epsilon|^2) = \left( \Delta |\tilde{v}_\epsilon| + \frac{|\tilde{v}_\epsilon|}{\epsilon^2} (1 - |\tilde{v}_\epsilon|^2) - |\nabla \tilde{\phi}_\epsilon|^2 |\tilde{v}_\epsilon| \right) e^{i\tilde{\phi}_\epsilon} + i \frac{1}{|\tilde{v}_\epsilon|} \text{div}(|\tilde{v}_\epsilon|^2 \nabla \tilde{\phi}_\epsilon) e^{i\tilde{\phi}_\epsilon}.$$ 

We would like to estimate this expression in $\tilde{\Omega}_{\epsilon/2}$.

A simple computation yields

$$\text{div} (\tilde{v}_\epsilon \land \nabla \tilde{v}_\epsilon) = |\tilde{v}_\epsilon|^2 \Delta (\tilde{K} - \tilde{K}) - \nabla |\tilde{v}_\epsilon|^2 \nabla \tilde{K}. \quad (10.8)$$

Observe that it follows from the construction of $v_\epsilon$ in Chapter 7 that, for all $k \geq 1$, there exists $c_k > 0$ such that

$$|\nabla^k |v_\epsilon|| \leq c_k \epsilon^2 r^{-2-k},$$

in $\tilde{\Omega}_{\epsilon/2} \setminus \tilde{\Omega}_{2\epsilon}$. We may now use the previous estimates together with the estimates for $\tilde{K}$, to get, for all $k \geq 0$, the existence of a constant $c_k > 0$ such that

$$|\nabla^k (\text{div} (\tilde{v} \land \nabla \tilde{v}))| \leq c_k \epsilon^2 d r^{-3-k}, \quad (10.9)$$

in $\tilde{\Omega}_{\epsilon/2}$.

We set

$$\hat{v}_\epsilon = |\hat{v}_\epsilon| e^{i\hat{\phi}_\epsilon}.$$ 

Since $|\hat{v}| = |\tilde{v}|$, we have the identity

$$\Delta |\hat{v}_\epsilon| + \frac{|\hat{v}_\epsilon|}{\epsilon^2} (1 - |\hat{v}_\epsilon|^2) - |\nabla \hat{\phi}_\epsilon|^2 |\hat{v}_\epsilon| = \Delta |\hat{v}_\epsilon| + \frac{|\hat{v}_\epsilon|}{\epsilon^2} (1 - |\hat{v}_\epsilon|^2) - |\nabla \tilde{\phi}_\epsilon|^2 |\tilde{v}_\epsilon|.$$ 

Since $\hat{v}$ is a solution of the Ginzburg-Landau equation in $\Omega_\sigma \setminus \tilde{\Omega}_{\sigma/2}$, we get

$$\Delta |\hat{v}_\epsilon| + \frac{|\hat{v}_\epsilon|}{\epsilon^2} (1 - |\hat{v}_\epsilon|^2) - |\nabla \hat{\phi}_\epsilon|^2 |\hat{v}_\epsilon| = (|\nabla \hat{\phi}_\epsilon|^2 - |\nabla \tilde{\phi}_\epsilon|^2) |\hat{v}_\epsilon|,$$

in $\tilde{\Omega}_\sigma \setminus \tilde{\Omega}_{\sigma/2}$. It is then not very difficult to see that, for all $k \geq 0$, there exists $c_k > 0$ such that

$$|\nabla^k \left( \Delta |\hat{v}_\epsilon| + \frac{|\hat{v}_\epsilon|}{\epsilon^2} (1 - |\hat{v}_\epsilon|^2) - |\nabla \hat{\phi}_\epsilon|^2 |\hat{v}_\epsilon| \right) | \leq c_k d r^{-1-k}. \quad (10.10)$$

in $\tilde{\Omega}_{\epsilon/2}$.

We are now going to obtain some estimates measuring how far $u_\epsilon$ is from $\tilde{v}_\epsilon$. 

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10.2.2 Estimating the $L^2$ norm of $|u_\varepsilon| - |\tilde{v}_\varepsilon|$

In the next Proposition, whose proof follows closely the strategy of the proof of Theorem 9.1 in the last Chapter, we estimate the $L^2$ norm of $|u_\varepsilon| - |\tilde{v}_\varepsilon|$. Since $\tilde{v}_\varepsilon$ is not assumed to be a solution of the Ginzburg-Landau equation the strategy developed in the last Chapter does not lead directly to the conclusion $u_\varepsilon = \tilde{v}_\varepsilon$ but provides us with an estimate.

**Proposition 10.1** Assume that the assumptions of Theorem 10.1 hold. Then, there exists $c > 0$ such that

$$\int_\Omega (|u| - |\tilde{v}|)^2 \leq c \varepsilon^4 d^2 \log 1/\varepsilon,$$

for all $\varepsilon$ small enough.

As already mentioned, we will follow step by step the proof of Theorem 9.1. The new difficulty here comes from the fact that $\tilde{v}_\varepsilon$ is not assumed to be a solution of the Ginzburg-Landau equation. Hence, we will have to compute carefully the error term we have introduced when we have translated the zeros of $v_\varepsilon$ and changed the phase of $v_\varepsilon$ near each zero.

As in Chapter 10, we define the function $\psi_{u_\varepsilon}$ by

$$\nabla \perp \psi_{u_\varepsilon} := u_\varepsilon \land \nabla u_\varepsilon.$$

This function is well defined since $u$ is a solution of the Ginzburg-Landau equation and hence $\text{div}(u \land \nabla u) = 0$.

Let $H_\varepsilon$ be the solution of

$$\begin{cases}
\Delta H_\varepsilon = \text{div}(\tilde{v} \land \nabla \tilde{v}) & \text{in } \Omega \\
H_\varepsilon = 0 & \text{on } \partial \Omega.
\end{cases}$$

(10.12)

We also obtain the existence of $\tilde{\psi}_{\tilde{v}_\varepsilon}$ such that

$$\nabla \perp \tilde{\psi}_{\tilde{v}_\varepsilon} := \tilde{v}_\varepsilon \land \nabla \tilde{v}_\varepsilon - \nabla H_\varepsilon.$$

**Lemma 10.1** Assume that $\nu \in (0, 1)$ and further assume that the hypothesis of Theorem 10.1 hold. Then, there exists $c > 0$ and $\bar{\varepsilon}_0 > 0$ such that, for all $\varepsilon \in (0, \bar{\varepsilon}_0)$, we have

$$\int_\Omega r^{2-2\nu} |\nabla H_\varepsilon|^2 \leq c \varepsilon^{A-2\nu} d^2.$$

(10.13)

In addition, we have

$$\|r \nabla H_\varepsilon\|_{L^\infty} \leq c \varepsilon d.$$

(10.14)
Proof: Combining (10.9) and Lemma 7.1 we get for all \( \nu \in (0,1) \)

\[
\int_{\Omega} r^{2-2\nu} |\nabla H|^2 \leq c \varepsilon^{4-2\nu} d^2 + c \sum_{j=1}^{N} \left( \int_{B_{r/4}(a_j(\varepsilon))} \Delta H \right)^2.
\]

(10.15)

In addition we have

\[
\int_{B_{r/4}(a_j(\varepsilon))} \Delta H = \int_{B_{r/4}(a_j(\varepsilon))} \text{div} (\tilde{v} \wedge \nabla \tilde{v}).
\]

Since \( \tilde{v} = \hat{v} e^{-i \tilde{K}} \) and since we have \( \tilde{K} = K \) on \( \partial B_{r/4}(a_j(\varepsilon)) \), we obtain

\[
\tilde{v} \wedge \partial_{\nu} \tilde{v} = |\tilde{v}|^2 \partial_{\nu} K + \hat{v} \wedge \partial_{\nu} \hat{v}.
\]

Since \( \text{div}(\hat{v} \wedge \nabla \hat{v}) = 0 \) in \( B_{r/2}(a_j(\varepsilon)) \), we can write the previous equality as

\[
\int_{B_{r/4}(a_j(\varepsilon))} \Delta H = - \int_{\partial B_{r/4}(a_j(\varepsilon))} |\hat{v}|^2 \partial_{\nu} K.
\]

Since \( \Delta K = 0 \) in \( B_{r/2}(a_j(\varepsilon)) \), we finally get that

\[
\int_{B_{r/4}(a_j(\varepsilon))} \Delta H = \int_{\partial B_{r/4}(a_j(\varepsilon))} (|\hat{v}|^2 - 1) \partial_{\nu} K.
\]

Finally, observe that, on \( B_{r/2}(a_j(\varepsilon)) \), the function \( \hat{v} \) is just a translation of \( v \) and using the estimates obtained for \( |v| \) and the estimate (10.6) for \( K \), we conclude that

\[
\int_{\Omega} r^{2-2\nu} |\nabla H|^2 \leq c \varepsilon^{4-2\nu} d^2 + c \varepsilon^4 d^2 \leq c \varepsilon^{4-2\nu} d^2.
\]

In order to derive the last estimate, we make use of the fact that for all \( p > 1 \) and all function \( U \in W^{2,p}(B_2 \setminus B_{1/2}) \), we have

\[
\sup_{\partial B_1} |\nabla U| \leq c \left( \int_{B_2 \setminus B_{1/2}} |\Delta U|^p \right)^{1/p} + c \left( \int_{B_2 \setminus B_{1/2}} |U|^2 \right)^{1/2}.
\]

We refer to Theorem 9.11 in [26] for a proof of this result. Using a simple scaling argument we get

\[
r \sup_{\partial B_r} |\nabla U| \leq c \left( r^{2p-2} \int_{B_{2r} \setminus B_{r/2}} |\Delta U|^p \right)^{1/p} + c \left( \int_{B_{2r} \setminus B_{r/2}} |U|^2 \right)^{1/2},
\]

for all \( U \in W^{2,p}(B_{2r} \setminus B_{r/2}) \). We apply this inequality to \( H \) and use the fact that \( |\Delta H| \leq c \varepsilon^2 r^{-3} \) together with (10.13) to get the pointwise bound

\[
r |\nabla H| \leq c (\varepsilon^2 r^{-1} + \varepsilon^{4-2\nu} r^{\nu-1}) \leq c d \varepsilon,
\]

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in each $B_\varepsilon(a_j(\varepsilon))$.

Finally, we also have

$$\sup_{\bar{\Omega}_\sigma} |\nabla H| \leq c \left( \left( \int_{\bar{\Omega}_{\sigma/2}} |\Delta H|^p \right)^{1/p} + \left( \int_{\bar{\Omega}_{\sigma/2}} |H|^2 \right)^{1/2} \right).$$

Which easily leads to

$$\sup_{\bar{\Omega}_\sigma} |\nabla H| \leq c \, d \, \varepsilon.$$

The proof of the Lemma is therefore complete. \(\square\)

Paralleling Lemma 9.7, we have

**Lemma 10.2** Assume that $\nu \in (1 - 2^{-1/2}, 1)$ and further assume that the hypothesis of Theorem 10.1 hold. Then, there exists $c > 0$ and $\tilde{\varepsilon}_0 > 0$ such that, for all $\varepsilon \in (0, \tilde{\varepsilon}_0)$, we have

$$\int_{\Omega} r^{2 - 2\nu} \frac{|\nabla (\psi_{u_\varepsilon} - \tilde{\psi}_{v_\varepsilon})|^2}{|\nabla u_\varepsilon|^2} \leq c \int_{\bar{\Omega}} r^{-2\nu} (|u_\varepsilon| - |\tilde{v}_\varepsilon|)^2 + c \int_{\Omega \setminus \bar{\Omega}_{2\varepsilon}} |X|^2 \left| \partial X \left( \frac{u_\varepsilon}{\tilde{v}_\varepsilon} \right) \right|^2 + c \varepsilon^{4 - 2\nu} d^2. \tag{10.16}$$

Where $r$ denotes the distance from $z$ to the zero set of $u_\varepsilon$ and where $X$ is the vector field defined by

$$X := \frac{|\tilde{v}_\varepsilon|^2}{|\tilde{v}_\varepsilon|^2 \wedge |\nabla \tilde{v}_\varepsilon|^2}.$$

**Proof:** For the sake of simplicity in the notations, we drop the $\varepsilon$ indices.

Since $u$ and $\tilde{v}$ have the same zeros $\{a_1(\varepsilon), \ldots, a_N(\varepsilon)\}$ and the same multiplicities $\{d_1, \ldots, d_N\}$, the following holds

$$\text{div} \left( \frac{u \wedge \nabla u}{|u|^2} \right) = \text{div} \left( \frac{\tilde{v} \wedge \nabla \tilde{v}}{|\tilde{v}|^2} \right) = 2 \pi \sum_{j=1}^{N} d_j \delta_{a_j(\varepsilon)}, \tag{10.17}$$

hence $\psi_u$ and $\tilde{\psi}_v$ satisfy

$$\text{div} \left( \frac{\nabla \psi_u}{|u|^2} \right) = \text{div} \left( \frac{\nabla \tilde{\psi}_v}{|\tilde{v}|^2} \right) - \nabla \frac{H}{|\tilde{v}|^2}.$$ 

Moreover, we have on $\partial \Omega$

$$\partial_\nu \psi_u = u \wedge \partial_\tau u := g \wedge \partial_\tau g,$$

and

$$\partial_\nu \tilde{\psi}_v + \partial_\tau H = \tilde{v} \wedge \partial_\tau \tilde{v}.$$
Since $H = 0$ on $\partial \Omega$, we obtain
\[
\partial_\nu \tilde{\psi}_v = \tilde{v} \wedge \partial_\nu \tilde{v} := g \wedge \partial_\tau g.
\]
As in the proof of Lemma 9.7, we set $\xi = \tilde{\psi}_v - \psi_u$. This time $\xi$ satisfies
\[-\Delta \xi + 2\frac{\nabla |u|}{|u|} \nabla \xi = |u|^2 \div \left( \left( \frac{1}{|\tilde{v}|^2} - \frac{1}{|u|^2} \right) \nabla \tilde{\psi}_v \right) - |u|^2 \div \left( \frac{\nabla^\perp H}{|u|^2} \right),
\]
in $\Omega$ together with
\[
\partial_\nu \xi = 0,
\]
on $\partial \Omega$. Observe that
\[
|u|^2 \div \left( \frac{\nabla^\perp H}{|u|^2} \right) = -2 \frac{|u|^2}{|v|^2} \nabla^\perp H \nabla |v|.
\]
Since there exists $c > 0$ such that
\[
sup \left( r \frac{\nabla |\tilde{v}|}{|\tilde{v}|} \right) \leq c,
\]
we get
\[
\int_{\Omega} r^{4-2\nu} \left| \frac{\nabla H}{|v|^2} \right|^2 \leq c \int_{\Omega} r^{2-2\nu} |\nabla H|^2. \tag{10.18}
\]
The result follows at once from (10.14), (10.13) and (10.18), using the arguments developed in the proof of Lemma 9.7, and taking into account the new term involving $H$. \hfill \Box

As in the proof of Theorem 9.1, we will need results corresponding to Corollaries 9.1 and 9.2.

**Corollary 10.1** Assume that $\nu \in (1 - 2^{-1/2}, 1)$ and further assume that the assumptions of Theorem 9.1 hold. Then, there exists $c > 0$, $\gamma_0 > 0$ and $\tilde{\varepsilon}_0 > 0$ such that, for all $\varepsilon \in (0, \tilde{\varepsilon}_0)$ and for all $R \in [\gamma_0 \varepsilon, \sigma]$, we have
\[
\frac{1}{\varepsilon^2} \int_{\tilde{\Omega}_R} (|u| - |\tilde{v}|)^2 + \int_{\tilde{\Omega}_R} |\nabla (|u| - |\tilde{v}|)|^2 \leq c d^2 \varepsilon^2 (1 + |\log R|)
\]
\[
+ c \left( \frac{\varepsilon}{R} \right)^2 \left( \frac{1}{\varepsilon^2} \int_{\tilde{\Omega}_R} (|u| - |\tilde{v}|)^2 + \int_{\tilde{\Omega}_R} \frac{1}{|X|^2} \left| \partial_X \left( \frac{u}{\tilde{v}} \right) \right|^2 \right), \tag{10.19}
\]
and also
\[
\int_{\tilde{\Omega}_R} |\nabla (\psi_u - \tilde{\psi}_v)|^2 \leq c \left( \frac{\varepsilon}{R} \right)^{2-2\nu} d^2 \varepsilon^2 (1 + |\log R|)
\]
\[
+ c \left( \frac{\varepsilon}{R} \right)^2 \left( \frac{1}{\varepsilon^2} \int_{\tilde{\Omega}_R} (|u| - |\tilde{v}|)^2 + \int_{\tilde{\Omega}_R} \frac{1}{|X|^2} \left| \partial_X \left( \frac{u}{\tilde{v}} \right) \right|^2 \right).
\]

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Where \( X \) is the vector field defined by

\[
X := |\tilde{\nu}|^2 \frac{\tilde{\nu} \wedge \nabla \perp \tilde{\nu}}{|\tilde{\nu} \wedge \nabla \perp \tilde{\nu}|^2}.
\]

**Proof:** The proof follows at once from a straightforward modification of Lemma 9.8. Indeed, we still have

\[
\Delta |u| + |u| \left( \frac{1 - |u|^2}{\varepsilon^2} - \frac{|\nabla \psi u|^2}{|u|^4} \right) = 0. \tag{10.20}
\]

However, this time

\[
R := \Delta |\tilde{v}| + |\tilde{v}| \left( \frac{1 - |\tilde{v}|^2}{\varepsilon^2} - \frac{|\nabla \tilde{\psi} \tilde{v}|^2}{|\tilde{v}|^4} \right), \tag{10.21}
\]

is not equal to 0 anymore, since \( \tilde{v} \) is not a solution of the Ginzburg-Landau equation and also since

\[
\nabla \perp \tilde{\psi} \tilde{v} \neq \tilde{v} \wedge \nabla \tilde{v}.
\]

This induces an extra term on the right hand side of the inequalities of Lemma 9.8.

This extra term can be estimated using (10.10) we know that

\[
|R| \leq \frac{c}{r} d + |\nabla H| + |\nabla H|^2. \tag{10.22}
\]

Hence

\[
\int_{\tilde{\Omega}_{n/2}} |R|^2 \leq c d^2 (1 + |\log R|) + \int_{\tilde{\Omega}_{n/2}} \frac{|\nabla H|^2}{r^2} + \int_{\tilde{\Omega}_{n/2}} |\nabla H|^4.
\]

Using (10.13), we estimate

\[
\int_{\tilde{\Omega}_{n/2}} \frac{|\nabla H|^2}{r^2} \leq c R^{2\nu - 4} \int_{\tilde{\Omega}} r^{2 - 2\nu} |\nabla H|^2 \leq c d^2 \left( \frac{\varepsilon}{R} \right)^{4 - 2\nu}.
\]

Finally, (10.2) implies that

\[
\int_{\Omega} |\Delta H|^2 \leq c d^2.
\]

Therefore, we obtain

\[
\int_{\Omega} |\nabla H|^4 \leq c d^4.
\]

Collecting these estimates we can conclude that

\[
\int_{\tilde{\Omega}_{n/2}} |R|^2 \leq c d^2 (1 + |\log R|).
\]

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Observe that we have implicitly assumed that $d$ is bounded! The remaining of the proof is identical to the proof of Lemma 9.8. We set

$$ D := |u| - |	ilde{v}|, $$

and find that

$$ \Delta \tilde{D} - \tilde{V} \tilde{D} = \tilde{F}, $$

where

$$ \tilde{V} := \frac{1}{\varepsilon^2} \left( |\nabla \psi_u|^2 + |\nabla \tilde{v}|^2 - |\nabla \tilde{v}|^2 + |u|^2 + |\tilde{v}|^2 \right), $$

and

$$ \tilde{F} := \frac{|\tilde{v}|}{|u|^4} \left( |\nabla \psi_u|^2 - |\nabla \tilde{v}|^2 \right) - \mathcal{R}. $$

The remaining of the proof is now straightforward and is left to the reader. \qed

**Proof of Proposition 10.1** *A priori* the function $\tilde{v}$ is not a solution of the Ginzburg-Landau equation. Hence, instead of the Pohozaev formula of Proposition 8.3, we need to apply to $w := u \tilde{v}$, the Pohozaev formula we have obtained in Proposition 8.4. This provides us with the following identity

$$ \frac{1}{4\varepsilon^2} \int_{\omega} \text{div}(|\tilde{v}|^2 X) (1 - |w|^2)^2 + 2 \int_{\omega} \frac{1}{|X|^2} \frac{\partial_X |\tilde{v}|}{|\tilde{v}|} |\partial_X w|^2 $$

$$ = - \int_{\partial \omega} \partial_X w \cdot \partial_{\nu} w + \frac{1}{2} \int_{\partial \omega} |\nabla w|^2 X \cdot \nu $$

$$ + \frac{1}{4\varepsilon^2} \int_{\partial \omega} (1 - |w|^2)^2 |\tilde{v}|^2 X \cdot \nu - \int_{\partial \omega} w \wedge \partial_{\nu} w $$

$$ + \int_{\omega} \frac{1}{|\tilde{v}|^2} \text{div}(\tilde{v} \wedge \nabla \tilde{v}) \partial_X w \cdot \partial_{\tilde{v}} \tilde{v} $$

$$ - \int_{\omega} \partial_X w \cdot \frac{w}{\tilde{v}} \left( \Delta \tilde{v} + \tilde{v} (\frac{1 - |\tilde{v}|^2}{\varepsilon^2}) \right), $$

(10.23)

where $\omega$ is any of the $B_{\tilde{\sigma}}(a_j(\varepsilon))$ and where $\tilde{\sigma} \in [\sigma, 2\sigma]$. As we have already done in the proof of Theorem 9.1, we are going to estimate all the terms on the right hand side of this identity. To simplify the formula, we avoid the mention of $a_j(\varepsilon)$ in $B_{\sigma}(a_j(\varepsilon))$.

As usual, it will be convenient to write

$$ u := |u| e^{i\phi_u}, \quad \text{and} \quad \tilde{v} := |\tilde{v}| e^{i\phi_{\tilde{v}}}. $$

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Step 1. Arguing as in Step 1 and Step 2 of the proof of Theorem 9.1, it is an easy exercise to see that (10.23) leads to

\[
\frac{d_j}{\varepsilon^2} \int_{B_x} \text{div}(\bar{\varepsilon}^2 X) (1 - |w|^2)^2 + 2d_j \int_{B_x} \frac{1}{|X|^2} \frac{\partial X|\bar{v}|}{|\bar{v}|} |\partial_X w|^2 \\
\leq c \frac{1}{\varepsilon^2} \int_{\tilde{\Omega}_\sigma} (|u| - |\bar{v}|)^2 + c \int_{\tilde{\Omega}_\sigma} |\nabla(|u| - |\bar{v}|)|^2 \\
+ c \int_{\tilde{\Omega}_\sigma} |\nabla(\psi_u - \tilde{\psi}_\varepsilon)|^2 + c \int_{\tilde{\Omega}_\sigma} |\nabla H|^2 \\
+ \left| \int_{B_{2\varepsilon}} \frac{1}{|\bar{v}|^2} \text{div}(\bar{v} \cdot \nabla \bar{v}) \partial_X w \cdot \partial_X w \right| \\
+ \left| \int_{B_{2\varepsilon}} \partial_X w \cdot \frac{w}{\bar{v}} \left( \Delta \tilde{v} + \tilde{v} \frac{(1 - |\bar{v}|^2)}{\varepsilon^2} \right) \right|. \\
\quad (10.24)
\]

Notice that this time we have estimated the $L^2$ norm of $\nabla(\phi_u - \phi_{\varepsilon})$ in $\tilde{\Omega}_\sigma$ by the $L^2$ norm of $\nabla(\psi_u - \tilde{\psi}_\varepsilon)$, $|u| - |\bar{v}|$ and $\nabla(|u| - |\bar{v}|)$ as before, but also the $L^2$ norm of $\nabla H$ on the same set.

In addition, let us recall that $\tilde{\sigma}$ is chosen in $[\sigma, 2\sigma]$ in such a way that

\[
\int_{\partial B_{2\varepsilon}} (|u| - |\bar{v}|)^2 \leq \varepsilon \int_{B_{2\varepsilon} \setminus B_{\varepsilon}} (|u| - |\bar{v}|)^2. \quad (10.25)
\]

Step 2. We claim that

\[
\int_{B_{2\varepsilon} \setminus B_{\varepsilon/2}} |\nabla(|u| - |\bar{v}|)|^2 \leq \varepsilon \int_{B_{2\varepsilon} \setminus B_{\varepsilon/4}} (|u| - |\bar{v}|)^2 \\
+ \int_{B_{4\varepsilon} \setminus B_{\varepsilon/4}} |\nabla(\psi_u - \tilde{\psi}_\varepsilon)|^2 + c d^2 \varepsilon^2 \log 1/\varepsilon.
\]

This inequality is obtained following closely the strategy of the proof of Lemma 9.8. As in the proof of Corollary 10.1, we set $\tilde{D} := |u| - |\bar{v}|$, we have already seen that $\tilde{D}$ solves

\[
\Delta \tilde{D} - \tilde{V} \tilde{D} = \tilde{F}.
\]

By opposition with what we have done in the proof of Lemma 9.8, we consider the function $\chi$ defined by

\[
\chi := \eta \left( \frac{z - a_j(\varepsilon)}{4\sigma} \right) - \eta \left( \frac{z - a_j(\varepsilon)}{\varepsilon/2} \right),
\]

where $\eta$ is a cutoff function equal to 1 in $B_{1/2}$ and equal to 0 outside $B_1$. Following closely the arguments of Lemma 9.8 and using the estimates for $\mathcal{R}$ which are derived in the proof of Corollary 10.1, we obtain

\[
\int_{\Omega} \chi^2 |\nabla(|u| - |\bar{v}|)|^2 \leq \varepsilon \int_{B_{2\varepsilon} \setminus B_{\varepsilon/4}} (|u| - |\bar{v}|)^2 \\
+ c \int_{B_{4\varepsilon} \setminus B_{\varepsilon/4}} |\nabla(\psi_u - \tilde{\psi}_\varepsilon)|^2 + c \varepsilon^2 d^2 \log 1/\varepsilon.
\]

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The proof of the claim is therefore complete.

**Step 3.** Using (10.9), we can write

\[
\left| \int_{B_{2\sigma}} \frac{1}{|v|^2} \text{div} (\tilde{v} \wedge \nabla \tilde{v}) \partial_X w \cdot \partial_X \tilde{w} \right| \leq c d \int_{B_{2\sigma} \setminus B_{\gamma \varepsilon}} \frac{\varepsilon^2}{r} |\nabla w|^2. \tag{10.26}
\]

Notice that \(\text{div} (\tilde{v} \wedge \nabla \tilde{v}) = 0\) in \(B_{\gamma \varepsilon/2}\). Thanks to (9.19), we have

\[|\nabla u| \leq c \frac{\varepsilon^2}{r^3}.\]

Moreover, both \(u\) and \(\tilde{v}\) are uniformly bounded from below in \(B_{4\sigma} \setminus B_{\varepsilon/4}\). Hence, we have the bound

\[|\nabla w|^2 \leq c \left( \frac{\varepsilon^4}{r^6} (|u| - |v|)^2 + |\nabla (|u| - |v|)|^2 + |\nabla (\phi u - \phi \tilde{v})|^2 \right),\]

in \(B_{4\sigma} \setminus B_{\varepsilon/4}\). Using the claim proved in Step 2, we conclude that

\[
\left| \int_{B_{2\sigma}} \frac{1}{|v|^2} \text{div}(\tilde{v} \wedge \nabla \tilde{v}) \partial_X w \cdot \partial_X \tilde{w} \right| \leq c \frac{d \varepsilon}{\varepsilon} \int_{B_{4\sigma} \setminus B_{\varepsilon/4}} (|u| - |v|)^2 + c d \varepsilon \int_{B_{4\sigma} \setminus B_{\varepsilon/4}} |\nabla (\psi u - \tilde{\psi} v)|^2 + c d^3 \varepsilon^3 \log 1/\varepsilon. \tag{10.27}
\]

**Step 4.** Observe that, for all fixed \(\gamma\), we may decompose

\[
\int_{B_\sigma} \frac{1}{|X|^2} |\nabla |X| |\partial_X w|^2 = \int_{B_{\gamma \varepsilon}} \frac{1}{|X|^2} |\nabla |X| |\partial_X w|^2 + \int_{B_\sigma \setminus B_{\gamma \varepsilon}} \frac{1}{|X|^2} |\nabla |X| |\partial_X w|^2.
\]

Now, the last term can be estimated as follows

\[
\left| \int_{B_\sigma \setminus B_{\gamma \varepsilon}} \frac{1}{|X|^2} |\nabla |X| |\partial_X w|^2 \right| \leq c \int_{B_\sigma \setminus B_{\gamma \varepsilon}} \frac{\varepsilon^2}{r^2} |\nabla w|^2
\]

\[
\leq \frac{c}{\gamma \varepsilon} \int_{B_\sigma \setminus B_{\gamma \varepsilon}} (|u| - |v|)^2
\]

\[+ \frac{c}{\gamma^3 \varepsilon^2} \int_{B_\sigma \setminus B_{\gamma \varepsilon}} (|u| - |v|)^2
\]

\[+ \frac{c}{\gamma^2} \int_{B_\sigma \setminus B_{\gamma \varepsilon}} |\nabla (\psi u - \tilde{\psi} v)|^2
\]

\[+ \frac{c}{\gamma^2} \int_{B_\sigma \setminus B_{\gamma \varepsilon}} |\nabla H|^2.
\]
We conclude using the result of Step 2, together with (10.13) that
\[
\left| \int_{B_{\epsilon} \setminus B_{\gamma \epsilon}} \frac{1}{|X|^2} \frac{\partial_X |\tilde{v}|}{|\tilde{v}|} |\partial_X w|^2 \right| \leq \frac{c}{\gamma^2} \epsilon^2 \int_{B_{4\epsilon} \setminus B_{\gamma \epsilon}/4} (|u| - |\tilde{v}|)^2 \\
+ \frac{c}{\gamma^2} \int_{B_{4\epsilon} \setminus B_{\gamma \epsilon}/4} |\nabla (\psi u - \tilde{\psi} \tilde{v})|^2 \\
+ \frac{c}{\gamma^2} d^2 \epsilon^2. 
\]

(10.28)

**Step 5.** We bound
\[
\int_{\tilde{\Omega}_2} |\nabla H|^2 \leq c d^2 \epsilon^{4-2\nu},
\]
thanks to (10.13). Further observe that we have from Lemma 10.2
\[
\int_{B_{\epsilon} \setminus B_{\gamma \epsilon}/4} |\nabla (\psi u - \tilde{\psi} \tilde{v})|^2 \leq \frac{c}{\epsilon^2} \int_{\tilde{\Omega}_2} (|u| - |\tilde{v}|)^2 \\
+ c \int_{\Omega \setminus \tilde{\Omega}_2} \frac{1}{|X|^2} |\partial_X w|^2 + c d^2 \epsilon^2.
\]

Now, we return to (10.24) and use all the estimates we have obtained in Step 2, 3 and 4, together with the results of Corollary 10.1 and Lemma 10.2. Arguing as in the proof of Theorem 9.1, we conclude that
\[
\frac{d_j}{4 \epsilon^2} \int_{B_{\epsilon}} \text{div}(|\tilde{v}|^2 X) (1 - |w|^2)^2 + 2 d_j \int_{B_{\gamma \epsilon}/4} \frac{1}{|X|^2} \frac{\partial_X |\tilde{v}|}{|\tilde{v}|} |\partial_X w|^2 \\
\leq c (\epsilon^{2-2\nu} + \frac{\gamma^{2\nu-4} + d \epsilon}{\epsilon^2} \int_{B_{\gamma \epsilon}/2 \setminus B_{\epsilon}} (|u| - |\tilde{v}|)^2 \\
+ c (\epsilon^{2-2\nu} + \frac{\gamma^{2\nu-4} + d \epsilon}{\epsilon^2} \int_{\Omega \setminus \tilde{\Omega}_2} \frac{1}{|X|^2} |\partial_X w|^2 \\
+ c d^2 \epsilon^2 + \left| \int_{B_{\epsilon}} \partial_X w \cdot \frac{w}{\tilde{v}} \left( \Delta \tilde{v} + \tilde{v} \left( \frac{1 - |\tilde{v}|^2}{\epsilon^2} \right) \right) \right|. 
\]

(10.29)

**Step 6.** It remains to estimate the last term of the right hand side of (10.29).

To this aim, we write
\[
\frac{1}{|\tilde{v}|} \left( \Delta \tilde{v} + \tilde{v} \left( \frac{1 - |\tilde{v}|^2}{\epsilon^2} \right) \right) = \frac{1}{|\tilde{v}|} \left( \Delta |\tilde{v}| + |\tilde{v}| \left( \frac{1 - |\tilde{v}|^2}{\epsilon^2} - |\nabla \phi \tilde{v}|^2 \right) \right) \\
+ \frac{i}{|\tilde{v}|^2} \text{div}(\tilde{v} \wedge \nabla \tilde{v}).
\]

For notational convenience, the first term on the right hand side of this identity will be denoted by $R_1$ and the second one by $R_2$. 

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Observe that \( R_1 \equiv 0 \) in all \( B_{\epsilon/2} \) and also that \( |R_1| + r |\nabla R_1| \leq c d r^{-1} \) in \( B_\sigma \setminus B_{\epsilon/2} \). Hence, we can integrate by parts and obtain

\[
\int_{B_\sigma \setminus B_{\epsilon/2}} \partial X w \cdot w R_1 = \int_{B_\sigma} (|w|^2 - 1) \text{div}(X R_1) + \int_{\partial B_\sigma} (|w|^2 - 1) R_1 X \cdot \nu
\]

\[
\leq c d \int_{B_\sigma \setminus B_{\epsilon/2}} \frac{1}{r} (|w|^2 - 1) + c d \int_{\partial B_\sigma} (|w|^2 - 1)
\]

\[
\leq c d \int_{B_\sigma \setminus B_{\epsilon/2}} \frac{1}{r} (|w|^2 - 1).
\]

Here, we have used (10.25) to obtain the last estimate. Now, for all \( \kappa_1 > 0 \), we use the inequality

\[
a b \leq \kappa_1 a^2 + \frac{1}{4\kappa_1} b^2,
\]

(10.30)

to obtain

\[
\int_{B_\sigma \setminus B_{\epsilon/2}} \partial X w \cdot w R_1 \leq c \kappa_1 \epsilon^2 d^2 \log 1/\epsilon + c \kappa_1 \epsilon^{-2} \int_{B_\sigma \setminus B_{\epsilon/2}} (|u| - |\tilde{v}|)^2.
\]

(10.31)

Recall that, by definition of \( \wedge \), we have

\[
\partial X w \cdot (i w) = w \wedge \partial X w.
\]

Furthermore, we have

\[
w \wedge \nabla w = \frac{|u|^2}{|\tilde{v}|^2} \nabla (\phi u - \phi \tilde{v}).
\]

Using this together with (10.9), we already obtain the estimate

\[
\int_{B_\sigma \setminus B_{\epsilon/2}} w \wedge \nabla w R_2 \leq c d \int_{B_\sigma \setminus B_{\epsilon/2}} \frac{\epsilon^2}{r^2} |\nabla (\phi u - \phi \tilde{v})|
\]

\[
\leq c d \int_{B_\sigma \setminus B_{\epsilon/2}} \frac{\epsilon^2}{r^2} \left( |\nabla (\psi u - \psi \tilde{v})| + \frac{1}{r} ||u| - ||\tilde{v}||\right)
\]

\[
+ c d \int_{B_\sigma \setminus B_{\epsilon/2}} \frac{\epsilon^2}{r^2} |\nabla H|.
\]

Making use of (10.30) and Cauchy-Schwarz inequality, we obtain, for all \( \kappa_1 > 0 \)

\[
\int_{B_\sigma \setminus B_{\epsilon/2}} w \wedge \nabla w R_2 \leq c \kappa_1 \int_{B_\sigma \setminus B_{\epsilon/2}} |\nabla (\psi u - \psi \tilde{v})|^2
\]

\[
+ c \kappa_1 \int_{B_\sigma \setminus B_{\epsilon/2}} (|u| - |\tilde{v}|)^2 + c \kappa_1 d^2 \epsilon^2
\]

\[
+ c d \epsilon^2 \left( \int_{B_\sigma \setminus B_{\epsilon/2}} r^{2-2\mu} |\nabla H|^2 \right)^{1/2}
\]

\[
\leq c \kappa_1 \int_{B_\sigma \setminus B_{\epsilon/2}} |\nabla (\psi u - \psi \tilde{v})|^2
\]

\[
+ c \kappa_1 \int_{B_\sigma \setminus B_{\epsilon/2}} (|u| - |\tilde{v}|)^2 + c \kappa_1 d^2 \epsilon^2.
\]
Here, we have used (10.13) to obtain the last estimate.

We conclude, using (10.29) that

$$\frac{d_j}{4\varepsilon^2} \int_{B_r} \text{div}(\varepsilon^2 X) (1 - |w|^2)^2 + 2 d_j \int_{B_{2r}} \frac{1}{|X|^2} \frac{\partial X}{|\varepsilon|} |\partial_X w|^2$$

$$\leq c (\varepsilon^{2-2\nu} + \gamma^{2\nu-4} + d\varepsilon + \kappa_1) \frac{1}{\varepsilon^2} \int_{\bar{\Omega}_{\varepsilon} \setminus \Omega_\varepsilon} (|u| - |\varepsilon|)^2$$

$$+ c (\varepsilon^{2-2\nu} + \gamma^{2\nu-4} + d\varepsilon + \kappa_1) \int_{\bar{\Omega}_{\varepsilon} \setminus \Omega_\varepsilon} \frac{1}{|X|^2} |\partial_X w|^2$$

$$+ d\varepsilon^2 \log 1/\varepsilon. \quad (10.32)$$

To complete the proof it suffices to choose \(\kappa_1\) small enough but fixed, \(\gamma\) large enough but fixed and then \(\varepsilon\) small enough. The end of the proof being identical to Step 3 in the proof of Theorem 9.1, we omit it.

Observe that, as a byproduct of the previous proof we have also obtained the following inequality

$$\int_{B_{2r}(a_j(\varepsilon))} \frac{1}{|X|^2} \left| \partial_X \left( \frac{u_\varepsilon}{\varepsilon^2} \right) \right|^2 \leq c d^2 \varepsilon^2 \log 1/\varepsilon. \quad (10.33)$$

This implies, using Lemma 10.2, that

$$\int_{\Omega} \frac{x^{2-2\nu}}{|x|^2} |\nabla (\psi_{u_\varepsilon} - \psi_{\varepsilon})|^2 \leq c d^2 \varepsilon^{4-2\nu} \log 1/\varepsilon. \quad (10.34)$$

Combining this together with the result of Corollary 10.1 yields

$$\int_{\bar{\Omega}_\varepsilon} |\nabla (|u| - |\varepsilon|)|^2 \leq c d^2 \varepsilon^2 \log 1/\varepsilon.$$

Moreover, arguing as in Step 2 of the proof of Proposition 10.1, we also get, for all \(\lambda > 0\), the existence of a constant \(c_\lambda > 0\) such that

$$\int_{\bar{\Omega}_\lambda \setminus \Omega_\varepsilon} |\nabla (|u| - |\varepsilon|)|^2 \leq c_\lambda d^2 \varepsilon^2 \log 1/\varepsilon.$$

Hence, we conclude that

$$\int_{\bar{\Omega}_\lambda \setminus \Omega_\varepsilon} |\nabla (|u| - |\varepsilon|)|^2 \leq c_\lambda d^2 \varepsilon^2 \log 1/\varepsilon. \quad (10.35)$$

### 10.2.3 Pointwise estimates for \(u_\varepsilon - \tilde{v}_\varepsilon\) and \(|u_\varepsilon| - |\tilde{v}_\varepsilon|\).

Based on the results of the previous section, we derive pointwise estimates for \(u_\varepsilon - \tilde{v}_\varepsilon\) and \(|u_\varepsilon| - |\tilde{v}_\varepsilon|\). This is the content of the following:
Proposition 10.2 Assume that the assumptions of Theorem 10.1 hold. Then, for all \( k \in \mathbb{N} \), for all \( \nu \in (0, 1) \) and for all \( p \in (2, 4/(\nu + 1)) \), there exists \( c > 0 \) such that
\[
\sup_{\Omega} \left( r^k |\nabla^k (u_\varepsilon - \tilde{v}_\varepsilon)| \right) + \sup_{\Omega} \left( r^k |\nabla^k (\phi u_\varepsilon - \phi \tilde{v}_\varepsilon)| \right) \leq c d^{2/p} \varepsilon^\nu, \tag{10.36}
\]
and
\[
\sup_{\Omega} \left( r^{2+k-\nu} |\nabla^k (|u_\varepsilon| - |	ilde{v}_\varepsilon|)| \right) \leq c d^{2/p} \varepsilon^{\nu - k}, \tag{10.37}
\]
Moreover we have
\[
\sup_{\Omega \setminus \tilde{\Omega}_\varepsilon} |\nabla^k (u_\varepsilon - \tilde{v}_\varepsilon)| \leq c d^{2/p} \varepsilon^{\nu - k}. \tag{10.38}
\]

Proof: We choose \( \mu \in (\nu, 1) \). Let \( \lambda \in (0, 1/2) \) be fixed small enough, namely we want \( \lambda^2 \) to be smaller than \( 1/8 \) times the first eigenvalue of \( \nabla H \) on \( B_1 \), for the Dirichlet boundary condition.

Step 1. We first establish the estimates in \( \tilde{\Omega}_{\lambda \varepsilon} \). As usual, we set
\[
u := |u| e^{i\phi_u} \quad \text{and} \quad \tilde{v} := |\tilde{v}| e^{i\phi_{\tilde{v}}}. \tag{10.39}
\]
With these notations, we may write
\[
u - \tilde{v} = \left( |u| \left( e^{(\phi_u - \phi_{\tilde{v}})} - 1 \right) + (|u| - |\tilde{v}|) \right) e^{i\phi_{\tilde{v}}}. \tag{10.40}
\]
Recall that, by definition, we have
\[
\nabla (\phi u - \phi \tilde{v}) = \frac{1}{|u|^2} \nabla \perp \psi u - \frac{1}{|\tilde{v}|^2} \nabla \perp \tilde{\psi} \tilde{v} + \frac{1}{|\tilde{v}|^2} \nabla H. \tag{10.41}
\]
We already know from (10.14) that
\[
\sup_{\Omega} (r |\nabla H|) \leq c d \varepsilon.
\]
In order to prove the result, it remains to obtain a pointwise estimate for \( \nabla (\psi u - \tilde{\psi} \tilde{v}) \) and also for \( |u| - |\tilde{v}| \).

Step 2. We claim that the following pointwise estimate holds
\[
\sup_{\tilde{\Omega}_{\lambda \varepsilon}} (r |\nabla (\psi u - \tilde{\psi} \tilde{v})|) \leq c_p d^{2/p} \varepsilon^\mu, \tag{10.42}
\]
provided \( p > 2 \) satisfies
\[
\mu < \frac{2}{p}.
\]
We set \( \xi := \tilde{\psi} \tilde{v} - \psi u \). We have already seen that
\[
-\Delta \xi + 2 \frac{\nabla |u|}{|u|} \nabla \xi = |u|^2 \text{div} \left( \left( \frac{1}{|\tilde{v}|^2} - \frac{1}{|u|^2} \right) \nabla \tilde{\psi} \tilde{v} \right) - |u|^2 \text{div} \left( \frac{\nabla \perp H}{|\tilde{v}|^2} \right). \tag{227}
\]
Using the fact that $|\nabla|u|| + |\nabla|\tilde{v}|| \leq c \varepsilon^2 r^{-3}$, $|\nabla \tilde{w}| \leq c r^{-1}$, together with the pointwise estimate for $\nabla H$, we get

$$|\Delta \xi| \leq c \varepsilon^2 r^{-3} |\nabla \xi| + c \varepsilon^2 r^{-1} |\nabla ((|u| - |\tilde{v}|)| + c \varepsilon^2 d^3 r^{-4}.$$ 

We make use of the trick we have already used in the proof of Lemma 10.1. Namely, for all $p > 2$ and all $r \in [\varepsilon, \sigma]$, we have

$$\sup_{\partial B_r} r |\nabla \xi| \leq c \left( r^{2p-2} \int_{B_{2r}} |\Delta \xi|^p \right)^{1/p} + c \left( \int_{B_{2r} \setminus B_{r/2}} |\nabla \xi|^2 \right)^{1/2}.$$

Using the fact that $|\nabla \xi| \leq c r^{-1}$,

we easily get

$$\left( r^{2p-2} \int_{B_{2r} \setminus B_{r/2}} \left( r^2 r^{-1} |\nabla \xi| \right)^p \right)^{1/p} \leq c \varepsilon \left( \int_{B_{2r} \setminus B_{r/2}} |\nabla \xi|^2 \right)^{1/p}.$$

Now we make use of (10.34) to conclude that

$$\left( r^{2p-2} \int_{B_{2r} \setminus B_{r/2}} \left( r^2 r^{-1} |\nabla \xi| \right)^p \right)^{1/p} \leq c \left( d^2 \varepsilon^2 \log 1/\varepsilon \right)^{1/p}.$$

Using similar arguments, (10.35), Proposition 10.1 and also

$$||u| - |\tilde{v}|| + r |\nabla ((|u| - |\tilde{v}|)| \leq c \varepsilon^2 r^{-2},$$

we conclude easily that, for all $r \geq \varepsilon$

$$\left( r^{2p-2} \int_{B_{2r} \setminus B_{r/2}} |\Delta \xi|^p \right)^{1/p} \leq c \left( d^2 \varepsilon^2 \log 1/\varepsilon \right)^{1/p}.$$

Finally, we also have

$$\left( \int_{B_{2r} \setminus B_{r/2}} |\nabla \xi|^2 \right)^{1/2} \leq c d^2 \varepsilon^2 \log 1/\varepsilon.$$

The claim follows at once since

$$(\varepsilon^2 \log 1/\varepsilon)^{1/p} \leq c d^{2/p} \varepsilon \mu,$$

if $\mu < 2/p$.

**Step 3.** We claim now that we have

$$\sup_{\tilde{\Omega}_{3r}} (r^{2p-\mu} ||u| - |\tilde{v}||) \leq \varepsilon^2 d^{2/p},$$

if $\mu < 2/p$. 

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provided $p > 2$ satisfies

$$\mu < \frac{4 - p}{p}.$$  

In particular, this will show that (10.37) holds in $\tilde{\Omega}_{\lambda \varepsilon}$, for $k = 0$.

Indeed, let us denote by

$$\chi := 1 - \sum_{j=1}^{N} \eta \left( \frac{2(z - a_{j}(\varepsilon) \lambda \varepsilon)}{\lambda \varepsilon} \right)$$

where $\eta$ is a cutoff function equal to 1 in $B_{1}$ and equal to 0 outside $B_{2}$. Thanks to (10.35), Proposition 10.1 and also to the fact that

$$\|u - \tilde{v}\| + r | \nabla (|u| - |\tilde{v}|) | \leq c \varepsilon^{2} r^{-2},$$

we get

$$\|\chi(|u| - |\tilde{v}|)\|_{W^{1,p}(\Omega)} \leq c \left( d^{2} \varepsilon^{4-p} \log 1/\varepsilon \right)^{1/p}. \quad (10.43)$$

In view of (10.44) we can conclude that the function

$$z \in \Omega \longrightarrow c \varepsilon^{2} \left( (d^{2} \varepsilon^{2} \log 1/\varepsilon)^{1/p} r^{-2} + d r^{-1} \right) + \left( d^{2} \varepsilon^{4-p} \log 1/\varepsilon \right)^{1/p} \left( e^{(\lambda_{0} \varepsilon - r)/\varepsilon} + e^{(r-2\sigma)/\varepsilon} \right),$$

can be used as a barrier function in $B_{2\sigma} \setminus B_{\lambda \varepsilon}$ to show that

$$|\tilde{D}| \leq c \varepsilon^{2} r^{2-\mu},$$

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in \( B_\sigma \setminus B_{\lambda \varepsilon} \), provided
\[
\mu < \frac{4 - p}{p}.
\]
In particular \(|\tilde{D}| \leq c \varepsilon^2\) on \( \partial \tilde{\Omega}_\sigma \).

Finally, the function
\[
z \in \Omega \longrightarrow c \varepsilon^2,
\]
can be used as a barrier function in \( \tilde{\Omega}_\sigma \) to show that
\[
|\tilde{D}| \leq c \varepsilon^2,
\]
in \( \tilde{\Omega}_\sigma \). This ends the proof of the claim.

**Step 4.** It follows from (10.41) and (10.42) that
\[
\sup_{\Omega_{\lambda \varepsilon}} (r |\nabla (\phi_u - \phi_{\tilde{v}})|) \leq c d^{2/p} \varepsilon^\mu.
\]
Since \( \phi_u - \phi_{\tilde{v}} = 0 \) on \( \partial \Omega \), this estimates yields
\[
\sup_{\Omega_{\lambda \varepsilon}} |\phi_u - \phi_{\tilde{v}}| \leq c d^{2/p} \varepsilon^\mu \log 1/\varepsilon.
\]
In particular we get
\[
\sup_{\Omega_{\lambda \varepsilon}} |u - \tilde{v}| \leq c d^{2/p} \varepsilon^\mu \log 1/\varepsilon \leq c d^{2/p} \varepsilon^\nu.
\]
To conclude, we have already proved (10.36) in \( \tilde{\Omega}_{\lambda \varepsilon} \), for \( k = 0 \).

**Step 5.** Observe that
\[
\|u - \tilde{v}\|_{L^\infty(\partial \Omega_{\lambda \varepsilon})} \leq c d^{2/p} \varepsilon^\mu. \tag{10.45}
\]
Further observe that \( \tilde{v} \) is a solution of the Ginzburg-Landau equation in each \( B_{\varepsilon/2} \). We set
\[
U := u(\lambda \varepsilon z) \quad \text{and} \quad V := v(\lambda \varepsilon z).
\]
These functions are solutions of
\[
\Delta U + \lambda^2 U(1 - |U|^2) = 0, \tag{10.46}
\]
and
\[
\Delta V + \lambda^2 V(1 - |V|^2) = 0,
\]
in \( B_1 \).

Let \( W \) denote the harmonic extension of \( (V - U)_{\partial B_1} \) in \( B_1 \). We denote
\[
T := \Delta (V + W) + \lambda^2 (V + W)(1 - |V + W|^2). \tag{10.47}
\]
Thanks to (10.45), we know that
\[ \|W\|_{L^\infty(B_1)} \leq c \, d^{2/p} \varepsilon^\mu. \]
Hence,
\[ |T| \leq c \, d^{2/p} \varepsilon^\mu. \]
in \( B_1 \). Taking the scalar product of \( V + W - U \) with the difference between (10.47) and (10.46), we obtain after an integration by part
\[
\int_{B_1} |\nabla (V + W - U)|^2 - \lambda^2 \int_{B_1} |V + W - U|^2 \\
+ \lambda^2 \int_{B_1} (V + W - U) \cdot (U|U|^2 - (V + W)|V + W|^2) \\
\leq c \{d^{2/p} \varepsilon^\mu\}^{1/2} \left( \int_{B_1} |V + W - U|^2 \right)^{1/2}.
\]
Since \( \lambda^2 \) has been chosen less than \( 1/8 \) times the first eigenvalue of \( \Delta \) in \( B_1 \), it is an easy exercise to conclude that
\[
\int_{B_1} |V + W - U|^2 \leq c \, d^{2/p} \varepsilon^\mu.
\]
A classical bootstrap argument then shows that
\[ \|V + W - U\|_{L^\infty(B_1)} \leq c \, d^{2/p} \varepsilon^\mu. \]
Performing the scaling backward, we obtain
\[ \|u - \tilde{v}\|_{L^\infty(B_{\lambda^2})} \leq d^{2/p} \varepsilon^\mu. \]
which proves (10.38) and completes the proof of (10.36) and (10.37) when \( k = 0 \).

The proof of the estimates for arbitrary \( k \) relies on an induction argument. \( \square \)

10.2.4 The final arguments to prove that \( u_\varepsilon = v_\varepsilon \).

We keep the notations of Chapter 7 and use all the function spaces defined in Chapter 4, Chapter 6 and Chapter 7. As in Chapter 7, the parameter \( \mu \) is chosen in \((1,2)\).

To begin with, let us recall that the solution \( v_\varepsilon \) we have constructed, can be written in the following way
\[
v_\varepsilon = e^{i\psi_0} \left( e^{\tilde{\phi}} \left( (|\tilde{u}| + w_\varepsilon) e^{i(1-\chi_\varepsilon) \tilde{\phi}} + i \chi_\varepsilon w_\varepsilon \right) \right) \circ \varphi_{d_{\pm 1}}, \tag{10.48}
\]
where \( d_0 \in \mathbb{R}^N \), \( d_{\pm 1} \in (\mathbb{R}^2)^N \) and \( w \in \mathcal{E} \). In addition, we have the bound
\[
|d_0|_{\mathbb{R}^N} + |d_{\pm 1}|_{\mathbb{R}^N \times \mathbb{R}^N} + \|w\|_{\mathcal{E}} \leq c e^{1-\alpha} \tag{10.49}
\]
231
where $\alpha$ is any fixed number $\alpha \in (0, 1)$ (see Theorem 6.1).

Observe that we can also write
\[
u = e^{i\varphi_0} \left( e^{i\theta} \left( (|\tilde{\nu}| + \tilde{\nu}) e^{i(1-\chi_0)\tilde{\nu}} + i \chi_0 \tilde{\nu} \right) \right) \circ \varphi_{\pm 1} \tag{10.50}
\]
for some $\tilde{d}_0 \in \mathbb{R}^N$, $\tilde{d}_{\pm 1} \in (\mathbb{R}^2)^N$ and $\tilde{\nu} \in \mathcal{E}$. However, this time we do not have any a priori control on the parameters. Notice that the same would be true for any complex valued map $u$ defined in $\Omega$ whose zero set has exactly $N$ points close to the $a_j$, with corresponding degree $d_j$.

Though we have not made this explicit in the notation, the parameters $d_0, d_{\pm 1}, w$ and $\tilde{d}_0, \tilde{d}_{\pm 1}, \tilde{w}$ all depend on $\varepsilon$ since $u_\varepsilon$ and $v_\varepsilon$ do.

Since $u_\varepsilon$ and $v_\varepsilon$ are solutions of the Ginzburg-Landau problem, we obviously have
\[
(d_0, d_{\pm 1}, w) = - (DM(0,0,0))^{-1} (M(0,0,0) + Q_1(d_0, d_{\pm 1}) + Q_2(d_0, d_{\pm 1}, w) + Q_3(d_0, d_{\pm 1}, w)),
\]
and
\[
(\tilde{d}_0, \tilde{d}_{\pm 1}, \tilde{w}) = - (DM(0,0,0))^{-1} (M(0,0,0) + Q_1(\tilde{d}_0, \tilde{d}_{\pm 1}) + Q_2(\tilde{d}_0, \tilde{d}_{\pm 1}, \tilde{w}) + Q_3(\tilde{d}_0, \tilde{d}_{\pm 1}, \tilde{w})).
\]
Hence
\[
(d_0, d_{\pm 1}, w) - (\tilde{d}_0, \tilde{d}_{\pm 1}, \tilde{w}) = - (DM(0,0,0))^{-1} (Q_1(d_0, d_{\pm 1}) - Q_1(\tilde{d}_0, \tilde{d}_{\pm 1}) + Q_2(d_0, d_{\pm 1}, w) - Q_2(\tilde{d}_0, \tilde{d}_{\pm 1}, \tilde{w}) + Q_3(d_0, d_{\pm 1}, w) - Q_3(\tilde{d}_0, \tilde{d}_{\pm 1}, \tilde{w})).
\]

The strategy is now to prove that the parameters $l_1, l_2$ and $l_3$ which are defined in Lemma 6.1, Lemma 6.2 and Lemma 6.3 can be assumed to be arbitrarily small provided $\varepsilon$ is small enough. As a consequence, we will obtain using (10.51)
\[
\|(d_0, d_{\pm 1}, w) - (\tilde{d}_0, \tilde{d}_{\pm 1}, \tilde{w})\|_{\mathbb{R}^N \times (\mathbb{R}^N \times \mathbb{R}^N) \times \mathcal{E}} \leq \frac{1}{2} \|(d_0, d_{\pm 1}, w) - (\tilde{d}_0, \tilde{d}_{\pm 1}, \tilde{w})\|_{\mathbb{R}^N \times (\mathbb{R}^N \times \mathbb{R}^N) \times \mathcal{E}},
\]
provided $\varepsilon$ is small enough. In particular, this will imply that
\[
(d_0, d_{\pm 1}, w) = (\tilde{d}_0, \tilde{d}_{\pm 1}, \tilde{w}).
\]
Hence
\[
u_\varepsilon = v_\varepsilon,
\]
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as wished.

**Estimate for \( \tilde{d}_{\pm 1} \).** By definition, the zero set of the function \( u_\varepsilon \) is given by \( \{a_1(\varepsilon), \ldots, a_N(\varepsilon)\} \) but also by

\[
\{a_j - (\tilde{d}_{+1,j} + i \tilde{d}_{-1,j}) : j = 1, \ldots, N\}.
\]

Similarly, the zero set of \( v_\varepsilon \) is given by \( \{b_1(\varepsilon), \ldots, b_N(\varepsilon)\} \) but also by

\[
\{a_j - (d_{+1,j} + id_{-1,j}) : j = 1, \ldots, N\}.
\]

Hence we have

\[
a_j(\varepsilon) - b_j(\varepsilon) = (d_{+1,j} + id_{-1,j}) - (\tilde{d}_{+1,j} + i \tilde{d}_{-1,j}).
\]

Since by assumption

\[
\lim_{\varepsilon \to 0} a_j(\varepsilon) = a_j,
\]
we conclude that

\[
\lim_{\varepsilon \to 0} \tilde{d}_{\pm 1} = 0. \tag{10.52}
\]

**Estimate for \( \tilde{d}_0 \).** By definition, we have

\[
\tilde{v}_\varepsilon := e^{-i\tilde{K}} v_\varepsilon \circ \varphi_{b(\varepsilon) - a(\varepsilon)}.
\]

Moreover, since \( \chi \equiv 1 \) in \( B_\varepsilon(a_j(\varepsilon)) \), we can write

\[
u_\varepsilon - \tilde{v}_\varepsilon = e^{i\tilde{d}_0} \left( e^{i\tilde{d}_0}(|\tilde{u}| + \tilde{w}) \right) - e^{i(\tilde{d}_0 - \tilde{K})} \left( e^{i\tilde{d}_0}(|\tilde{u}| + w) \right). \tag{10.53}
\]

For the sake of convenience, we have dropped all the \( j \) indices and we will implicitly assume that \( a_j(\varepsilon) = 0 \).

Thanks to Proposition 9.1, we already know that the sequence of functions \( u_\varepsilon(\cdot + a_j(\varepsilon)) \) converges uniformly to \( S e^{i(\theta + \tau_j)} \) in \( B_{2\varepsilon} \). Similarly, the sequence of functions \( v_\varepsilon(\cdot + a_j(\varepsilon)) \) converges uniformly to \( S e^{i(\theta + \tau_j)} \) in \( B_{2\varepsilon} \). Hence, we have

\[
\lim_{\varepsilon \to 0} \frac{|\tilde{u}| + w}{|\tilde{u}|} = \lim_{\varepsilon \to 0} \frac{|\tilde{u}| + \tilde{w}}{|\tilde{u}|} = 1,
\]

uniformly in \( B_{2\varepsilon} \).

Integrating (10.53) divided by \( \tilde{u} \) over \( \partial B_\varepsilon \), we find

\[
\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\partial B_\varepsilon} \frac{u_\varepsilon - \tilde{v}_\varepsilon}{\tilde{u}} = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_{\partial B_\varepsilon} (e^{i\tilde{d}_0} - e^{i(d_0 - \tilde{K}))}.
\]

It follows from Proposition 10.2 that the left hand side of this expression is bounded by

\[
\frac{1}{\varepsilon} \int_{\partial B_\varepsilon} \frac{u_\varepsilon - \tilde{v}_\varepsilon}{\tilde{u}} \leq c d^{2/p} \varepsilon^\nu.
\]
So this expression tends to 0 as $\varepsilon$ does. Moreover, we have from (10.5)
\[
\sup_{\partial B_\varepsilon} |\tilde{K}| \leq c d.
\]
Since $d$ tends to 0 as $\varepsilon$ does, this implies that
\[
\lim_{\varepsilon \to 0} |d_0 - \tilde{d}_0| = 0.
\]
However, we already know that $d_0$ tends to 0 hence
\[
\lim_{\varepsilon \to 0} \tilde{d}_0 = 0.
\]

**Estimate of $w$ in $\Omega_{2\varepsilon}$.** In this set, $\chi \equiv 0$, hence we have
\[
u = e^{i\psi_0} \left( e^{i\tilde{\phi}} (|\bar{u}| + \bar{w}) e^{i\psi_0} \right) \circ \varphi_{a-\alpha(\varepsilon)},
\]
and
\[
\tilde{\nu} = e^{-i\tilde{K}} e^{i\psi_0} \left( e^{i\tilde{\phi}} (|\bar{u}| + w) e^{i\psi_0} \right) \circ \varphi_{a-\alpha(\varepsilon)}.
\]
In particular, we get
\[
\bar{w}_r - w_r = (|\bar{u}| - |\bar{v}|) \circ \varphi_{a(\varepsilon)-a}.
\]
Using the pointwise bound of Proposition 10.2, we conclude easily that
\[
\|w_r - \bar{w}_r\|_{C^1_0(\Omega_{2\varepsilon})} + \|w_r - \bar{w}_r\|_{C^2_{m-2}(\Pi_{2\varepsilon}\setminus\Omega_{\sigma})} \leq c d^{2/2} \varepsilon^2.
\]
Therefore
\[
\lim_{\varepsilon \to 0} \|w_r - \bar{w}_r\|_{C^1_0(\Omega_{2\varepsilon})} = \lim_{\varepsilon \to 0} \|w_r - \bar{w}_r\|_{C^2_{m-2}(\Pi_{2\varepsilon}\setminus\Omega_{\sigma})} = 0.
\]

In addition, we also have
\[
\phi_u - \phi_{\tilde{u}} = \psi_{d_0} - \psi_0 \circ \left( \frac{\bar{u}_r - w_r}{|\bar{u}|} \right) \circ \varphi_{a-\alpha(\varepsilon)} + i\tilde{K}.
\]
Applying once more the result of Proposition 10.2, we get
\[
\|w_r - \bar{w}_r\|_{C^1_0(\Omega_{2\varepsilon})} + \|w_r - \bar{w}_r\|_{C^2_{m-2}(\Pi_{2\varepsilon}\setminus\Omega_{\sigma})} \leq c (d + |d_0 - \tilde{d}_0| + d^{2/2} \varepsilon^2).
\]
Therefore
\[
\lim_{\varepsilon \to 0} \|w_r - \bar{w}_r\|_{C^1_0(\Omega_{2\varepsilon})} = \lim_{\varepsilon \to 0} \|w_r - \bar{w}_r\|_{C^2_{m-2}(\Pi_{2\varepsilon}\setminus\Omega_{\sigma})} = 0.
\]

**Estimates for $w$ in $B_{2\varepsilon}$.** In $B_\varepsilon$, $\chi \equiv 1$, hence we have
\[
\bar{w} - w = (e^{-i\tilde{K}} - 1) w + e^{-i\tilde{\phi}} (u - \bar{v}) - (e^{i\psi_0} - e^{i(d_0-\tilde{K})}) |\bar{u}|.
\]
Applying one last time Proposition 10.2, it is easy to see that
\[ \| w - \tilde{w} \|_{C^2_0(\Omega \setminus \Omega_\epsilon)} \leq c(d \| w \|_{C^2_0(\Omega \setminus \Omega_\epsilon)} + d^{2/p} \epsilon^\nu + d + |d_0 - \tilde{d}_0|). \]
Hence, we conclude that
\[ \lim_{\epsilon \to 0} \| w - \tilde{w} \|_{C^2_0(\Omega \setminus \Omega_\epsilon)} = 0. \]

Though things are slightly more involved, we can prove using the same strategy that
\[ \lim_{\epsilon \to 0} \| w - \tilde{w} \|_{C^2_0(\Omega \setminus \Omega_\epsilon)} = 0. \] 
(10.57)

Clearly, (10.54) and (10.52) imply that \( l_1 \) (which is defined in Lemma 6.1) tends to 0 as \( \epsilon \) does while (10.55), (10.56) and (10.57) imply that \( l_2 \) (which is defined in Lemma 6.2) and \( l_3 \) (which is defined in Lemma 6.3) also tend to 0 as \( \epsilon \) does. This proves the desired result.

10.3 A conjecture of F. Bethuel, H. Brezis and F. Hélein

In the special case where \( \Omega = B_1 \) and \( g = e^{i\theta} \) the previous Theorem allows us to give a positive (partial) answer to Brezis’s conjecture.

**Theorem 10.2** Any critical point of the Ginzburg-Landau functional on the unit disk which is equal to \( e^{i\theta} \) on the boundary is, for \( \epsilon \) sufficiently small, the axially symmetric solution defined in Theorem 8.2.

Otherwise stated, any solution of
\[
\begin{align*}
\Delta u + \frac{u}{\epsilon^2} (1 - |u|^2) &= 0 \quad \text{in} \quad B_1 \\
u &= e^{i\theta} \quad \text{on} \quad \partial B_1
\end{align*}
\] 
(10.58)
is, for \( \epsilon \) small enough, the axially symmetric solution defined in Theorem 8.2. This result was conjectured by F. Bethuel, H. Brezis and F. Hélein in [11].

The main difference between Theorem 10.1 and Theorem 10.2 is that, in the latter, we do not assume any \emph{a priori} bound on the energy of the solution. Recall that this bound is needed in theorem 10.1 as it was already needed in Theorem 9.1, to control the number of vortices of the solutions. However, since the unit ball is a starshaped domain, we can integrate over \( B_1 \) the Pohozaev formula which was derived in Chapter 10, Lemma 9.1. As a result, we get
\[ \int_{\partial B_1} \left( \partial X u \cdot \partial u - \frac{1}{2} \nabla u |^2 X \cdot \nu \right) + \frac{1}{2\epsilon^2} \int_{B_1} (1 - |u|^2)^2 = 0, \]
where the vector field $X$ is given by $X := (x, y)$. Hence
\[
\frac{1}{2} \int_{\partial B_1} |\partial_r u|^2 - \frac{1}{2} \int_{\partial B_1} |\partial_t u|^2 + \frac{1}{2\varepsilon^2} \int_{B_1} (1 - |u|^2)^2 = 0.
\]

Since $\partial_r u = \partial_r e^{i\theta}$ on $\partial B_1$ we conclude that
\[
\int_{\partial B_1} |\partial_r u|^2 + \frac{1}{\varepsilon^2} \int_{B_1} (1 - |u|^2)^2 = 2\pi.
\]

It is also proved in Chapter X of [11] that given any sequence of solutions of (10.58), one can always extract a subsequence converging to a $S^1$ valued harmonic map of the form
\[
u_* = \prod_{j=1}^N \left( \frac{z - a_j}{|z - a_j|} \right)^{d_j} e^{i\phi},
\]
where $\phi$ is an harmonic function in $B_1$. We then invoke Theorem X.5 of [11] which implies that
\[
\lim_{\varepsilon \to 0} \int_{B_1} \frac{|1 - |u|^2|^2}{\varepsilon^2} = 2\pi \sum_{j}^{N} d_j^2.
\]

However, using (10.59), we deduce then that necessarily $N = 1$ and $d_1 = \pm 1$. Using a degree argument, we conclude that $d_j = 1$.

In addition, it is proved in [11] (see pages 96-97) that necessarily the unique vortex $a_1$ is located at the origin and also that it is a nondegenerate critical point of the corresponding renormalized energy $W_g$ (see also [42] where a simple expression of the renormalized energy in the axially symmetric setting is derived). Hence, $\nu_* = \frac{z}{|z|}$.

Using all these informations, we see that Theorem 10.2 is now a Corollary of Theorem 10.1.
Chapter 11

Towards Jaffe and Taubes conjectures

We prove that, under some natural energy bound, all solutions of the gauge invariant Ginzburg-Landau equation are gauge equivalent to the radially symmetric solution.

11.1 Statement of the result

In this last Chapter, we explain how the results we have established in the previous Chapters may be applied to study the gauge invariant Ginzburg-Landau functional

\[ G_\varepsilon(u, A) = \int_\Omega |d_A u|^2 + \frac{1}{2\varepsilon^2} \int_\Omega (1 - |u|^2)^2 + \int_\Omega |dA|^2, \]

for large values of the coupling parameter $1/\varepsilon$. Here $u$ is a complex valued function and $A$ can be understood either as a vector field

\[ A := (A_x, A_y), \]

or as a 1-form

\[ A := A_x \, dx + A_y \, dy. \]

We recall that, by definition

\[ d_A u := da - i A u. \]

We will always restrict our attention to the case where

\[ \Omega = \mathbb{C}. \]
11.1.1 Preliminary remarks

It is an easy exercise to show that, if \((u, A)\) is a critical point of \(G_\varepsilon\), then \((u, A)\) is a solution of the following Euler-Lagrange equation

\[
\begin{cases}
\Delta u - 2i \partial_A u - |A|^2 u - i \text{div}\ A u = -\frac{u}{\varepsilon^2} (1 - |u|^2) & \text{in } \mathbb{C} \\
-\star d \star d A = u \wedge du - |u|^2 A & \text{in } \mathbb{C}.
\end{cases}
\] (11.1)

Where \(\star\) is the Hodge operator defined on 2-forms by

\[
\star dx dy := 1,
\]
on 1 forms by

\[
\star dx := dy \quad \text{and} \quad \star dy := -dx,
\]
and on 0-forms by

\[
\star 1 := dx dy.
\]

Given \((u, A)\) solution of (11.1), we define the induced magnetic field \(h\) by

\[
h := \star dA.
\]

Observe that \(h\) is a 0-form, namely a function. It will be useful to notice that, if \((u, A)\) is a solution of (11.1), then \(h\) satisfies

\[
-\text{div} \left( \frac{\nabla h}{|u|^2} \right) + h = 0,
\] (11.2)

away from the zero set of \(u\). Indeed, we have

\[
-\star dh = u \wedge dA u = u \wedge (du - iAu).
\]

Hence

\[
\star d \left( \frac{1}{|u|^2} dh \right) = -\star d \left( \frac{u}{|u|^2} \wedge (du - iAu) \right) = \star d \left( \frac{u}{|u|^2} \wedge (iAu) \right) = \star dA = h.
\]

Here we have used the fact that, away from the zeros of \(u\), we have the identity

\[
d \left( \frac{u}{|u|^2} \wedge du \right) = 0,
\]

which may be proved by setting, at least locally away from the zero set of \(u\), we can write

\[
u := |u| e^{i\phi},
\] (11.3)
and thus, we compute

\[ \frac{u}{|u|^2} \wedge du = \frac{e^{i\phi}}{|u|} \wedge (e^{i\phi} d|u| + i |u| e^{i\phi} d\phi) = -d\phi. \]

Moreover, if \((u, A)\) is a solution of (11.1), we can also prove that \(|u|\) satisfies

\[ \Delta |u| + \frac{1}{\varepsilon^2} |u| (1 - |u|^2) = \frac{|\nabla h|^2}{|u|^3}, \]  

(11.4)

away from the zero set of \(u\). Indeed, using (11.3), we can multiply the first equation of (11.1) by \(e^{-i\phi}\) and take the real part. We find

\[ \Delta |u| + \frac{1}{\varepsilon^2} |u| (1 - |u|^2) = |A - \nabla \phi|^2 |u|. \]

Moreover, we can use the second equation of (11.1) which yields

\[ |\nabla h|^2 = |u \wedge (\nabla u - iAu)|^2 = |\nabla \phi - A|^2, \]

and the result follows at once.

One of the main properties of the functional \(G_\varepsilon\) is the fact that it is invariant under gauge transformation. More precisely, for any sufficiently smooth function \(\phi\), if we define

\[ u_\phi := e^{i\phi} u, \]

and

\[ A_\phi := A + d\phi. \]

Then

\[ G_\varepsilon(u, A) = G_\varepsilon(u_\phi, A_\phi). \]

In particular, if \((u, A)\) is a solution of (11.1) so is \((u_\phi, A_\phi)\).

Some quantities do not depend on the particular choice of gauge. For example, the induced magnetic field \(h := \star dA\), the current \(u \wedge dA\) and \(|u|\) which do not depend on the choice of gauge.

Very often, it will be convenient to choose a particular gauge to work with: the Coulomb gauge. To define it on some open simply connected subset \(\omega \subset \mathbb{C}\), we solve the following elliptic problem

\[ \begin{cases} 
\Delta \phi &= -\star d \star A & \text{in } \omega \\
\partial_\nu \phi &= -A \cdot \nu & \text{on } \partial \omega.
\end{cases} \]  

(11.5)

And then, we set

\[ (\tilde{u}, \tilde{A}) := (e^{i\phi} u, A + d\phi). \]
Observe that with this particular choice, we have
\[
\begin{cases}
\star d \star \tilde{A} = 0 & \text{in } \omega \\
\tilde{A} \cdot \nu = 0 & \text{on } \partial \omega.
\end{cases}
\]

It is interesting to notice that, since \( \omega \) is simply connected, one may equivalently define \( \xi \) as the solution of
\[
\begin{cases}
\Delta \xi = \star dA & \text{in } \omega \\
\xi = 0 & \text{on } \partial \omega,
\end{cases}
\]
and then define \( \tilde{A} = \star d\xi \). Obviously, we have the relation
\[
\star d\xi = A + d\phi.
\]
And \( \Delta \xi = 0 \) follows easily. Furthermore, since \( \partial \nu + A \cdot \nu = 0 \) on \( \partial \omega \) we see that \( \partial_r \xi = 0 \) on \( \partial \omega \). Hence, \( \xi \) is constant on \( \partial \omega \). Since \( \star d\xi \) does not depend on the particular choice of this constant, we may take \( \xi = 0 \) on \( \partial \omega \).

### 11.1.2 The uniqueness result

The existence of a radially symmetric solution (up to gauge transformation) has been proved by M.S. Berger and Y.Y. Chen.

**Theorem 11.1** [9] There exists \((v, B)\) a non constant solution of (11.1) which is of the form
\[
v := \hat{S}_\varepsilon e^{i\theta} \quad \text{and} \quad B := \hat{T}_\varepsilon d\theta,
\]
where \( \hat{S}_\varepsilon \) and \( \hat{T}_\varepsilon \) only depend on \( r \). The functions \( \hat{S}_\varepsilon \) and \( \hat{T}_\varepsilon \) are strictly increasing and satisfy
\[
\hat{S}_\varepsilon(0) = \hat{T}_\varepsilon(0) = 0, \quad \lim_{r \to +\infty} \hat{S}_\varepsilon = \lim_{r \to +\infty} \hat{T}_\varepsilon = 1.
\]
Finally \( 0 < \hat{S}_\varepsilon < 1 \) and \( 0 < \hat{T}_\varepsilon < 1 \) for all \( r > 0 \).

Observe that the degree of \( v/|v| \) at infinity is equal to \(+1\). To obtain the radially symmetric solution whose degree at infinity is \(-1\) is suffices to choose
\[
\tilde{v} := \hat{S} e^{-i\theta} \quad \text{and} \quad \tilde{B} := -\hat{T} d\theta.
\]

The main observation is that, if we look for solutions of the form
\[
v := \hat{S}_\varepsilon e^{i\theta} \quad \text{and} \quad B := \hat{T}_\varepsilon d\theta,
\]
then \( \hat{S}_\varepsilon \) and \( \hat{T}_\varepsilon \) have to solve the following coupled system of ordinary differential equations
\[
\begin{cases}
\frac{d^2 \hat{S}}{dr^2} + \frac{1}{r} \frac{d\hat{S}}{dr} + \frac{\hat{S}}{\varepsilon^2} (1 - \hat{S}^2) - \left( \hat{T} - \frac{1}{r} \right)^2 \hat{S} = 0 \\
\frac{d^2 \hat{T}}{dr^2} - \hat{S}^2 (\hat{T} - 1) = 0.
\end{cases}
\]
The existence of a solution can be obtained as follows. First, for any $R > 1$, we obtain a solution of this system defined in $[1/R, R]$, with the boundary conditions $\tilde{S}(1/R) = \tilde{T}(1/R) = 0$ and $\tilde{S}(R) = \tilde{T}(R) = 1$. For example, this solution can be obtained by minimizing the corresponding energy. Then, we obtain a priori estimates which allow one to pass to the limit as $R$ tend to $+\infty$.

One of the main qualitative difference between the function $S_\varepsilon := S(\cdot/\varepsilon)$, where $S$ has been defined in Theorem 3.1, and the function $\tilde{S}_\varepsilon$ defined in the last Theorem is that while $S_\varepsilon$ converges quadratically to 1 at $\infty$, the function $\tilde{S}_\varepsilon$ converges exponentially to 1 at $\infty$. This is an important observation which has many physical consequences [36]. Furthermore, observe that the identity
\[ S_\varepsilon = S_{1}(\cdot/\varepsilon), \]
which is obviously true, is no longer true for $\tilde{S}_\varepsilon$.

We will need the following:

**Lemma 11.1** There exists a constant $c > 0$ such that, for all $\varepsilon \in (0, 1)$ we have
\[ \frac{r \partial_r |v|}{|v|} \leq c, \]
in $C$ where $v$ is the solution defined in the previous Theorem.

Our main result asserts that any solution of (11.1) which satisfies some energy bound, is gauge equivalent to the radially symmetric solution defined in the previous Theorem. More precisely, we prove the

**Theorem 11.2** Let $c_0 > 0$ be given. Then, there exists $\varepsilon_0 > 0$ such that, any $(u_\varepsilon, A_\varepsilon)$ solution of (11.1) satisfying
\[ G_\varepsilon(u_\varepsilon, A_\varepsilon) \leq 2\pi \log 1/\varepsilon + c_0, \] (11.7)
then $(u_\varepsilon, A_\varepsilon)$ is gauge equivalent to one of the two axially symmetric solutions of (11.1) having respectively degree +1 or −1 at infinity.

### 11.2 Gauge-invariant Ginzburg-Landau critical points with one zero.

Before we proceed to the proof of this result, we will give a thorough description of the general shape of any solution of (11.1). What we want to obtain is a result close to the result we have obtained in Proposition 9.1 for the non-gauge invariant Ginzburg-Landau equation.

**Proposition 11.1** Let $c_0 > 0$ be fixed. Assume that $(u_\varepsilon, A_\varepsilon)$ is a solution of (11.1) which satisfies (11.7). Further assume that $u_\varepsilon(0) = 0$. Then, for $\varepsilon$ small
enough, \( u_ε^{-1}(\{0\}) = \{0\} \), the degree of \( u_ε \) at 0 is equal to \( \pm 1 \) and there exists \( \lambda > 0 \) such that

\[
\inf_{C \setminus B_\lambda} |u_ε| \geq \frac{1}{2}.
\]

Moreover, up to a subsequence, there exists a gauge \( \phi_ε \) such that the sequence of rescaled functions

\[
\tilde{u}_ε := e^{\phi_ε} u_ε(ε \cdot),
\]

converges in \( C^{k,\alpha}_{\text{loc}} \) topology to \( S e^{i(\theta + \tau)} \) for some \( \tau \in \mathbb{R} \), for all \( k \in \mathbb{N} \).

The proof of this result is very close to the proof of the corresponding result in Chapter 10, the main difference being that, this time, we have to prove also that the upper-bound (11.7) implies that \( u_ε \) has a unique zero. Again, one of the main ingredients is the Pohozaev formula, which, in the case of the gauge invariant Ginzburg-Landau equation, reads:

**Lemma 11.2** [36], [83]. Assume that \((u, A)\) is a solution of (11.1) and define the vector field \( X \) by

\[
X := (x, y).
\]

Then for all \( R > 0 \) and all \( z_0 \in \mathbb{C} \), we have

\[
\frac{R}{2} \int_{\partial B_R(z_0)} (|(\nabla u - iAu) \cdot \tau|^2 - |(\nabla u - iAu) \cdot \nu|^2)
+ \frac{R}{4\varepsilon^2} \int_{\partial B_R(z_0)} (1 - |u|^2)^2 X \cdot \nu - \frac{R}{2} \int_{\partial B_R(z_0)} |dA|^2
- \frac{1}{2\varepsilon^2} \int_{B_R(z_0)} (1 - |u|^2)^2 + \int_{B_R(z_0)} |dA|^2 = 0.
\]

**Proof**: Notice that, in order to simplify the notations, we can assume that \( z_0 = 0 \). In order to prove this formula, the simplest technique is to consider \((u, A)\) to be a critical point of \( G_ε \) with respect to the variations of the domain as we have done in the first section of Chapter 9.

Assume that \( \eta \) is a cutoff function. We consider, for all \( t \) small enough, the one parameter family of functions

\[
u_t(z) := u(z + t \eta z),
\]

and of one forms

\[
A_t(z) = (1 + t \eta) A(z + t \eta z) + t (z \cdot A(z + t \eta z)) \, d\eta.
\]

and write that

\[
\frac{d}{dt} G_ε(u_t, A_t)_{|t=0} = 0.
\]

The reason why we have done such choices can be motivated as follows. With these formulas we get explicitly

\[
\nabla u_t - iA_t u_t = (1 + t \eta) (\nabla u - iAu)(z + t \eta z)
+ t \nabla \eta z \cdot ((\nabla u - iAu)(z + t \eta z)).
\]

(11.8)
And observe that, under a gauge transformation

\[ u' := e^{i\phi} u \quad \text{and} \quad A' := A + d\phi, \]

the identity (11.8) is transformed into

\[ \nabla u_t' - i A_t' u_t' = e^{i\phi(z + t\eta z)} (\nabla u_t - i A_t u_t). \]

Hence the gauge invariance is preserved.

Now we set

\[ v := \eta \partial_X u, \]

and

\[ B := \eta A + \eta \partial_X A + (X \cdot A) \nabla \eta, \]

where, as above, \( \delta > 0 \) is some small parameter and \( \eta \) is a cutoff function which is radial, identically equal to 1 in \( B_R \) and equal to 0 outside \( B_{R+\delta} \). Notice that, in order to simplify the notations, we have assumed that \( z_0 = 0 \).

Since \((u, A)\) is assumed to be a critical point of \( G_\epsilon \), we may write

\[ \int_C (\nabla u - iAu) \cdot (\nabla v - iAv - iBu) - \frac{1}{\epsilon^2} \int_C (1 - |u|^2) u \cdot v + \int_C dA dB = 0. \]

We now evaluate each term involved in this identity and then let \( \delta \) tend to 0.

We find, with little work,

\[ \int_C (\nabla u - iAu) \cdot (\nabla v - iAv - iBu) = \int_C (\nabla u - iAu) \cdot (\partial_X u - i(X \cdot A) u) \nabla \eta - \frac{1}{2} \int_C \partial_X \eta |\nabla u - iAu|^2 \]

Letting \( \delta \) tend to 0 in this expression, we obtain

\[ \lim_{\delta \to 0} \int_C (\nabla u - iAu) \cdot (\nabla v - iAv - iBu) = \frac{R}{2} \int_{\partial B_R} |\nabla u - iAu|^2 \]

\[ - R \int_{\partial B_R} (\nabla u - iAu) \cdot (\partial_X u - i(X \cdot A) u). \]

Next, we find

\[ \frac{1}{\epsilon^2} \int_C (1 - |u|^2) u \cdot v + \int_C dA dB = - \frac{1}{2\epsilon^2} \int_C \eta (1 - |u|^2)^2 + \frac{1}{4\epsilon^2} \int_C \partial_X \eta (1 - |u|^2)^2. \]
And, letting \( \delta \) tend to 0, we get
\[
\lim_{\delta \to 0} \frac{1}{\varepsilon^2} \int_C (1 - |u|^2) u \cdot v + \int_C dA dB = -\frac{1}{2\varepsilon^2} \int_{B_R} (1 - |u|^2)^2 + \frac{R}{4\varepsilon^2} \int_{\partial B_R} (1 - |u|^2)^2.
\]

Finally, we obtain, after a few efforts
\[
\int_C dA dB = \int_C \eta |dA|^2 + \frac{1}{2} \int_C \partial_X \eta |dA|^2.
\]
Which leads to
\[
\lim_{\delta \to 0} \int_C dA dB = \int_{B_R} |dA|^2 - \frac{R}{2} \int_{\partial B_R} |dA|^2.
\]
The result then follows at once.

Let us assume that \((u, A)\) is a solution of (11.1) which satisfies (11.7). First of all, combining the equation (11.1) and the energy upper-bound (11.7), we get as in the proof of Proposition II.6 of [12]
\[
\|d u - i A u\|_{L^\infty} \leq c \varepsilon^{-1}. \quad (11.9)
\]
Furthermore, observe that, we can write, at least locally away from the zeros of \(u\),
\[
u := |u| e^{i\phi},
\]
in which case, we find
\[
d u - i A u = (d|u| + i (d\phi - A) u)) e^{i\phi}.
\]
Therefore
\[
|d u - i A u|^2 = |d|u|^2 + |d\phi - A|^2.
\]
Hence we also have, as a consequence of the previous estimate
\[
\|\nabla u\|_{L^\infty} \leq \|d u - i A u\|_{L^\infty} \leq c \varepsilon^{-1}.
\]
Using the second equation of (11.1) we deduce the pointwise estimate
\[
|\nabla h| \leq |d u - i A u|. \quad (11.10)
\]
In particular, this implies that
\[
\|\nabla h\|_{L^\infty} \leq c \varepsilon^{-1}, \quad (11.11)
\]
and, thanks to (11.7), we also get
\[
\int_C |\nabla h|^2 + \int_C |h|^2 \leq \int_C |d u - i A u|^2 + \int_C |d A|^2 \leq 4\pi \log 1/\varepsilon + 2c_0. \quad (11.12)
\]
We will also need a local estimate which plays a crucial rôle in this Chapter. For all $p > 2$, all $z_0 \in \mathbb{C}$ and all $r \in (0, 1)$ we have by Hölder estimate
\[
\int_{B_r(z_0)} |h|^2 \leq |B_r(z_0)|^{(p-2)/p} \left( \int_{B_r(z_0)} |h|^p \right)^{2/p}.
\]
Using Sobolev embedding together with (11.12), we find
\[
\int_{B_r(z_0)} |h|^2 \leq c_p r^{(2p-4)/p} \log 1/\varepsilon,
\]
(11.13) for all $\varepsilon < 1$. Observe that $4\pi \log 1/\varepsilon + 2c_0 \leq c \log 1/\varepsilon$ for all $\varepsilon \in (0, 1)$!

We will also need the $\eta$-compactness Lemma for the gauge-invariant Ginzburg-Landau equation. This is a modified version of Lemma 9.2.

**Lemma 11.3** [$\eta$-compactness Lemma] Let $c_1 > 0$ be given. There exists $\eta > 0$, $R_0 \in (0, 1)$ and $\varepsilon_0 > 0$ such that, if $(u, A)$ is a solution of (11.1), with $\varepsilon \in (0, \varepsilon_0)$, which satisfies
\[
|\nabla|u|| \leq \frac{c_1}{4\varepsilon},
\]
and if there exists $z \in \mathbb{C}$ and $R \in (4\varepsilon, R_0)$ such that
\[
\int_{B_R(z)} \left( |\nabla u - iAu|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 + |dA|^2 \right) \leq \eta \log \frac{R}{2\varepsilon},
\]
then
\[
|u| \geq \frac{1}{2} \quad \text{in} \quad B_{R/2}(z).
\]
(11.16)

**Proof:** We choose $\eta = \frac{\pi}{32c_1^2}$ and argue by contradiction. Assume that, for some $z' \in B_{R/2}(z)$, we have $|u|(z') < 1/2$. For the sake of simplicity in the notations, let us assume that $z' = 0$. Thanks to (11.14) we see that
\[
|u| < \frac{3}{4}, \quad \text{in} \quad B_{\varepsilon/c_1}.
\]
Hence, we have
\[
\frac{1}{\varepsilon^2} \int_{B_{\varepsilon/c_1}} (1 - |u|^2)^2 > \frac{\pi}{16c_1^2} = 2\eta.
\]
(11.17)

Now, we integrate the Pohozaev formula which was derived in the previous Lemma, over the ball of radius $r$, we find using (11.13)
\[
\frac{1}{\varepsilon^2} \int_{B_r} (1 - |u|^2)^2 \leq r \int_{\partial B_r} g(u, A) + c_p r^{(2p-4)/p} \log 1/\varepsilon.
\]
Where we have set
\[
g(u, A) := |\nabla u - iAu|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 + |dA|^2.
\]
Integrating this inequality over $r \in (\varepsilon, R/2)$, we find using (11.15)

$$
\int_\varepsilon^{R/2} \frac{1}{r} \left( \int_{B_r} \frac{(1 - |u|^2)^2}{\varepsilon^2} \right) dr \leq \eta \log \frac{R}{2\varepsilon} + c_p R^{(2p-4)/p} \log 1/\varepsilon.
$$

Thanks to the mean value formula, we can conclude that there exists $r_0 \in [\varepsilon, R/2]$ such that

$$
\frac{1}{\varepsilon^2} \int_{B_{r_0}} (1 - |u|^2)^2 \leq \eta + c_p R^{(2p-4)/p} (\log 1/\varepsilon) (\log \frac{R}{2\varepsilon})^{-1}.
$$

If we assume that $R_0$ and $\varepsilon_0$ are chosen so that

$$c_p R^{(2p-4)/p} \log 1/\varepsilon (\log \frac{R}{2\varepsilon})^{-1} \leq \eta,$$

for all $R \in (4\varepsilon, R_0)$ and all $\varepsilon \in (0, \varepsilon_0)$, we see that (11.18) clearly contradicts (11.17). Hence $|u|'(z') \geq 1/2$.

**Proof of Proposition 11.1**: To begin with, let us observe that the result of Lemma 9.3 remains true. Namely, there exist $\tilde{\varepsilon}_0 > 0$, $n \in \mathbb{N}$ and $\lambda \geq 1$ only depending on $c_0$ and, for all $\varepsilon \in (0, \varepsilon_0)$, there exist $n_\varepsilon$ points $z_1, \ldots, z_{n_\varepsilon} \in \mathbb{C}$ such that:

1. $\{z \in \mathbb{C} : |u| < 1/2\} \cap B_{\lambda \varepsilon}(z_j) \neq \emptyset$.
2. $\{z \in \mathbb{C} : |u| < 1/2\} \subset \bigcup_{j=1}^{n_\varepsilon} B_{\lambda \varepsilon}(z_j)$.
3. For all $\varepsilon \in (0, \varepsilon_0)$, $n_\varepsilon \leq n$.
4. For all $k \neq l$, $B_{2\lambda \varepsilon}(z_k) \cap B_{2\lambda \varepsilon}(z_l) = \emptyset$.

As in the proof of Lemma 9.3, the proof of this result relies on the extensive use of the Pohozaev formula together with the $\eta$-compactness Lemma.

Recall that we may also assume that $\lambda$ is chosen large enough in order to ensure that $S > 3/4$ for all $r > \lambda$.

We prove now that $n_\varepsilon = 1$ provided $\varepsilon$ is chosen small enough. We argue by contradiction and assume that this is not the case. Hence, we can find a sequence $\varepsilon_i \to 0$ such that $n_{\varepsilon_i} \geq 2$. To keep the notations short, we will drop the $i$ indices and write $n_\varepsilon$ instead of $n_{\varepsilon_i}$, $u_\varepsilon$ instead of $u_{\varepsilon_i}$, ...

Up to a subsequence, we may always assume that either

$$
\sup_{k \neq l} \frac{|z_k - z_l|}{\varepsilon},
$$

tends to $+\infty$ as $\varepsilon$ tends to 0 or

$$
\sup_{k \neq l} \frac{|z_k - z_l|}{\varepsilon},
$$

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stays bounded as $\varepsilon$ tends to 0.

**Step 1.** We first consider the case where

$$\sup_{k \neq l} \frac{|z_k - z_l|}{\varepsilon},$$

tends to $+\infty$ as $\varepsilon$ tends to 0. Observe that, increasing the value of $\lambda$ if this is necessary, we may assume that for all $k \neq l$

$$\frac{|z_k - z_l|}{\varepsilon},$$

tends to $+\infty$ as $\varepsilon$ tends to 0. Obviously, this value of $\lambda$ may depend on the sequence we are dealing with but this is not relevant for the remaining of the argument. We keep on assuming that the number of balls $B_{\lambda \varepsilon}(z_k)$ needed to cover the set where $|u_\varepsilon| \leq 1/2$ is larger than or equal to two.

Given any $B_r(z)$, we will use the Coulomb gauge on it. Namely, we will replace $(u_\varepsilon, A_\varepsilon)$ by the gauge equivalent solution $(\tilde{u}_\varepsilon, \tilde{A}_\varepsilon)$ where

$$\tilde{A}_\varepsilon := \ast d\xi$$

and where $\xi$ is the solution of

$$\begin{cases}
\Delta \xi = \ast dA_\varepsilon & \text{in } B_r(z) \\
\xi = 0 & \text{on } \partial B_r(z).
\end{cases}$$

(11.19)

For all $p > 2$, classical elliptic estimates lead to

$$\|\tilde{A}_\varepsilon\|_{L^\infty(B_r)} \leq c r^{p-2} \int_{B_r} |h|^p.$$  

However, Sobolev embedding implies that

$$\int_{B_r} |h|^p \leq c \|h\|_{H^1(C)}^p.$$

We now make use of (11.12) to conclude that

$$\|\tilde{A}_\varepsilon\|_{L^\infty(B_r)} \leq c r^{(p-2)/p} (\log 1/\varepsilon)^{1/2}. \quad (11.20)$$

To keep notations short, we will write $(u_\varepsilon, A_\varepsilon)$ instead of $(\tilde{u}_\varepsilon, \tilde{A}_\varepsilon)$, referring to the latter one as the **Coulomb gauge** in $B_r$.

We claim that, for all $l = 1, \ldots, n_\varepsilon$, we have

$$\deg \left( \frac{u_\varepsilon}{|u_\varepsilon|}; \partial B_{\lambda \varepsilon}(z_l) \right) \neq 0, \quad (11.21)$$
provided \( \varepsilon \) is small enough. Indeed, fix some index \( l \in \{1, \ldots, n\} \). By assumption, we can find a sequence \( R_{\varepsilon} \in (0, 1/\varepsilon) \) which tends to \( +\infty \) such that

\[
B_{R_{\varepsilon}}(z_l) \cap B_{\lambda \varepsilon}(z_k) = \emptyset,
\]

for all \( k \neq l \). We make use of the Coulomb gauge in \( B_{R_{\varepsilon}}(z_l) \) and define the sequences of rescaled functions

\[
\hat{u}_{\varepsilon} := u_{\varepsilon}(\varepsilon \cdot + z_l) \quad \text{and} \quad \hat{A}_{\varepsilon} := A_{\varepsilon}(\varepsilon \cdot + z_l),
\]

in \( B_{R_{\varepsilon}}(0) \). Obviously, \((\hat{u}_{\varepsilon}, \hat{A}_{\varepsilon})\) satisfy the following equations

\[
\Delta \hat{u}_{\varepsilon} + \hat{u}_{\varepsilon}(1 - |\hat{u}_{\varepsilon}|^2) - \varepsilon^2 |\hat{A}_{\varepsilon}|^2 \hat{u}_{\varepsilon} - 2 i \varepsilon \partial_{\hat{A}_{\varepsilon}} \hat{u}_{\varepsilon} = 0.
\]

Observe that we do not have the term \(-i \varepsilon \text{div} \hat{A}_{\varepsilon} \hat{u}_{\varepsilon}\) since we have chosen the Coulomb gauge.

Using (11.20), (11.7) together with the fact that \( |u| \leq 1 \), we deduce that, given any \( R > 0 \), the sequence \( \Delta \hat{u}_{\varepsilon} \) is uniformly bounded in \( L^2(B_{2R}) \). Hence, \( \nabla \hat{u}_{\varepsilon} \) is bounded in any \( L^p(B_{3R/2}) \), for \( p > 1 \). Using this information back into the equation, we see that, for all \( p > 1 \), the sequence \( \nabla^2 \hat{u}_{\varepsilon} \) is also bounded in any \( L^p(B_{R}) \). In particular, we can conclude that, up to a subsequence, \( \hat{u}_{\varepsilon} \) converges at least in \( C^0_{\text{loc}}(\mathbb{C}) \) topology to \( \hat{u} \) which is a solution of

\[
\Delta \hat{u} + \hat{u}(1 - |\hat{u}|^2) = 0 \quad \text{in} \quad \mathbb{C},
\]

\[
\int_{\mathbb{C}} (1 - |\hat{u}|^2)^2 < +\infty.
\]

Moreover, \(|\hat{u}| > \frac{1}{2}\) in \( \mathbb{C} \setminus B_{\lambda} \).

It is proved in [15] that either

\[
\text{deg} \left( \frac{\hat{u}}{|\hat{u}|}; \partial B_{\lambda}(0) \right) \neq 0,
\]

or \( \hat{u} \) is constant with \(|\hat{u}| = 1\) in \( \mathbb{C} \). However, in the latter case, because of the \( C^0 \) convergence, this would imply that \(|\hat{u}_{\varepsilon}| \geq 1/2\) in \( B_{\lambda} \) for \( \varepsilon \) small enough, which contradicts the definition of \( z_l \). Hence

\[
\text{deg} \left( \frac{\hat{u}}{|\hat{u}|}; \partial B_{\lambda}(0) \right) \neq 0,
\]

and, since degrees do converge, we have proved (11.21).

Let us recall that, by definition \( h = * dA_{\varepsilon} \). Arguing as in Theorem V.1 of [12] or as in part 5 of [83], we deduce that, for all \( p \in (1, 2) \)

\[
\| * dA_{\varepsilon} \|_{W^{1, p}(\mathbb{C})} \leq c,
\]

(11.23)
for some constant $c > 0$ which does not depend on $\varepsilon$. Moreover, since (11.7) holds, we may find a sequence of radii $r_i \to +\infty$ such that

$$
\lim_{i \to +\infty} r_i \int_{\partial B_{r_i}} \left( |\partial_A u|^2 + \frac{(1 - |u|^2)^2}{2\varepsilon^2} + |dA|^2 \right) = 0.
$$

This, together with Pohozaev identity proved in Lemma 11.2, implies that

$$
\frac{1}{2\varepsilon^2} \int_C (1 - |u\varepsilon|^2)^2 = \int_C |dA\varepsilon|^2.
$$

(11.24)

Hence we also conclude that

$$
\frac{1}{2\varepsilon^2} \int_C (1 - |u\varepsilon|^2)^2 \leq c,
$$

(11.25)

for some constant $c > 0$ which does not depend on $\varepsilon$.

It is easy to see that, up to a subsequence, there exists $R > 0$, which does not depend on $\varepsilon$, such that either $B_R(z_l) \cap B_R(z_k) = \emptyset$ or $|z_k - z_l| \leq R/2$. Assuming that, for each $l$, we are working with the Coulomb gauge in $B_R(z_l)$, we obtain from (11.23) that

$$
\int_{B_R(z_l)} |A\varepsilon|^2 \leq c,
$$

(11.26)

where, once more, $c > 0$ is independent of $\varepsilon$. Combining this upper-bound together with (11.7) we deduce that

$$
\int_{B_R(z_l)} |\nabla u\varepsilon|^2 \leq 2\pi \log 1/\varepsilon + c.
$$

(11.27)

Let us denote by $\tilde{z}_1, \ldots, \tilde{z}_q$ the collection of points $z_k$ which are included in $B_R(z_l)$. Because of the choice of $R$, of (11.25) and because $|u\varepsilon| \geq 1/2$ in $\mathbb{C} \setminus \bigcup_{k=1}^{n\varepsilon} B_{\lambda\varepsilon}(z_k)$, we can apply Theorem 4 and Theorem 5 of [15] and deduce that

$$
\int_{B_R(z_l)} |\nabla u\varepsilon|^2 \geq 2\pi \sum_{j=1}^{q} d_j^2 \log 1/\varepsilon - \sum_{i \neq j} d_i d_j \log |\tilde{z}_k - \tilde{z}_l| - c,
$$

(11.28)

where $d_i$ is the degree of $u\varepsilon$ at the point $\tilde{z}_i$.

Since, by assumption

$$
\lim_{\varepsilon \to 0} \frac{|x_i - x_j|}{\varepsilon} = +\infty,
$$

we have

$$
\sum_{i \neq j} d_i d_j \log |x_k - x_l| = o(\log 1/\varepsilon).
$$
We can now combine (11.27) together with (11.28), to conclude that (for $\varepsilon$ small enough) $q = 1$ and also that the degree of $u_\varepsilon$ at $z_l$ is $\pm 1$. In particular we have, thanks to (11.28)

$$
\int_{B_R(z_l)} |\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon|^2 \geq 2\pi \log \frac{1}{\varepsilon} - c.
$$

(11.29)

Since all the $B_R(z_l)$ have been assumed to be disjoint, this last inequality together with (11.7) gives the desired contradiction for $\varepsilon$ small enough.

**Step 2.** We now assume that

$$
\sup_{k \neq l} \frac{|z_k - z_l|}{\varepsilon},
$$

remains bounded. We fix $R > 0$ (independently of $\varepsilon$) in such a way that $B_{R\varepsilon}(z_1)$ contains all the $z_k$. Using a blow up argument as in Step 1, it is easy to see that

$$
\deg \left( \frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_{R\varepsilon}(z_1) \right) \neq 0.
$$

Moreover, we have already seen in (11.25) that

$$
\frac{1}{\varepsilon^2} \int_C (1 - |u_\varepsilon|^2)^2 < c,
$$

for some constant $c > 0$ which is independent of $\varepsilon$. Using lower bounds for the energy which was obtained in [15], we get that

$$
\int_{B_1(z_1) \setminus B_{R\varepsilon}(z_1)} |\nabla u_\varepsilon|^2 \geq 2\pi \deg \left( \frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_{R\varepsilon}(z_1) \right)^2 \log \frac{1}{\varepsilon} - c,
$$

(11.30)

which is valid independently of the choice of the gauge. In particular, if we take the Coulomb gauge in $B_1(z_1)$, we also have

$$
\int_{B_1(z_1)} |A_\varepsilon|^2 \leq c,
$$

as we have already seen in (11.26). This last inequality, together with (11.25) implies that

$$
\int_{B_1(z_1) \setminus B_{R\varepsilon}(z_1)} |\nabla u_\varepsilon - iA_\varepsilon u_\varepsilon|^2 \geq 2\pi \deg \left( \frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_{R\varepsilon}(z_1) \right)^2 \log \frac{1}{\varepsilon} - c.
$$

Comparing this inequality with (11.7), we conclude that

$$
\deg \left( \frac{u_\varepsilon}{|u_\varepsilon|}, \partial B_{R\varepsilon}(z_1) \right) = \pm 1,
$$

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since we have already seen that the degree is non zero. In particular, there exists a point where $u_\varepsilon$ vanishes. Relabeling the points $z_k$ if this is necessary, we can assume that $u_\varepsilon(z_1) = 0$.

In $B_{RE}(z_1)$, we take once more the Coulomb gauge. Let us define the rescaled functions

$$\hat{u}_\varepsilon := u(\varepsilon \cdot + z_1) \quad \text{and} \quad \hat{A}_\varepsilon := A_\varepsilon(\varepsilon \cdot + z_1).$$

As we have already seen in Step 1, $\hat{u}_\varepsilon$ is a solution of

$$\Delta \hat{u}_\varepsilon + \hat{u}_\varepsilon (1 - |\hat{u}_\varepsilon|^2) - \varepsilon^2 |\hat{A}_\varepsilon|^2 \hat{u}_\varepsilon - 2 i \varepsilon \hat{A}_\varepsilon \cdot \nabla \hat{u}_\varepsilon = 0,$$

in $\mathbb{C}$, since we are working with the Coulomb gauge. Moreover $\hat{u}_\varepsilon(0) = 0$,

$$\deg \left( \frac{\hat{u}_\varepsilon}{|\hat{u}_\varepsilon|}, \partial B_R \right) = \pm 1,$$

and finally

$$\int_{\mathbb{C}} (1 - |\hat{u}_\varepsilon|^2)^2 \leq c,$$

where $c > 0$ does not depend on $\varepsilon$. We can argue as in Step 1 to show that, up to a subsequence, the sequence $\hat{u}_\varepsilon$ converges at least in $C^0_\text{loc}$ norm, to $\hat{u}$ solution of

$$\Delta \hat{u} + \hat{u} (1 - |\hat{u}|^2) = 0 \quad \text{in} \quad \mathbb{C}$$

$$\int_{\mathbb{C}} (1 - |\hat{u}|^2)^2 < +\infty$$

$$\hat{u}(0) = 0$$

$$\deg \left( \frac{\hat{u}}{|\hat{u}|}, \partial B_R \right) = \pm 1.$$

Using Corollary 8.3, we conclude that $\hat{u}$ is equal to $Se^{i(\theta + \tau)}$, for some constant $\tau \in \mathbb{R}$. In particular, this implies that $|\hat{u}| \geq 3/4$ in $\mathbb{C} \setminus B_\lambda$. Therefore, for $\varepsilon$ small enough, $|u_\varepsilon| \geq 1/2$ in $B_{RE}(z_1) \setminus B_{\lambda \varepsilon}(z_1)$. This is again a contradiction.

Since we have obtained a contradiction in both cases, we have proved that $n_\varepsilon = 1$.

The remaining of the proof is now similar to what we have already done in Chapter 10, so we omit it.

One of the main observation established in [36] is the exponential decay at infinity of various intrinsic (gauge invariant) quantities defined with any $(u, A)$ solution of (11.1). More precisely, we have

**Proposition 11.2** For all $\alpha \in (0, 1)$ and all $k \in \mathbb{N}$, there exists $c > 0$ and $\varepsilon_0 > 0$, such that, for all $\varepsilon \in (0, \varepsilon_0)$ and for all $(u, A)$ solution of (11.1) which satisfies (11.7) and $u(0) = 0$

$$\sup_{r \geq 1} (e^{\alpha r} |\nabla^k h|) \leq c$$

$$\sup_{r \geq 1} (e^{2\alpha r} |\nabla^k (1 - |u|)|) \leq c \varepsilon^2.$$

(11.31)
where as usual \( h = \ast dA \).

**Proof:** To begin with, we know from the result of Proposition 11.1, that \( |u| \geq 1/2 \) in \( \mathbb{C} \setminus B_{\lambda \varepsilon} \). Where the constant \( \lambda \) is independent of \( \varepsilon \). Moreover, as we have seen in Step 1 of the proof of Proposition 11.1, for all \( p \in (1, 2) \), there exists \( c > 0 \) such that

\[
\|h\|_{W^{1,p}(\mathbb{C})} \leq c. \tag{11.32}
\]

Recall that \( h = \ast dA \) satisfies

\[
-\text{div} \left( \frac{\nabla h}{|u|^2} \right) + h = 0, \tag{11.33}
\]

in \( \mathbb{C} \setminus \{0\} \) and, thanks to (11.4), we see that \( 1 - |u| \) satisfies

\[
-\Delta (1 - |u|) + \left( \frac{|u| + |u|^2}{\varepsilon^2} \right)(1 - |u|) = \frac{|
abla h|^2}{|u|^3}, \tag{11.34}
\]

in \( \mathbb{C} \setminus \{0\} \).

Using (11.33) and arguing as in Step 6 of the proof of Theorem 1 in [10], we obtain for any \( k \in \mathbb{N} \), the existence of a constant \( c_k > 0 \) such that

\[
\|\nabla^k h\|_{L^\infty(\mathbb{C} \setminus B_{\lambda \varepsilon})} \leq c_k
\]

\[
\|\nabla^k (|u| - 1)\|_{L^\infty(\mathbb{C} \setminus B_{\lambda \varepsilon})} \leq c_k. \tag{11.35}
\]

Now, we can improve the second estimates. Indeed, it is not difficult to see that the potential in the equation satisfied by \( 1 - |u| \) satisfies

\[
\left( \frac{|u| + |u|^2}{\varepsilon^2} \right) \geq \frac{1}{4\varepsilon^2},
\]

in \( \mathbb{C} \setminus B_{\lambda \varepsilon} \). And then that the function

\[
r \longrightarrow c \left( e^{-r/(2\varepsilon)} + \varepsilon^2 \right),
\]

can be used as a barrier function to show that

\[
|1 - |u|| \leq c \left( e^{-r/(2\varepsilon)} + \varepsilon^2 \right),
\]

in \( \mathbb{C} \setminus B_{\lambda \varepsilon} \). In particular, this implies that

\[
|1 - |u|| \leq c \varepsilon^2, \tag{11.36}
\]

in \( \mathbb{C} \setminus B_{1/4} \), for all \( \varepsilon \) small enough. Using rescaled Schauder’s estimates we also conclude that

\[
|
abla (1 - |u|)| \leq c \varepsilon, \tag{11.37}
\]

in \( \mathbb{C} \setminus B_{1/2} \).
For all $\beta \in (0, 1)$, we define $G_\beta$ to be the solution of
\[ -\Delta G_\beta + \beta^2 G_\beta = \delta_0 \quad \text{in} \quad D'(\mathbb{C}). \tag{11.38} \]

It is well known that
\[ \lim_{r \to +\infty} \log G_\beta = -\beta. \tag{11.39} \]

Let us now fix $\beta \in (0, 1)$ in such a way that
\[ \alpha < \beta^2 < 1. \]

Next, we choose $M > 0$ large enough so that $|h| \leq MG_\beta$ on $\partial B_{1/2}$ and we set
\[ K_{\pm} := \pm h - MG_\beta. \]

Obviously, we have
\[ -\text{div} \left( \frac{\nabla K_{\pm}}{|u|^2} \right) + K_{\pm} = M \left( \frac{\beta^2}{|u|^2} G_\beta - 2 \frac{\nabla |u|}{|u|^3} \nabla G_\beta - G_\beta \right). \]

Using (11.36) and (11.37), it is an easy exercise to show that, provided $\varepsilon$ is chosen small enough, the right hand side of this identity is negative in $\mathbb{C} \setminus B_{1/2}$. Hence, we conclude that both $K_{\pm}$ verify
\[ \begin{cases} 
-\text{div} \left( \frac{\nabla K_{\pm}}{|u|^2} \right) + K_{\pm} \leq 0 & \text{on} \quad \mathbb{C} \setminus B_{1/2} \\
K_{\pm} \leq 0 & \text{on} \quad \partial B_{1/2}. \end{cases} \tag{11.40} \]

Using the maximum principle, we deduce that $|h| \leq MG_\beta$ in $\mathbb{C} \setminus B_{1/2}$. This already proves the first estimate in (11.31) when $k = 0$. The general case follows at once by induction.

In order to prove the second estimate, we apply similar arguments. First observe, using (11.39), that for all $\beta' \in (\alpha, \beta)$, we can find a constant $c > 0$ such that
\[ G_{2\beta'} \geq c G^2_\beta, \]
in $\mathbb{C} \setminus B_{1/2}$. Since we have already proved $|h| \leq MG_\beta$, we see that it is possible to choose $M > 0$ in such a way that
\[ \frac{|
abla h|^2}{|u|^3} \leq MG_{2\beta'} \]
in $\mathbb{C} \setminus B_{1/2}$ and, thanks to (11.36), in such a way that we also have
\[ MG_{2\beta'} \geq \frac{1 - |u|^2}{\varepsilon^2}, \]
on $\partial B_{1/2}$. We set
\[ H := 1 - |u| - \varepsilon^2 MG_{2\beta'}. \]

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A simple computation shows that
\[-\Delta H + \frac{(|u| + |u|^2)}{\varepsilon^2} H = \frac{\nabla h|^2}{|u|^3} - M \left(\frac{|u| + |u|^2}{|u|^3} - 4\varepsilon^2 (\beta')^2\right) G_{2\beta}.\]

For \(\varepsilon\) small enough, we can ensure that
\[
\begin{cases}
-\Delta H + \frac{(|u| + |u|^2)}{\varepsilon^2} H &\leq 0 \quad \text{in} \quad \mathbb{C} \setminus B_{1/2} \\
H &\leq 0 \quad \text{on} \quad \partial B_{1/2}. 
\end{cases}
\]

Applying the maximum principle once more, we deduce that
\[1 - |u| \leq M \varepsilon G_{2\beta}\]
in \(\mathbb{C} \setminus B_{1/2}\) which already proves the second estimate of (11.31) for \(k = 0\). The estimate for general \(k\) is obtained by induction. \(\square\)

11.3 Proof of Theorem 11.1

Let \((u, A)\) be a solution of (11.1) which satisfies (11.7). Thanks to Proposition 11.1 we know that, if \(\varepsilon\) is small enough, then the degree of \(u/|u|\) at \(\infty\) is equal to \(\pm 1\). In the remaining of this section, we will assume that the degree is always 1.

11.3.1 The Coulomb gauge

To begin with, let us make the following observation.

**Lemma 11.4** Under the assumptions of Theorem 11.1, if the degree of \(u/|u|\) at \(\infty\) is equal to 1, then
\[\int_{\mathbb{C}} h = 2\pi.\]

**Proof:** Indeed, we know from Proposition 11.2 that \(dh\) tends exponentially fast to 0 at \(\infty\). This implies that \(u \wedge (du - iAu)\) also tends to 0 exponentially fast at \(\infty\). Now write
\[u := |u| e^{i\phi},\]
to conclude that \(\nabla \phi - A\) tends to 0 exponentially fast at \(\infty\).

Since we have assumed that the degree of \(u/|u|\) at \(\infty\) is 1, we have
\[\int_{\partial B_R} \partial_{\theta} \phi = 2\pi R.\]

Observe that \(\partial_{\theta} \phi = A^\perp \cdot \nu\). Hence, we conclude that
\[\lim_{R \to +\infty} \frac{1}{R} \int_{\partial B_R} A^\perp \cdot \nu = 2\pi.\]
Finally, integration by parts yields
\[ \int_{\partial B} A^\perp \cdot \nu = \int_{B} \text{div} A^\perp = \int_{B} h. \]
And the result follows.

Let \((u, A)\) be a solution of (11.1) which satisfies (11.7). We will use the following gauge in \(C\)
\[
A(z) := \frac{1}{2\pi} \left( \begin{array}{c}
- \partial_y \int_C \log |z - z'| h(z') + c_x \\
\partial_x \int_C \log |z - z'| h(z') + c_y
\end{array} \right),
\] (11.43)
where \(c_x\) and \(c_y\) are chosen in such a way that \(A(0) = 0\). Otherwise stated,
\[ A + \nabla^\perp \left( \frac{1}{2\pi} \log r * h \right), \]
is a constant vector field.

Standard results on singular integrals [95], yields for any \(p \in (1, 2)\)
\[
\left( \int_C |\nabla A|^p \right)^{\frac{1}{p}} \leq c \left( \int_C |\nabla h|^p \right)^{\frac{1}{p}},
\]
Where \(\frac{1}{p^*} := \frac{1}{p} - \frac{1}{2}\). In addition, Morrey estimates yields [26], [95]
\[
|A(z) - A(z')| \leq c |z - z'|^{\alpha} \left( \int_C |\nabla A|^p \right)^{1/p^*},
\]
where \(\alpha := 1 - \frac{2}{p^*}\).

Recall that we have proven in Step 1 of the proof of Proposition 11.1, that
\[
\| \ast dA \|_{W^{1, p}(C)} \leq c, \quad (11.44)
\]
provided \(p \in (1, 2)\). Combining this information with the above classical estimates, we conclude that, for all \(p \in (1, 2)\), there exists \(c > 0\), independent of \(\varepsilon\) such that
\[
|A(z) - A(0)| \leq c |z|^{\alpha}, \quad (11.45)
\]
where
\[
\alpha = 2 - \frac{2}{p}.
\]

In the following Lemma, we give a precise description of the behavior of \(A\) at \(\infty\).
Lemma 11.5 Under the assumptions of Theorem 11.1, there exists $A_{\infty} \in \mathbb{R}^2$ such that the following estimate holds in $\mathbb{C} \setminus B_1$

$$|A(z) - A_{\infty} - \nabla \theta - \frac{1}{2\pi} \nabla^\perp \left( \frac{z}{|z|^2} \cdot \int_{\mathbb{C}} z' h(z') \right)| \leq c r^{-3}, \quad (11.46)$$

where the constant $c > 0$ does not depend on $\varepsilon$.

Proof: We define $\zeta = \frac{1}{2\pi} \log r * h$. Thanks to (11.42), we may write

$$\zeta(z) - \log r = \frac{1}{2\pi} \int_{\mathbb{C}} \log \frac{|z - z'|}{|z|} h(z').$$

With this definition, we see that $A + \nabla^\perp \zeta$, is a constant vector field.

Using the result of Proposition 11.2, we can estimate, for any $\alpha \in (0, 1)$

$$\left| \int_{|z'| \leq |z|/2} \partial_z \log \frac{|z - z'|}{|z|} h(z') \right| \leq c \int_{|z'| \leq |z|/2} \partial_z \log \frac{|z - z'|}{|z|} e^{-\alpha|z|} \leq c \left| \frac{z'}{|z'|^2} e^{-\alpha|z|} \right|,$$

where $c > 0$ depends on $\alpha$.

Now, for all $|z'| \leq |z|/2$, we use the expansion

$$\left| \partial_z \log \frac{|z - z'|}{|z|^2} + 2\partial_z \frac{z \cdot z'}{|z|^2} \right| \leq c \frac{|z'|^2}{|z|^5}.$$

Therefore, we can estimate

$$\left| \int_{|z'| \leq |z|/2} \partial_z \log \frac{|z - z'|}{|z|} h(z') + 2\partial_z \frac{z}{|z|^2} \cdot \int_{\mathbb{C}} z' h(z') \right| \leq c |z|^{-3}. \quad (11.48)$$

Finally, in $\omega$ where $|z'| \geq |z|/2$ and $|z' - z| \geq |z|/2$, we use the fact that

$$\left| \partial_z \log \frac{|z' - z|}{|z|} \right| \leq c \frac{1}{|z|},$$

to get

$$\left| \int_{\omega} \partial_z \log \frac{|z - z'|}{|z|} h(z') \right| \leq c e^{-\alpha|z|}. \quad (11.49)$$

The result then follows from (11.47)-(11.49).

As a simple Corollary of the previous Lemma, we have
Corollary 11.1 Under the assumptions of Theorem 11.1, we have
\[
\lim_{R \to +\infty} R \int_{\partial B_R} (|A \cdot \tau|^2 - |A \cdot \nu|^2) = 2\pi + \pi a_\infty \cdot A_\infty^\perp.
\]
Where
\[
a_\infty := \int_C z' h(z'),
\]
is identified with the corresponding vector of \( \mathbb{R}^2 \).

**Proof:** This simply follows from the fact that, thanks to the previous result, we can estimate
\[
A \cdot \nu = A_\infty \cdot \nu - \frac{1}{2\pi r^2} a_\infty \cdot \tau + O\left(\frac{1}{r^3}\right),
\]
and
\[
A \cdot \nu = A_\infty \cdot \nu - \frac{1}{2\pi r^2} a_\infty \cdot \nu + O\left(\frac{1}{r^3}\right).
\]
The proof is then straightforward. \(\square\)

11.3.2 Preliminary results

We already know from (11.9) that \( |\nabla u - iAu| \leq c \varepsilon^{-1} \). It follows from Proposition 11.1 that
\[
\{u\} = \{0\}.
\]
We have also seen in Proposition 11.1, that, up to a gauge transformation, the rescaled function \( \tilde{u} := u(\varepsilon \cdot) \) converge in any \( C^k(B_R) \) topology to \( S e^{i(\theta + \tau)} \).

Arguing as in the proof of Lemma 9.6, we obtain that, for any fixed \( \gamma > 0 \), we have
\[
|u| \geq c \frac{|z|}{\varepsilon},
\]
in \( B_{\gamma \varepsilon} \).

Notice that, by construction, \( \text{div} A = 0 \), hence, \( u \) satisfies the equation
\[
\Delta u + u \left(1 - \frac{|u|^2}{\varepsilon^2}\right) - |A|^2 u - 2i \partial_A u = 0.
\]
in \( \mathbb{C} \) and we also have, thanks to Lemma 11.4
\[
-\text{div} \left( \frac{\nabla h}{|u|^2} \right) + h = 2\pi \delta_0,
\]
in \( \mathbb{C} \).

We now use the radially symmetric solutions defined in Theorem 11.1. In order to simplify the equations it will be convenient to drop the \( \varepsilon \) indices in the definition of \( \hat{S}_\varepsilon \) and \( \hat{T}_\varepsilon \) and simply write \( \hat{S} \) and \( \hat{T} \). Also, we define
\[
v = \hat{S} e^{i\theta} \quad B := \hat{T} d\theta,
\]

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and
\[ k := \ast dB, \]
the magnetic field of the axially symmetric solution.

To begin with, we have the

**Lemma 11.6** Under the assumptions of Theorem 11.1, there exists \( \varepsilon_0 > 0 \) and for all \( l \geq 0 \), there exists \( c_l > 0 \) such that
\[ |\nabla^l k| + |\nabla^l h| \leq c_l \sup(r, \varepsilon)^{-l}, \]
and
\[ |\nabla^l|u|| + |\nabla^l|v|| \leq c_l \varepsilon^2 \sup(r, \varepsilon)^{-l-2}, \]
in \( \mathbb{C} \setminus \{0\} \).

**Proof:** We give the proof of these estimates for \( u \). It follows from Proposition 11.1 that \( u \) has a unique \( 0 \), which may be assumed to be the origin. Moreover \( |u| \geq 1/2 \) in \( \mathbb{C} \setminus B_{\lambda \varepsilon} \), for some \( \lambda > 0 \) independent of \( \varepsilon \).

It follows from [11], [12] and [82] that the result holds in any \( B_{R \varepsilon} \cup (\mathbb{C} \setminus B_R) \), for any fixed \( R > 0 \) independent of \( \varepsilon \).

We claim that there exists \( c > 0 \), independent of \( \varepsilon \), such that, for all \( r \in (\varepsilon, 1) \), we have
\[ \int_{B_{2r} \setminus B_r} g_c(u, A) \leq c. \]  
(11.53)

where
\[ g_c(u, A) := |dA|^2 + \frac{1}{2\varepsilon^2} (1 - |u|^2)^2 + \int_{\Omega} |dA|^2. \]
As usual, we consider the Coulomb gauge
\[ A = -\nabla \left( \frac{1}{2\pi} \ast h \right) + A_0 \]
where \( A_0 \) is chosen so that \( A(0) = 0 \). We know that \( \|h\|_{W^{1,p}(\mathbb{C})} \leq c \), for all \( p \in (1, 2) \), and also that
\[ \frac{1}{\varepsilon^2} \int_{\mathbb{C}} (1 - |u|^2)^2 \leq c. \]

Therefore, in order to prove (11.53), it suffices to prove that, for all \( r \in (\varepsilon, 1) \), we have
\[ \int_{B_{2r} \setminus B_r} |\nabla u|^2 \leq c. \]

We write, away from 0
\[ u := |u|e^{i\theta + \zeta}. \]

Hence, we obtain
\[ \int_{B_{2r} \setminus B_r} |\nabla u|^2 = \int_{B_{2r} \setminus B_r} (|\nabla|u||^2 + |u|^2 |\nabla(\theta + \zeta)|)^2). \]

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So,
\[ \int_{B_{2r} \setminus B_r} |\nabla u|^2 \geq \int_{B_{2r} \setminus B_r} (|\nabla \theta|^2 + |u|^2 |\nabla \zeta|^2 + 2 |u|^2 \nabla \theta \nabla \zeta) \]
\[ + \int_{B_{2r} \setminus B_r} \frac{1}{r^2} (|u|^2 - 1). \]

Moreover, since we have
\[ \int_{\partial B_r} \nabla \theta \nabla \zeta = \frac{1}{r^2} \int_{\partial B_r} \partial \theta \zeta = 0, \]

hence, we can write
\[ \int_{B_{2r} \setminus B_r} |u|^2 \nabla \theta \nabla \zeta = \int_{B_{2r} \setminus B_r} (|u|^2 - 1) \nabla \theta \nabla \zeta. \]

Hence, we conclude that
\[ \int_{B_{2r} \setminus B_r} |\nabla u|^2 \geq \int_{B_{2r} \setminus B_r} |\nabla \theta|^2 + \int_{B_{2r} \setminus B_r} \left( |u| \nabla \zeta + \frac{(|u|^2 - 1) \nabla \theta}{|u|} \right)^2 \]
\[ + \int_{B_{2r} \setminus B_r} \frac{(1 - |u|^2)^2}{|u|^2} |\nabla \theta|^2 + \int_{B_{2r} \setminus B_r} \frac{1}{r^2} (|u|^2 - 1). \quad (11.54) \]

Using the inequality \( 2ab \leq a^2 + b^2 \), we find that
\[ \int_{B_{2r} \setminus B_r} (1 - |u|^2)^2 \frac{1}{r^2} \leq \frac{\varepsilon^2}{2} \int_{B_{2r} \setminus B_r} \frac{1}{r^4} + \frac{1}{2\varepsilon^2} \int_{B_{2r} \setminus B_r} (1 - |u|^2)^2. \]

Therefore, we finally get from (11.54)
\[ \int_{B_{2r} \setminus B_r} |\nabla u|^2 \geq 2 \pi \log 1/\varepsilon - c, \quad (11.55) \]

for some constant \( c > 0 \) only depending on \( c_0 \).

Observe that
\[ \int_{B_{2r} \setminus B_r} |\nabla u|^2 \leq \int_{B_1 \setminus B_{2r}} |\nabla u|^2 + \int_{B_{2r} \setminus B_r} |\nabla u|^2 + \int_{B_r \setminus B_r} |\nabla u|^2, \]

and, thanks to (11.7) and (11.55), we can say that
\[ \int_{B_{2r} \setminus B_r} |\nabla u|^2 \leq c, \]

for some constant \( c > 0 \) independent of \( \varepsilon \). This ends the proof of the claim.

Now that (11.53) is proven, we observe that, in \( B_{2r} \setminus B_r \) the equations read
\[ \Delta |u| + |u| \left( \frac{1 - |u|^2}{\varepsilon^2} - \frac{|\nabla h|^2}{|u|^4} \right) = 0 \]

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and
\[-\text{div}(|u|^2 \nabla h) + h = 0,\]
and we know from (11.53) that
\[
\int_{B_{2r} \setminus B_r} \left( |\nabla h|^2 + |\nabla |u||^2 + \frac{1}{\varepsilon^2} (1 - |u|^2)^2 \right) \leq c.
\]
We fix $R \in (\lambda \varepsilon, 1)$ and define $\tilde{\varepsilon} := \varepsilon / R$,
\[
\tilde{h}(z) := h(Rz) \quad \text{and} \quad \tilde{s}(z) = |u|(Rz).
\]
Obviously, we have
\[
\Delta \tilde{s} + \tilde{s} \left( \frac{1 - \tilde{s}^2}{\tilde{\varepsilon}^2} - \frac{|\nabla \tilde{h}|^2}{\tilde{\varepsilon}^4} \right) = 0,
\]
and
\[-\text{div}(\tilde{s}^{-2} \nabla \tilde{h}) + R^2 \tilde{h} = 0,
\]
in $B_2 \setminus B_1$. Furthermore,
\[
\int_{B_{2r} \setminus B_r} \left( |\nabla \tilde{h}|^2 + |\nabla \tilde{s}|^2 + \frac{1}{\tilde{\varepsilon}^2} (1 - \tilde{s}^2)^2 \right) \leq c.
\]
It is proven in [10] that, under such assumptions, we have
\[
|\nabla \tilde{h}| \leq c_t,
\]
and
\[
|\nabla \tilde{s}| \leq c t \tilde{\varepsilon}^2.
\]
The result then follows by simply performing the scaling backward.

As we have done in Lemma 9.7, we would like to compare $h$ and $k$. This is the content of the following

**Lemma 11.7** Assume that $\nu \in (1 - 2^{-1/2}, 1)$ and further assume that the hypothesis of Theorem 11.1 hold. Then, there exists $c > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, we have
\[
\int_{C_{\varepsilon^{1/2}}} r^{-2\nu} \left( |\nabla \tilde{h}|^2 + |\nabla \tilde{s}|^2 + \frac{1}{\tilde{\varepsilon}^2} (1 - \tilde{s}^2)^2 \right) \leq c.
\]

**Proof:** The strategy of the proof is very similar to what we have already done in the proof of Lemma 9.7. We define $\xi = h - k$. A simple computation shows that $\xi$ satisfies
\[
-\Delta \xi + 2 \frac{\nabla |v|}{|v|} \nabla \xi + |v|^2 \xi = |v|^2 \left( \text{div} \left( \left( \frac{1}{|u|^2} - \frac{1}{|v|^2} \right) \nabla h \right) \right). \tag{11.57}
\]
The proof now follows exactly as in the proof of Lemma 9.7. However, this time, instead of Proposition 7.5, we apply Proposition 7.6 and we find

\[
\int_C r^{2-2\nu} \frac{1}{|v|^2} |\nabla \xi|^2 \leq c \int_C r^{4-2\nu} \frac{1}{|v|^2} |f|^2 + c \int_C r^{2-2\nu} |v|^2 |g|^2 \\
+ c \left( \int_{\partial B_c} \frac{1}{|v|^2} \partial_r \xi - \int_{\partial B_c} \xi \right)^2 \\
+ c \left( \int_{\partial B_c} g \cdot x \right),
\]

where we have set

\[
g := \text{div} \left( (1 - \eta_c) \frac{|v|^2 - |u|^2}{|u|^2|v|^2} \nabla h \right),
\]

and

\[
f := \nabla h \nabla \left( \eta_c \frac{|v|^2 - |u|^2}{|u|^2} \right).
\]

Here, as in the proof of Lemma 9.7, \( \eta \) is a cutoff function equal to 1 in \( B_1 \) and equal to 0 outside of \( B_2 \) and, for all \( \varepsilon > 0 \) we have defined

\[
\eta_c := \eta(2 \cdot \varepsilon).
\]

Finally, \( c \in [\sigma_0/2, \sigma_0] \).

**Step 1.** To begin with, we bound

\[
\int_C r^{2-2\nu} |v|^2 |g|^2 \leq c \int_{C \setminus B_{1/2}} r^{2-2\nu} |\nabla h|^2 (|u| - |v|)^2.
\]

**Step 2.** Integrating the equation satisfied by \( \xi \) over \( B_c \), we find

\[
c \left( \int_{\partial B_c} \frac{1}{|v|^2} \partial_r \xi - \int_{\partial B_c} \xi \right)^2 \leq \left( \int_{\partial B_c} \frac{|v|^2 - |u|^2}{|u|^2|v|^2} |\nabla h| \right)^2.
\]

Using Cauchy-Schwarz inequality, we get

\[
c \left( \int_{\partial B_c} \frac{1}{|v|^2} \partial_r \xi - \int_{\partial B_c} \xi \right)^2 \leq \int_{\partial B_c} (|v| - |u|)^2 |\nabla h|^2.
\]

Now we may use the mean value formula to conclude that

\[
c \left( \int_{\partial B_c} \frac{1}{|v|^2} \partial_r \xi - \int_{\partial B_c} \xi \right)^2 + c \left( \int_{\partial B_c} g \cdot x \right) \leq c \int_{B_{\sigma_0} \setminus B_{\sigma_0/2}} |\nabla h|^2 (|u| - |v|)^2 \\
+ c \int_{C \setminus B_{3/2}} r^{2-2\nu} |\nabla h|^2 (|u| - |v|)^2.
\]
Step 3. Finally,

\[
\int_{\mathcal{C}} r^{4-2\nu} \frac{1}{|v|^2} |f|^2 \leq c \int_{B_{\varepsilon}} r^{4-2\nu} \frac{1}{|v|^2} \left| \nabla h \nabla \left( \frac{|u|}{|v|} \right) \right|^2 + c \int_{B_{\varepsilon} \setminus B_{\varepsilon/2}} r^{4-2\nu} \frac{1}{|v|^2} |\nabla h|^2 (|u| - |v|)^2. \tag{11.60}
\]

Obviously,

\[
\int_{B_{\varepsilon} \setminus B_{\varepsilon/2}} r^{4-2\nu} \frac{1}{|v|^2} |\nabla h|^2 (|u| - |v|)^2 \leq c \int_{\mathcal{C} \setminus B_{\varepsilon}} r^{2-2\nu} |\nabla h|^2 (|u| - |v|)^2.
\]

While, for the first term on the right hand side of (11.60), we have

\[
\int_{B_{\varepsilon}} r^{4-2\nu} \frac{1}{|v|^2} \left| \nabla h \nabla \left( \frac{|u|}{|v|} \right) \right|^2 \leq c \int_{B_{\varepsilon}} r^{4-2\nu} \frac{1}{|v|^2} \left| \nabla k \nabla \left( \frac{|u|}{|v|} \right) \right|^2 + c \int_{B_{\varepsilon}} r^{4-2\nu} \frac{1}{|v|^2} \left| \nabla (h - k) \right|^2 \left| \nabla \left( \frac{|u|}{|v|} \right) \right|^2.
\]

Since \( B = \tilde{T} d\theta \) where \( \tilde{T} \) only depends on \( r \), we can state that

\[
\left| \nabla k \nabla \left( \frac{|u|}{|v|} \right) \right|^2 = |\nabla k|^2 \left( \partial_r \left( \frac{|u|}{|v|} \right) \right)^2.
\]

Moreover, using a blow up argument, it is easy to show that

\[
\lim_{\varepsilon \to 0} \left( \varepsilon \| \nabla |u| - |v| \|_{L^\infty(B_{\varepsilon})} + \left\| \frac{|u| - |v|}{|v|} \right\|_{L^\infty(B_{\varepsilon})} \right) = 0,
\]

which in particular implies that

\[
\lim_{\varepsilon \to 0} \sup_{B_{\varepsilon}} \left( r \left| \nabla \left( \frac{|u|}{|v|} \right) \right| \right) = 0.
\]

Hence, for \( \varepsilon \) small enough we have

\[
\int_{B_{\varepsilon}} r^{4-2\nu} \frac{1}{|v|^2} \left| \nabla (h - k) \right|^2 \left| \nabla \left( \frac{|u|}{|v|} \right) \right|^2 \leq \frac{1}{2} \int_{B_{\varepsilon}} r^{2-2\nu} \frac{1}{|v|^2} |\nabla \xi|^2.
\]

We conclude that

\[
\int_{B_{\varepsilon}} r^{4-2\nu} \frac{1}{|v|^2} \left| \nabla h \nabla \left( \frac{|u|}{|v|} \right) \right|^2 \leq c \int_{B_{\varepsilon}} r^{4-2\nu} \frac{1}{|v|^2} |\nabla k|^2 \left| \partial_r \left( \frac{|u|}{|v|} \right) \right|^2 + \frac{1}{2} \int_{B_{\varepsilon}} r^{4-2\nu} \frac{1}{|v|^2} |\nabla \xi|^2. \tag{11.61}
\]

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Step 4. Collecting all these estimates, we obtain
\[ \int_{C} r^{2-2\nu} \frac{1}{|v|^2} |\nabla \xi|^2 \leq c \int_{C \setminus B_{\epsilon/2}} r^{2-2\nu} |\nabla h|^2 (|u| - |v|)^2 + c \int_{B_\epsilon} r^{4+2\nu} \frac{1}{|v|^2} |\nabla k|^2 \left| \partial_r \left( \frac{|u|}{|v|} \right) \right|^2. \]

The result then follows from the fact that we know from Lemma 11.6 that
\[ |\nabla h| \leq c |u|/r \quad \text{and} \quad |\nabla k| \leq c |v|/r, \]
in \( C \setminus \{0\} \).

Without saying so, we have extensively used the fact that, for \( \epsilon \) small enough
\[ |u|/2 \leq |v| \leq 2 |u|. \]
and also that \( |u| \geq c \) and \( |v| \geq c \) in \( C \setminus B_{\epsilon/2} \), provided \( \epsilon \) is small enough. \( \square \)

We will also need the following simpler:

Lemma 11.8 For any \( p \geq 0 \) the following bound holds
\[ \int_{C \setminus B_2} r^p \left( (|u| - |v|)^2 + |\nabla (h - k)|^2 \right) \leq c_p \int_{C \setminus B_1} (|u| - |v|)^2 + |\nabla (h - k)|^2. \]

Proof: Outside \( B_1 \) we have on one hand
\[ -\Delta h + 2 \frac{\nabla |u|}{|u|} \nabla h + |u|^2 h = 0, \]
\[ -\Delta k + 2 \frac{\nabla |v|}{|v|} \nabla k + |v|^2 k = 0, \]
and on the other hand
\[ \Delta |v| + |v| \left( \frac{1 - |v|^2}{\epsilon^2} - \frac{|\nabla k|^2}{|v|^4} \right) = 0, \]
\[ \Delta |u| + |u| \left( \frac{1 - |u|^2}{\epsilon^2} - \frac{|\nabla h|^2}{|u|^4} \right) = 0. \]

We define \( \xi = h - k \) and \( D = |u| - |v| \). We have already seen that
\[ -\Delta \xi + 2 \frac{\nabla |v|}{|v|} \nabla \xi + |v|^2 \xi = -|v|^2 \text{div} \left( \frac{|u| + |v|}{|u||v|} D \nabla h \right), \quad (11.62) \]
and we have now
\[ -\Delta D + \frac{c_\epsilon}{\epsilon^2} D = -\frac{1}{|u|^2} \nabla \xi \nabla (h + k). \quad (11.63) \]
where
\[ c_\epsilon := |u|^2 + |v|^2 + |v| |u| - 1 - \frac{\epsilon^2}{|u|^3 |v|^3} |\nabla k|^2 \frac{|u|^3 - |v|^3}{|u| - |v|}. \]
Observe that the last term in $c_\varepsilon$ is uniformly bounded in $C \setminus B_1$ by a constant times $\varepsilon^2$, thus we can state that $c_\varepsilon \geq 2$ in $C \setminus B_1$ provided $\varepsilon$ is chosen small enough.

Let $\eta$ be a cutoff function equal to 0 in $B_1$ and equal to 1 outside $B_2$. We multiplying (11.62) by $\chi \sigma(p) \xi$ and integrate the results $C$. We get

$$
\int_C \eta^2 \sigma(p) (|\nabla \xi|^2 + |\xi|^2) \leq c \int_C |D||\nabla (\xi \sigma(p) \xi)| + c \int_C |\xi||\nabla \xi||\nabla (\sigma(p) \eta^2)|.
$$

(11.64)

As we have done in Lemma 9.8, we now use the inequality $a b \leq \kappa a^2 + \frac{1}{4\kappa} b^2$ to estimate all the terms in the right hand side of this inequality. For example, we estimate

$$
c \int_C |\xi||\nabla \xi| \sigma(p) \eta |\nabla \eta|^2 \leq \frac{1}{4} \int_C \sigma(p) \eta^2 |\xi|^2 + c \int_C |\nabla \xi|^2 |\nabla \eta|^2
$$

so that the first term on the right hand side, which involves $|\xi|^2$ can be absorbed in the left hand side of (11.64). We obtain

$$
\int_C \eta^2 \sigma(p) (|\nabla \xi|^2 + |\xi|^2) \leq c \int_C |D| \eta^2 \sigma(p) + c \int_{B_2 \setminus B_1} |\nabla \xi|^2
$$

(11.65)

Next, we multiplying (11.63) by $\eta^2 \sigma(p) D$ and integrate the results $C$. We get

$$
\int_C \eta^2 \sigma(p) \left(|\nabla D|^2 + \frac{1}{\varepsilon^2} |D|^2\right) \leq c \int_C |D||\nabla D||\nabla (\sigma(p) \eta^2)| + c \int_C \eta^2 \sigma(p) |D||\nabla \xi|.
$$

Again, we use the inequality $a b \leq \kappa a^2 + \frac{1}{4\kappa} b^2$ to get

$$
\int_C \eta^2 \sigma(p) \left(|\nabla D|^2 + \frac{1}{\varepsilon^2} |D|^2\right) \leq c \int_{B_2 \setminus B_1} |D|^2 + \frac{1}{2} \int_C \eta^2 \sigma(p) |\nabla \xi|^2.
$$

(11.66)

The result follows at once from the combination of the two inequalities (11.65) and (11.66).

11.3.3 The Pohozaev formula

We now use the radially symmetric solutions defined in Theorem 11.1. And consider the auxiliary function

$$
w := \frac{u}{|v|}.
$$

Observe that, in opposition to what we have done in Chapter 9, we do not consider the quotient of the two solutions $u$ and $v$ but rather the quotient of $u$ and $|v|$. In order to simplify the equations it will be convenient to drop the $\varepsilon$ indices in the definition of $\hat{S}_\varepsilon$ and $\hat{T}_\varepsilon$ and simply write $\hat{S}$ and $\hat{T}$.

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It verifies the following equation
\[
\Delta w + 2 \frac{\partial_r |v|}{|v|} \partial_r w + \frac{w}{r^2} + w |v|^2 \left( \frac{1 - |w|^2}{\varepsilon^2} \right) - w (|A|^2 - |B|^2) \\
- 2 w \partial_B \theta - 2 i w A_r \frac{\partial_r |v|}{|v|} - 2 i \partial_A w = 0,
\]
(11.67)
in $\mathbb{C} \setminus \{0\}$. Where $A_r$ is the first component of the $1$-form $A$ in cylindrical coordinates, namely $A = A_r \, dr + r A_\theta \, d\theta$.

The Pohozaev argument applied to this equation reads
\[
\frac{1}{4} \int_{B_R} \frac{1}{r} \partial_r (r^2 |v|^2) \left( \frac{1 - |w|^2}{\varepsilon^2} \right) + 2 \int_{B_R} \frac{r \partial_r |v|}{|v|} |\partial_r w|^2 = \\
- \frac{1}{2R} \int_{\partial B_R} (|w|^2 - 1) + \frac{R}{2} \int_{\partial B_R} (|\partial_r w|^2 - |\partial_r w|^2) \\
+ \frac{R}{4} \int_{\partial B_R} |v|^2 \frac{(1 - |w|^2)^2}{\varepsilon^2} + \frac{1}{2} \int_{B_R} r \partial_r |w|^2 (|A|^2 - |B|^2) \\
+ 2 \int_{B_R} r |w|^2 \partial_r \Phi \frac{\partial_r |v|}{|v|} + \int_{B_R} r \partial_r |w|^2 (\partial_B \theta - \partial_A \Phi) \\
+ \int_{B_R} r \partial_r \Phi \partial_A |w|^2,
\]
(11.68)
where $\Phi$ is the multivalued function on $\mathbb{C} \setminus \{0\}$ defined by
\[
\nabla \Phi = \frac{u \wedge \nabla u}{|u|^2}.
\]

In particular, we have $u = |u| e^{i \Phi}$ away from the origin. In order to obtain the above Pohozaev formula, we have just taken the scalar product of (11.67) with $r \partial_r w$ and integrated the result over $B_R \setminus B_s$ and finally let $s$ tend to 0. The boundary term on $\partial B_s$ vanishes here also exactly like for the corresponding Pohozaev identity established in Chapter 9. Everything is very close to what we have already done in Chapter 9, except the derivation of the last term.

Indeed, the last term comes from the identity
\[
2 r w \partial_A \Phi \cdot \partial_r w + r (2 i \partial_A w) \cdot \partial_r w = r \partial_r \Phi \partial_A |w|^2.
\]
In order to prove this identity, we observe that we always have
\[
\partial_r w \cdot (i \partial_\theta w) = \frac{1}{2} \left( \partial_r \Phi \partial_\theta |w|^2 - \partial_r |w| \partial_\theta \Phi \right).
\]
Using this, we obtain
\[
2 r w \partial_A \Phi \cdot \partial_r w + r (2 i \partial_A w) \cdot \partial_r w = r \partial_A \Phi \partial_r |w|^2 - r (2 i A_\theta \partial_\theta w) \cdot \partial_r w \\
= r \partial_r \Phi (A_r \partial_r |w|^2 + A_\theta \partial_\theta |w|^2) \\
= r \partial_r \Phi \partial_A |w|^2,
\]
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as desired.

It will be convenient to observe that the following identities hold
\[ \nabla \perp h = |u|^2 (\nabla \Phi - A) \quad \text{and} \quad \nabla \perp k = |v|^2 (\nabla \theta - B). \quad (11.69) \]
These follow at once from the second equation of (11.1).

We would like to pass to the limit in the Pohozaev formula as \( R \) tends to \(+\infty\). To this aim, observe that \( 1 - |w|^2, |\nabla |w|| \) and \( dh \) decay exponentially at \( \infty \). Since \(- \ast d \ast h = u \wedge (du - iAu) \) this implies that \((iu) \cdot (du - iAu)\) also tends to 0 exponentially. Hence
\[ (iw) \cdot (dw - iAw) = \frac{1}{|v|^2} (iu) \wedge (du - iAu), \]
also tends to 0 exponentially at \( \infty \). Using this, we see that
\[
\lim_{R \to +\infty} \left( \frac{1}{2R} \int_{\partial B_R} |w|^2 \right) - \frac{R}{2} \int_{\partial B_R} (|\partial_r w|^2 - |\partial_\theta w|^2)
\]
\[ - \frac{R}{4} \int_{\partial B_R} |v|^2 \left( \frac{1 - |w|^2}{\varepsilon^2} \right) \]
\[ = \lim_{R \to +\infty} \left( \pi - \frac{R}{2} \int_{\partial B_R} |A \cdot \tau|^2 - |A \cdot \nu|^2 \right). \quad (11.70) \]

Using the result of Corollary 11.1, we see that (11.68) becomes
\[ \frac{1}{4} \int_C \frac{1}{r} \partial_r (r^2 |v|^2) \left( \frac{1 - |w|^2}{\varepsilon^2} \right) + 2 \int_C r \frac{\partial_r |v|}{|v|} |\partial_r w|^2 \]
\[ = \frac{1}{2} \int_C r \partial_r |w|^2 (|A|^2 - |B|^2) + 2 \int_C r |w|^2 \partial_r \Phi A_r \frac{\partial_r |v|}{|v|} \]
\[ + \int_C r \partial_r |w|^2 (\partial_\theta \Phi - \partial_\Phi A_r) - \int_C r \partial_\Phi \partial_A |w|^2 + \frac{\pi}{2} a_\infty \cdot A_\infty. \quad (11.71) \]

11.3.4 The end of the proof

We set
\[ Q := \frac{1}{4} \int_C \frac{1}{r} \partial_r (r^2 |v|^2) \left( \frac{1 - |w|^2}{\varepsilon^2} \right) + 2 \int_C r \frac{\partial_r |v|}{|v|} |\partial_r w|^2. \]
In the remaining of the proof, we will show that each term, which is on the right hand side of (11.70) is bounded by \(1/100\) times \( Q \). This will imply that \( Q = 0 \) and then this will imply that
\[ u = |v| e^{i(\theta + \tau)}, \]
for some \( \tau \in \mathbb{R} \). This will be the end of the proof of Theorem 11.1. To do so, it is useful to observe that there exists a constant \( c > 0 \) such that, for all \( \varepsilon \) small enough we have
\[ Q \geq c \int_C \frac{(|v| - |w|)^2}{\varepsilon^2} + c \int_{B_{\varepsilon}} |\partial_r w|^2. \]

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Let us derive two useful estimates. By definition, $B = \hat{T} d\theta$, hence $B(0) = 0$. Moreover, we also have $A(0) = 0$ thus, using estimate (11.45) (which holds also for $B$ which is the Coulomb gauge), we conclude that

$$|A| + |B| \leq c_\alpha |z|^\alpha,$$

for any $\alpha \in (0, 1)$. Using the fact that $A(0) - B(0) = 0$, since

$$A - B = \frac{1}{2\pi} \log r \ast \nabla^\perp (h - k),$$

is a constant vector field, we may apply the result of Proposition 7.4 to obtain, for all $\nu \in (0, 1)$

$$\sum_{j=0}^{2} \int_\mathcal{C} r^{-2(\nu-j)-2} |\nabla^j (A - B)|^2 \leq c \int_\mathcal{C} r^{2-2\nu} |\nabla (h - k)|^2.$$

Assuming from now on that $\nu \in (1-2^{-1/2}, 1)$ and using the result of Lemma 11.7, we conclude that

$$\sum_{j=0}^{2} \int_\mathcal{C} r^{-2(\nu-j)-2} |\nabla^j (A - B)|^2 \leq c \int_{\mathcal{C}\setminus B_{1/2}} r^{-2\nu} (|u| - |v|)^2$$

$$+ c \int_{B_{1/2}} r^{2-2\nu} |\partial_r w|^2 \leq \epsilon^{2-2\nu} Q.$$

Now Lemma 11.7 yields, for all $\nu \in (1-2^{-1/2}, 1)$

$$\int_\mathcal{C} r^{2-2\nu} \frac{|\nabla (h - k)|^2}{|u|^2} \leq c \epsilon^{2-2\nu} Q.$$

Combining this with Lemma 11.8, we also get, for all $p \geq 0$

$$\int_{\mathcal{C}\setminus B_{2}} r^p (|u| - |v|)^2 \leq c \epsilon^{2-2\nu} Q,$$

$$\int_{\mathcal{C}\setminus B_{2}} r^p |\nabla (h - k)|^2 \leq c \epsilon^{2-2\nu} Q.$$

In the remaining, we will assume that the constant $\nu \in (1-2^{-1/2}, 1)$ and that $\alpha \in (1 - \nu, 1)$.

**Step 1.** Let us decompose the first term on the right hand side of (11.71) in the following way

$$\int_\mathcal{C} r \partial_r |w|^2 (|A|^2 - |B|^2) = \int_{B_{1/2}} r \partial_r |w|^2 (|A|^2 - |B|^2)$$

$$+ \int_{\mathcal{C}\setminus B_{1/2}} r \partial_r |w|^2 (|A|^2 - |B|^2),$$

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where \( \gamma \in (1/2, 1) \).

**Step 1.1.** Let us recall that \( 0 < c \leq |w| \leq C \) in \( B_\varepsilon \), for all \( \varepsilon \) small enough. Using (11.72) together with Cauchy-Schwarz inequality, we can estimate

\[
\int_{B_{\gamma \varepsilon}} r \partial_r |w|^2 (|A|^2 - |B|^2) \leq c \varepsilon^{1+\alpha} \left( \int_{B_\varepsilon} |\partial_r w|^2 \right)^{1/2} \left( \int_{B_\varepsilon} |A - B|^2 \right)^{1/2}.
\]

Thanks to (11.73), we get

\[
\int_{B_{\gamma \varepsilon}} r \partial_r |w|^2 (|A|^2 - |B|^2) \leq c \varepsilon^{3+\alpha} Q.
\]

Hence, this quantity is bounded by \( Q/100 \), provided \( \varepsilon \) is chosen small enough.

**Step 1.2.** On the other hand, outside \( B_{\gamma \varepsilon} \) integrating by part we have

\[
\int_{C \setminus B_{\gamma \varepsilon}} r \partial_r |w|^2 (|A|^2 - |B|^2) \leq c \int_{\partial B_{\gamma \varepsilon}} \varepsilon ||u| - |v|| |A|^2 - |B|^2|
+ c \int_{C \setminus B_{\gamma \varepsilon}} \frac{|u| - |v|}{r} |\partial_r (r^2(|A|^2 - |B|^2))|.
\]

Making use of the mean value formula, we conclude that we may choose \( \gamma \) in such a way that

\[
\frac{\varepsilon}{2} \int_{\partial B_{\gamma \varepsilon}} ||u| - |v|| |A|^2 - |B|^2| \leq \int_{B_{\epsilon} \setminus B_{\gamma \varepsilon}} ||u| - |v|| |A|^2 - |B|^2|
\leq c \varepsilon^{\alpha} \left( \int_{B_{\epsilon} \setminus B_{\gamma \varepsilon}} (|u| - |v|)^2 \right)^{1/2} \left( \int_{B_\varepsilon} |A - B|^2 \right)^{1/2},
\]

where we have also used (11.72). Using (11.73) we conclude that

\[
c \int_{\partial B_{\gamma \varepsilon}} \varepsilon ||u| - |v|| |A|^2 - |B|^2| \leq c \varepsilon^{3+\alpha} Q.
\]

Hence, this term is also bounded by \( Q/100 \) for all \( \varepsilon \) small enough.

In order to estimate the second term in the right hand side of (11.77), we use (11.72) to get

\[
\left| \int_{C \setminus B_{\gamma \varepsilon}} \frac{|u| - |v|}{r} |\partial_r (r^2(|A|^2 - |B|^2))| \right|^2 \leq
c \left( \int_{C \setminus B_{\gamma \varepsilon}} r^{2(1+\nu+\alpha)} (|u| - |v|)^2 \right) \left( \sum_{j=0}^1 \int_{C} r^{-2(\nu-j+1)} |\nabla^j (A - B)|^2 \right).
\]

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In order to evaluate the first term on the right hand side of this inequality, we decompose this term into two integrals and use (11.75) to get
\[
\int_{C^2 \setminus B_{r/2}} r^{2+2\nu+2\alpha} (|u| - |v|)^2 \leq \int_{B_1 \setminus B_{r/2}} (|u| - |v|)^2 \\
+ \int_{C^2 \setminus B_1} r^{2+2\nu+2\alpha} (|u| - |v|)^2 \\
\leq c \varepsilon^2 Q + c \varepsilon^{2-2\nu} Q.
\]

Next, we make use of (11.73) to obtain
\[
c \left| \int_{C \setminus B_r} \frac{|u| - |v|}{r} \left| \partial_r (r^2 (|A|^2 - |B|^2)) \right| \right| \leq c \varepsilon^{2-2\nu} Q.
\]

Again, we can conclude that the second term on the right hand side of (11.77) is bounded by \( Q/100 \) for all \( \varepsilon \) small enough.

**Step 2.** We bound now the second term in the right hand side of (11.71). Again, we decompose the integral as the sum of the integral over \( B_\varepsilon \) and the integral over \( C \setminus B_\varepsilon \).

**Step 2.1.** We use the fact that
\[
r \frac{\partial_r |v|}{|v|},
\]
is uniformly bounded in \( B_\varepsilon \) together with Cauchy-Schwarz inequality to obtain
\[
\int_{B_\varepsilon} r \ |w|^2 \partial_r \Phi \ A_r \frac{\partial_r |v|}{|v|} \leq c \int_{B_\varepsilon} \frac{r \partial_r |v|}{|v|} \left| \partial_r \Phi \right| |A - B| \\
\leq c \left( \int_{B_\varepsilon} |\partial_r \Phi|^2 \right)^{1/2} \left( \int_{B_\varepsilon} |A - B|^2 \right)^{1/2}.
\]

Next, observe that
\[
|\partial_r \Phi| = \frac{1}{|u|^2} |u \wedge \partial_r u| \leq \frac{c}{|v|^2} |u \wedge \partial_r u| \leq c |\partial_r u|.
\]

Thus we may estimate, using (11.73)
\[
\int_{B_\varepsilon} r \ |w|^2 \partial_r \Phi \ A_r \frac{\partial_r |v|}{|v|} \leq c \left( \int_{B_\varepsilon} |\partial_r u|^2 \right)^{1/2} \left( \int_{B_\varepsilon} |A - B|^2 \right)^{1/2} \\
\leq c \varepsilon^2 Q.
\]

Again, this term is bounded by \( Q/100 \) for all \( \varepsilon \) small enough.

**Step 2.2.** Observe that (11.69) implies that
\[
|\partial_r \Phi|^2 \leq \frac{|\partial_r h - \partial_r k|^2}{|u|^2} + |A_r - B_r|^2,
\]

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since
\[ \partial_t k = 0 \quad \text{and} \quad B_r = 0. \]
Hence
\[ |\partial_r \Phi| \leq c |\nabla (h - k)| + c |A - B|, \]
in \( \mathbb{C} \setminus B_r \). Furthermore, we may bound that \(|A_r| = |A_r - B_r| \leq |A - B|\). Using these simple inequalities, together with Cauchy-Schwarz inequality, we get
\[ \int_{\mathbb{C} \setminus B_r} r |w|^2 \partial_r \Phi \frac{\partial_r |v|}{|v|} \leq c \int_{\mathbb{C} \setminus B_r} r \frac{\partial_r |v|}{|v|} |A - B|^2. \]

To obtain this estimate, we have implicitly used the result of Lemma 11.1, which asserts that
\[ \frac{r \partial_r |v|}{|v|}, \]
is bounded in \( \mathbb{C} \), by some constant which is independent of \( \varepsilon \). Now, the first two terms on the right hand side of (11.78) are seen to be bounded by \( c \varepsilon^{2-2\nu} Q \), using (11.73), (11.74) and (11.75). While for the last term we decompose it as
\[ \int_{\mathbb{C} \setminus B_r} r \frac{\partial_r |v|}{|v|} |A - B|^2 = \int_{\mathbb{C} \setminus B_1} r \frac{\partial_r |v|}{|v|} |A - B|^2 + \int_{B_1 \setminus B_r} r \frac{\partial_r |v|}{|v|} |A - B|^2. \]

And, using Proposition 11.2, we get
\[ \int_{\mathbb{C} \setminus B_r} r \frac{\partial_r |v|}{|v|} |A - B|^2 \leq c \int_{\mathbb{C} \setminus B_1} \varepsilon^2 e^{-2\alpha r} |A - B|^2 + \int_{B_1 \setminus B_r} r \frac{\partial_r |v|}{|v|} |A - B|^2. \]

Finally, we use (11.73) to conclude that
\[ \int_{\mathbb{C} \setminus B_r} r \frac{\partial_r |v|}{|v|} |A - B|^2 \leq c \varepsilon^{2-2\nu} Q. \]

To summarize, we have proved that
\[ \int_{\mathbb{C} \setminus B_r} r |w|^2 \partial_r \Phi A_r \frac{\partial_r |v|}{|v|} \leq c \varepsilon^{2-2\nu} Q, \]
and thus is bounded by \( Q/100 \) for all \( \varepsilon \) small enough.

Step 3. The third term on the right hand side of (11.71) is decomposed in
the following way, using (11.69).

\[
\int_C r \partial_r |w|^2 (\partial_B \theta - \partial_A \Phi) = \int_C r \partial_r |w|^2 (|B|^2 - |A|^2)
+ \int_C r \partial_r |w|^2 \frac{\partial_B h}{|v|^2} (|w|^2 - 1)
+ \int_C r \partial_r |w|^2 \frac{\nabla \tilde{h} - A h}{|u|^2}.
\]  

(11.79)

Let us denote by \( I, II, III \) and \( IV \) the different terms which appear on the right hand side of this identity.

In order to estimate \( I - IV \) we argue as before and decompose each of these integrals into the integral over \( B_{\gamma \varepsilon} \) and the integral over \( C \setminus B_{\gamma \varepsilon} \), for some \( \gamma \in (1/2, 1) \).

**Step 3.1.1.** For the first term \( I \), we get using (11.72), together with Cauchy-Schwarz inequality

\[
\int_{B_{\gamma \varepsilon}} r \partial_r |w|^2 (|B|^2 - |A|^2) \leq c \varepsilon^{\alpha+1} \left( \int_{B_{\gamma \varepsilon}} |\partial_r w|^2 \right)^{1/2} \left( \int_{B_{\gamma \varepsilon}} |A - B|^2 \right)^{1/2}.
\]

which, thanks to (11.73) is bounded by \( c \varepsilon^{3+\alpha} Q \) and hence is smaller than \( Q/100 \), provided \( \varepsilon \) is chosen small enough.

**Step 3.1.2.** Integrating by part now the integral of the same quantity but over \( C \setminus B_{\gamma \varepsilon} \) yields

\[
\int_{C \setminus B_{\gamma \varepsilon}} r \partial_r |w|^2 (|B|^2 - |A|^2) = -\gamma \int_{\partial B_{\gamma \varepsilon}} \varepsilon (|w|^2 - 1)(|B|^2 - |A|^2)
- \int_{C \setminus B_{\gamma \varepsilon}} (|w|^2 - 1) \frac{1}{r} \partial_r (r^2 (|B|^2 - |A|^2)).
\]

Using the mean value formula, we obtain the existence of \( \gamma \in (1/2, 1) \) such that

\[
\frac{\varepsilon}{2} \int_{\partial B_{\gamma \varepsilon}} ||w|^2 - 1||^2 \leq \int_{B_{\gamma \varepsilon} \setminus B_{1/2}} ||w|^2 - 1|| (|B|^2 - |A|^2)
\leq c \varepsilon^{\alpha} \left( \int_{B_{\gamma \varepsilon} \setminus B_{1/2}} (|u| - |v|)^2 \right)^{1/2}
\times \left( \int_{B_1} |B - A|^2 \right)^{1/2},
\]

Now we use (11.73) to conclude that

\[
\gamma \int_{\partial B_{\gamma \varepsilon}} \varepsilon (|w|^2 - 1)(|B|^2 - |A|^2) \leq c \varepsilon^{3+\alpha} Q.
\]  

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which is also bounded by $Q/100$ for all $\varepsilon$ small enough.

Finally, we have

\[
\left| \int_{C \setminus B_{\varepsilon}} (|w|^2 - 1) \frac{1}{r} \partial_r \left( r^2(|B|^2 - |A|^2) \right) \right|^2 \leq \\
\left( \int_{C \setminus B_{\varepsilon}} r^{2(\nu+\alpha)} (|u| - |v|)^2 \right) \left( \sum_{j=0}^{1} \int_{C \setminus B_{\varepsilon/2}} r^{-2(\nu-j+1)} |\nabla^j (A - B)|^2 \right) .
\]

(11.80)

Which, thanks to (11.75) and (11.73) is bounded by $c \varepsilon^{2-\nu} Q$ and hence, is less than $Q/100$, for $\varepsilon$ small enough.

**Step 3.2.1.** The second integral $II$ which appears on the right hand side of (11.79) is also decomposed into the sum of the integral over $B_{\varepsilon}$ and the integral over $C \setminus B_{\varepsilon}$, for some $\gamma \in (1/2, 1)$. First we have, by Cauchy-Schwarz inequality

\[
\int_{B_{\varepsilon}} r \partial_r |w|^2 \frac{\partial_h (k - h)}{|v|^2} \leq \left( \int_{B_{\varepsilon}} |\partial_r w|^2 \right)^{1/2} \left( \int_{B_{\varepsilon}} r^2 |B|^2 \frac{|\nabla (h - k)|^2}{|v|^2} \right)^{1/2} .
\]

(11.81)

Thanks to (11.74), we easily conclude, as we have already done, that

\[
\int_{B_{\varepsilon}} r \partial_r |w|^2 \frac{\partial_h (k - h)}{|v|^2} \leq c \varepsilon^{1+\alpha} Q,
\]

which in turn is bounded by $Q/100$, provided $\varepsilon$ is small enough. Observe that we have used the fact that, thanks to (11.72), we know that

\[
r^2 \frac{|B|^2}{|v|^2} \leq c \varepsilon^2 r^{2\alpha} ,
\]

in $B_{\gamma \varepsilon}$ and we have used the fact that $\alpha > 1 - \nu$.

**Step 3.2.2.** As usual, we integrate by part

\[
\int_{C \setminus B_{\varepsilon}} r \partial_r |w|^2 \frac{\partial_h (k - h)}{|v|^2} = - \gamma \varepsilon \int_{\partial B_{\varepsilon}} \partial_r |w|^2 \frac{\partial_h (k - h)}{|v|^2} \\
- \int_{C \setminus B_{\varepsilon}} (|w|^2 - 1) \frac{1}{r} \partial_r \left( r^2 \frac{\partial_h (k - h)}{|v|^2} \right) .
\]

(11.82)

The mean value formula allows us to state that

\[
\left| \frac{\varepsilon}{2} \int_{\partial B_{\varepsilon}} r \partial_r |w|^2 \frac{\partial_h (k - h)}{|v|^2} \right| \leq c \left( \int_{B_{\varepsilon} \setminus B_{\varepsilon/2}} |\partial_r w|^2 \right)^{1/2} \\
\times \left( \int_{B_{\varepsilon} \setminus B_{\varepsilon/2}} |B|^2 |\nabla (h - k)|^2 \right)^{1/2} .
\]

(11.83)
which, thanks to (11.72) and (11.74), is bounded by $c \epsilon^\alpha Q$ and hence, is less than $Q/100$ provided $\epsilon$ is small enough.

The second integral on the right hand side of (11.82) is decomposed in the following way

$$\int_{C \setminus B_x} (|w|^2 - 1) \frac{1}{r} \frac{\partial_r B}{|v|^2} \left( r^2 \frac{\partial^2 \nu}{|v|^2} (k - h) \right) = 2 \int_{C \setminus B_x} (|w|^2 - 1) \frac{\partial^2 \nu}{|v|^2} (k - h)$$

$$+ \int_{C \setminus B_x} (|w|^2 - 1) r \frac{\partial_r B}{|v|^2} \sum_{\nu} (k - h)$$

$$- 2 \int_{C \setminus B_x} (|w|^2 - 1) \frac{\partial_r |v|}{|v|^2} r \frac{\partial^2 \nu}{|v|^2} (k - h)$$

$$+ \int_{C \setminus B_x} (|w|^2 - 1) \frac{r B}{|v|^2} \partial_r \nabla^\perp (k - h).$$

(11.84)

All the terms which appear in this equality can be handled as before to show that they are bounded by $Q/100$, provided $\epsilon$ is small enough. For example, using (11.72), we have

$$\left| \int_{C \setminus B_x} (|w|^2 - 1) \frac{\partial^2 \nu}{|v|^2} (k - h) \right|$$

$$\leq \left| \int_{C \setminus B_1} (|w|^2 - 1) \frac{\partial^2 \nu}{|v|^2} (k - h) \right| + \left| \int_{B_1 \setminus B_x} (|w|^2 - 1) \frac{\partial^2 \nu}{|v|^2} (k - h) \right|$$

$$\leq \left( \int_{C \setminus B_1} r^{2\alpha} (|u| - |v|)^2 \right)^{1/2} \left( \int_{C \setminus B_1} |\nabla (k - h)|^2 \right)^{1/2}$$

$$+ \left( \int_{B_1 \setminus B_x} r^{2\alpha+1} (|u| - |v|)^2 \right)^{1/2} \left( \int_{C \setminus B_1} r^{2-2\nu} |\nabla (k - h)|^2 \right)^{1/2}.$$

And using (11.75) together with (11.74), we see that this term is bounded by $c \epsilon^{2-2\nu} Q$. The same arguments work for the next two terms. Let us now focus on the last term

$$\int_{C \setminus B_x} (|w|^2 - 1) \frac{r B}{|v|^2} \partial_r \nabla^\perp (k - h).$$

Obviously this term can be estimated by

$$\int_{C \setminus B_x} (|w|^2 - 1) \frac{r A}{|v|^2} \partial_r \nabla^\perp (k - h) \leq c \int_{C \setminus B_{x/2}} r^{\alpha+1} ||u| - |v|| |\nabla^2 (h - k)||,$$

where we have used (11.72). We claim that

$$\int_{C \setminus B_x} r^{\alpha+1} ||u| - |v|| |\nabla^2 (h - k)|| \leq c \epsilon^{1-\nu} Q.$$  

(11.85)

Assuming that the claim is already proved, we conclude, using (11.75) that

$$\int_{C \setminus B_x} (|w|^2 - 1) \frac{r A}{|v|^2} \partial_r \nabla^\perp (k - h) \leq c \epsilon^{(3-3\nu)/2} Q,$$

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and hence is bounded by $Q/100$, provided $\varepsilon$ is small enough.

It remains to prove the claim. To begin with, standard elliptic estimates yield for all $r > 0$

$$\int_{B_{2r} \setminus B_r} |\nabla^2 (h - k)|^2 \leq c \int_{B_{4r} \setminus B_{2r/2}} |\Delta (h - k)|^2 + c r^{-4} \int_{B_{4r} \setminus B_{2r/2}} |h - k|^2.$$ 

Hence, we obtain

$$\int_{C \setminus B_{2r/2}} r^{4-2\nu} |\nabla^2 (h - k)|^2 \leq c \int_{C \setminus B_{2r/4}} r^{4-2\nu} |\Delta (h - k)|^2 + c \int_{C \setminus B_{2r/4}} r^{-2\nu} |h - k|^2.$$ 

However, we know that $h - k$ satisfies the equation

$$-\Delta (h - k) + 2 \frac{\nabla |v|}{|v|} \nabla (h - k) + |v|^2 (h - k) + 2 \frac{\nabla (|u| - |v|)}{|u|} \nabla h + \frac{2}{|u|} (|v| - |u|) + (|u|^2 - |v|^2) h = 0.$$ 

Now, we use the fact that

$$r \frac{|\nabla v|}{|v|} \leq c,$$

and also that

$$r \frac{|\nabla h|}{|u|} \leq c,$$

to conclude that we can estimate

$$\int_{C \setminus B_{2r/2}} r^{4-2\nu} |\nabla^2 (h - k)|^2 \leq c \int_{C \setminus B_{2r/4}} r^{2-2\nu} |\nabla (h - k)|^2 + c \int_{C \setminus B_{2r/4}} (r^{4-2\nu} + r^{-2\nu}) |h - k|^2 + c \int_{C \setminus B_{2r/4}} r^{2-2\nu} |\nabla (|u| - |v|)|^2 + c \int_{C \setminus B_{2r/4}} r^{-2\nu} (|u| - |v|)^2 + c \int_{C \setminus B_{2r/4}} r^{4-2\nu} (|u| - |v|)^2 |h|^2.$$ 

Thanks to (11.73) we can evaluate the first term by $c \varepsilon^{2-2\nu} Q$. The fourth term is easily seen to be bounded by $c \varepsilon^{2-2\nu} Q$ too.

Let us now evaluate the last term. Thanks to Proposition 11.2, we know that $h$ is bounded (an even decays exponentially fast to 0) in $C \setminus B_1$. Moreover,
it follows from Lemma 11.6 that $|\nabla h| \leq c r^{-1}$ in $\mathbb{C} \setminus B_{r/4}$. Therefore we can bound

$$
\int_{\mathbb{C} \setminus B_{r/4}} r^{4-2\nu} (|u| - |v|)^2 |h|^2 \leq c \int_{\mathbb{C} \setminus B_1} r^{4-2\nu} (|u| - |v|)^2 + c \int_{\mathbb{C} \setminus B_{r/4}} r^{4-2\nu} (1 - \log r) (|u| - |v|)^2.
$$

Finally, applying (11.75), we see that this term is bounded by $c \varepsilon^{2-2\nu} Q$.

We now turn to the evaluation of the third term, which involves

$$
\int_{\mathbb{C} \setminus B_{r/4}} r^{2-2\nu} |\nabla (|u| - |v|)|^2.
$$

We write, as in the proof of Lemma 11.8, $D := |u| - |v|$. Using the result of Lemma 11.6, it is an easy exercise to see that

$$
|\Delta D| \leq c \left( \frac{|D|}{\varepsilon^2} + \frac{1}{r} \frac{|\nabla (h - k)|}{|u|} \right),
$$

in $\mathbb{C} \setminus B_{r/8}$.

To obtain the relevant estimate in $\mathbb{C} \setminus B_{Re}$, we use a strategy close to the one used in the proof of Lemma 11.8. Let us recall that $D := |u| - |v|$ satisfies

$$
-\Delta D + c_{\varepsilon} D = -\frac{1}{|u|^2} \nabla \xi \nabla (h + k),
$$

where $c_{\varepsilon} \geq 2$ in $\mathbb{C} \setminus B_{Re}$ provided $\varepsilon$ is chosen small enough and $R$ is chosen large enough, but fixed.

Let $\chi$ be a cutoff function equal to 0 in $B_{Re/2}$ and equal to 1 outside $B_{Re}$. Further assume that $|\nabla \chi| \leq c (R \varepsilon)^{-1}$, for some constant $c$ independent of $\varepsilon$ and $R$. We multiply the above equation by $\chi^2 r^{2-2\nu} D$ and integrate the results on $\mathbb{C}$

$$
\int_{\mathbb{C}} r^{2-2\nu} \chi^2 |\nabla D|^2 + \frac{2}{\varepsilon^2} \int_{\mathbb{C}} \chi^{2r^{1-2\nu}} |D|^2 \leq c \int_{\mathbb{C}} \chi^{r^{1-2\nu}} |\nabla (h - k)||D| + c \int_{\mathbb{C}} r^{1-2\nu} \chi^2 |D||\nabla D| + c \int_{\mathbb{C}} r^{2-2\nu} \chi |\nabla \chi||D||\nabla D|.
$$

Making use of the inequality $a b \leq \kappa a^2 + \frac{1}{4\kappa} b^2$, we can bound

$$
c \int_{\mathbb{C}} r^{1-2\nu} \chi^2 |D||\nabla D| \leq \frac{1}{2} \int_{\mathbb{C}} r^{2-2\nu} \chi^2 |\nabla D|^2 + c \int_{\mathbb{C}} r^{-2} \chi^2 |D|^2.
$$
Hence, for $R$ chosen large enough, we conclude that
\[
c \int_C r^{1-2\nu} \chi^2 |D| |\nabla D| \leq \frac{1}{2} \int_C r^{2-2\nu} \chi^2 |\nabla D|^2 + \frac{1}{2\nu^2} \int_C \chi^2 |D|^2.
\]

Similarly,
\[
c \int_C r^{2-2\nu} \chi |\nabla \chi| |\nabla D| \leq \frac{1}{2} \int_C r^{2-2\nu} \chi^2 |\nabla D|^2 + \frac{c}{\nu^2} \int_{B_{Rr}/B_{r/2}} r^{2-2\nu} |\nabla D|^2.
\]

Hence we obtain
\[
\int_C r^{2-2\nu} \chi^2 |\nabla D|^2 + \frac{2}{\nu^2} \int_C \chi^2 r^{2-2\nu} |D|^2 \leq c \int_{B_{Rr} \setminus B_{r/2}} \frac{r^{2-2\nu} |D|^2}{B_{r/4}} + c \left( \int_{C \setminus B_{r/4}} r^{2-2\nu} |\nabla (h - k)|^2 \right)^{1/2} \left( \int_{C \setminus B_{r/4}} r^{-2} |D|^2 \right)^{1/2}.
\]

It is now an easy exercise, using (11.73) to show that the right hand side of the last inequality is bounded by $c \epsilon^{1-\nu} Q$.

Let $R > 0$ be chosen large enough, but fixed. We can use Sobolev embedding together with classical elliptic estimates to show that, for all $r > 0$
\[
\int_{B_{2r} \setminus B_{r/2}} |\nabla D|^2 \leq c r^2 \int_{B_{4r} \setminus B_{r/2}} |\Delta D|^2 + c r^{-2} \int_{B_{4r} \setminus B_{r/2}} |D|^2.
\]

Multiplying by $r^{2-2\nu}$, we get easily
\[
\int_{B_{2r} \setminus B_{r/2}} r^{2-2\nu} |\nabla D|^2 \leq c \int_{B_{2r} \setminus B_{r/2}} r^{4-2\nu} |\Delta D|^2 + c \int_{B_{2r} \setminus B_{r/2}} r^{-2} |D|^2.
\]

Using (11.86), we conclude that
\[
\int_{B_{2r} \setminus B_{r/2}} r^{2-2\nu} |\nabla (|u| - |v|)|^2 \leq c \int_{B_{2r} \setminus B_{r/2}} e^{-2\nu} (|u| - |v|)^2 + c \int_{B_{2r} \setminus B_{r/2}} r^{2-2\nu} |\nabla (h - k)|^2.
\]

Finally, we use (11.74) to evaluate the second term and we get
\[
\int_{B_{2r} \setminus B_{r/2}} r^{2-2\nu} |\nabla (|u| - |v|)|^2 \leq c \epsilon^{2-2\nu} Q.
\]

Hence, we have proved that
\[
c \int_{C \setminus B_{r/4}} r^{2-2\nu} |\nabla (|u| - |v|)|^2 \leq c \epsilon^{1-\nu} Q.
\]
Finally, it remains to estimate
\[ \int_{\mathbb{C} \setminus B_{\varepsilon/4}} (r^{-2\nu} + r^{4-2\nu}) |h - k|^2. \]

Let us recall that \( \xi := h - k \) satisfies
\[ -\text{div} \left( \frac{\nabla \xi}{|v|^2} \right) + \xi = -\text{div} \left( \frac{|u| + |v|}{|u| |v|} D \nabla h \right), \]
where, as above \( D := |u| - |v| \).

First we consider a cutoff function \( \chi \) identically equal to 1 outside \( B_1 \) and equal to 0 in \( B_1/2 \). We multiply the above equation by \( r^{4-2\nu} \chi^2 \xi \) and integrate the result over \( \mathbb{C} \). Using the fact that \( |u| \) and \( |v| \) are uniformly bounded from below in \( \mathbb{C} \setminus B_1/2 \) we get
\[ \int_{\mathbb{C}} \chi^2 r^{4-2\nu} |\nabla \xi|^2 + \int_{\mathbb{C}} \chi^2 r^{4-2\nu} |\xi|^2 \leq c \int_{\mathbb{C}} |D| r^{-1} |\nabla (\chi^2 r^{4-2\nu} \xi)| + c \int_{\mathbb{C}} |\nabla (\chi^2 r^{4-2\nu})| |\xi| |\nabla \xi|. \]

Arguing as above, we easily conclude that
\[ \int_{\mathbb{C} \setminus B_1} r^{4-2\nu} |h - k|^2 \leq c \int_{\mathbb{C} \setminus B_{1/2}} r^{2-2\nu} |\nabla (h - k)|^2 + c \int_{\mathbb{C} \setminus B_{1/2}} \left( \int_{\mathbb{C} \setminus B_{1/2}} |D|^2 r^{4-2\nu} \right)^{1/2} \left( \int_{\mathbb{C} \setminus B_{1/2}} r^{2-2\nu} |\nabla (h - k)|^2 \right)^{1/2} \]
\[ + c \int_{\mathbb{C} \setminus B_{1/2}} r^{2-2\nu} |\nabla (h - k)|^2 + c \int_{\mathbb{C} \setminus B_{1/2}} r^{-2\nu} |D|^2. \]

In order to conclude, we use (11.74) and (11.75), and obtain
\[ \int_{\mathbb{C} \setminus B_1} r^{4-2\nu} |h - k|^2 \leq c \varepsilon^{2-2\nu} Q. \]

It remains to estimate
\[ \int_{B_1 \setminus B_{\varepsilon/4}} r^{-2\nu} |h - k|^2. \]

However, since by construction \( h - k = \ast d (A - B) \), it immediately follows from (11.74) that
\[ \int_{B_1 \setminus B_{\varepsilon/4}} r^{-2\nu} |h - k|^2 \leq c \varepsilon^{2-2\nu}. \]

This ends the proof of the claim.
Step 3.3.1. Now we need to bound the third term $III$ on the right hand side of (11.79). Again we decompose this integral as the sum of the integral over $B_{\gamma\varepsilon}$ and $\mathbb{C} \setminus B_{\gamma\varepsilon}$

$$\left| \int_{B_{\gamma\varepsilon}} r \partial_r |w|^2 \frac{\partial h}{|w|^2} (|w|^2 - 1) \right|^2 \leq c \left( \int_{B_{\gamma\varepsilon}} r^{2\alpha} (1 - |w|)^2 \right) \left( \int_{B_{\gamma\varepsilon}} |\partial_r w|^2 \right),$$

(11.87)

where we have use (11.72) together with the fact that

$$|\nabla h| \leq c r^{-1} |u|^2,$$

in $B_{\gamma\varepsilon}$. Indeed, we know for (11.1) that

$$|\nabla h| \leq |u| |\nabla u - iA|.$$

Now, we use the fact that $|u| \geq c r/\varepsilon$ in $B_{\gamma\varepsilon}$ together with the fact that $|\nabla u| \leq c \varepsilon^{-1}$ and since $|A| \leq c r^{\alpha}$, to get the desired inequality.

We claim that

$$\int_{B_{\gamma}} r^{2\alpha} |1 - |w||^2 \leq c \varepsilon^{2\alpha} Q,$$

(11.88)

for all $\varepsilon$ small enough. In order to prove this identity, we use the fact that

$$\left( |w(r, \theta)| - |w(\tilde{\gamma}\varepsilon, \theta)\right)^2 = \left( \int_{r^\gamma \varepsilon} \partial_r |w| \right)^2 \leq c \log r \int_{B_r} |\partial_r w|^2.$$

for any $\tilde{\gamma} \in [1/2, 1]$.

Integrating this inequality over $B_{\gamma\varepsilon}$, we already conclude that

$$\int_{B_{\gamma}} r^{2\alpha} (w(r, \theta) - w(\tilde{\gamma}\varepsilon, \theta))^2 \leq c \varepsilon^{2+2\alpha} \log \varepsilon Q.$$

(11.89)

In addition, the mean value formula yields the existence of $\tilde{\gamma} \in (1/2, 1)$ such that

$$\int_{B_{\gamma}} r^{2\alpha} |1 - |w(\tilde{\gamma}\varepsilon, \theta)||^2 \leq c \varepsilon^{1+2\alpha} \int_{\partial B_{\gamma\varepsilon}} |1 - |w||^2$$

$$\leq c \varepsilon^{2\alpha} \int_{B_{\gamma} \setminus B_{\varepsilon/2}} |1 - |w||^2.$$

The claim then follows at once from this last inequality and from (11.89).

It is now an easy exercise to show that

$$\int_{B_{\gamma\varepsilon}} r \partial_r |w|^2 \frac{\partial h}{|w|^2} (|w|^2 - 1) \leq c \varepsilon^\alpha Q,$$

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and hence is bounded by $\mathcal{Q}/100$ provided $\varepsilon$ is small enough.

**Step 3.3.2.** Next, we integrate by part to get

$$
\int_{C \setminus B_{\varepsilon}} r \partial_{r} |w|^{2} \frac{\partial \frac{1}{2} h}{|w|^{2}} (|w|^{2} - 1) = - \int_{C \setminus B_{\varepsilon}} \frac{r}{2} \frac{\partial h}{|w|^{2}} \partial_{r} (|w|^{2} - 1)^{2} = - \int_{\partial B_{\varepsilon}} \frac{1}{|w|^{2}} (|w|^{2} - 1)^{2} - \int_{C \setminus B_{\varepsilon}} \frac{1}{r} \partial_{r} \left( \frac{r^{2}}{2} \frac{\partial h}{|w|^{2}} \right) (|w|^{2} - 1)^{2}.
$$

To begin with, we have, thanks to Lemma 11.6

$$
\left| \int_{\partial B_{\varepsilon}} \varepsilon \frac{\partial \frac{1}{2} h}{|u|^{2}} (|w|^{2} - 1)^{2} \right| \leq c \varepsilon^{1/2+2\alpha} \int_{\partial B_{\varepsilon}} (|w|^{2} - 1)^{2} \frac{1}{\varepsilon^{2}}.
$$

Using the mean value formula, we conclude as we have already done

$$
\left| \int_{\partial B_{\varepsilon}} \varepsilon \frac{\partial \frac{1}{2} h}{|u|^{2}} (|w|^{2} - 1)^{2} \right| \leq c \varepsilon^{1+2\alpha} \mathcal{Q}.
$$

Now, in view of Lemma 11.6, it should be clear that,

$$
\sup_{C \setminus B_{\varepsilon/2}} \frac{1}{r} \partial_{r} \left( \frac{r^{2}}{2} \frac{A \nabla \frac{1}{2} h}{|u|^{2}} \right),
$$

is bounded by some constant independent of $\varepsilon$. Hence we have

$$
\left| \int_{C \setminus B_{\varepsilon}} \frac{1}{r} \partial_{r} \left( \frac{r^{2}}{2} \frac{A \nabla \frac{1}{2} h}{|u|^{2}} \right) (|w|^{2} - 1)^{2} \right| \leq c \varepsilon^{2} \mathcal{Q}.
$$

We conclude that the third term on the right hand side of (11.79) is also bounded by $\mathcal{Q}/100$, provided $\varepsilon$ is small enough.

**Step 3.4.1** We now deal with $IV$ the fourth term in (11.79). Using Lemma 11.6, together with Cauchy-Schwarz inequality, we get

$$
\int_{B_{\varepsilon}} r \partial_{r} |w|^{2} (B - A) \frac{\nabla \frac{1}{2} h}{|u|^{2}} \leq c \left( \int_{B_{\varepsilon}} r^{2\nu+1} \varepsilon |\partial_{r} w|^{2} \right)^{1/2} \times \left( \int_{B_{\varepsilon}} r^{-2-2\nu} |A - B|^{2} \right)^{1/2}.
$$

which, thanks to (11.73) is bounded by $c \varepsilon^{2} \mathcal{Q}$ and hence is smaller than $\mathcal{Q}/100$, provided $\varepsilon$ is chosen small enough.
Step 3.4.2. Integrating by part now the integral of the same quantity but over $C \setminus B_{\gamma_{\varepsilon}}$ yields

$$
\int_{C \setminus B_{\gamma_{\varepsilon}}} r \partial_r |w|^2 (B - A) \frac{\nabla^\perp h}{|u|^2} = -\gamma \int_{\partial B_{\gamma_{\varepsilon}}} \varepsilon(|w|^2 - 1)(B - A) \frac{\nabla^\perp h}{|u|^2} - \int_{C \setminus B_{\gamma_{\varepsilon}}} (|w|^2 - 1) \frac{1}{r} \partial_r \left( r^2 (B - A) \frac{\nabla^\perp h}{|u|^2} \right).
$$

Using the mean value formula, we obtain the existence of $\gamma \in (1/2, 1)$ such that

$$
\frac{\varepsilon}{2} \int_{\partial B_{\gamma_{\varepsilon}}} ||w|^2 - 1|| (B - A) \frac{\nabla^\perp h}{|u|^2} \leq \int_{B_{\gamma_{\varepsilon}}/2} \frac{1}{r} ||u| - |v|| (B - A) \nabla^\perp h \int_{B_{\gamma_{\varepsilon}}/2} |B - A|^2 \frac{1}{|u|^2},
$$

which is also bounded by $Q/100$ for all $\varepsilon$ small enough.

Finally, we have

$$
\left( \int_{C \setminus B_{\gamma_{\varepsilon}}} (|w|^2 - 1) \frac{1}{r} \partial_r \left( r^2 (B - A) \frac{\nabla^\perp h}{|u|^2} \right) \right)^2 \leq \left( \int_{C \setminus B_{\gamma_{\varepsilon}}} r^{2\nu} ||u| - |v||^2 \right) \left( \sum_{j=0}^1 \int_{C \setminus B_{\gamma_{\varepsilon}}} r^{-2(\nu - j) - 2\|\nabla^j (A - B)||^2} \right),
$$

(11.90)

Which, thanks to (11.75) and (11.73) is bounded by $c\varepsilon^{2-2\nu} Q$ and hence, is less than $Q/100$, for $\varepsilon$ small enough. Observe that, to obtain the last estimate, we have used the result of Lemma 11.6 which implies that

$$
||\partial_r \left( r^2 \frac{\nabla h}{|u|^2} \right) || \leq c,
$$

in $C \setminus B_{\gamma_{\varepsilon}/2}$.

Step 4. Let us bound now the fourth term on the right hand side of (11.71). Once again we write this integral as the sum of the integral over $B_{\gamma_{\varepsilon}}$ and the integral over $C \setminus B_{\gamma_{\varepsilon}}$, for some $\gamma \in (1/2, 1)$.

Step 4.1. Clearly, we have

$$
\int_{B_{\gamma_{\varepsilon}}} r \partial_r \Phi \partial_A |w|^2 \leq c \left( \int_{B_{\gamma_{\varepsilon}}} |\partial_r w|^2 \right)^{1/2} \left( \int_{B_{\gamma_{\varepsilon}}} r^{2+2\alpha} |\nabla |w||^2 \right)^{1/2},
$$

(11.91)
where we have used the fact that \( \partial_r \Phi \leq c|\partial_r w| \) in \( B_{\gamma \varepsilon} \) and also (11.72).

However, we have
\[
\nabla |w| = \frac{\nabla (|u| - |v|)}{|v|} + \frac{|\nabla v|}{|v|^2} (|v| - |u|),
\]

This together with Lemma 11.1 implies that (11.91) becomes
\[
\left| \int_{B_{\gamma \varepsilon}} r \partial_r \Phi \partial_A |w|^2 \right|^2 \leq c \left( \int_{B_{\gamma \varepsilon}} |\partial_r w|^2 \right) \left( \int_{B_{\gamma \varepsilon}} r^{2\alpha} (|u| - |v|)^2 \right) \leq c \left( \int_{B_{\gamma \varepsilon}} |\partial_r w|^2 \right) \left( \int_{B_{\gamma \varepsilon}} r^{2\alpha} \varepsilon^2 |\nabla (|u| - |v|)|^2 \right).
\]

Using (11.75) we see that the first term on the right hand side of this inequality is bounded by \( c \varepsilon^{1-\alpha} Q \).

It remains to estimate
\[
\int_{B_{\gamma \varepsilon}} r^{2\alpha} \varepsilon^2 |\nabla (|u| - |v|)|^2.
\]

We write, as in the proof of Lemma 11.8, \( D := |u| - |v| \). Using the result of Lemma 11.6, it is an easy exercise to see that
\[
|\Delta D| \leq c \left( \frac{|D|}{r^2} + \frac{1}{r} \frac{|\nabla (h - k)|}{|u|} \right), \tag{11.92}
\]
in \( B_{2\varepsilon} \). Now, we can use Sobolev embedding together with classical elliptic estimates to show that, for all \( r > 0 \)
\[
\int_{B_{2\varepsilon} \setminus B_{r/2}} |\nabla D|^2 \leq c r^2 \int_{B_{4\varepsilon} \setminus B_{r/2}} |\Delta D|^2 + c r^{-2} \int_{B_{4\varepsilon} \setminus B_{r/2}} |D|^2.
\]

Multiplying by \( r^{2\alpha} \), we get easily
\[
\int_{B_{\gamma \varepsilon}} r^{2\alpha} |\nabla D|^2 \leq c \int_{B_{2\varepsilon}} r^{2\alpha + 2} |\Delta D|^2 + c \int_{B_{2\varepsilon}} r^{2\alpha - 2} |D|^2.
\]

Using (11.92), we conclude that
\[
\int_{B_{\gamma \varepsilon}} r^{2\alpha} \varepsilon^2 |\nabla (|u| - |v|)|^2 \leq c \int_{B_{2\varepsilon}} r^{2\alpha - 2} \varepsilon^2 (|u| - |v|)^2 + c \int_{B_{2\varepsilon}} r^{2\alpha} \varepsilon^2 \frac{|\nabla (h - k)|^2}{|u|^2}.
\]

Finally, we use (11.74) to evaluate the second term and (11.88) to evaluate the first term and we get
\[
\int_{B_{\gamma \varepsilon}} r^{2\alpha} \varepsilon^2 |\nabla (|u| - |v|)|^2 \leq c \varepsilon^{2\alpha} Q.
\]

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Therefore, we have proved that
\[ \int_{B_{\gamma \varepsilon}} r \partial_r \Phi \partial A |w|^2 \leq c (\varepsilon^\alpha + \varepsilon^{1-\alpha}) Q, \]
and in particular, is bounded by \( Q/100 \) provided \( \varepsilon \) is chosen small enough.

**Step 4.2.** We now turn to the evaluation of the same quantity \( B_{\gamma \varepsilon} \). Again, we integrate by parts
\[ \int_{C \setminus B_{\gamma \varepsilon}} r \partial_r \Phi \partial A |w|^2 = -\int_{\partial B_{\gamma \varepsilon}} \gamma \varepsilon \partial_r \Phi (|w|^2 - 1) A_r \]
\[ - \int_{C \setminus B_{\gamma \varepsilon}} \partial_A (r \partial_r \Phi) (|w|^2 - 1). \]
This first term is bounded as usual using the mean value formula and we find easily, using (11.72) that
\[ \left| \int_{\partial B_{\gamma \varepsilon}} \gamma \varepsilon \partial_r \Phi (|w|^2 - 1) A_r \right| \leq c \varepsilon^{2+\alpha-\nu} Q. \]
In order to estimate the second term, we recall that
\[ \partial_r \Phi = (A_r - B_r) + \frac{\partial_r (h - k)}{|u|^2}. \]
And hence we can bound
\[ \int_{C \setminus B_{\gamma \varepsilon}} \partial_A (r \partial_r \Phi) (|w|^2 - 1) \leq c \int_{C \setminus B_{\gamma \varepsilon}} |u| - |v| |r^\alpha |\nabla (h - k)| \]
\[ + c \int_{C \setminus B_{\gamma \varepsilon}} |u| - |v| |r^{\alpha+1} (|A - B| + r |\nabla (A - B)|) \]
\[ + c \int_{C \setminus B_{\gamma \varepsilon}} |u| - |v| |r^{\alpha+1} |\nabla^2 (h - k)|. \]
We use Cauchy-Schwarz inequality, (11.73)-(11.75) and (11.85) to conclude that
\[ \int_{C \setminus B_{\gamma \varepsilon}} \partial_A (r \partial_r \Phi) (|w|^2 - 1) \leq c \varepsilon^{(3-3\nu)/2} Q. \]
Again this quantity is bounded by \( Q/100 \) if \( \varepsilon \) is chosen small enough.

**Step 5.** So it just remains to bound the last term in the right hand side of (11.71). First observe that we can write
\[ \frac{\pi}{2} |a_{\infty} A_{\infty}^1| = \left| \left( \int_{C} \log r \nabla h \right) \cdot \left( \int_{C} z' h(z') \right) \right|. \]
However, since
\[ \int_{C} z' k(z') = 0, \quad \text{and} \quad B_{\infty} = 0, \]
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we can also write
\[ \frac{\pi}{2} |a_\infty A_\infty^1| = \left| \left( \int_C \log r \nabla (h - k) \right) \cdot \left( \int_C z' (h(z') - k(z')) \right) \right|. \]

We can bound the first term on the right hand side as follows. First, using (11.74) we get
\[
\left| \int_{B_1} \log r \nabla (h - k) \right|^2 \leq \left( \int_{B_1} r^{2\nu - 2} \log^2 r |u|^2 \right) \left( \int_{B_1} r^{2 - 2\nu} \frac{|\nabla (h - k)|^2}{|u|^2} \right) \\
\leq c \varepsilon^{2 - 2\nu} Q,
\]
(provided \(2\nu > 1\), which is true) and then, using (11.75), we obtain
\[
\left| \int_{C \setminus B_1} \log r \nabla (h - k) \right|^2 \leq \left( \int_{C \setminus B_1} \frac{1}{r^3} \log^2 r \right) \left( \int_{C \setminus B_1} r^3 |\nabla (h - k)|^2 \right) \\
\leq c \varepsilon^{2 - 2\nu} Q.
\]

The second term can be bounded as follows. First we have, using an integration by parts
\[
\left| \int_C z' (h - k) \right| \leq c \int_C r^2 |\nabla (h - k)|.
\]
Next we bound, using (11.74)
\[
\left| \int_{B_1} r^2 |\nabla (h - k)| \right|^2 \leq \left( \int_{B_1} r^{2\nu + 2} |u|^2 \right) \left( \int_{B_1} r^{2 - 2\nu} \frac{|\nabla (h - k)|^2}{|u|^2} \right) \\
\leq c \varepsilon^{2 - 2\nu} Q,
\]
and then, using (11.75)
\[
\left| \int_{C \setminus B_1} \log r \nabla (h - k) \right|^2 \leq \left( \int_{C \setminus B_1} \frac{1}{r^7} \right) \left( \int_{C \setminus B_1} r^7 |\nabla (h - k)|^2 \right) \\
\leq c \varepsilon^{2 - 2\nu} Q.
\]

Once more we conclude that \(\pi/2 a_\infty A_\infty^1\) is bounded by \(Q/100\) provided \(\varepsilon\) is small enough. And this finishes the proof.
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