VIII. Gabber's modification theorem (absolute case)

Luc Illusie and Michael Temkin [(i)]

version du 2016-11-14 à 13h36 TU (19c1b56)

Contents

In this exposé we state and prove Gabber's modification theorem mentioned in the introduction (see step (C)). Its main application is to Gabber's refined — i.e. prime to ℓ — local uniformization theorem. This is treated in exposé IX. A relative variant of the modification theorem, also due to Gabber, has applications to prime to ℓ refinements of theorems of de Jong on alterations of schemes of finite type over a field or a trait. This is discussed in exposé X. In §1, we state Gabber's modification theorem in its absolute form (Theorem 1.1). The proof of this theorem occupies §§4—5. A key ingredient is the existence of functorial (with respect to regular morphisms) resolutions in characteristic zero; the relevant material is collected in §2. We apply it in §3 to get resolutions of log regular log schemes, using the language of Kato's fans and Ogus's monoschemes. The main results, on which the proof of 1.1 is based, are Theorems 3.3.16 and 3.4.15. §§2 and 3 can be read independently of §§1, 4, 5.

Though we basically follow the lines of Gabber's original proof, our approach differs from it at several places, especially in our use of associated points and saturated desingularization towers, whose idea is due to the second author. In 2.3.13 and 2.4 we discuss material from Gabber's original proof.

We wish to thank Sophie Morel for sharing with us her notes on resolution of log regular log schemes and Gabber's magic box. They were quite useful.

1. Statement of the main theorem

Theorem 1.1. — *Let X be a noetherian, qe, separated, log regular fs log scheme (exp.* ***VI****, 1.2), endowed with an action of a finite group G. We assume that G acts tamely (exp.* ***VI****, 3.1) and generically freely on X (i.e. there exists a G-stable, dense open subset of X where the inertia groups $G_{\overline{x}}$ are trivial). Let Z be the complement of the open subset of triviality of the log structure of X, and let T be the complement of the largest G-stable open subset of X on which G acts freely. Then there exists an fs log scheme X' and a G-equivariant morphism $f = f_{(G,X,Z)}: X' \to X$ of log schemes having the following properties:*

(i) As a morphism of schemes, f is a projective modification, i.e. f is projective and induces an isomorphism of dense open subsets.

(ii) X' is log regular and $Z' = f^{-1}(Z \cup T)$ is the complement of the open subset of triviality of the log structure of X'.

(iii) The action of G on X' is very tame (exp. ***VI****, 3.1).*

When proving the theorem we will construct $f_{(G,X,Z)}$ that satisfies a few more nice properties that will be listed in Theorem 5.6.1. We remark that Gabber also proves the theorem, more generally, when X is not assumed to be qe. However, the quasi-excellence assumption simplifies the proof so we impose it here. Most of the proof works for a general noetherian X, so we will assume that X is qe only when this will be needed in §5.

(i) The research of M.T. was partially supported by the European Union Seventh Framework Programme (FP7/2007-2013) under grant agreement 268182. Part of it was done when M.T. was visiting the department of mathematics of the University of Paris-Sud and he is grateful for the hospitality.

1.2. — (a) Note that we do not demand that f is log smooth. In general, it is not. Here is an example. Let k be an algebraically closed field of characteristic $\neq 2$. Let $G = \{\pm 1\}$ act on the affine plane $X = \mathbf{A}^2_k$, endowed with the trivial log structure, by $x \mapsto \pm x$. Then X is regular and log regular, and $T = \{0\}$. The action of G on X is tame, but not very tame, as $G_{\{0\}}$ ($= G$) does not act trivially on the (only) stratum X of the stratification by the rank of $M^{gp}/\mathscr{O}^*$. Let $f\colon X' \to X$ be the blow up of T, with its natural action of G. Then the pair $(X', Z' = f^{-1}(T))$ is log regular, f is a G-equivariant morphism of log schemes, $X' - Z'$ is at the same time the open subset of triviality of the log structure and the largest G-stable open subset of X' where G acts freely, and G acts very tamely on X'. However, f is not log smooth (the fiber of f at $\{0\}$ is the line Z' with the log structure associated to $\mathbf{N} \to \mathscr{O}, 1 \mapsto 0$, which is not log smooth over $\operatorname{Spec} k$ with the trivial log structure).

(b) In the above example, let D_1, D_2 be distinct lines in X crossing at $\{0\}$, and put the log structure $M(D)$ on X defined by the divisor with normal crossings $D = D_1 \cup D_2$. Then $\tilde{X} = (X, M(D))$ is log smooth over $\operatorname{Spec} k$ endowed with the trivial log structure, and G acts very tamely on $(X, M(D))$ (exp. **VI**, 4.6). Moreover, the modification f considered above underlies the log blow up $\tilde{f}\colon \tilde{X}' \to \tilde{X}$ of X at (the ideal in $M(D)$ of) $\{0\}$. While f depends only on X, the log étale morphism $\tilde{f}$ is not canonical, as it depends on the choice of D. However, one can recover f from the *canonical* resolutions of toric singularities (discussed in the next section). Namely, as G acts very tamely on $\tilde{X}$, the quotient $\tilde{Y} = \tilde{X}/G$ is log regular (exp. **VI**, 3.2): $\tilde{Y} = \operatorname{Spec} k[P]$, where P is the submonoid of $\mathbf{Z}^2$ generated by $(2,0)$, $(1,1)$ and $(0,2)$, and the projection $p\colon \tilde{X} \to \tilde{Y}$ is a Kummer étale cover of group G, in particular, U is a G-étale cover of $V = p(U)$, where $U = X - D$. Let $g\colon \tilde{Y}' \to \tilde{Y}$ be the log blow up of $\{0\} = p(\{0\})$ in $\tilde{Y}$. We then have a cartesian diagram of log schemes

$$\begin{tikzcd} \tilde{X}' \arrow[r] \arrow[d] & \tilde{Y}' \arrow[d, "g"] \\ \tilde{X} \arrow[r, "p"] & \tilde{Y} \end{tikzcd},$$

where the horizontal maps are Kummer étale covers of group G. Now, as a morphism of schemes, $\tilde{Y}' \to \tilde{Y}$ is the *canonical resolution* of $\tilde{Y}$, and the underlying scheme X' of $\tilde{X}'$ is the *normalization* of $\tilde{Y}'$ in the G-étale cover $p\colon U \to V$. This observation, suitably generalized, plays a key role in the proof of 1.1.

2. Functorial resolutions

For simplicity, all schemes considered in this section are quasi-compact and quasi-separated. In particular, all morphisms are quasi-compact and quasi-separated.

2.1. Towers of blow ups. — In this section we review various known results on the following related topics: blow ups and their towers, various operations on towers, such as strict transforms and pushforwards, associated points of schemes and schematic closure.

2.1.1. Blow ups. — We start with recalling basic properties of blow ups; a good reference is [**Conrad, 2007**, §1]. Let X be a scheme. By a **blow up** of X we mean a triple consisting of a morphism $f\colon Y \to X$, a finitely presented closed subscheme V of X (the **center**), and an X-isomorphism $\alpha\colon Y \xrightarrow{\sim} \operatorname{Proj}(\bigoplus_{n\in\mathbf{N}} \mathscr{I}^n)$, where $\mathscr{I} = \mathscr{I}(V)$ is the ideal of V. We will write $Y = \mathrm{Bl}_V(X)$. When there is no risk of confusion we will omit V and α from the notation. A blow up (f, V, α) is said to be **empty** if $V = \varnothing$. In this case, $Y = X$ and f is the identity. The only X-automorphism of Y is the identity. Also, it is well known that Y is the universal X-scheme such that $V \times_X Y$ is a Cartier divisor (i.e. the ideal $\mathscr{I}\mathscr{O}_Y$ is invertible).

2.1.2. Total and strict transforms. — Given a blow up $f\colon Y = \mathrm{Bl}_V(X) \to X$, there are two natural ways to pullback closed subschemes $i\colon Z \hookrightarrow X$. The **total transform** of Z under f is the scheme-theoretic preimage $f^{\mathrm{tot}}(Z) = Z \times_X Y$. The **strict transform** $f^{\mathrm{st}}(Z)$ is defined as the schematic closure of $f^{-1}(Z - V) \xrightarrow{\sim} Z - V$.

Remark 2.1.3. — (i) The strict transform depends on the centers and not only on Z and the morphism $Y \to X$. For example, if $D \hookrightarrow X$ is a Cartier divisor then the morphism $\mathrm{Bl}_D(X) \to X$ is an isomorphism but the strict transform of D is empty.

(ii) While $f^{\mathrm{tot}}(Z) \to Z$ is just a proper morphism, the morphism $f^{\mathrm{st}}(Z) \to Z$ can be provided with the blow up structure because $f^{\mathrm{st}}(Z) \xrightarrow{\sim} \mathrm{Bl}_{V \times_X Z}(Z)$ (e.g., if $Z \hookrightarrow V$ then $f^{\mathrm{st}}(Z) = \varnothing = \mathrm{Bl}_Z(Z)$). Thus, the strict transform can be viewed as a genuine blow up pullback of f with respect to i.

2.1.4. Towers of blow ups. — Next we introduce blow up towers and study various operations with them (see also [**Temkin, 2012**, §2.2]). By a **tower of blow ups** of X we mean a finite sequence of length $n \geq 0$

$$X_\bullet = (\ X_n \xrightarrow{f_{n-1}} X_{n-1} \longrightarrow \cdots \xrightarrow{f_0} X_0 = X\ )$$

of blow ups. In particular, this data includes the centers $V_i \hookrightarrow X_i$ for $0 \leq i \leq n-1$. Usually, we will denote the tower as $X_\bullet$ or $(X_\bullet, V_\bullet)$. Also, we will often use notation $X_n \dashrightarrow X_0$ to denote a sequence of morphisms.

If $n = 0$ then we say that the tower is **trivial** . Note that the morphism $X_n \to X$ is projective, and it is a modification if and only if the centers V_i are nowhere dense. If $X_\bullet$ is a tower of blow ups, we denote by $(X_\bullet)_c$ the **contracted** tower deduced from $X_\bullet$ by omitting the empty blow ups.

2.1.5. Strict transform of a tower. — Assume that $\mathscr{X} = (X_\bullet, V_\bullet)$ is a blow up tower of X and $h\colon Y \to X$ is a morphism. We claim that there exists a unique blow up tower $\mathscr{Y} = (Y_\bullet, W_\bullet)$ of Y such that $Y_i \to Y \to X$ factors through X_i and $W_i = V_i \times_{X_i} Y_i$. Indeed, this defines Y_0, W_0 and Y_1 uniquely. Since $V_0 \times_X Y_1 = W_0 \times_{Y_0} Y_1$ is a Cartier divisor, $Y_1 \to X$ factors uniquely through X_1. The morphism $Y_1 \to X_1$ uniquely defines W_1 and Y_2, etc. We call $\mathscr{Y}$ the **strict transform** of $\mathscr{X}$ with respect to h and denote it $h^{st}(\mathscr{X})$.

Remark 2.1.6. — The following observation motivates our terminology: if $h\colon Y \hookrightarrow X$ is a closed immersion then $Y_i \hookrightarrow X_i$ is a closed immersion and Y_{i+1} is the strict transform of Y_i under the blow up $X_{i+1} \to X_i$.

2.1.7. Pullbacks. — One can also define a naive base change of $\mathscr{X}$ with respect to h simply as $(Y_\bullet, W_\bullet) = \mathscr{X} \times_X Y$. This produces a sequence of proper morphisms $Y_n \dashrightarrow Y_0$ and closed subschemes $W_i \hookrightarrow Y_i$ for $0 \leq i \leq n-1$. If this datum is a blow up sequence, i.e. $Y_{i+1} \xrightarrow{\sim} \mathrm{Bl}_{W_i}(Y_i)$, then we say that $\mathscr{X} \times_X Y$ is the **pullback** of $\mathscr{X}$ and use the notation $h^*(\mathscr{X}) = \mathscr{X} \times_X Y$.

Remark 2.1.8. — The pullback exists if and only if $h^{st}(\mathscr{X}) \xrightarrow{\sim} \mathscr{X} \times_X Y$. Indeed, this is obvious for towers of length one, and the general case follows by induction on the length.

2.1.9. Flat pullbacks. — Blow ups are compatible with flat base changes $h\colon Y \to X$ in the sense that $\mathrm{Bl}_{V \times_X Y}(Y) \xrightarrow{\sim} \mathrm{Bl}_V(X) \times_X Y$ (e.g. just compute these blow ups in the terms of Proj). By induction on length of blow up towers it follows that pullbacks of blow up towers with respect to flat morphisms always exist. One can slightly strengthen this fact as follows.

Remark 2.1.10. — Assume that $X_\bullet$ is a blow up tower of X and $h\colon Y \to X$ is a morphism. If there exists a flat morphism $f\colon X \to S$ such that the composition $g\colon Y \to S$ is flat and the blow up tower $X_\bullet$ is the pullback of a blow up tower $S_\bullet$ then the pullback $h^*(X_\bullet)$ exists and equals to $g^*(S_\bullet)$.

2.1.11. Equivariant blow ups. — Assume that X is an S-scheme acted on by a flat S-group scheme G. We will denote by $p, m\colon X_0 = G \times_S X \to X$ the projection and the action morphisms. Assume that $V \hookrightarrow X$ is a G-equivariant closed subscheme (i.e., $V \times_X (X_0; m)$ coincides with $V_0 = V \times_X (X_0; p)$) then the action of G lifts to the blow up $Y = \mathrm{Bl}_V(X)$. Indeed, the blow up $Y_0 = \mathrm{Bl}_{V_0}(X_0) \to X_0$ is the pullback of $Y \to X$ with respect to both m and p, i.e. there is a pair of cartesian squares

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d, shift right, "p'"'] \arrow[d, shift left, "m'"] & X_0 \arrow[d, shift right, "p"'] \arrow[d, shift left, "m"] \\ Y \arrow[r] & X \end{tikzcd}$$

So, we obtain an isomorphism $Y_0 \xrightarrow{\sim} G \times_S Y$ (giving rise to the projection $p'\colon Y_0 \to Y$) and a group action morphism $m'\colon Y_0 \to Y$ compatible with m. Furthermore, the unit map $e\colon X \to X_0$ satisfies the condition of Remark 2.1.10 (with $X = S$), hence we obtain the base change $e'\colon Y \to Y_0$ of e.

It is now straightforward to check that m' and e' satisfy the group action axioms, but let us briefly spell this out using simplicial nerves. The action of G on V defines a cartesian sub-simplicial scheme $\mathrm{Ner}(G, V) \hookrightarrow \mathrm{Ner}(G, X)$. By the flatness of G over S and Remark 2.1.10, $\mathrm{Bl}_{\mathrm{Ner}(G,V)}(\mathrm{Ner}(G, X))$ is cartesian over $\mathrm{Ner}(G)$, hence corresponds to an action of G on $\mathrm{Bl}_V(X)$.

2.1.12. Flat monomorphism. — Flat monomorphisms are studied in [**Raynaud, 1967**]. In particular, it is proved in [**Raynaud, 1967**, Prop. 1.1] that $i : Y \hookrightarrow X$ is a flat monomorphism if and only if i is injective and for any $y \in Y$ the homomorphism $\mathscr{O}_{X,y} \to \mathscr{O}_{X,x}$ is an isomorphism. Moreover, it is proved in [**Raynaud, 1967**, Prop. 1.2] that in this case i is a topological embedding and $\mathscr{O}_Y = \mathscr{O}_X|_Y$. In addition to open immersions, the main source of flat monomorphisms for us will be morphisms of the form $\mathrm{Spec}(\mathscr{O}_{X,x}) \hookrightarrow X$ and their base changes.

2.1.13. Pushforwards of ideals. — Let $i\colon Y \to X$ be a flat monomorphism, e.g. $\mathrm{Spec}(\mathcal{O}_{X,x})\hookrightarrow X$. By the **pushforward** $U = i_\star(V)$ of a closed subscheme $V\hookrightarrow Y$ we mean its schematic image in X, i.e. U is the minimal closed subscheme such that $V\hookrightarrow X$ factors through U. It exists by [**ÉGA** I 9.5.1].

Lemma 2.1.14. — *The pushforward* $U = i_\star(V)$ *extends* V *in the sense that* $U \times_X Y = V$.

Proof. — See [**Raynaud, 1967**, Proof of Proposition 1.2]. □

2.1.15. Pushforwards of blow up towers. — Given a blow up $f\colon \mathrm{Bl}_V(Y) \to Y$ and a flat monomorphism $i\colon Y\hookrightarrow X$ we define the pushforward $i_\star(f)$ as the blow up along $U = i_\star(V)$, assuming that U is finitely presented over X. Using Lemma 2.1.14 and flat pullbacks we see that $i^\star i_\star(f) = f$ and $\mathrm{Bl}_V(Y)\hookrightarrow \mathrm{Bl}_U(X)$ is a flat monomorphism. So, we can iterate this procedure to construct pushforward with respect to i of any blow up tower $Y_\bullet$ of Y. It will be denoted $(X_\bullet, U_\bullet) = i_\star(Y_\bullet, V_\bullet)$.

Remark 2.1.16. — (i) Clearly, $i^\star i_\star(Y_\bullet) = Y_\bullet$.

(ii) In the opposite direction, a blow up tower $(X_\bullet, U_\bullet)$ of X satisfies $i_\star i^\star(X_\bullet) = X_\bullet$ if and only if the preimage of Y in each center U_i of the tower is schematically dense.

2.1.17. Associated points of a scheme. — Assume that X is a noetherian scheme. Recall that a point $x \in X$ is called associated if m_x is an associated prime of $\mathcal{O}_{X,x}$, i.e. $\mathcal{O}_{X,x}$ contains an element whose annihilator is m_x, see [**ÉGA** IV$_2$ 3.1.1]. The set of all such points will be denoted $\mathrm{Ass}(X)$. The following result is well known but difficult to find in the literature.

Lemma 2.1.18. — *Let* $i\colon Y\hookrightarrow X$ *be a flat monomorphism. Then the schematic image of* i *coincides with* X *if and only if* $\mathrm{Ass}(X) \subset i(Y)$.

Proof. — Note that the schematic image of i can be described as $\mathrm{Spec}(\mathscr{F})$, where $\mathscr{F}$ is the image of the homomorphism $\phi\colon \mathcal{O}_X \to i_\star(\mathcal{O}_Y)$. Thus, the schematic image coincides with X if and only if $\mathrm{Ker}(\phi) = 0$.

If $i(Y)$ omits a point x then any m_x-torsion element $s \in \mathcal{O}_{X,x}$ is in the kernel of $\phi_x\colon \mathcal{O}_{X,x} \to i_\star(\mathcal{O}_Y)_x$ (we use that $\mathcal{O}_Y = \mathcal{O}_X|_Y$ by 2.1.12 and since the closure of x is the support of an extension of s to a sufficiently small neighborhood, the restriction $s|_Y$ vanishes). So, if there exists $x \in \mathrm{Ass}(X)$ with $x \notin i(Y)$ then $\mathrm{Ker}(\phi) \neq 0$.

Conversely, if the kernel is non-zero then we take x to be any maximal point of its support and choose any non-zero $s \in \mathrm{Ker}(\phi_x)$. In particular, $s|_{Y\cap X_x} = 0$ and hence $x \notin i(Y)$. For any non-trivial generization y of x the image of s in $\mathcal{O}_{X,y}$ vanishes because $\mathrm{Ker}(\mathcal{O}_{X,y} \to i_\star(\mathcal{O}_Y)_y) = 0$ by maximality of x. Thus, the closure of x is the support of s, and hence s is annihilated by a power of m_x. Since X is noetherian, we can find a multiple of s whose annihilator is m_x, thereby obtaining that $x \in \mathrm{Ass}(X)$. □

2.1.19. Associated points of blow up towers. — If $(X_\bullet, V_\bullet)$ is a blow up tower and all X_i's are noetherian then by the set $\mathrm{Ass}(X_\bullet)$ of its associated points we mean the union of the images of $\mathrm{Ass}(V_i)$ in X. Combining Remark 2.1.16(i) and Lemma 2.1.18 we obtain the following:

Lemma 2.1.20. — *Let* $i\colon Y\hookrightarrow X$ *be a flat monomorphism and let* $X_\bullet$ *be a blow up tower of* X*. Then* $i_\star i^\star(X_\bullet) = X_\bullet$ *if and only if* $\mathrm{Ass}(X_\bullet) \subset i(Y)$.

2.2. Normalized blow up towers. — For reduced schemes most of the notions, constructions and results of §2.1 have normalized analogs. We develop such a "normalized" theory in this section.

2.2.1. Normalization. — The normalization of a reduced scheme X with finitely many irreducible components, as defined in [**ÉGA** II 6.3.8], will be denoted X^{nor}. Recall that normalization is compatible with open immersions and for an affine $X = \mathrm{Spec}(A)$ its normalization is $X^{\mathrm{nor}} = \mathrm{Spec}(B)$ where B is the integral closure of A in its total ring of fractions (which is a finite product of fields). The normalization morphism $X^{\mathrm{nor}} \to X$ is integral but not necessarily finite.

2.2.2. Functoriality. — Recall (exp. **II**, 1.1.2) that a morphism $f\colon Y \to X$ is called **maximally dominating** if it takes generic points of Y to generic points of X. Normalization is a functor on the category whose objects are reduced schemes having finitely many irreducible components and whose morphisms are the maximally dominating ones. Furthermore, it possesses the following universal property: any maximally dominating morphism $Y \to X$ with normal Y factors uniquely through X^{nor}. (By definition, Y is normal if its local rings are normal domains. Both claims are local on Y and X and are obvious for affine schemes.)

2.2.3. *Normalized blow ups.* — Assume that X is a reduced scheme with finitely many irreducible components. By the **normalized blow up** of X along a closed subscheme V of finite presentation we mean the morphism $f\colon \mathrm{Bl}_V(X)^{\mathrm{nor}} \to X$. The normalization is well defined since $\mathrm{Bl}_V(X)$ is reduced and has finitely many irreducible components. Note that f is universally closed but does not have to be of finite type. As in the case of usual blow ups, V is a part of the structure. In particular, $\mathrm{Bl}_V(X)^{\mathrm{nor}}$ has no X-automorphisms and we can talk about equality of normalized blow ups (as opposed to an isomorphism).

Proposition 2.2.4. — *(i) Keep the above notation. Then $\mathrm{Bl}_V(X)^{\mathrm{nor}} \to X$ is the universal maximally dominating morphism $Y \to X$ such that Y is normal and $V \times_X Y$ is a Cartier divisor.*

(ii) For any blow up $f\colon Y = \mathrm{Bl}_V(X) \to X$ its normalization $f^{\mathrm{nor}}\colon Y^{\mathrm{nor}} \to X^{\mathrm{nor}}$ is the normalized blow up along $V \times_X X^{\mathrm{nor}}$.

Proof. — Combining the universal properties of blow ups and normalizations we obtain (i), and (ii) is its immediate corollary. □

Towers of normalized blow ups and their transforms can now be defined similarly to their non-normalized analogs.

2.2.5. *Towers of normalized blow ups.* — A **tower of normalized blow ups** is a finite sequence $X_n \dashrightarrow X_{-1}$ with $n \geq 0$ of normalized blow ups with centers $V_i \hookrightarrow X_i$ for $-1 \leq i \leq n-1$ and $V_{-1} = \varnothing$. The centers are part of the datum. Note that the map $X_0 \to X_{-1}$ is just the normalization map. The **contraction** of a normalized blow up tower removes the normalized blow ups with empty centers for $i \geq 0$. It follows from [**ÉGA** IV$_3$ 8.6.3] that each X_i with $i \geq 0$ is a normalization of a reduced projective X_{-1}-scheme. A tower is called **noetherian** if all X_i are noetherian.

2.2.6. *Normalization of a blow up tower.* — Using induction on length and Proposition 2.2.4(ii), we can associate to a blow up tower $\mathscr{X} = (X_\bullet, V_\bullet)$ of a reduced scheme X with finitely many irreducible components a normalized blow up tower $\mathscr{X}^{\mathrm{nor}} = (Y_\bullet, W_\bullet)$, where $Y_{-1} = X$ and $Y_i = X_i^{\mathrm{nor}}$, $W_i = V_i \times_{X_i} X_i^{\mathrm{nor}}$ for $i \geq 0$. We call $\mathscr{X}^{\mathrm{nor}}$ the **normalization** of $\mathscr{X}$.

2.2.7. *Strict transforms.* — If $\mathscr{X} = (X_\bullet, V_\bullet)$ is a normalized blow up tower of $X = X_{-1}$ and $f\colon Y \to X$ is a morphism between reduced schemes with finitely many irreducible components then we define the strict transform $f^{\mathrm{st}}(\mathscr{X})$ as the normalized blow up tower $(Y_\bullet, W_\bullet)$ such that $Y_{-1} = Y$ and $W_i = V_i \times_{X_i} Y_i$. Using induction on the length and the universal property of normalized blow ups, see 2.2.4 (i), one shows that such a tower exists and is the universal normalized blow up tower of Y such that $f = f_{-1}$ extends to a compatible sequence of morphisms $f_i\colon Y_i \to X_i$.

2.2.8. *Pullbacks.* — The strict transform $f^{\mathrm{st}}(\mathscr{X})$ as above will be called the **pullback** and denoted $f^*(\mathscr{X})$ if $Y_i \xrightarrow{\sim} X_i \times_X Y$ for any $-1 \leq i \leq n$. Recall that a morphism is **regular** if it is flat and has geometrically regular fibers, see [**ÉGA** IV$_2$ 6.8.1].

Lemma 2.2.9. — *If $f\colon Y \to X$ is a regular morphism between reduced noetherian schemes then any normalized blow up tower $\mathscr{X}$ of X admits a pullback $f^*(\mathscr{X})$.*

Proof. — Blow ups are compatible with flat morphisms hence we should only show that normalizations in our tower are compatible with regular morphisms: if $f\colon Y \to X$ is a regular morphism of reduced noetherian schemes then the morphism $h\colon Y^{\mathrm{nor}} \to X^{\mathrm{nor}} \times_X Y$ is an isomorphism. Since h is an integral morphism which is generically an isomorphism, it suffices to show that $X^{\mathrm{nor}} \times_X Y$ is normal.

To prove the latter we can assume that X and Y are affine. Then f is a filtered limit of smooth morphisms $h_i\colon Y_i \to X$ by Popescu's theorem. If the claim holds for h_i then it holds for h, so we can assume that h is smooth. We claim that, more generally, if A is a normal domain and $\phi\colon A \to B$ is a smooth homomorphism then B is normal. Indeed, A is a filtered colimit of noetherian normal subdomains A_i and by [**ÉGA** IV$_3$ 8.8.2] and [**ÉGA** IV$_4$ 17.7.8] ϕ is the base change of a smooth homomorphism $\phi_i\colon A_i \to B_i$ for large enough i. For each $j \geq i$ let $\phi_j\colon A_j \to B_j$ be the base change of ϕ_i. Each B_j is normal by [**Matsumura, 1980**, 21.E (iii)] and B is the colimit of B_j, hence B is normal. □

2.2.10. *Fpqc descent of blow up towers.* — The classical fpqc descent of ideals (and modules) implies that there is also an fpqc descent for blow up towers. Namely, if $Y \to X$ is an fpqc covering and $Y_\bullet$ is a blow up tower of Y whose both pullbacks to $Y \times_X Y$ are equal then $Y_\bullet$ canonically descends to a blow up tower of X because the centers descend. In the same way, normalized blow up towers descend with respect to quasi-compact surjective regular morphisms.

2.2.11. Associated points. — The material of 2.1.15–2.1.19 extends to noetherian normalized blow up towers almost verbatim. In particular, if $\mathscr{X} = (X_\bullet, V_\bullet)$ is such a tower then $\mathrm{Ass}(\mathscr{X})$ is the union of the images of $\mathrm{Ass}(V_j)$ and for any flat monomorphism $i\colon Y \hookrightarrow X$ (which is a regular morphism by 2.1.12) with a blow up tower $\mathscr{Y}$ of Y we always have that $i^* i_* \mathscr{Y} = \mathscr{Y}$, and we have that $i_* i^* \mathscr{X} = \mathscr{X}$ if and only if $\mathrm{Ass}(\mathscr{X}) \subset i(Y)$.

2.3. Functorial desingularization. — In this section we will formulate the desingularization result about toric varieties that will be used later in the proof of Theorem 1.1. Then we will show how it is obtained from known desingularization results.

2.3.1. Desingularization of a scheme. — By a **resolution** (or **desingularization**) **tower** of a noetherian scheme X we mean a tower of blow ups with nowhere dense centers $X_\bullet$ such that $X = X_0$, X_n is regular and no f_i is an empty blow up. For example, the trivial tower is a desingularization if and only if X itself is regular.

2.3.2. Normalized desingularization. — We will also consider normalized blow up towers such that each center is non-empty and nowhere dense, $X = X_{-1}$ and X_n is regular. Such a tower will be called a **normalized desingularization tower** of X.

Remark 2.3.3. — (i) For any desingularization tower $\mathscr{X}$ of X its normalization $\mathscr{X}^{\mathrm{nor}}$ is a normalized desingularization tower of X.

(ii) Usually one works with non-normalized towers; they are subtler objects that possess more good properties. All known constructions of functorial desingularization (see below) produce blow up towers by an inductive procedure, and one cannot work with normalized towers instead. However, it will be easier for us to deal with normalized towers in log geometry because in this case one may work only with fs log schemes.

2.3.4. Functoriality of desingularization. — For concreteness, we consider desingularizations in the current section, but all what we say holds for normalized desingularizations too. Assume that a class $\mathscr{S}^0$ of noetherian S-schemes is provided with desingularizations $\mathscr{F}(X) = X_\bullet$ for any $X \in \mathscr{S}^0$. We say that the desingularization (family) $\mathscr{F}$ is **functorial** with respect to a class $\mathscr{S}^1$ of S-morphisms between the elements of $\mathscr{S}^0$ if for any $f\colon Y \to X$ from $\mathscr{S}^1$ the desingularization of X induces that of Y in the sense that $f^*\mathscr{F}(X)$ is defined and its contraction coincides with $\mathscr{F}(Y)$ (so, $\mathscr{F}(Y) = (Y \times_X \mathscr{F}(X))_c$). Note that we put the $=$ sign instead of an isomorphism sign, which causes no ambiguity by the fact that any automorphism of a blow up is the identity as we observed above.

Remark 2.3.5. — (i) Contractions in the pulled back tower appear when some centers of $\mathscr{F}(X)$ are mapped to the complement of $f(Y)$ in X. In particular, if $f \in \mathscr{S}^1$ is surjective then the precise equality $\mathscr{F}(Y) = Y \times_X \mathscr{F}(X)$ holds.

(ii) Assume that $X = \bigcup_{i=1}^n X_i$ is a Zariski covering and the morphisms $X_i \hookrightarrow X$ and $\coprod_{i=1}^n X_i \to X$ are in $\mathscr{S}^1$. In general, one cannot reconstruct $\mathscr{F}(X)$ from the $\mathscr{F}(X_i)$'s because the latter are contracted pullbacks and it is not clear how to glue them with correct synchronization. However, all information about $\mathscr{F}(X)$ is kept in $\mathscr{F}(\coprod_{i=1}^n X_i)$. The latter is the pullback of $\mathscr{F}(X)$ hence we can reconstruct $\mathscr{F}(X)$ by gluing the restricted blow up towers $\mathscr{F}(\coprod_{i=1}^n X_i)|_{X_i}$. Note that $\mathscr{F}(\coprod_{i=1}^n X_i)|_{X_i}$ can be obtained from $\mathscr{F}(X_i)$ by inserting empty blow ups, and these empty blow ups make the gluing possible. This trick with synchronization of the towers $\mathscr{F}(X_i)$ by desingularizing disjoint unions is often used in the modern desingularization theory, and one can formally show (see [Temkin, 2012, Rem. 2.3.4(iv)]) that such approach is equivalent to the classical synchronization of the algorithm with an invariant.

(iii) Assume that $\mathscr{S}^1$ contains all identities Id_X with $X \in \mathscr{S}^0$ and for any $Y, Z \in \mathscr{S}^0$ there exists $T = Y \coprod Z$ in $\mathscr{S}^0$ such that for any pair of morphisms $a\colon Y \to X$, $b\colon Z \to X$ in $\mathscr{S}^1$ the morphism $(a, b)\colon T \to X$ is in $\mathscr{S}^1$. As an illustration of the above trick, let us show that even if $f, g\colon Y \to X$ are in $\mathscr{S}^1$ but not surjective, we have an equality $\mathscr{F}(X) \times_X (Y, f) = \mathscr{F}(X) \times_X (Y, g)$ of non-contracted towers. Indeed, set $Y' = Y \coprod X$ and consider the morphisms $f', g'\colon Y' \to X$ that agree with f and g and map X by the identity. Then $\mathscr{F}(X) \times_X (Y', f')$ and $\mathscr{F}(X) \times_X (Y', g')$ are equal because f' and g' are surjective, hence their restrictions onto Y are also equal, but these are precisely $\mathscr{F}(X) \times_X (Y, f)$ and $\mathscr{F}(X) \times_X (Y, g)$.

2.3.6. Gabber's magic box. — Now we have tools to formulate the aforementioned desingularization result.

Theorem 2.3.7. — *Let $\mathscr{S}^0$ denote the class of finite disjoint unions of affine toric varieties over $\mathbf{Q}$, i.e. $\mathscr{S}^0 = \{\coprod_{i=1}^n \mathrm{Spec}(\mathbf{Q}[P_i])\}$, where $P_1, \dots, P_n$ are fs torsion free monoids. Let $\mathscr{S}^1$ denote the class of smooth morphisms*

$$f\colon \coprod_{j=1}^m \mathrm{Spec}(\mathbf{Q}[Q_j]) \to \coprod_{i=1}^n \mathrm{Spec}(\mathbf{Q}[P_i])$$

such that for each $1 \leq j \leq m$ *there exists* $1 \leq i = i(j) \leq n$ *and a homomorphism of monoids* $\phi_j \colon P_i \to Q_j$ *so that the restriction of* f *onto* $\mathrm{Spec}(\mathbf{Q}[Q_j])$ *factors through the toric morphism* $\mathrm{Spec}(\mathbf{Q}[\phi_j])$*. Then there exists a desingularization* $\mathcal{F}$ *on* $\mathcal{S}^0$ *which is functorial with respect to* $\mathcal{S}^1$ *and, in addition, satisfies the following compatibility condition: if* $\mathcal{O}_1, \dots, \mathcal{O}_l$ *are complete noetherian local rings containing* $\mathbf{Q}$, $Z = \coprod_{i=1}^{l} \mathrm{Spec}(\mathcal{O}_i)$*, and* $g, h \colon Z \to X$ *are two regular morphisms with* $X \in \mathcal{S}^0$ *then*

$$(Z, g) \times_X \mathcal{F}(X) = (Z, h) \times_X \mathcal{F}(X). \tag{2.3.7.1}$$

In the above theorem, we use the convention that different tuples $P_1, \dots, P_n$ give rise to different schemes $\coprod_{i=1}^{n} \mathrm{Spec}(\mathbf{Q}[P_i])$. Before showing how this theorem follows from known desingularization results, let us make a few comments.

Remark 2.3.8. — (i) Gabber's original magic box also requires that the centers are smooth schemes. This (and much more) can also be achieved as will be explained later, but we prefer to emphasize the minimal list of properties that will be used in the proof of Theorem 1.1.

(ii) It is very important to allow disjoint unions in the theorem in order to deal with synchronization issues, as explained in Remark 2.3.5(ii). This theme will show up repeatedly throughout the exposé.

2.3.9. Desingularization of qe schemes over $\mathbf{Q}$. — In practice, all known functorial desingularization families are constructed in an explicit algorithmic way, so one often says a **desingularization algorithm** instead of a desingularization family. We adopt this terminology below.

Gabber's magic box 2.3.7 is a particular case of the following theorem, see [**Temkin, 2012**, Th. 1.2.1]. Indeed, due to Remark 2.3.5(iii), functoriality with respect to regular morphisms implies (2.3.7.1).

Theorem 2.3.10. — *There exists a desingularization algorithm* $\mathcal{F}$ *defined for all reduced noetherian quasi-excellent schemes over* $\mathbf{Q}$ *and functorial with respect to all* regular *morphisms. In addition,* $\mathcal{F}$ *blows up only regular centers.*

Remark 2.3.11. — Although this is not stated in [**Temkin, 2012**], one can strengthen Theorem 2.3.10 by requiring that $\mathcal{F}$ blows up only regular centers contained in the singular locus. An algorithm $\mathcal{F}$ is constructed in [**Temkin, 2012**] from an algorithm $\mathcal{F}_{\mathrm{Var}}$ that desingularizes varieties of characteristic zero, and one can check that if the centers of $\mathcal{F}_{\mathrm{Var}}$ lie in the singular loci (of the intermediate varieties) then the same is true for $\mathcal{F}$. Let us explain how one can choose an appropriate $\mathcal{F}_{\mathrm{Var}}$. In [**Temkin, 2012**], one uses the algorithm of Bierstone-Milman to construct $\mathcal{F}$, see Theorem 6.1 and its Addendum in [**Bierstone et al., 2011**] for a description of this algorithm and its properties. It follows from the Addendum that the algorithm blows up centers lying in the singular loci until X becomes smooth, and then it performs some additional blow ups to make the exceptional divisor snc. Eliminating the latter blow ups we obtain a desingularization algorithm $\mathcal{F}_{\mathrm{Var}}$ which only blows up regular centers lying in the singular locus.

It will be convenient for us to use the algorithm $\mathcal{F}$ from Theorem 2.3.10 in the sequel. Also, to simplify the exposition we will freely use all properties of $\mathcal{F}$ but the careful reader will notice that only the properties of Gabber's magic box will be crucial in the end. Also, instead of working with $\mathcal{F}$ itself we will work with its normalization $\mathcal{F}^{\mathrm{nor}}$ which assigns to a reduced qe scheme over $\mathbf{Q}$ the normalized blow up tower $\mathcal{F}(X)^{\mathrm{nor}}$. It will be convenient to use the notation $\widetilde{\mathcal{F}} = \mathcal{F}^{\mathrm{nor}}$ in the sequel.

Remark 2.3.12. — (i) Since normalized blow ups are compatible with regular morphisms, it follows from Theorem 2.3.10 that the normalized desingularization $\widetilde{\mathcal{F}}$ is functorial with respect to all regular morphisms.

(ii) The feature which is lost under normalization (and which is not needed for our purposes) is some control on the centers. The centers $\widetilde{V}_i$ of $\widetilde{\mathcal{F}}(X)$ are preimages of the centers $V_i \hookrightarrow X_i$ of $\mathcal{F}(X)$ under the normalization morphisms $X_i^{\mathrm{nor}} \to X_i$, so they do not have to be even reduced. It will only be important that $\widetilde{V}_i$'s are equivariant when a smooth group acts on X. In Gabber's original argument it was important to blow up only regular centers because they were not part of the blow up data, and one used that a regular center without codimension one components intersecting the regular locus is determined already by the underlying morphism of the blow up.

2.3.13. Alternative desingularization inputs. — For the sake of completeness, we discuss how other algorithms could be used instead of $\mathcal{F}$. Some desingularization algorithms for reduced varieties over $\mathbf{Q}$ are constructed in [**Bierstone & Milman, 1997**], [**Włodarczyk, 2005**], [**Bravo et al., 2005**], and [**Kollár, 2007**]. They all are functorial with respect to equidimensional smooth morphisms (though usually one "forgets" to mention the equidimensionality restriction). It is shown in [**Bierstone et al., 2011**, §6.3] how to make the algorithm of [**Bierstone & Milman, 1997**] fully functorial by a slight adjusting of the synchronization of its blow ups. All these algorithms can be used to produce a desingularization of log regular schemes (see §3), so the only difficulty is in establishing the compatibility (2.3.7.1).

For the algorithm of [**Bierstone et al., 2011**] it was shown by Bierstone-Milman (unpublished, see [**Bierstone et al., 2011**, Rem. 7.1(2)]) that the induced desingularization of a formal completion at a point depends only on the formal completion as a scheme. This is precisely what we need in (2.3.7.1).

Finally, there is a much more general result by Gabber, see Theorem 2.4.1, whose proof uses Popescu's theorem and the cotangent complex. It implies that, actually, any desingularization of reduced varieties over $\mathbf{Q}$ which is functorial with respect to smooth morphisms automatically satisfies (2.3.7.1). So, in principle, any functorial desingularization of varieties over $\mathbf{Q}$ could be used for our purposes. Since Gabber's result and its proof are powerful and novel for the desingularization theory (and were missed in [**Bierstone et al., 2011**], mainly due to a not so trivial involvement of the cotangent complex), we include them in §2.4.

2.3.14. Invariance of the regular locus. — Until the end of §2.3 we consider only qe schemes of characteristic zero, and our aim is to establish a few useful properties of $\mathscr{F}$ (and $\widetilde{\mathscr{F}}$) that are consequences of the functoriality property $\mathscr{F}$ satisfies. First, we claim that $\mathscr{F}$ does not modify the regular locus of X, and even slightly more than that:

Corollary 2.3.15. — *All centers of $\mathscr{F}(X)$ and $\widetilde{\mathscr{F}}(X)$ sit over the singular locus of X. In particular, X is regular if and only if $\mathscr{F}(X)$ is the trivial tower.*

Proof. — It suffices to study $\mathscr{F}$. The claim is obvious for $S = \mathrm{Spec}(\mathbf{Q})$ because S does not contain non-dense non-empty subschemes. By functoriality, $\mathscr{F}(T)$ is trivial for any regular T of characteristic zero, because it is regular over S. Finally, if T is the regular locus of X then $\mathscr{F}(T) = (\mathscr{F}(X) \times_X T)_c$ and hence any center $V_i \hookrightarrow X_i$ of $\mathscr{F}(X)$ does not intersect the preimage of T. □

2.3.16. Equivariance of the desingularization. — It is well known that functorial desingularization is equivariant with respect to any smooth group action (and, moreover, extends to functorial desingularization of stacks, see [**Temkin, 2012**, Th. 5.1.1]). For the reader's convenience we provide an elementary argument.

Corollary 2.3.17. — *Let S be a qe scheme over $\mathbf{Q}$, G be a smooth S-group and X be a reduced S-scheme of finite type acted on by G. Then there exists a unique action of G on $\mathscr{F}(X)$ and $\widetilde{\mathscr{F}}(X)$ that agrees with the given action on X.*

Proof. — Again, it suffices to study $\mathscr{F}$. Let $\mathscr{F}(X)$ be given by $X_n \dashrightarrow X_0 = X$ and $V_i \hookrightarrow X_i$ for $0 \le i \le n-1$. By $p, m \colon Y = G \times_S X \to X$ we denote the projection and the action morphisms. Note that m is smooth (e.g. m is the composition of the automorphism $(g, x) \to (g, gx)$ of $G \times_S X$ and p). Therefore, $\mathscr{F}(X) \times_X (Y; m) = \mathscr{F}(Y) = \mathscr{F}(X) \times_S G$ by Theorem 2.3.10 and Remark 2.3.5(i). In particular, $V_0 \times_X (Y; p) = V_0 \times_X (Y; m)$, i.e. V_0 is G-equivariant. By 2.1.11, X_1 inherits a G-action. Then the same argument implies that V_1 is G-equivariant and X_2 inherits a G-action, etc. □

2.4. Complements on functorial desingularizations. — This section is devoted to Gabber's result on a certain non-trivial compatibility property that any functorial desingularization satisfies. It will not be used in the sequel, so an uninterested reader may safely skip it.

Theorem 2.4.1. — *Assume that S is a noetherian scheme, $\mathscr{S}^0$ is a class of reduced S-schemes of finite type and $\mathscr{S}^1$ is a class of morphisms between elements of $\mathscr{S}^0$ such that if $f \colon Y \to X$ is smooth and $X \in \mathscr{S}^0$ then $Y \in \mathscr{S}^0$ and $f \in \mathscr{S}^1$. Let $\mathscr{F}$ be a desingularization on $\mathscr{S}^0$ which is functorial with respect to all morphisms of $\mathscr{S}^1$. Then any pair of regular morphisms $g \colon Z \to X$ and $h \colon Z \to Y$ with targets in $\mathscr{S}^0$ induces the same desingularization of Z; namely, $\mathscr{F}(X) \times_X Z = \mathscr{F}(Y) \times_Y Z$.*

Note that the theorem has no restrictions on the characteristic (because no such restriction appears in Popescu's theorem). Before proving the theorem let us formulate its important corollary, whose main case is when $S = \mathrm{Spec}(k)$ for a field k and $\mathscr{S}^0$ is the class of all reduced k-schemes of finite type.

Corollary 2.4.2. — *Keep the notation of Theorem 2.4.1. Then $\mathscr{F}$ canonically extends to the class $\widehat{\mathscr{S}^0}$ of all schemes that admit a regular morphism to a scheme from $\mathscr{S}^0$ and the extension is functorial with respect to all regular morphisms between schemes of $\widehat{\mathscr{S}^0}$.*

The main ingredient of the proof will be the following result that we are going to establish first.

Proposition 2.4.3. — *Consider a commutative diagram of noetherian schemes*

$$\begin{tikzcd} & Z \arrow[dl, "g"'] \arrow[d, "f"] \arrow[dr, "h"] & \\ X \arrow[dr, "a"'] & Z' \arrow[l, "g'"'] \arrow[r, "h'"] & Y \arrow[dl, "b"] \\ & S & \end{tikzcd}$$

such that a and b are of finite type, g and h are regular and g' is smooth. Then h' is smooth around the image of f.

For the proof we will need the following three lemmas. In the first one we recall the Jacobian criterion of smoothness, rephrased in terms of the cotangent complex.

Lemma 2.4.4. — *Let $f\colon X \to S$ be a morphism which is locally of finite presentation, and let $x \in X$. Then the following conditions are equivalent:*

(i) f is smooth at x;

(ii) $H_1(L_{X/S} \overset{L}{\otimes} k(x)) = 0$.

In the lemma we use the convention $H_i = H^{-i}$, and $L_{X/S}$ denotes the cotangent complex of X/S.

Proof. — (i) $\Rightarrow$ (ii) is trivial: as f is smooth at x, up to shrinking X we may assume f smooth, then $L_{X/S}$ is cohomologically concentrated in degree zero and locally free [**Illusie, 1972**, III 3.1.2]. Let us prove (ii) $\Rightarrow$ (i). We may assume that we have a factorization

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[d, "f"'] & Z \arrow[dl, "g"] \\ S & \end{tikzcd},$$

where i is a closed immersion of ideal I and g is smooth. Consider the standard exact sequence

$$(*) \qquad I/I^2 \to i^*\Omega^1_{Z/S} \to \Omega^1_{X/S} \to 0.$$

By the Jacobian criterion [**ÉGA** IV_4 17.12.1] and [**ÉGA** 0_{IV} 19.1.12], the smoothness of f at x is equivalent to the fact that the morphism

$$(**) \qquad (I/I^2) \otimes k(x) \to \Omega^1_{Z/S} \otimes k(x)$$

deduced from the left one in $(*)$ is injective. Now, $(I/I^2) \otimes k(x) = H_1(L_{X/Z} \overset{L}{\otimes} k(x))$ [**Illusie, 1972**, III 3.1.3], and $(**)$ is a morphism in the exact sequence associated with the triangle deduced from the transitivity triangle $Li^*L_{Z/S} \to L_{X/S} \to L_{X/Z} \to Li^*L_{Z/S}[1]$ by applying $\overset{L}{\otimes} k(x)$:

$$H_1(L_{X/S} \overset{L}{\otimes} k(x)) \to H_1(L_{X/Z} \overset{L}{\otimes} k(x))(= (I/I^2) \otimes k(x)) \to \Omega^1_{Z/S} \otimes k(x).$$

By (ii), $H_1(L_{X/S} \overset{L}{\otimes} k(x)) = 0$, hence $(**)$ is injective, which completes the proof. □

Lemma 2.4.5. — *Consider morphisms $f\colon X \to Y$, $g\colon Y \to S$, $h = gf\colon X \to S$, and let $x \in X$, $y = f(x) \in Y$. Assume that*

(i) $H_1(L_{X/S} \overset{L}{\otimes} k(x)) = 0$

(ii) $H_2(L_{X/Y} \overset{L}{\otimes} k(x)) = 0$.

Then $H_1(L_{Y/S} \overset{L}{\otimes} k(y)) = 0$. In particular, if g is locally of finite presentation then g is smooth at y.

Proof. — It is equivalent to show that $H_1(L_{Y/S} \overset{L}{\otimes} k(x)) = 0$, and this follows trivially from the exact sequence

$$H_2(L_{X/Y} \overset{L}{\otimes} k(x)) \to H_1(L_{Y/S} \overset{L}{\otimes} k(x)) \to H_1(L_{X/S} \overset{L}{\otimes} k(x)).$$

□

Lemma 2.4.6. — *Let $f\colon X \to S$ be a regular morphism between noetherian schemes. Then $L_{X/S}$ is cohomologically concentrated in degree zero and $H_0(L_{X/S}) = \Omega^1_{X/S}$ is flat.*

Proof. — We may assume $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$ affine. Then, by Popescu's theorem [**Swan, 1998**, 1.1], X is a filtering projective limit of smooth affine S-schemes $X_\alpha = \operatorname{Spec} B_\alpha$. By [**Illusie, 1972**, II (1.2.3.4)], we have

$$L_{B/A} = \operatorname{colim}_\alpha L_{B_\alpha/A}.$$

By [**Illusie, 1972**, III 3.1.2 and II 2.3.6.3], $L_{B_\alpha/A}$ is cohomologically concentrated in degree zero and $H_0(L_{B_\alpha/A}) = \Omega^1_{B_\alpha/A}$ is projective of finite type over B_α, so the conclusion follows. □

Proof of Proposition 2.4.3. — The composition $ag'\colon Z' \to S$ is locally of finite type. Since S is noetherian, ag' is locally of finite presentation, and so h' is locally of finite presentation too.

Next, we note that the question is local around a point $y = f(x) \in Z'$, $x \in Z$. In view of Lemma 2.4.4, by Lemma 2.4.5 applied to $Z \to Z' \to Y$ it suffices to show that $H_1(L_{Z/Y} \overset{L}{\otimes} k(x)) = 0$ and $H_2(L_{Z/Z'} \overset{L}{\otimes} k(x)) = 0$. As Z is regular over Y, the first vanishing follows from Lemma 2.4.6. For the second one, consider the exact sequence

$$H_2(L_{Z/X} \overset{L}{\otimes} k(x)) \to H_2(L_{Z/Z'} \overset{L}{\otimes} k(x)) \to H_1(L_{Z'/X} \overset{L}{\otimes} k(x)).$$

By the regularity of Z/X and Lemma 2.4.6, $H_2(L_{Z/X} \overset{L}{\otimes} k(x)) = 0$. As Z' is smooth over X, $H_1(L_{Z'/X} \overset{L}{\otimes} k(x)) = 0$ by Lemma 2.4.4, which proves the desired vanishing and finishes the proof. □

Proof of Theorem 2.4.1. — Find finite affine coverings $X = \bigcup_i X_i$, $Y = \bigcup_i Y_i$ and $Z = \bigcup_i Z_i$ such that $g(Z_i) \subset X_i$ and $h(Z_i) \subset Y_i$. Set $X' = \coprod_i X_i$, $Y' = \coprod_i Y_i$ and $Z' = \coprod_i Z_i$ and let $Z' \to X'$ and $Z' \to Y'$ be the induced morphisms. It suffices to check that $\mathscr{F}(X) \times_X Z$ and $\mathscr{F}(Y) \times_Y Z$ become equal after pulling them back to Z'. So, we should check that $(\mathscr{F}(X) \times_X X') \times_{X'} Z'$ coincides with $(\mathscr{F}(Y) \times_Y Y') \times_{Y'} Z'$. The morphisms $X' \to X$ and $Y' \to Y$ are smooth and hence contained in $\mathscr{S}^1$. So, $\mathscr{F}(X) \times_X X' = \mathscr{F}(X')$ and similarly for Y. In particular, it suffices to prove that $\mathscr{F}(X') \times_{X'} Z' = \mathscr{F}(Y') \times_{Y'} Z'$. This reduces the problem to the case when all schemes are affine, so in the sequel we assume that X, Y and Z are affine.

Next, note that it suffices to find factorizations $g = g_0 f$ and $h = h_0 f$, where $f\colon Z \to Z_0$ is a morphism with target in $\mathscr{S}^0$ and $g_0\colon Z_0 \to X$, $h_0\colon Z_0 \to X$ are smooth. By Popescu's theorem, one can write $g\colon Z \to X$ as a filtering projective limit of affine smooth morphisms $g_\alpha\colon Z_\alpha \to X$, $\alpha \in A$. As Y is of finite type over S, h will factor through one of the Z_α's ([**ÉGA** IV$_3$ 8.8.2.3]): there exists $\alpha \in A$, $f_\alpha\colon Z \to Z_\alpha$, $h_\alpha\colon Z_\alpha \to Y$ such that $g = g_\alpha f_\alpha$, $h = h_\alpha f_\alpha$. By Proposition 2.4.3, h_α is smooth around the image of f_α, so we can take Z_0 to be a sufficiently small neighborhood of the image of f_α. □

3. Resolution of log regular log schemes

All schemes considered from now on will be assumed to be noetherian. Unless said to the contrary, by log structure we mean a log structure with respect to the étale topology. We will say that a log structure M_X on a scheme X is **Zariski** if $\varepsilon^*\varepsilon_* M_X \xrightarrow{\sim} M_X$, where $\varepsilon\colon X_{\text{ét}} \to X$ is the morphism between the étale and Zariski sites. In this case, we can safely view the log structure as a Zariski log structure. A similar convention will hold also for log schemes.

3.1. Fans. — Many definitions/constructions on log schemes are of "combinatorial nature". Roughly speaking, these constructions use only multiplication and ignore addition. Naturally, there exists a category of geometric spaces whose structure sheaves are monoids, and most of combinatorial constructions can be described as "pullbacks" of analogous "monoidal" operations. The first definition of such a category was done by Kato in [**Kato, 1994**]. Kato called his spaces **fans** to stress their relation to the classical combinatorial fans obtained by gluing polyhedral cones. For example, to any combinatorial fan C one can naturally associate a fan $F(C)$ whose set of points is the set of faces of C. The main motivation for the definition is that fans can be naturally associated to various log schemes.

It took some time to discover that fans are sort of "piecewise schemes" rather than a monoidal version of schemes. A more geometric version of combinatorial schemes was introduced by Deitmar in [**Deitmar, 2005**]. He called them $\mathbf{F}_1$-schemes, but we prefer the terminology of monoschemes introduced by Ogus in his book project [**Ogus, 2013**]. Note that when working with a log scheme X, we use the sheaf M_X in some constructions and we use its sharpening $\overline{M}_X$ (see 3.1.1) in other constructions. Roughly speaking, monoschemes naturally arise when we work with M_X while fans naturally arise when we work with $\overline{M}_X$.

In §3, we will show that: (a) a functorial desingularization of toric varieties over $\mathbf{Q}$ descends to a desingularization of monoschemes, (b) to give the latter is more or less equivalent to give a desingularization of fans, (c) a desingularization of fans can be used to induce a monoidal desingularization of log schemes, (d) the latter induces a desingularization of log regular schemes, which (at least in some cases) depends only on the underlying scheme.

In principle, we could work locally, using desingularization of disjoint unions of all charts for synchronization. In this case, we could almost ignore the intermediate categories by working only with fine monoids and blow up towers of their spectra. However, we decided to emphasize the actual geometric objects beyond the constructions, and, especially, stress the difference between fans and monoschemes.

3.1.1. Sharpening. — For a monoid M, by M^* we denote the group of its invertible elements, and its **sharpening** $\overline{M}$ is defined as M/M^*.

3.1.2. Localization. — By **localization** of a monoid M along a subset S we mean the universal M-monoid M_S such that the image of S in M_S is contained in M_S^*. If M is integral then M_S is simply the submonoid $M[S^{-1}] \subset M^{gp}$ generated by M and S^{-1}. If M is a fine then any localization is isomorphic to a localization at a single element f, and will be denoted M_f.

3.1.3. Spectra of fine monoids. — All our combinatorial objects will be glued from finitely many spectra of fine monoids. Recall that with any fine monoid P one can associate the set $\mathrm{Spec}(P)$ of prime ideals (with the convention that $\varnothing$ is also a prime ideal) equipped with the Zariski topology whose basis is formed by the sets $D(f) = \{p \in \mathrm{Spec}(P) | f \notin p\}$ for $f \in P$, see, for example, [**Kato, 1994**, §9]. The structure sheaf M_P is defined by $M_P(D(f)) = P_f$, and the sharp structure sheaf $\overline{M}_P = M_P/M_P^*$ is the sharpening of M_P (we will see in Remark 3.1.4(iii) that actually $\overline{M}_P(D(f)) = \overline{P_f} = P_f/(P_f^*)$, i.e. no sheafification is needed).

Remark 3.1.4. — (i) Since $P \setminus P^*$ and $\varnothing$ are the maximal and the minimal prime ideals of P, $\mathrm{Spec}(P)$ possesses unique closed and generic points s and η. The latter is the only point whose stalk $M_{P,\eta} = P^{gp}$ is a group.

(ii) The set $\mathrm{Spec}(P)$ is finite and its topology is the specialization topology, i.e. U is open if and only if it is closed under generizations. (More generally, this is true for any finite sober topological space, such as a scheme that has finitely many points.)

(iii) A subset $U \subset \mathrm{Spec}(P)$ is affine (and even of the form $D(f)$) if and only if it is the localization of $\mathrm{Spec}(P)$ at a point x (i.e. the set of all generizations of x). Any open covering $U = \bigcup_i U_i$ of an affine set is trivial (i.e. U is equal to some U_i), therefore any functor $\mathscr{F}(U)$ on affine sets uniquely extends to a sheaf on $\mathrm{Spec}(P)$. In particular, this explains why no sheafification is needed when defining $\overline{M}_P$. Furthermore, we see that, roughly speaking, any notion/construction that is "defined in terms of" localizations X_x and stalks M_x or $\overline{M}_x$ is Zariski local. This is very different from the situation with schemes.

3.1.5. Local homomorphisms of monoids. — Any monoid M is local because $M \setminus M^*$ is its unique maximal ideal. A homomorphism $f\colon M \to N$ of monoids is **local** if it takes the maximal ideal of M to the maximal ideal of N. This happens if and only if $f^{-1}(N^*) = M^*$.

3.1.6. Monoidal spaces. — A **monoidal space** is a topological space X provided with a sheaf of monoids M_X. A morphism of monoidal spaces $(f, f^\#)\colon (Y, M_Y) \to (X, M_X)$ is a continuous map $f\colon Y \to X$ and a homomorphism $f^\#\colon f^{-1}(M_X) \to M_Y$ such that for any $y \in Y$ the homomorphism of monoids $f^\#_y\colon M_{X,f(y)} \to M_{Y,y}$ is local.

Remark 3.1.7. — Strictly speaking one should have called the above category the category of locally monoidal spaces and allow non-local homomorphisms in the general category of monoidal spaces. However, we will not use the larger category, so we prefer to abuse the terminology slightly.

Spectra of monoids possess the usual universal property, namely:

Lemma 3.1.8. — *Let (X, M_X) be a monoidal space and let P be a monoid.*

(i) The global sections functor Γ induces a bijection between morphisms of monoidal spaces $(f, f^\#)\colon (X, M_X) \to (\mathrm{Spec}(P), M_P)$ and homomorphisms $\phi\colon P \to \Gamma(M_X)$.

(ii) If M_X has sharp stalks then Γ induces a bijection between morphisms of monoidal spaces $(f, f^\#)\colon (X, M_X) \to (\mathrm{Spec}(P), \overline{M}_P)$ and homomorphisms $\phi\colon P \to \Gamma(M_X)$.

Proof. — (i) Let us construct the opposite map. Given a homomorphism ϕ, for any $x \in X$ we obtain a homomorphism $\phi_x\colon P \to M_{X,x}$. Clearly, $m = P \setminus \phi_x^{-1}(M^*_{X,x})$ is a prime ideal and hence ϕ_x factors through a uniquely defined local homomorphism $P_m \to M_{X,x}$. Setting $f(x) = m$ we obtain a map $f\colon X \to \mathrm{Spec}(P)$, and the rest of the proof of (i) is straightforward.

If the stalks of M_X are sharp then any morphism $(X, M_X) \to (\mathrm{Spec}(P), M_P)$ factors uniquely through $(\mathrm{Spec}(P), \overline{M}_P)$. Also, $\Gamma(M_X)$ is sharp, hence any homomorphism to it from P factors uniquely through $\overline{P}$. Therefore, (ii) follows from (i). □

3.1.9. Fine fans and monoschemes. — A fine **monoscheme** (resp. a fine **fan**) is a monoidal space (X, M_X) that is locally isomorphic to $A_P = (\mathrm{Spec}(P), M_P)$ (resp. $\overline{A}_P = (\mathrm{Spec}(P), \overline{M}_P)$), where P is a fine monoid. We say that (X, M_X) is **affine** if it is isomorphic to A_P (resp. $\overline{A}_P$). A morphism of monoschemes (resp. fans) is a morphism of monoidal spaces. A monoscheme (resp. a fan) is called **torsion free** if it is covered by spectra of P's with torsion free P^{gp}'s. It follows from Remark 3.1.4(iii) that this happens if and only if all groups $M^{gp}_{X,x}$ are torsion free.

Remark 3.1.10. — (i) Any fs fan is torsion free because if an fs monoid is torsion free then any of its localization is so. In particular, if P is fs and sharp then P^{gp} is torsion free. This is not true for general fine fans. For example, if $\mu_2 = \{\pm 1\}$ then $P = \mathbf{N} \oplus \mu_2 \setminus \{(0, -1)\}$ is a sharp monoid with $P^{gp} = \mathbf{Z} \oplus \mu_2$.

(ii) For any point x of a fine monoscheme (resp. fan) X the localization X_x that consists of all generizations of x is affine. In particular, by Remark 3.1.4(i) there exists a unique maximal point generizing x, and hence X is a disjoint union of irreducible components.

3.1.11. Comparison of monoschemes and fans. — There is an obvious sharpening functor $(X, M_X) \mapsto (X, \overline{M}_X)$ from monoschemes to fans, and there is a natural morphism of monoidal spaces $(X, \overline{M}_X) \to (X, M_X)$. The sharpening functor loses information, and one needs to know M^{gp}_X to reconstruct M_X from $\overline{M}_X$ as a fibred product (see [**Ogus, 2013**]). Actually, there are much more fans than monoschemes. For example, the generic point $\eta \in \mathrm{Spec}(P)$ is open and $\overline{M}_{P,\eta}$ is trivial hence for any pair of fine monoids P and Q we can glue their sharpened spectra along the generic points. What one gets is sort of "piecewise scheme" and, in general, it does not correspond to standard geometric objects, such as schemes or monoschemes. We conclude that, in general, fans can be lifted to monoschemes only locally.

Remark 3.1.12. — As a side remark we note that sharpened monoids naturally appear as structure sheaves of piecewise linear spaces (a work in progress of the second author on skeletons of Berkovich spaces). In particular, PL functions can be naturally interpreted as sections of the sharpened sheaf of linear functions on polytopes.

3.1.13. Local smoothness. — A local homomorphism of fine monoids $\phi\colon P \to Q$ is called **smooth** if it can be extended to an isomorphism $P \oplus \mathbf{N}^r \oplus \mathbf{Z}^s \xrightarrow{\sim} Q$. The following lemma checks that this property is stable under localizations.

Lemma 3.1.14. — *Assume that $\phi\colon P \to Q$ is smooth and P', Q' are localizations of P, Q such that ϕ extends to a local homomorphism $\phi'\colon P' \to Q'$. Then ϕ' is smooth.*

Proof. — Recall that $P' = P_a$ for $a \in P$ (notation of 3.1.2), and ϕ' factors through the homomorphism $\phi_a\colon P' \to Q_a$, which is obviously smooth. Therefore, replacing ϕ with ϕ_a we can assume that $P = P'$. Let $b = (p, n, z) \in Q$ be such that $Q' = Q_b$. Then $p \in P^*$ because $P \to Q'$ is local, and hence Q' is isomorphic to $P \oplus (\mathbf{N}^r)_n \oplus \mathbf{Z}^s$. It remains to note that any localization of $\mathbf{N}^r$ is of the form $\mathbf{N}^{r-t} \oplus \mathbf{Z}^t$. □

3.1.15. Smoothness. — The lemma enables us to globalize the notion of smoothness: a morphism $f\colon Y \to X$ of monoschemes is called **smooth** if the homomorphisms of stalks $M_{X,f(y)} \to M_{Y,y}$ are smooth. In particular, X is smooth if its morphism to $\mathrm{Spec}(1)$ is smooth, that is, the stalks $M_{X,x}$ are of the form $\mathbf{N}^r \oplus \mathbf{Z}^s$. In particular, a smooth monoscheme is torsion free.

Analogous smoothness definitions are given for fans. Moreover, in this case we can consider only sharp monoids, and then the group component $\mathbf{Z}^s$ is automatically trivial. It follows that we can rewrite the above paragraph almost *verbatim* but with $s = 0$. Obviously, a morphism of torsion free fs monoschemes is smooth if and only if its sharpening is a smooth morphism of fans.

Remark 3.1.16. — (i) Recall that any fine monoscheme X admits an open affine covering $X = \bigcup_{x \in X} \mathrm{Spec}(M_{X,x})$. It follows that a morphism of fine monoschemes $f\colon Y \to X$ is smooth if and only if it is covered by open affine charts of the form $\mathrm{Spec}(P \oplus \mathbf{N}^r \oplus \mathbf{Z}^s) \to \mathrm{Spec}(P)$.

(ii) Smooth morphisms of fine fans admit a similar local description, and we leave the details to the reader.

3.1.17. Saturation. — As usually, for a fine monoid P we denote its saturation by P^{sat} (it consists of all $x \in P^{gp}$ with $x^n \in P$ for some $n > 0$). Saturation is compatible with localizations and sharpening and hence extends to a saturation functor $F \mapsto F^{sat}$ on the categories of fine monoschemes (resp. fine fans). We also have a natural morphism $F^{sat} \to F$, which is easily seen to be bijective. So, actually, $(F, M_F)^{sat} = (F, M^{sat}_F)$.

3.1.18. Ideals. — A subsheaf of ideals $\mathscr{I} \subset M_X$ on a monoscheme (X, M_X) is called a **coherent ideal** if for any point $x \in X$ the restriction of $\mathscr{I}$ on X_x coincides with $\mathscr{I}_x M_{X_x}$. (Due to Remark 3.1.4(iii) this means that $\mathscr{I}$ is coherent in the usual sense, i.e. its restriction on an open affine submonoscheme U is generated by the global sections over U.) We will consider only coherent ideals, so we will omit the word "coherent" as a rule. An ideal $\mathscr{I} \subset M_X$ is **invertible** if it is locally generated by a single element.

3.1.19. Blow ups. — Similarly to schemes, for any ideal $\mathscr{I} \subset \mathscr{O}_X$ there exists a universal morphism of monoschemes $h\colon X' \to X$ such that the *pullback ideal* $h^{-1}\mathscr{I} = \mathscr{I} M_{X'}$ is invertible. We call $\mathscr{I}$ the **center** of the blow up. (One does not have an adequate notion of closed submonoscheme, so unlike blow ups of the scheme it would not make sense to say that "$V(\mathscr{I})$" is the center.) An explicit construction of X' copies its scheme analog: it is local on the base and for an affine $X = \mathrm{Spec}(P)$ with an ideal $I \subset P$ corresponding to $\mathscr{I}$ one glues X' from the charts $\mathrm{Spec}(P[a^{-1}I])$, where $a \in I$ and $P[a^{-1}I]$ is the submonoid of P^{gp} generated by the fractions b/a for $b \in I$ (see [**Ogus, 2013**] for details).

Remark 3.1.20. — Blow ups induce isomorphisms on the stalks of M^{gp}; this is an analog of the fact that blow ups of schemes along nowhere dense subschemes are birational morphisms.

3.1.21. Saturated blow ups. — Analogously to normalized blow ups, one defines **saturated blow up** of a monoscheme X along an ideal $\mathscr{I} \subset M_X$ as the saturation of $\mathrm{Bl}_{\mathscr{I}}(X)$. The same argument as for schemes shows that $\mathrm{Bl}_{\mathscr{I}}(X)^{\mathrm{sat}}$ is the universal saturated X-monoscheme such that the pullback of $\mathscr{I}$ is invertible.

3.1.22. Towers and pullbacks. — Towers of (saturated) blow ups of a monoscheme X are defined analogously to towers of (normalized) blow ups. In particular, saturated towers start with the saturation morphism $X_0 \to X_{-1}$. Given such a tower $X_\bullet$ with $X = X_0$ (resp. $X_{-1} = X$) and a morphism $f\colon Y \to X$ we define the **pullback tower** $Y_\bullet = f^*(X_\bullet)$ as follows: $Y_0 = Y$ (resp. $Y_{-1} = Y$) and Y_{i+1} is the (saturated) blow up of Y_i along the pullback of the center $\mathscr{I}_i$ of $X_{i+1} \to X_i$. Due to the universal property of (saturated) blow ups this definition makes sense and $Y_\bullet$ is the universal (saturated) blow up tower of Y that admits a morphism to $X_\bullet$ extending f.

Remark 3.1.23. — Unlike pullbacks of (normalized) blow up towers of schemes, see 2.1.7 and 2.2.8, we do not distinguish strict transforms and pullbacks. The above definition of pullback covers our needs, and we do not have to study the base change of monoschemes. For the sake of completeness, we note that fibred products of monoschemes exist and in the affine case are defined by amalgamated sums of monoids, see [**Deitmar, 2005**]. Also, it is easy to check that for a smooth f (which is the only case we will use) one indeed has that $f^*(X_\bullet) = X_\bullet \times_X Y$ for any (saturated) blow up tower $X_\bullet$. For blow ups one checks this with charts and in the saturated case one also uses that saturation is compatible with a smooth morphism $f\colon Y \to X$, i.e. $Y^{\mathrm{sat}} \xrightarrow{\sim} X^{\mathrm{sat}} \times_X Y$.

3.1.24. Compatibility with sharpening. — Ideals and blow ups of fans are defined in the same way, but with $\overline{M}_X$ used instead of M_X (Kato defines their saturated version in [**Kato, 1994**, 9.7]). Towers of blow ups of fans are defined in the obvious way.

Lemma 3.1.25. — *Let $X = (X, M_X)$ be a monoscheme, let $(F, M_F) = (X, \overline{M}_X)$ be the corresponding fan and let $\lambda\colon M_X \to M_F$ denote the sharpening homomorphism.*

(i) $\mathscr{I} \mapsto \lambda(\mathscr{I})$ induces a natural bijection between the ideals on X and on F.

(ii) Blow ups are compatible with sharpening, that is, the sharpening of $\mathrm{Bl}_{\mathscr{I}}(X))$ is naturally isomorphic to $\mathrm{Bl}_{\lambda(\mathscr{I})}(F)$. The same statement holds for saturated blow ups.

(iii) Sharpening induces a natural bijection between the (saturated) blow up towers of X and F.

Proof. — (i) is obvious. (ii) is shown by comparing the blow up charts. Combining (i) and (ii), we obtain (iii). $\square$

3.1.26. Desingularization. — Using the above notions of smoothness and blow ups, one can copy other definitions of the desingularization theory. By a **desingularization** (resp. **saturated desingularization**) of a fine monoscheme X we mean a blow up tower (resp. saturated blow up tower) $X_n \dashrightarrow X_0 = X$ (resp. $X_n \dashrightarrow X_{-1} = X$) with smooth X_n. By Remark 3.1.20, if X admits a desingularization then it is torsion free, and we will later see that the converse is also true.

For concreteness, we consider below non-saturated desingularizations, but everything extends to the saturated case verbatim. A family $\mathscr{F}^{\mathrm{mono}}(X)$ of desingularizations of torsion free monoschemes is called **functorial** (with respect to smooth morphisms) if for any smooth $f\colon Y \to X$ the desingularization $\mathscr{F}^{\mathrm{mono}}(Y)$ is the contracted pullback of $\mathscr{F}^{\mathrm{mono}}(X)$. The same argument as for schemes (see Remark 2.3.5(ii)) shows that $\mathscr{F}^{\mathrm{mono}}$ is already determined by its restriction to the family of finite disjoint unions of affine monoschemes.

The definition of a functorial desingularization $\mathscr{F}^{\mathrm{fan}}$ of fine torsion free fans is similar. Since blow up towers and smoothness are compatible with the sharpening functor, it follows that a desingularization of a monoscheme X induces a desingularization of its sharpening. Moreover, any affine fan can be lifted to an affine monoscheme and $\mathscr{F}^{\mathrm{fan}}$ is determined by its restriction onto disjoint unions of affine fans, hence we obtain the following result.

Theorem 3.1.27. — *The sharpening functor induces a natural bijection between functorial desingularizations of quasi-compact fine torsion free monoschemes and functorial desingularizations of quasi-compact fine torsion free fans. A similar statement holds for saturated desingularizations.*

Remark 3.1.28. — Similarly to the normalization of a desingularization tower, to any desingularization $\mathscr{F}$ of monoschemes or fans one can associate a saturated desingularization $\mathscr{F}^{\mathrm{sat}}$: one replaces all levels of the towers, except the zero level, with their saturations. In this case blow ups are replaced with saturated blow ups along the pulled back ideals. If $\mathscr{F}$ is functorial with respect to all smooth morphisms then the same is true for $\mathscr{F}^{\mathrm{sat}}$. Indeed, for any smooth $Y \to X$ the centers of $\mathscr{F}(Y)$, $\mathscr{F}^{\mathrm{sat}}(X)$ and $\mathscr{F}^{\mathrm{sat}}(Y)$ are the pullbacks of those of $\mathscr{F}(X)$. In addition, the saturation construction is compatible with the bijections from Theorem 3.1.27 in the obvious way.

Remark 3.1.29. — In principle, (saturated) desingularization of fans or monoschemes can be described in purely combinatorial terms of fans and their subdivisions (e.g. see [**Kato, 1994**, 9.6] or [**Nizioł, 2006**, §4]). However, it is not easy to construct a functorial one directly. We will instead use a relation between monoschemes and toric varieties to descend desingularization of toric varieties to monoschemes and fans.

3.2. Monoschemes and toric varieties. —

3.2.1. Base change from monoschemes to schemes. — Let S be a scheme. The following proposition introduces a functor from monoschemes to S-schemes that can be intuitively viewed as a base change with respect to a "morphism" $S \to \mathrm{Spec}(1)$.

Proposition 3.2.2. — *Let S be a scheme and F be a monoscheme. Then there exists an S-scheme $X = S[F]$ with a morphism of monoidal spaces $f\colon (X, \mathscr{O}_X) \to (F, M_F)$ such that any morphism $(Y, \mathscr{O}_Y) \to (F, M_F)$, where Y is an S-scheme, factors uniquely through f.*

Proof. — Assume, first, that $F = \mathrm{Spec}(P)$ is affine. By Lemma 3.1.8(i), to give a morphism $(Y, \mathscr{O}_Y) \to (F, M_F)$ is equivalent to give a homomorphism of monoids $\phi\colon P \to \Gamma(\mathscr{O}_Y)$, and the latter factors uniquely through a homomorphism of sheaves of rings $\mathscr{O}_S[P] \to \mathscr{O}_Y$. It follows that $S[F] = \mathrm{Spec}(\mathscr{O}_S[P])$ in this case. Since the above formula is compatible with localizations by elements $a \in P$, i.e. $S[F_a] \widetilde{\to} S[F]_{\phi(a)}$, it globalizes to the case of an arbitrary monoscheme. Thus, for a general monoscheme F covered by $F_i = \mathrm{Spec}(P_i)$, the scheme $S[F]$ is glued from $S[F_i]$. □

Remark 3.2.3. — Note that if $S = \mathrm{Spec}(R)$ and $F = \mathrm{Spec}(P)$ then $S[F] = \mathrm{Spec}(R[P])$. However, we will often consider an "intermediate" situation where $S = \mathrm{Spec}(R)$ is affine and F is a general monoscheme. To simplify notation, we will abuse them by writing $R[F]$ instead of $\mathrm{Spec}(R[F])$. Such "mixed" notation will always refer to a scheme.

3.2.4. Toric schemes. — If F is torsion free and connected then we call $S[F]$ a **toric scheme** over S. Recall that by Remark 3.1.10(ii), F possesses a unique maximal point $\eta = \mathrm{Spec}(P^{\mathrm{gp}})$, where $\mathrm{Spec}(P)$ is any affine open submonoscheme. Hence $X = S[F]$ possesses a dense open subscheme $T = S[\eta]$, which is a split torus over S, and the action of T on itself naturally extends to the action of T on X.

Remark 3.2.5. — Assume that $S = \mathrm{Spec}(k)$ where k is a field. Classically, a toric k-variety is defined as a normal finite type separated k-scheme X that contains a split torus T as a dense open subscheme such that the action of T on itself extends to the whole X. If F is saturated then our definition above is equivalent to the classical one. However, we also consider non-normal toric varieties corresponding to non-saturated monoids.

3.2.6. Canonical log structures. — For any monoscheme F, the S-scheme $X = S[F]$ possesses a natural log structure induced by the universal morphism $f\colon (X, \mathscr{O}_X) \to (F, M_F)$. Namely, M_X is the log structure associated with the pre-log structure $g^{-1}M_F \to \mathscr{O}_{X_{\text{ét}}}$, where $g\colon (X_{\text{ét}}, \mathscr{O}_{X_{\text{ét}}}) \to (X, \mathscr{O}_X) \to (F, M_F)$ is the composition. We call M_X the **canonical** log structure of $X = S[F]$.

Remark 3.2.7. — (i) The canonical log structure is Zariski, as $\varepsilon_\star M_X$ coincides with the Zariski log structure associated with the pre-log structure $f^{-1}M_F \to \mathscr{O}_X$.

(ii) The log scheme $(X = S[F], M_X)$ is log smooth over the scheme S provided with the trivial log structure. In particular, if S is regular and F is saturated then (X, M_X) is fs and log regular. Without the saturation assumption we still have that $X^{\mathrm{sat}} \xrightarrow{\sim} S[F^{\mathrm{sat}}]$ is log regular, hence X is log regular in the sense of Gabber (see §3.5).

(iii) If η is the set of generic points of F then $T = S[\eta]$ is the open subset of X which is the triviality set of its log structure. However, the map $M_X \to \mathscr{O}_X \cap j_\star \mathscr{O}_T^*$ is not an isomorphism in general, as the case where $T = \operatorname{Spec} \mathbf{C}[t, t^{-1}] \subset X = \operatorname{Spec} \mathbf{C}[t^2, t^3]$ already shows: the image of $t^2 + t^3$ in $\mathscr{O}_{X,\{0\}}$ belongs to $(j_\star \mathscr{O}_T^*)_{\{0\}}$, but does not belong to $M_{\{0\}} = t^P \mathscr{O}^*_{X,\{0\}}$, where P is the (fine, but not saturated) submonoid of $\mathbf{N}$ generated by 2 and 3. (We use that $\mathscr{O}_{X,\{0\}}$ is strictly smaller than its normalization $\mathbf{C}[t]_{(t)} = \mathscr{O}_{X,\{0\}}[t]$ and hence $\frac{t^2+t^3}{t^2} = 1 + t$ is not contained in $\mathscr{O}_{X,\{0\}}$.)

3.2.8. *Toric saturation.* — Saturation of monoschemes corresponds to normalization of schemes. This will play an essential role later, since we get a combinatorial description of the normalization.

Lemma 3.2.9. — *If S is a normal scheme and F is a fine torsion free monoscheme then there is a natural isomorphism $S[F]^{\mathrm{nor}} \xrightarrow{\sim} S[F^{\mathrm{sat}}]$.*

Proof. — Note that $f\colon \mathbf{Z}[F^{\mathrm{sat}}] \to \operatorname{Spec}(\mathbf{Z})$ is a flat morphism and its fibers are normal because they are classical toric varieties $\mathbf{F}_p[F^{\mathrm{sat}}]$. So, $f \times S\colon S[F^{\mathrm{sat}}] \to S$ is a flat morphism with normal fibers and normal target, and we obtain that its source is normal by [**Matsumura, 1980**, 21.E(iii)]. It remains to note that $S[F^{\mathrm{sat}}] \to S[F]$ is a finite morphism inducing isomorphism of dense open subschemes $S[F^{\mathrm{gp}}]$, hence $S[F^{\mathrm{sat}}]$ is the normalization of $S[F]$. □

Remark 3.2.10. — The same argument shows that if S is Cohen-Macaulay then so is $S[F^{\mathrm{sat}}]$.

3.2.11. *Toric smoothness.* — Next, let us compare smoothness of morphisms of monoschemes as defined in 3.1.15 and classical smoothness of toric morphisms. The following lemma slightly extends the classical result (e.g. see [**Fulton, 1993**, §2.1]) that if P is fs and $\mathbf{C}[P]$ is regular then $P \xrightarrow{\sim} \mathbf{N}^r \oplus \mathbf{Z}^s$.

Lemma 3.2.12. — *Let $f\colon F \to F'$ be a morphism of fine monoschemes and let S be a non-empty scheme.*
(i) If f is smooth then $S[f]$ is smooth (as a morphism of schemes).
(ii) If F is torsion free and the morphism $S[F] \to S$ is smooth then F is smooth.

Proof. — Part (i) is obvious, so let us check (ii). We can also assume that $F = \operatorname{Spec}(P)$ is affine. Also, we can replace S with any of its points achieving that $S = \operatorname{Spec}(k)$. Then P is a fine torsion free monoid and $k[P] \subset k[P^{\mathrm{gp}}] \xrightarrow{\sim} k[\mathbf{Z}^n]$. It follows that $X = \operatorname{Spec}(k[P])$ is an integral smooth k-variety of dimension n. Note that $\operatorname{Spec}(k[P^{\mathrm{sat}}])$ is a finite modification of X which is generically an isomorphism. Since X is normal we have that $\operatorname{Spec}(k[P^{\mathrm{sat}}]) \xrightarrow{\sim} X$, and it follows that P is saturated. Now, $P \xrightarrow{\sim} \overline{P} \oplus \mathbf{Z}^l$ and hence $X \xrightarrow{\sim} \operatorname{Spec}(k[\overline{P}]) \times_k \mathbf{G}_m^l$. Obviously, $\operatorname{Spec}(k[\overline{P}])$ is smooth of dimension $r = n - l$ and our task reduces to showing that $\overline{P} \xrightarrow{\sim} \mathbf{N}^r$.

Let $m = \overline{P} \setminus \{1\}$ be the maximal ideal of $\overline{P}$. Then $I = k[m]$ is a maximal ideal of $k[\overline{P}]$ with residue field k. In particular, by k-smoothness of $k[\overline{P}]$ we have that $\dim_k(I/I^2) = r$. On the other hand, $I = \bigoplus_{x \in m} xk$ and $I^2 = \bigoplus_{x \in m^2} xk$, hence $I/I^2 \xrightarrow{\sim} \bigoplus_{x \in m \setminus m^2} xk$ and we obtain that $m \setminus m^2$ consists of r elements $t_1, \dots, t_r$. Note that these elements generate $\overline{P}$ as a monoid and hence they generate $\overline{P}^{\mathrm{gp}}$ as a group. Since $\overline{P}^{\mathrm{gp}} \xrightarrow{\sim} \mathbf{Z}^r$, the elements $t_1, \dots, t_r$ are linearly independent in $\overline{P}^{\mathrm{gp}}$, and we obtain that the surjection $\bigoplus_{i=1}^r t_i^{\mathbf{N}} \to \overline{P}$ is an isomorphism. □

Remark 3.2.13. — It seems very probable that, much more generally, f is smooth whenever $S[f]$ is smooth as a morphism of schemes and one of the following conditions holds: (a) S has points in all characteristics, (b) the homomorphisms $M^{\mathrm{gp}}_{F',x'} \to M^{\mathrm{gp}}_{F,x}$ induced by f have torsion free kernels and cokernels. We could prove this either in the saturated case or under some milder but unnatural restrictions. The main ideas are similar but the proof becomes more technical. We do not develop this direction here since the lemma covers our needs.

3.2.14. *Toric ideals.* — Let F be a torsion free monoscheme, k be a field and $X = k[F]$. For any ideal $\mathscr{I}$ on F one naturally defines an ideal $k[\mathscr{I}]$ on X: in local charts, an ideal $I \subset P$ goes to the ideal $Ik[P] = k[I] = \bigoplus_{a \in I} ak$ in $k[P]$. We say that $\mathscr{J} = k[\mathscr{I}]$ is a **monoidal ideal** on $k[F]$. Note that $\mathscr{J}$ determines $\mathscr{I}$ uniquely via the following equality, which is obvious in local charts: $f^\star \mathscr{I} = \mathscr{J} \cap M_X$, where $f\colon X \to F$ is the natural map.

Lemma 3.2.15. — *Assume that F is connected, η is its maximal point, $X = k[F]$, and $T = k[\eta]$ is the torus of the toric scheme X. A coherent ideal $\mathscr{J} \subset \mathscr{O}_X$ is T-equivariant if and only if it is monoidal.*

Proof. — Any monoidal ideal is obviously T-equivariant, so let us prove the inverse implication. The claim is local on F, so we should prove that any T-equivariant ideal $J \subset A = k[P]$ is of the form $k[I]$ for a unique ideal I of P. Consider the coaction homomorphism $\mu\colon A \to A \otimes_k k[P^{gp}] = B$. The equivariance of J means that JB (with respect to the embedding $A \hookrightarrow A \otimes_k k[P^{gp}]$) is equal to $\mu(J)B$. In particular, $\mu|_J\colon J \to JB = J \otimes_k k[P^{gp}]$ induces a P^{gp}-grading on J compatible with the P^{gp}-grading $A = \bigoplus_{\gamma\in P} A_\gamma$. Thus, J is a homogeneous ideal in A and, since $A_1 = k$ is a field and each k-module A_γ is of rank one, we obtain that $J = \bigoplus_{\gamma\in I} A_\gamma$ for a subset $I \subset P$. Thus, $J = k[I]$, and clearly I is an ideal. □

3.2.16. Toric blow ups. — We will also need the well known fact that toric blow ups are of combinatorial origin, i.e. they are induced from blow ups of monoschemes.

Lemma 3.2.17. — *Assume that* F *is a monoscheme,* $\mathscr{I} \subset M_F$ *is an ideal,* $X = k[F]$, *and* $\mathscr{J} = k[\mathscr{I}]$. *Then there is a canonical isomorphism* $Bl_{\mathscr{J}}(X) \xrightarrow{\sim} k[Bl_{\mathscr{I}}(F)]$.

Proof. — Assume first that $F = Spec(P)$. Then $\mathscr{I}$ corresponds to an ideal $I \subset P$ and we can simply compare charts: $Bl_{\mathscr{I}}(F)$ is covered by the charts $Spec(P[a^{-1}I])$ for $a \in I$, and, since I generates J, the charts $k[a^{-1}J] = k[P[a^{-1}I]]$ cover $Bl_J(X)$. This construction is compatible with localizations $(P, I) \mapsto (P_b, b^{-1}I)$ hence it globalizes to the case of a general fine monoscheme with an ideal. □

Using Lemma 3.2.9 we obtain a similar relation between saturated blow ups and normalized toric blow ups.

Corollary 3.2.18. — *Keep notation of Lemma 3.2.17 and assume that* F *is torsion free. Then* $Bl_{\mathscr{J}}(X)^{nor} \xrightarrow{\sim} k[Bl_{\mathscr{I}}(F)^{sat}]$.

3.2.19. Desingularization of monoschemes. — Let F be a torsion free monoscheme and $X = k[F]$ for a field k of characteristic zero (e.g. $k = \mathbf{Q}$). Recall that the normalized desingularization functor $\widetilde{\mathscr{F}}$ from 2.3.9 is compatible with the action of any smooth k-group, hence the centers of $\widetilde{\mathscr{F}}(X)\colon X_n \dashrightarrow X_{-1} = X$ are T-equivariant ideals. By Lemma 3.2.15, the blown up ideal of $X_0 = k[F^{sat}]$ is of the form $k[\mathscr{I}]$ for an ideal $\mathscr{I} \subset M_{F^{sat}}$, hence $X_1 = k[F_1]$ for $F_1 = Bl_{\mathscr{I}}(F^{sat})^{sat}$ by Lemma 3.2.18. Applying this argument inductively we obtain that the entire normalized blow up tower $\widetilde{\mathscr{F}}(X)$ descends to a saturated blow up tower of F, which we denote as $\widetilde{\mathscr{F}}^{mono}(F)$ (in other words, $\widetilde{\mathscr{F}}(X) = k[\widetilde{\mathscr{F}}^{mono}(F)]$). Since F_n is smooth by Lemma 3.2.12(ii), the tower $\widetilde{\mathscr{F}}^{mono}(F)$ is a desingularization of F. Moreover, part (i) of the same lemma implies that $\widetilde{\mathscr{F}}^{mono}$ is functorial with respect to smooth morphisms of monoschemes. Namely, for any smooth morphism $F' \to F$, $\widetilde{\mathscr{F}}^{mono}(F')$ is the contracted pullback of $\widetilde{\mathscr{F}}^{mono}(F)$ (see 3.1.22). Inspecting what is needed for 2.3.17, one obtains:

Theorem 3.2.20. — *Let* k *be a field and let* $\widetilde{\mathscr{F}}$ *be a normalized desingularization of finite disjoint unions of toric* k*-varieties which is functorial with respect to smooth toric morphisms. Then each normalized blow up tower* $\widetilde{\mathscr{F}}(k[F])$ *is the pullback of a uniquely defined saturated blow up tower of monoschemes* $\widetilde{\mathscr{F}}^{mono}(F)$. *This construction produces a saturated desingularization of quasi-compact fine torsion free monoschemes which is functorial with respect to smooth morphisms.*

Combining theorems 3.1.27 and 3.2.20 we obtain a functorial saturated desingularization of fans that will be denoted $\widetilde{\mathscr{F}}^{fan}$.

Remark 3.2.21. — We will work with normalized and saturated desingularizations, so we formulated the theorem for $\widetilde{\mathscr{F}}$. The same argument shows that $\mathscr{F}$ induces desingularizations $\mathscr{F}^{mono}$ and $\mathscr{F}^{fan}$ that are functorial with respect to all smooth morphisms. Moreover, the descent from toric desingularization is compatible with normalization/saturation, i.e. $\widetilde{\mathscr{F}}^{mono} = (\mathscr{F}^{mono})^{sat}$ and similarly for fans.

3.3. Monoidal desingularization. — In this section we will establish, what we call, monoidal desingularization of fine log schemes (X, M_X). This operation "resolves" the sheaf $\overline{M}_X$ but does not "improve" the log strata of X.

3.3.1. Log stratification. — Using charts one immediately checks that for any fine log scheme (X, M_X) the monoidal rank function $x \mapsto rank(\overline{M}^{gp}_x)$ is upper semicontinuous. The corresponding stratification of X, whose strata $X^{(i)}$ are the locally closed subsets on which the monoidal rank function equals to i, will be called the **log stratification**. We remark that the analogous stratification in exp. **VI**, 1.5 was called canonical or stratification by rank.

Remark 3.3.2. — Sometimes it is more convenient to work with the **local log stratification** of (X, M_X) whose strata are the maximal connected non-empty locally closed subsets of X on which the monoidal rank function is constant. This stratification is obtained from the log stratification by replacing each stratum with the set of its connected components, in particular, all empty strata are discarded. For example, this stratification showed up in exp. **VI**, 3.9.

3.3.3. Monoscheme charts of log schemes. — A (global) **monoscheme chart** of a Zariski log scheme (X, M_X) is a morphism of monoidal spaces $c\colon (X, \varepsilon_* M_X) \to (F, M_F)$ such that the target is a monoscheme and $\varepsilon_* M_X$ is isomorphic to the Zariski log structure associated with the pre-log structure $c^{-1}M_F \to \mathscr{O}_X$ (obtained as $c^{-1}M_F \to M_X \to \mathscr{O}_X$). In particular, M_X is the log structure associated with $(c \circ \varepsilon)^{-1}M_F \to \mathscr{O}_{X_{\text{ét}}}$. We say that the chart is **fine** if (F, M_F) is so. For example, any toric scheme $R[F]$, where F is a monoscheme, possesses a canonical chart $R[F] \to F$.

Lemma 3.3.4. — *Let (X, M_X) be a log scheme and let (F, M_F) be a monoscheme. Then any morphism of monoidal spaces $f\colon (X, M_X) \to (F, M_F)$ factors uniquely into the composition of a morphism of log schemes $(X, M_X) \to \mathbf{Z}[F]$ and the canonical chart $\mathbf{Z}[F] \to F$.*

Proof. — Note that $(\mathrm{Id}_X, \alpha)\colon (X, \mathscr{O}_X) \to (X, M_X)$ is a morphism of monoidal spaces, hence so is the composition $h\colon (X, \mathscr{O}_X) \to (F, M_F)$. If $F = \mathrm{Spec}(P)$ then h is determined by the homomorphism $P \to \Gamma(\mathscr{O}_X)$ by Lemma 3.1.8. Since the latter factors uniquely into the composition of the homomorphism of monoids $P \to \mathbf{Z}[P]$ and the homomorphism of rings $\mathbf{Z}[P] \to \Gamma(\mathscr{O}_X)$, we obtain a canonical factoring $X \to \mathrm{Spec}(\mathbf{Z}[P]) \to \mathrm{Spec}(P)$. Furthermore, this affine construction is compatible with localizations of P, hence it globalizes to the case when the monoscheme F is arbitrary. $\square$

Remark 3.3.5. — (i) Usually, one works with log schemes using local charts $(X, M_X) \to \mathrm{Spec}(\mathbf{Z}[P])$. By Lemma 3.3.4 this is equivalent to working with affine monoscheme charts.

(ii) In particular, any fine log scheme (X, M_X) admits a fine monoscheme chart étale locally, i.e., there exists a strict (in the log sense) étale covering $(Y, M_Y) \to (X, M_X)$ whose source possesses a fine monoscheme chart. Similarly, any Zariski fine log scheme admits a fine monoscheme chart Zariski locally.

3.3.6. Chart base change. — Given a fine monoscheme chart $(X, M_X) \to F$ and a morphism of monoschemes $F' \to F$ we will write $(X, M_X) \times_F F'$ instead of $(X, M_X) \times_{\mathbf{Z}[F]} \mathbf{Z}[F']$, where the second product is taken in the category of fine log schemes. This notation is partially justified by the following result.

Lemma 3.3.7. — *Keep the above notation and let $(X', M_{X'}) = (X, M_X) \times_F F'$.*

(i) The morphism $c'\colon (X', M_{X'}) \to F'$ is a monoscheme chart.

(ii) If (Y, M_Y) is a log scheme over (X, M_X) and $d\colon (Y, M_Y) \to F$ is the induced morphism of monoidal spaces, then any lifting of d to a morphism $(Y, M_Y) \to F'$ factors uniquely through c'.

Proof. — Strictness is stable under base changes, hence $(X', M_{X'}) \to \mathbf{Z}[F']$ is strict and we obtain (i). The assertion of (ii) is a consequence of Lemma 3.3.4. $\square$

3.3.8. Log ideals. — By a **log ideal** on a fine log scheme (X, M_X) we mean any ideal $\mathscr{I} \subset M_X$ that étale locally on X admits a coherent chart as follows: there exists a strict étale covering $f\colon (Y, M_Y) \to (X, M_X)$, a fine monoscheme chart $c\colon (Y, M_Y) \to F$ and a coherent ideal $\mathscr{I}_F \subset M_F$ such that $f^{-1}(\mathscr{I})M_Y = c^{-1}(\mathscr{I}_F)M_Y$.

3.3.9. Log blow ups. — It is proved in [**Niziol, 2006**, 4.2] that there exists a universal morphism $f\colon (X', M_{X'}) \to (X, M_X)$ such that the ideal $f^{-1}(\mathscr{I})M_{X'}$ is invertible, i.e. locally (in the étale topology) generated by one element. (We use here that, unlike rings, any principal ideal aM of an integral monoid M is invertible in the usual sense, i.e. $M \xrightarrow{\sim} aM$ as M-sets.) Actually, the formulation in [**Niziol, 2006**] refers only to saturated blow ups, but the proof deals also with the non-saturated ones.

The construction of log blow ups is standard and it also shows that they are compatible with arbitrary strict morphisms. If (X, M_X) and $\mathscr{I}$ admit a chart $F, \mathscr{J} \subset M_F$ then $(X', M_{X'}) = (X, M_X) \times_F \mathrm{Bl}_{\mathscr{J}}(F')$ is as required. If $(Y, M_Y) \to (X, M_X)$ is strict then F is also a chart of (Y, M_Y), hence the local construction is compatible with strict morphisms. The general case now follows by descent because any fine log scheme admits a chart étale locally. We call f the **log blow up** of (X, M_X) along $\mathscr{I}$ and denote it $\mathrm{LogBl}_{\mathscr{I}}(X, M_X)$ (it is called unsaturated log blow up in [**Niziol, 2006**]). Log blow up towers are defined in the obvious way. As usual, contraction of such a tower is obtained by removing all empty log blow ups (i.e. blow ups along $\mathscr{I} = M_X$).

3.3.10. Saturated log blow ups. — Saturated log blow up along a log ideal $\mathscr{I}$ is defined as $(\mathrm{LogBl}_{\mathscr{I}}(X, M_X))^{\mathrm{sat}}$. It satisfies an obvious universal property too. (It is called log blow up in [**Niziol, 2006**]). Towers of saturated log blow ups, their pullbacks, and saturation of a tower of log blow ups are defined in the obvious way.

3.3.11. Pullbacks. — Let $f\colon (Y, M_Y) \to (X, M_X)$ be a morphism of log schemes. By **pullback** of the log blow up $\mathrm{LogBl}_{\mathscr{I}}(X, M_X)$ along a log ideal $\mathscr{I} \subset M_X$ we mean the log blow up $\mathrm{LogBl}_{\mathscr{J}}(Y, M_Y)$, where $\mathscr{J} = f^{-1}(\mathscr{I})M_Y$. This is the universal log scheme over (Y, M_Y) whose morphism to (X, M_X) factors through $\mathrm{LogBl}_{\mathscr{I}}(X, M_X)$. The pullback of saturated blow ups is defined similarly, and these definitions extend inductively to pullbacks of towers of (saturated) log blow ups.

3.3.12. Basic properties. — Despite the similarity with usual blow ups of schemes, log blow ups (resp. saturated log blow ups) have nice properties that are not satisfied by usual blow ups. First, it is proved in [**Nizioł, 2006**, 4.8] that log blow ups are compatible with any log base change $f\colon Y \to X$, i.e. $\mathrm{LogBl}_{f^{-1}\mathscr{I}}(Y) \xrightarrow{\sim} \mathrm{LogBl}_{\mathscr{I}}(X) \times_X Y$ for a monoidal ideal $\mathscr{I}$ on X. In particular, saturated blow ups are compatible with saturated base changes. Second, log blow ups (resp. saturated log blow ups) are log étale morphisms because so are both saturation morphisms and charts of the form $\mathbf{Z}[\mathrm{Bl}_{\mathscr{I}}(F)] \to \mathbf{Z}[F]$.

3.3.13. Fan charts. — A **fan chart** of a Zariski log scheme (X, M_X) is a morphism $d\colon (X, \varepsilon_\star \overline{M}_X) \to (F, M_F)$ of monoidal spaces such that the target is a fan and $d^{-1}(M_F) \xrightarrow{\sim} \varepsilon_\star \overline{M}_X$. For example, for any monoscheme chart $c\colon (X, \varepsilon_\star M_X) \to (F, M_F)$, its sharpening $\overline{c}\colon (X, \varepsilon_\star \overline{M}_X) \to (F, \overline{M}_F)$ is a fan chart. Fan charts were considered by Kato (e.g., in [**Kato, 1994**, 9.9]). They contain less information than monoscheme charts, but "remember everything about ideals and blow ups" because there is a one-to-one correspondence between ideals and blow up towers of M_F and $M_{\overline{F}}$. Let us make this observation rigorous. For concreteness, we discuss only non-saturated (log) blow ups, but everything easily extends to the saturated case.

Remark 3.3.14. — (i) Assume that $\overline{c}\colon (X, \overline{M}_X) \to (F, M_F)$ is a fan chart. Any ideal $\mathscr{I}_F \subset M_F$ induces a log ideal $\mathscr{I} \subset M_X$, which is the preimage of $\overline{c}^{-1}(\mathscr{I}_F)\overline{M}_X$ under $M_X \to \overline{M}_X$. We say that the blow up $F' = \mathrm{Bl}(\mathscr{I}_F)$ induces the log blow up $(X', \overline{M}_{X'}) = \mathrm{LogBl}_{\mathscr{I}}(X, \overline{M}_X)$ or that the latter log blow up is the **pullback** of $\mathrm{Bl}(\mathscr{I}_F)$. Furthermore, $(X', \overline{M}_{X'}) \to F'$ is also a fan chart (see [**Kato, 1994**, 9.9], where the fs case is treated), hence this definition iterates to a tower $F_\bullet$ of blow ups of F. We will denote the pullback tower of log blow ups as $\overline{c}^\star(F_\bullet)$.

(ii) By a slight abuse of notation, Kato and Nizioł denote $\overline{c}^\star(F_\bullet)$ as $(X, M_X) \times_F F_\bullet$. One should be very careful with this notation because, in general, there is no morphism $(X, M_X) \to F$ that lifts $\overline{c}$. Also, one cannot define analogous "base change" for an arbitrary morphism of fans $F' \to F$. The reason is that there are many "unnatural" gluings in the category of fans (e.g. along generic points), and such gluings cannot be lifted to log schemes (and even to monoschemes).

(iii) For blow up towers, however, the base change notation is safe and agrees with the base change from the monoschemes. Namely, if $\overline{c}$ is the sharpening of a monoscheme chart $c\colon (X, M_X) \to (F, M_F)$ then there exists a one-to-one correspondence between blow up towers of the monoscheme $F = (F, M_F)$ and the fan $\overline{F} = (F, \overline{M}_F)$, see Lemma 3.1.25. Clearly, the matching towers induce the same log blow up tower of (X, M_X). In particular, $\mathscr{F}^{\mathrm{mono}}(F, M_F)$ and $\mathscr{F}^{\mathrm{fan}}(F, \overline{M}_F)$ (see Theorem 3.1.27) induce the same log blow up tower of (X, M_X).

3.3.15. Monoidal desingularization of log schemes. — Let (X, M) be a fine log scheme and assume that (X, M) is **monoidally torsion free** in the sense that the groups $\overline{M}^{\mathrm{gp}}_{\overline{x}}$ are torsion free. By a **monoidal desingularization** (resp. a **saturated monoidal desingularization**) of a fine log scheme (X, M) we mean a tower of log blow ups (resp. a tower of saturated log blow ups) $(X_n, M_n) \dashrightarrow (X_0, M_0) = (X, M)$ (resp. $(X_n, M_n) \dashrightarrow (X_{-1}, M_{-1}) = (X, M)$) such that for any geometric point $\overline{x} \to X_n$ the stalk of $\overline{M}_n$ at $\overline{x}$ is a free monoid. A morphism $(Y, N) \to (X, M)$ is called **monoidally smooth** if each induced homomorphism of stalks of monoids $\overline{M}_{\overline{x}} \to \overline{N}_{\overline{y}}$ can be extended to an isomorphism $\overline{M}_{\overline{x}} \oplus \mathbf{N}^r \xrightarrow{\sim} \overline{N}_{\overline{y}}$.

Theorem 3.3.16. — *Let $\widetilde{\mathscr{F}}^{\mathrm{fan}}$ be a saturated desingularization of quasi-compact fine torsion free fans which is functorial with respect to smooth morphisms. Then there exists unique saturated monoidal desingularization $\widetilde{\mathscr{F}}^{\log}(X, M)$ of monoidally torsion free fine log schemes (X, M), such that $\widetilde{\mathscr{F}}^{\log}$ is functorial with respect to all monoidally smooth morphisms and $\widetilde{\mathscr{F}}^{\log}(X, M)$ is the contraction of $c^\star(\widetilde{\mathscr{F}}^{\mathrm{fan}}(F))$ for any log scheme (X, M) that admits a fan chart $c\colon (X, \overline{M}_X) \to F$. In the same way, a functorial desingularization $\mathscr{F}^{\mathrm{fan}}$ induces a monoidal desingularization $\mathscr{F}^{\log}$.*

Remark 3.3.17. — Since any monoscheme chart induces a fan chart, it then follows from Remark 3.3.14(iii) that $\widetilde{\mathscr{F}}^{\log}(X, M)$ is the contraction of $d^\star(\widetilde{\mathscr{F}}^{\mathrm{mono}}(F))$ for any log scheme (X, M) that admits a monoscheme chart $d\colon (X, \overline{M}_X) \to F$.

Proof of Theorem 3.3.16. — Both cases are established similarly, so we prefer to deal with $\mathscr{F}^{\mathrm{fan}}$ (to avoid mentioning saturations at any step of the proof). By descent, it suffices to show that the pullback from fans induces a functorial monoidal desingularization of those fine log schemes that admit a global fan chart. Thus, if $\mathscr{F}^{\log}$

exists then it is unique, and our aim is to establish existence and functoriality. Both are consequences of the following claim: assume that $f\colon (Y, M_Y) \to (X, M_X)$ is a monoidally smooth morphism whose source and target admit fan charts $d\colon (Y, \overline{M}_Y) \to G$ and $d'\colon (X, \overline{M}_X) \to F$, then the contractions of $d^*(\mathscr{F}^{\mathrm{fan}}(G))$ and $c^*(\mathscr{F}^{\mathrm{fan}}(F))$ are equal, where $c = d' \circ \bar{f}\colon (Y, \overline{M}_Y) \to (X, \overline{M}_X) \to F$. Note that $d^{-1}(\overline{M}_G) \xrightarrow{\sim} \overline{M}_Y$ and the homomorphism $c^{-1}(\overline{M}_F) \to \overline{M}_Y$ is smooth.

Choose a point $y \in Y$ and consider the localizations $Y' = \mathrm{Spec}(\mathscr{O}_{Y,y})$, $F' = \mathrm{Spec}(M_{F,c(y)})$ and $G' = \mathrm{Spec}(M_{G,d(y)})$ at y and its images in the fans. Since $\phi\colon M_{G,d(y)} \to \overline{M}_{Y,y}$ is an isomorphism and the homomorphism $\psi\colon M_{F,c(y)} \to \overline{M}_{Y,y}$ is smooth, we obtain a factorization of ψ into a composition of a homomorphism $\lambda\colon M_{F,c(y)} \to M_{G,d(y)}$ and ϕ, where λ is smooth. Set $U = c^{-1}(F') \cap d^{-1}(G')$. Then U is a neighborhood of y, and c and d induce homomorphisms $\phi_U\colon M_{G,d(y)} \to \overline{M}_Y(U)$ and $\psi_U\colon M_{F,c(y)} \to \overline{M}_Y(U)$. Since the monoids are fine, we can shrink U so that the equality $\psi_U = \phi_U \circ \lambda$ holds. It then follows from Lemma 3.1.8(ii) that $c|_U$ factors into a composition of $d|_U$ and the smooth morphism $\mathrm{Spec}(\lambda)\colon G' \to F'$.

By quasi-compactness of Y we can now find finite coverings $Y = \bigcup_{i=1}^n Y_i$, $F = \bigcup_{i=1}^n F_i$ and $G = \bigcup_{i=1}^n G_i$, and smooth morphisms $\lambda_i\colon G_i \to F_i$ such that Y_i is mapped to F_i and G_i by c and d, respectively, and the induced maps of monoidal spaces $c_i\colon Y_i \to F_i$ and $d_i\colon Y_i \to G_i$ satisfy $c_i = \lambda_i \circ d_i$. Set $Y' = \coprod_{i=1}^n Y_i$, $F' = \coprod_{i=1}^n F_i$, $G' = \coprod_{i=1}^n G_i$, $c'\colon Y' \to F$ and $d'\colon Y' \to G$. By descent, it suffices to check that contractions of $c'^*(\mathscr{F}^{\mathrm{fan}}(F))$ and $d'^*(\mathscr{F}^{\mathrm{fan}}(G))$ are equal. Since, the morphism $Y' \to F$ factors through the surjective smooth morphism $F' \to F$, and similarly for G, these two pullbacks are equal to the contracted pullbacks of $\mathscr{F}^{\mathrm{fan}}(F')$ and $\mathscr{F}^{\mathrm{fan}}(G')$, respectively. It remains to note that $Y' \to F'$ factors through the smooth morphism $\coprod_{i=1}^n \lambda_i\colon G' \to F'$. Hence $\mathscr{F}^{\mathrm{fan}}(G')$ is the contracted pullback of $\mathscr{F}^{\mathrm{fan}}(F')$, and their contracted pullbacks to Y' coincide. □

3.4. Desingularization of log regular log schemes. — In this section we will see how saturated monoidal desingularization leads to normalized desingularization of log regular log schemes. Up to now we freely considered saturated and unsaturated cases simultaneously, and did not feel any essential difference. This will not be the case in the present section because the notion of log regularity was developed by Kato and Nizioł in the saturated case. Gabber generalized the definition to the non-saturated case and extended to that case all main results about log regular log schemes. This was necessary for his original approach, but can be bypassed by use of saturated monoidal desingularization. So, we prefer to stick to the saturated case and simply refer to all foundational results about log regular fs log schemes to [**Kato, 1994**] and [**Nizioł, 2006**]. For the sake of completeness, we will outline Gabber's results about the general case in §3.5.

3.4.1. Conventions. — Recall that Kato's notion of log regular fs log schemes was already used in exp. **VI**, 1.2. Throughout §3.4 we assume that (X, M_X) is a log regular fs log scheme. Note that the homomorphism $\alpha_X\colon M_X \to \mathscr{O}_X$ of $X_{\text{ét}}$-sheaves is injective by [**Nizioł, 2006**, 2.6], and actually $M_X = \mathscr{O}_U^* \cap \mathscr{O}_X$, where $U \subset X$ is the triviality locus of M_X. So, we will freely identify M_X with a multiplicative submonoid of $\mathscr{O}_X$.

3.4.2. Monoidal ideals. — For any log ideal $\mathscr{I} \subset M_X$ consider the ideal $\mathscr{J} = \alpha(\mathscr{I})\mathscr{O}_X$ it generates. We call $\mathscr{J}$ a **monoidal ideal** and by a slight abuse of notation, we will write $\mathscr{J} = \mathscr{I}\mathscr{O}_X$.

Lemma 3.4.3*. — Let (X, M_X) be as in 3.4.1. The rules $\mathscr{I} \mapsto \mathscr{I}\mathscr{O}_X$ and $\mathscr{K} \mapsto \mathscr{K} \cap M_X$ give rise to a one-to-one correspondence between log ideals $\mathscr{I} \subset M_X$ and monoidal ideals $\mathscr{K} \subset \mathscr{O}_X$.*

Proof. — It suffices to show that any log ideal $\mathscr{I}$ coincides with $\mathscr{J} = \mathscr{I}\mathscr{O}_X \cap M_X$. Furthermore, it suffices to check the equality at the strict localizations of X, hence we can assume that $X = \mathrm{Spec}(A)$ for a strictly local ring A. Then the log structure admits a chart $X \to \mathrm{Spec}(\mathbf{Z}[P])$ and $\mathscr{I} = IM_X$ for an ideal $I \subset P$, and we should prove that $J := \mathscr{J}(X) = IA \cap P$ coincides with I.

Assume on the contrary that $I \subsetneq J$, and consider the exact sequence

$$I\mathbf{Z}[P] \otimes_{\mathbf{Z}[P]} A \to J\mathbf{Z}[P] \otimes_{\mathbf{Z}[P]} A \to J\mathbf{Z}[P]/I\mathbf{Z}[P] \otimes_{\mathbf{Z}[P]} A \to 0. \tag{3.4.3.1}$$

Since $\mathrm{Tor}_1^{\mathbf{Z}[P]}(\mathbf{Z}[P]/I\mathbf{Z}[P], A) = 0$ by [**Kato, 1994**, 6.1(ii)], $I\mathbf{Z}[P] \otimes_{\mathbf{Z}[P]} A = IA$ and similarly for J, and we obtain that the first morphism in the sequence (3.4.3.1) is $IA \to JA$. To obtain a contradiction, it suffices to show that $J\mathbf{Z}[P]/I\mathbf{Z}[P] \otimes_{\mathbf{Z}[P]} A \neq 0$. Note that $\mathbf{Z}[P]/\mathbf{Z}[m_P]$ is a quotient of $J\mathbf{Z}[P]/I\mathbf{Z}[P] = \mathbf{Z}[J]/\mathbf{Z}[I]$, so it remains to note that $m_P A \neq A$ and hence $\mathbf{Z}[P]/m_p\mathbf{Z}[P] \otimes_{\mathbf{Z}[P]} A \neq 0$. □

3.4.4. Interpretation of monoidal smoothness. — Note that by exp. **VI**, 1.7 (X, M_X) is monoidally smooth if and only if X is regular and in this case the non-triviality locus of M_X is a normal crossings divisor D.

3.4.5. Saturated log blow ups of log regular log schemes. — Using Kato's Tor-independence result [**Kato, 1994**, 6.1(ii)] Niziol proved in [**Niziol, 2006**, 4.3] that saturated log blow ups of (X, M_X) are compatible with normalized blow ups along monoidal ideals. Namely, if $(Y, M_Y) = \mathrm{LogBl}_{\mathscr{I}}(X, M_X)^{\mathrm{sat}}$ then $Y \xrightarrow{\sim} \mathrm{Bl}_{\mathscr{I}\mathscr{O}_X}(X)^{\mathrm{nor}}$. We will also need more specific results that showed up in the proof of loc.cit., so we collect them altogether in the following lemma.

Lemma 3.4.6. — *Let $f\colon (X, M_X) \to (Y, M_Y)$ be a strict morphism of fs log regular log schemes, let $\mathscr{I} \subset M_Y$ be a log ideal and $\mathscr{J} = f^{-1}\mathscr{I}M_X$. Set $(X', M_{X'}) = \mathrm{LogBl}_{\mathscr{J}}(X, M_X)$, $(X'', M_{X''}) = (X', M_{X'})^{\mathrm{sat}}$, $(Y', M_{Y'}) = \mathrm{LogBl}_{\mathscr{I}}(Y, M_Y)$ and $(Y'', M_{Y''}) = (Y', M_{Y'})^{\mathrm{sat}}$. Then*

(i) The (saturated) log blow up of (X, M_X) is compatible with (normalized) blow up of X: $X' \xrightarrow{\sim} \mathrm{Bl}_{\mathscr{J}\mathscr{O}_X}(X)$ and $X'' \xrightarrow{\sim} \mathrm{Bl}_{\mathscr{J}\mathscr{O}_X}(X)^{\mathrm{nor}}$.

(ii) The (normalized) blow up of X along $\mathscr{J}\mathscr{O}_X$ is the pullback of the (normalized) blow up of Y along $\mathscr{I}\mathscr{O}_Y$. In particular, $X' \xrightarrow{\sim} X \times_Y Y'$ and $X'' \xrightarrow{\sim} X \times_Y Y''$.

Proof. — All claims can be checked étale locally, hence we can assume that there exists a chart $g\colon (Y, M_Y) \to (\mathbf{Z}[P], P)$ and $\mathscr{I} = g^{-1}(I_0)M_Y$ for an ideal $I_0 \subset P$. Then it suffices to prove (ii) for g and the induced chart $g \circ f$ of (X, M_X). In particular, this reduces the lemma to the particular case when $X = \mathrm{Spec}(A)$ and f is a chart $(X, M_X) \to (\mathbf{Z}[P], P)$. It is shown in the first part of the proof of [**Niziol, 2006**, 4.3] that

$$X' \xrightarrow{\sim} \mathrm{Proj}(A \otimes_{\mathbf{Z}[P]} (\bigoplus_{n=0}^{\infty} I_0^n)) \xrightarrow{\sim} \mathrm{Proj}(\bigoplus_{n=0}^{\infty} \mathscr{J}^n)$$

The first isomorphism implies that $X' \to X$ is the base change of $\mathrm{Proj}(\bigoplus_{n=0}^{\infty} I_0^n) = Y' \to Y$, and the second isomorphism means that $X' \xrightarrow{\sim} \mathrm{Bl}_{\mathscr{J}\mathscr{O}_X}(X)$. This establishes the unsaturated and unnormalized part of the Lemma, and the second part follows in the same way from the second part of the proof of [**Niziol, 2006**, 4.3]. □

Remark 3.4.7. — It follows from the lemma that the unsaturated log blow up $(X', M_{X'})$ is log regular in the sense of Gabber, see §3.5. Thus, once log regularity is correctly defined in full generality, it becomes a property preserved by log blow ups (as it should be, since log blow ups are log smooth).

3.4.8. Desingularization of log regular log schemes. — By Lemma 3.4.6, any saturated log blow up tower $f\colon (X_n, M_n) \dashrightarrow (X, M_X)$ induces a normalized blow up tower $g\colon X_n \dashrightarrow X$ of the underlying schemes. Furthermore, g completely determines f as follows: if $X_{i+1} \to X_i$ is the normalized blow up along $\mathscr{I} \subset \mathscr{O}_{X_i}$ then $(X_{i+1}, M_{X_{i+1}}) \to (X_i, M_{X_i})$ is the saturated log blow up along $\mathscr{I} \cap M_{X_i}$ by Lemma 3.4.3. The convention that centers of blow ups are part of the data is used here essentially. Furthermore, by 3.4.4, f is a saturated monoidal desingularization if and only if g is a normalized desingularization. In particular, the saturated monoidal desingularization $\widetilde{\mathscr{F}}^{\log}(X, M_X)$ induces a normalized desingularization of the scheme X that depends on (X, M_X) and will be denoted $\widetilde{\mathscr{F}}(X, M_X)$.

Theorem 3.4.9. — *The saturated monoidal desingularization $\widetilde{\mathscr{F}}^{\log}$ (see Theorem 3.3.16) gives rise to desingularization $\widetilde{\mathscr{F}}$ of log regular log schemes that possesses the same functoriality properties: if $\phi\colon (Y, M_Y) \to (X, M_X)$ is a monoidally smooth morphism of log regular log schemes then $\widetilde{\mathscr{F}}(Y, M_Y)$ is the contraction of $\phi^{\mathrm{st}}(\widetilde{\mathscr{F}}(X, M_X))$. Furthermore, if ϕ is strict then $\phi^{\mathrm{st}}(\widetilde{\mathscr{F}}(X, M_X)) = \widetilde{\mathscr{F}}(X, M_X) \times_X Y$.*

Strictness of ϕ in the last claim is not needed. To remove it one should work out the assertion of Remark 3.1.23.

Proof. — We only need to establish functoriality. Let $(X', M_{X'}) = \mathrm{LogBl}_{\mathscr{I}}(X, M_X)^{\mathrm{sat}}$ be the first saturated blow up of $\widetilde{\mathscr{F}}^{\log}(X, M_X)$. Set $\mathscr{J} = \phi^{-1}\mathscr{I}M_Y$, then $(Y', M_{Y'}) = \mathrm{LogBl}_{\mathscr{J}}(Y, M_Y)^{\mathrm{sat}}$ is the first saturated blow up of $\widetilde{\mathscr{F}}^{\log}(Y, M_Y)$ by functoriality of $\widetilde{\mathscr{F}}^{\log}$. By Lemma 3.4.6, $X' = \mathrm{Bl}_{\mathscr{I}\mathscr{O}_X}(X)^{\mathrm{nor}}$ and $Y' = \mathrm{Bl}_{\mathscr{J}\mathscr{O}_Y}(Y)^{\mathrm{nor}}$, and using that $\phi^{-1}(\mathscr{I}\mathscr{O}_X)\mathscr{O}_Y = \mathscr{J}\mathscr{O}_Y$ we obtain that Y' is the strict transform of X'. It remains to inductively apply the same argument to the other levels of the towers. The last claim follows from Lemma 3.4.6(ii). □

Remark 3.4.10. — The same results hold for (non-saturated) monoidal desingularization, which induces a (usual) desingularization of log regular log schemes. For non-saturated log regular log schemes (see §3.5) one should first establish analogs of Lemmas 3.4.3 and 3.4.6 (where the input in the second one does not have to be saturated). Then the same proof as above applies.

3.4.11. Canonical fans and associated points. — By the **canonical fan** $\mathrm{Fan}(X)$ of (X, M_X) we mean the set of maximal points of the log stratification (i.e. the maximal points of the log strata). Alternatively, $\mathrm{Fan}(X)$ can be described as in [**Nizioł, 2006**, §2.2] as the set of points $x \in X$ such that $m_{\overline{x}}$ coincides with the ideal $I_{\overline{x}} \subset \mathscr{O}_{\overline{x}}$ generated by $\alpha(M_{\overline{x}} - M^*_{\overline{x}})$.

We provide $F = \mathrm{Fan}(X)$ with the induced topology and define M_F to be the restriction of $\overline{M}_X$ onto F. For example, for a toric k-variety $X = \mathrm{Spec}(k[Z])$, where Z is a monoscheme, (F, M_F) is isomorphic to the sharpening of Z. More generally, if a log scheme (X, M_X) is Zariski then (F, M_F) is a fan and the map $c\colon X \to F$ sending any point to the maximal point of its log stratum is a fan chart of X. This follows easily from the fact that such (X, M_X) admits monoscheme charts Zariski locally.

Remark 3.4.12. — In general, (F, M_F) does not have to be a fan, but it seems probable that it can be extended to a meaningful object playing the role of algebraic spaces in the category of fans. We will not investigate this direction here.

Lemma 3.4.13. — *Let $X = (X, M_X)$ be an fs log regular log scheme with a monoidal ideal $\mathscr{I} \subset \mathscr{O}_X$. Then:*

(i) The set of associated points of $\mathscr{O}_X/\mathscr{I}$ is contained in $\mathrm{Fan}(X)$.

(ii) The fans of $\mathrm{Bl}_{\mathscr{I}}(X)$ and $\mathrm{Bl}_{\mathscr{I}}(X)^{\mathrm{nor}}$ are contained in the preimage of $\mathrm{Fan}(X)$.

(iii) For any tower of monoidal blow ups (resp. normalized monoidal blow ups) $X_n \dashrightarrow X$, its set of associated points is contained in $\mathrm{Fan}(X)$.

Proof. — (i) Fix a point $x \in X - \mathrm{Fan}(X)$ and let us prove that it is not an associated point of $\mathscr{O}_X/\mathscr{I}$. Since associated points are compatible with flat morphisms, we can pass to the formal completion $\widehat{X}_x = \mathrm{Spec}(\widehat{\mathscr{O}}_{X,x})$. Let us consider first the more difficult case when $A = \widehat{\mathscr{O}}_{X,x}$ is of mixed characteristic $(0, p)$. By exp. **VI**, 1.6, $A \xrightarrow{\sim} B/(f)$ where $B = C(k)[[Q]][[\underline{t}]]$, Q is a sharp monoid defining the log structure, $\underline{t} = (t_1, \dots, t_n)$, and $f \in B$ reduces to p modulo $(Q \setminus \{1\}, \underline{t})$. Note that $n \geq 1$ as otherwise $Q \setminus \{1\}$ would generate the maximal ideal of A. In this case, x is the log stratum of $\widehat{X}_x$ and hence x is the maximal point of its log stratum in X_x, which contradicts that $x \notin \mathrm{Fan}(X)$. The completion $\widehat{\mathscr{I}}$ of $\mathscr{I}$ is of the form IA for an ideal $I \subset Q$. We should prove that $x \in \widehat{X}_x$ is not an associated point of $A/\widehat{\mathscr{I}}$, or, equivalently, $\mathrm{depth}(A/\widehat{\mathscr{I}}) \geq 1$. Since $B/(fB + \widehat{\mathscr{I}}B) = A/\widehat{\mathscr{I}}$, it suffices to show that $\mathrm{depth}(B/\widehat{\mathscr{I}}B) \geq 2$ and f is a regular element of $B/\widehat{\mathscr{I}}B$.

Regularity of f follows from the following easy claim by taking $C = C(k)$, $J = I$ and $R = Qt_1^{\mathbf{N}} \dots t_n^{\mathbf{N}}$: if C is a domain, R is a sharp fine monoid with an ideal J, f is an element of $C[[R]]$ with a non-zero free term (i.e. $c_1 \neq 0$, where $f = \sum_{r \in R} c_r r$), and $g \in C[[R]]$ satisfies $fg \in JC[[R]]$ then $g \in JC[[R]]$. Next, let us bound the depth of $B/\widehat{\mathscr{I}}B$. In view of [**Matsumura, 1980**, 21.C], depth is preserved by completions of local rings hence it suffices to show that $\mathrm{depth}(D_r/ID_r) \geq 2$, where $D = C(k)[Q][\underline{t}]$ and $r = (m_Q, p, \underline{t})$ is the ideal generated by $m_Q = Q \setminus \{1\}$, p and $\underline{t}$. Note that $D/ID \xrightarrow{\sim} C(k)[\underline{t}][Q \setminus I]$ as a $C(k)[\underline{t}]$-module, hence it is a flat $C(k)[\underline{t}]$-module and the local homomorphism $C(k)[\underline{t}]_{(p,\underline{t})} \to D_r/ID_r$ is flat. By [**Matsumura, 1980**, 21.C], $\mathrm{depth}(D_r/ID_r) \geq \mathrm{depth}(C(k)[\underline{t}]_{(p,\underline{t})}) = n+1 \geq 2$, so we have established the mixed characteristic case. In the equal characteristic case we have that $A = k[[Q]][[\underline{t}]]$, and the same argument shows that $\mathrm{depth}(A/\widehat{\mathscr{I}}) \geq \mathrm{depth}(k[\underline{t}]_{(\underline{t})}) = n \geq 1$.

To prove (ii) we should check that if $x \in X - \mathrm{Fan}(X)$ then no point of $\mathrm{Fan}(\mathrm{Bl}_{\mathscr{I}}(X))$ sits over x. We will only check the mixed characteristic case since it is more difficult. As earlier, $\widehat{X}_x = \mathrm{Spec}(A)$ where $A = B/(f)$ and $B = C(k)[[Q]][[\underline{t}]]$ with $n \geq 1$. Note that $\psi\colon \widehat{X}_x \to X$ is a flat strict morphism of log schemes. Hence ψ is compatible with blow ups and it maps the fans of $\widehat{X}_x$ and $\mathrm{Bl}_{\widehat{\mathscr{I}}}(\widehat{X}_x)$ to the fans of X_x and $\mathrm{Bl}_{\mathscr{I}}(X)$, respectively. Therefore, we should only check that $\mathrm{Fan}(\mathrm{Bl}_{\widehat{\mathscr{I}}}(\widehat{X}_x))$ is disjoint from the preimage of x. The latter blow up is covered by the charts $V_a = \mathrm{Spec}(A[a^{-1}\widehat{\mathscr{I}}])$ with $a \in I$. Set $P' = Q[a^{-1}I]$, $B' = C(k)[[P']][[t_1, \dots, t_n]]/(f)$ and $A' = B'/(f)$ (where f is as above). Then the m_x-adic completion of V_a is $\widehat{V}_a = \mathrm{Spec}(A')$. The ideal defining the closed point of $\mathrm{Fan}(\widehat{V}_a)$ is generated by the maximal ideal m' of P'. This ideal does not contain any t_i. Indeed, $t_i \notin m'B' + fB'$ because $t_i \notin fC(k)[[t_1, \dots, t_n]]$ as $f - p \in (t_1, \dots, t_n)$. Thus, $\mathrm{Fan}(\widehat{V}_a)$ is disjoint from the preimage of x, and hence the same is true for $\mathrm{Fan}(V_a)$. This proves (ii) in the non-saturated case, and the saturated case is dealt with similarly but with P' replaced by its saturation. Finally, (iii) follows immediately from (i) and (ii). □

3.4.14. Independence of the log structure. — Dependence of $\widetilde{\mathscr{F}}(X, M_X)$ on M_X is a subtle question. In this section we will use functoriality of $\widetilde{\mathscr{F}}$ to prove that $\widetilde{\mathscr{F}}(X, M_X)$ is independent of M_X in characteristic zero. This will cover our needs, but the restriction on the characteristic will complicate our arguments later. Conjecturally, $\widetilde{\mathscr{F}}(X, M_X)$ does not depend on M_X at all and the following result of Gabber supports this conjecture: if P

and Q are two fine sharp monoids and $k[[P]] \simeq k[[Q]]$ for a field k (of any characteristic!) then $P \simeq Q$. For completeness, we will give a proof of this in §3.6.

Theorem 3.4.15. — *Let $\widetilde{\mathcal{F}}$ be a functorial normalized desingularization of reduced qe schemes of characteristic zero, and let $\widetilde{\mathcal{F}}(X, M_X)$ be the normalized desingularization of log regular log schemes it induces (see Theorem 3.4.9). Assume that (X, M_X) and (Y, M_Y) are saturated log regular log schemes such that there exists an isomorphism $\phi\colon Y \xrightarrow{\sim} X$ of the underlying schemes. Assume also that the maximal points of the strata of the stratifications of X and Y by the rank of $\overline{M}^{gp}$ are of characteristic zero. Then $\widetilde{\mathcal{F}}(X, M_X)$ and $\widetilde{\mathcal{F}}(Y, M_Y)$ are compatible with ϕ, that is, $\widetilde{\mathcal{F}}(Y, M_Y) = \phi^*(\widetilde{\mathcal{F}}(X, M_X))$.*

Proof. — We can check the assertion of the theorem étale locally. Namely, we can replace X with a strict étale covering X' and replace Y with $Y' = Y \times_X X'$ with the log structure induced from Y. In particular, we can assume that the log structures are Zariski, and so the canonical fans $\mathrm{Fan}(X)$ and $\mathrm{Fan}(Y)$ are defined. Our assumption on the maximal points actually means that $\mathrm{Fan}(X)$ is contained in $X_{\mathbf{Q}} = X \otimes_{\mathbf{Z}} \mathbf{Q}$. By Lemma 3.4.13(iii) and 2.2.11, $\widetilde{\mathcal{F}}(X, M_X)$ is the pushforward of its restriction onto $X_{\mathbf{Q}}$, and similarly for Y. So, it suffices to prove that the normalized desingularizations of $X_{\mathbf{Q}}$ and $Y_{\mathbf{Q}}$ are compatible with respect to $\phi \otimes_{\mathbf{Z}} \mathbf{Q}\colon X_{\mathbf{Q}} \xrightarrow{\sim} Y_{\mathbf{Q}}$. Thus, it suffices to prove the theorem for X and Y of characteristic zero, and, in the sequel, we assume that this is the case.

To simplify notation we identify X and Y, and set $N_X = M_Y$. It suffices to check that the blow up towers $\widetilde{\mathcal{F}}(X, M_X)$ and $\widetilde{\mathcal{F}}(X, N_X)$ coincide after the base change to each completion $\widehat{X}_{\overline{x}} = \mathrm{Spec}(\widehat{\mathcal{O}}_{X,\overline{x}})$ at a geometric point $\overline{x}$. By exp. **VI**, 1.6, we have that $\widehat{X}_{\overline{x}} \xrightarrow{\sim} \mathrm{Spec}(k[[P]][[t_1, \dots, t_n]])$, where $P \xrightarrow{\sim} \overline{M}_{X,\overline{x}}$, and the morphism of fs log schemes $(\widehat{X}_{\overline{x}}, P) \to (X, M_X)$ is strict. By Theorem 3.3.16 the contracted pullback of $\widetilde{\mathcal{F}}(X, M_X)$ to $\widehat{X}_{\overline{x}}$ coincides with $\widetilde{\mathcal{F}}^{\log}(\widehat{X}_{\overline{x}}, P)$. In the same way, the contracted pullback of $\widetilde{\mathcal{F}}(X, N_X)$ coincides with $\widetilde{\mathcal{F}}^{\log}(\widehat{X}_{\overline{x}}, Q)$, where $\widehat{\mathcal{O}}_{X,\overline{x}} \xrightarrow{\sim} \mathrm{Spec}(k[[Q]][[t_1, \dots, t_m]])$ (we use that k is isomorphic to the residue field of this ring and hence depends only on the ring $\widehat{\mathcal{O}}_{X,\overline{x}}$).

Let us now recall how $\widetilde{\mathcal{F}}^{\log}(\widehat{X}_{\overline{x}}, P)$ is constructed (Theorems 3.1.27, 3.2.20 and 3.3.16). We have the obvious strict morphism $\widehat{X}_{\overline{x}} \to Z := \mathrm{Spec}(\mathbf{Q}[P][t_1, \dots, t_n])$, hence $\widetilde{\mathcal{F}}^{\log}(\widehat{X}_{\overline{x}}, P)$ is the pullback of $\widetilde{\mathcal{F}}^{\log}(Z, P)$. The latter is the pullback of $\widetilde{\mathcal{F}}^{\mathrm{mono}}(P) = \widetilde{\mathcal{F}}^{\mathrm{fan}}(P)$, which, in its turn, is induced from $\widetilde{\mathcal{F}}(Z)$. Therefore, $\widetilde{\mathcal{F}}(Z, P) = \widetilde{\mathcal{F}}(Z)$ and, by the functoriality of $\widetilde{\mathcal{F}}$, its pullback to $\widehat{X}_{\overline{x}}$ is $\widetilde{\mathcal{F}}(\widehat{X}_{\overline{x}})$. The same argument shows that $\widetilde{\mathcal{F}}(\widehat{X}_{\overline{x}})$ is the contracted pullback of $\widetilde{\mathcal{F}}(X, N_X)$.

In order to prove that $\widetilde{\mathcal{F}}(X, M_X) = \widetilde{\mathcal{F}}(X, N_X)$ it only remains to resolve the synchronization issues, i.e. to prove equality without contractions. For this one should take the union S of the fans of (X, M_X) and (X, N_X), and consider the morphism $\widehat{X}_S := \coprod_{x \in S} \widehat{X}_{\overline{x}}$ rather than the individual completions. The pullbacks of $\widetilde{\mathcal{F}}(X, M_X)$ and $\widetilde{\mathcal{F}}(X, N_X)$ to $\widehat{X}_S$ have no empty blow ups because the fans contain all associated points of the towers by Lemma 3.4.13(iii). Hence the same argument as above shows that they both coincide with $\widetilde{\mathcal{F}}(\widehat{X}_S)$. □

Remark 3.4.16. — (i) Without taking the completion, $\widetilde{\mathcal{F}}(X)$ does not even have to be defined as X may be non-qe. In order to pass to the completion we used functoriality of the monoidal desingularization with respect to strict morphisms, which may be very bad (e.g. non-flat) on the level of usual morphisms of schemes. Even when (X, M_X) is log regular, the formal completion morphism $\widehat{X}_x \to X$ can be non-regular in the non-qe case. However, one can show that it is regular over the fan, and this is enough to relate the (log) desingularization of X and $\widehat{X}_x$.

(ii) We used the very strong desingularization $\widetilde{\mathcal{F}}$ from Theorem 2.3.10. However, it is easy to see that only the properties listed in Theorem 2.3.7 were essential.

3.5. Complements on non-saturated log regular log schemes. — For the sake of completeness, we mention Gabber's results on non-saturated log regular log schemes that will not be used. We only formulate results but do not give proofs. Gabber defines a fine log scheme (X, M_X) to be **log regular** if its saturation is log regular in the usual sense. Assume now that (X, M_X) is fine and log regular. The key result that lifts the theory off the ground is that Kato's Tor independence extends to non-saturated log regular log schemes. Namely, if (X, M_X) admits a chart $\mathbf{Z}[P]$ and $I \subset P$ is an ideal then $\mathrm{Tor}_1^{\mathbf{Z}[P]}(\mathcal{O}_X, \mathbf{Z}[P]/I) = 0$. For fs log schemes this is due to Kato, and Gabber deduces the general case using a non-flat descent. It then follows similarly to the saturated case that if $(Y, M_Y) = \mathrm{LogBl}_{\mathcal{J}}(X, M_X)$ then $Y \xrightarrow{\sim} \mathrm{Bl}_{\mathcal{J}\mathcal{O}_X}(X)$ and (Y, M_Y) is log regular. In addition, one shows that (X, M_X) is saturated if and only if X is normal, and if $(Y, M_Y) = (X, M_X)^{\mathrm{sat}}$ then $Y \xrightarrow{\sim} X^{\mathrm{nor}}$. Using these foundational results on log regular fine log schemes one can imitate the method of §3.4 to extend Lemma 3.4.13 to the non-saturated case. As a corollary, one obtains an analog of Theorem 3.4.15 for $\mathscr{F}$ and $\mathscr{F}^{\log}$.

3.6. Reconstruction of the monoid. — This section will not be used in the sequel. Its aim is to prove that a fine torsion free monoid P can be reconstructed from a ring $A = k[[P]]$ where k is a field. The main idea of the proof is that an isomorphism $k[[P]] \xrightarrow{\sim} A$ defines an action of the torus $\mathrm{Spec}(k[P^{gp}])$ on A, and any two maximal tori in $\mathrm{Aut}_{k-\mathrm{aug}}(A)$ are conjugate.

3.6.1. Automorphism groups of complete rings. — Let k be a field and A be a complete local noetherian k-algebra with residue field k. Let m denote the maximal ideal and set $A_n = A/m^{n+1}$. Let $G_n = \mathrm{Aut}_{k-\mathrm{aug}}(A_n)$ be the automorphism group scheme of A_n viewed as an augmented k-algebra, i.e.

$$G_n(B) = \{\sigma \in \mathrm{Aut}_B(A_n \otimes_k B) | \ \sigma(mA_n \otimes_k B) = mA_n \otimes_k B\}.$$

This is a closed subgroup of the automorphism group scheme $\mathrm{Aut}_k(A_n)$ defined by a nilpotent ideal. The groups G_n form a filtering projective system $\dots \to G_2 \to G_1 \to G_0$; we call such a system an algebraic pro-group. Note that $G_\bullet$ induces a set-valued functor $G(B) = \lim_n G_n(B)$ on the category of k-algebras, and $G(k) = \mathrm{Aut}_k(A)$.

Remark 3.6.2. — Gabber also considered more complicated filtering families, but we stick to the simplest case we need.

3.6.3. Stabilization. — We say that an algebraic pro-group $G_\bullet$ is **stable** if the homomorphisms $\pi_n : G_{n+1} \to G_n$ are surjective for large enough n. Any algebraic group can be stabilized as follows. For each G_n and $i \geq 0$ let $G_{n,i}$ denote the scheme-theoretic image of G_{n+i} in G_n. Then $G_{n,0} \supset G_{n,1} \supset \dots$ is a decreasing sequence of algebraic subgroups of G, hence it stabilizes on a subgroup $G^{st}_n \subset G_n$. The family $G^{st}_\bullet$ with obvious transition morphisms is an algebraic pro-subgroup of $G_\bullet$, and it is clear from the definition that $G^{st}_\bullet$ is stable. Note also that $G^{st}_\bullet$ is isomorphic to $G_\bullet$ at least in the sense that $G(B) \xrightarrow{\sim} \lim_n G^{st}_n(B)$ for any k-algebra B.

3.6.4. Maximal pro-tori. — By a **pro-torus** $T_\bullet$ in $G_\bullet$ we mean a *compatible* family of tori $T_n \hookrightarrow G_n$ for all $n \geq 0$, in the sense that $\pi_n(T_{n+1}) = T_n$. It is called a **torus** if the π_n's are isomorphisms for $n \gg 0$. A pro-torus is **componentwise maximal** if for $n \gg 0$ all tori T_n are maximal. A pro-torus $T_\bullet$ is **maximal** if for any inclusion of pro-tori $T_\bullet \subset T'_\bullet$ we have that $T_n = T'_n$ for $n \gg 0$. In particular, any componentwise maximal pro-torus is maximal. If k is algebraically closed and $G_{n+1} \to G_n$ is surjective then any torus $S_n \subset G_n$ is the image of a torus $S_{n+1} \subset G_{n+1}$, and hence componentwise maximal pro-tori exist whenever $G_\bullet$ is stable. For shortness, given an element $c \in G_n(k)$ we will denote the corresponding conjugation by $c : G_n \to G_n$. Pro-tori $T_\bullet$ and $T'_\bullet$ are **conjugate** if there exists a *compatible* family of elements $c_n \in G_n(k)$ such that for $n \gg 0$ the conjugation $c_n : G_n \to G_n$ takes T_n to T'_n.

Proposition 3.6.5. — *Assume that k is an algebraically closed field, $G_\bullet$ is a stable pro-algebraic k-group, and $T_\bullet, T'_\bullet \hookrightarrow G_\bullet$ are pro-tori. If $T_\bullet$ is componentwise maximal then $T'_\bullet$ is conjugate to a sub-pro-torus of $T_\bullet$. In particular, a pro-torus is maximal if and only if it is componentwise maximal, and any two maximal pro-tori are conjugate.*

Proof. — It is a classical result that maximal tori in algebraic k-groups are conjugate. In particular, for each n we can move T'_n into T_n by a conjugation, and the only issue is compatibility of the conjugations. Naturally, to achieve compatibility we should lift conjugations inductively from G_n to G_{n+1}. It suffices to prove that if π_n is surjective and $c_n : G_n \to G_n$ conjugates T'_n into T_n then it lifts to $c_{n+1} : G_{n+1} \to G_{n+1}$ that conjugates T'_{n+1} into T_{n+1}. By the stability assumption, we can lift c_n to a conjugation c' of G_{n+1}. It takes T'_{n+1} to the subgroup $H = KT_{n+1}$, where K is the kernel of π_n. Since maximal tori in H are conjugate and conjugation by elements of T_{n+1} preserves T_{n+1}, we can find a conjugation c'' by an element of K that takes $c'(T'_{n+1})$ to T_{n+1}. Then $c_{n+1} = c''c'$ is a lifting of c_n as required. ☐

Corollary 3.6.6. — *Assume that k is an algebraically closed field, $G_\bullet$ is a pro-algebraic k-group, and $T_\bullet, T'_\bullet \hookrightarrow G_\bullet$ are pro-tori. If $T_\bullet$ is maximal then $T'_\bullet$ is conjugate to a sub-pro-torus of $T_\bullet$.*

Proof. — Obviously, $T_n, T'_n \subset G^{st}_n$ for all $n \geq 0$. So, $T'_\bullet$ is conjugate to a subtorus of $T_\bullet$ already inside of G^{st} by Proposition 3.6.5. ☐

3.6.7. Certain tori in $\mathrm{Aut}_{k-\mathrm{aug}}(A)$. — Any k-isomorphism $C[[P]] \xrightarrow{\sim} A$, where C is a complete local k-algebra and P is a sharp fine monoid, induces a pro-algebraic action of the split torus $T_P = \mathrm{Spec}(k[P^{gp}])$ on A: a character $\chi\colon P \to k^*$ acts on C trivially and acts on $p \in P$ by $p \mapsto \chi(p)p$ (and the action of B-points $\chi\colon P \to B^*$ is analogous). Thus we obtain homomorphisms $\psi\colon T_P \to G_n$ which are monomorphisms for $n > 0$. In particular, the image is a split torus of G. Furthermore, we claim that if $C = k$ then the torus is maximal (as a pro-torus). Indeed, if $\phi \in \mathrm{Aut}_k(A)$ commutes with T_P then its action on $k[[P]]$ preserves the T_P-eigenspaces pk for $p \in P$ and on each pk it acts by multiplication by a number $\lambda(p)$. Clearly, $\lambda\colon P \to k^*$ is a homomorphism and we obtain that ϕ belongs to $\psi(T_P(k))$ and corresponds to $\lambda \in T_P(k)$.

Theorem 3.6.8. — *Assume that k is an algebraically closed field, P is a sharp fine monoid and $A = k[[P]]$. If C is a complete local k-algebra, Q is a sharp fine monoid and $C[[Q]] \to A$ is a k-isomorphism then $C \simeq k[[R]]$ and $P \simeq Q \times R$ for a sharp fine monoid R. In particular, P is uniquely determined by A.*

Proof. — Consider $G = \mathrm{Aut}_{k-\mathrm{aug}}(A)$ with split tori $T_P, T_Q \hookrightarrow G$ corresponding to these isomorphisms. By maximality of T_P and Corollary 3.6.6 there exists a conjugation of G that maps T_Q into T_P. This produces a new isomorphism $C[[Q]] \xrightarrow{\sim} A = k[[P]]$ that respects the grading, i.e. each pk lies in some qC, and we obtain a surjective map $f\colon P \to Q$, which is clearly a homomorphism. If $C = k$ then f is an isomorphism and we obtain that P is determined by A.

Set $R_q = \prod_{p \in f^{-1}(q)} pk$. We have natural embeddings $\prod_{p\in f^{-1}(q)} pk \hookrightarrow qC$ which are all isomorphisms because $A = \prod_{q\in Q} R_q$ is isomorphic to $C[[Q]] = \prod_{q \in Q} qC$. In particular, for $R = f^{-1}(1)$ we have that $C = \prod_{q\in R} qk = k[[R]]$. Therefore, $A \simeq k[[R]][[Q]] \simeq k[[R \times Q]]$, and since the monoid is determined by A we obtain that $P \simeq Q \times R$. $\square$

Corollary 3.6.9. — *Assume that P and Q are sharp fine monoids and k is a field such that $k[[P]]$ is k-isomorphic to $k[[Q]]$. Then P is isomorphic to Q.*

Proof. — Observe that $\overline{k}[[P]]$ is isomorphic to $\overline{k}[[Q]]$ and use the above theorem. $\square$

4. Proof of Theorem 1.1 — preliminary steps

The goal of §4 is to reduce the proof of Theorem 1.1 to the case when the following conditions are satisfied: (1) X is regular, (2) the log structure is given by an snc divisor Z which is G-**strict** in the sense that for any $g \in G$ and a component Z_i either $gZ_i = Z_i$ or $gZ_i \cap Z_i = \varnothing$, (3) G acts freely on $X \setminus Z$ and for any geometric point $\overline{x} \to X$ the inertia group $G_{\overline{x}}$ is abelian.

4.1. Plan. — A general method for constructing a G-equivariant morphism f as in Theorem 1.1 is to construct a tower of G-equivariant morphisms of log schemes $X' = X_n \dashrightarrow X_0 = X$, where the underlying morphisms of schemes are normalized blow ups along G-stable centers sitting over $Z \cup T$, such that various properties of the log scheme X_i with the action of G gradually improve to match all assertions of the Theorem. To simplify notation, we will, as a rule, replace X with the new log scheme after each step. The three conditions above will be achieved in three steps as follows.

4.1.1. Step 1. Making X regular.— This is achieved by the saturated log blow up tower $\widetilde{\mathscr{F}}(X, Z)\colon X' \dashrightarrow X$ from Theorem 3.4.9. In particular, the morphism $X' \to X$ is even log smooth. In the sequel, we assume that X is regular, in particular, Z is a normal crossings divisor by exp. **VI**, 1.7. We will call Z the **boundary** of X. In the sequel, these conditions will be preserved, so let us describe an appropriate restriction on further modifications.

4.1.2. Permissible blow ups. — After Step 1 any modification in the remaining tower will be of the form $f\colon (X', Z') \to (X, Z)$ where $X' = Bl_V(X)$, $Z' = f^{-1}(Z \cup V)$ and V has **normal crossings** with Z, i.e. étale locally on X there exist regular parameters $t_1, \dots, t_d$ such that $Z = V(\prod_{i=1}^{l} t_i)$ and $V = V(t_{i_1}, \dots, t_{i_m})$. We call such modification of log schemes **permissible** and use the convention that the center V is part of the data. A blow up of schemes $f\colon X' \to X$ is called **permissible** (with respect to the boundary Z) if it underlies a permissible modification. Since there is an obvious bijective correspondence between permissible modifications and blow ups we will freely pass from one to another. Note that $Z' = f^{\mathrm{st}}(Z) \cup E'$, where $E' = f^{-1}(V)$ is the exceptional divisor.

4.1.3. Permissible towers. — A **permissible modification tower** $(X_d, Z_d) \dashrightarrow (X_0, Z_0) = (X, Z)$ is defined in the obvious way and we say that a blow up tower $X_d \dashrightarrow X$ is permissible if it underlies such a modification tower. Again, we will freely pass between permissible towers of these types. Note that $Z_i = Z_i^{st} \cup E_i$, where Z_i^{st} is the strict transform of Z under $h_i \colon X_i \dashrightarrow X$ and E_i is the exceptional divisor of h_i (i.e. the union of the preimages of the centers of h_i).

Remark 4.1.4. — (i) Consider a permissible tower as above. It is well known that for any i one has that X_i is regular, Z_i is normal crossings and E_i is even snc. For completeness, let us outline the proof. Both claims follow from the following: if Z is snc then Z_i is snc. Indeed, the claim about E_i follows by taking $Z = \varnothing$ and the claim about Z_i can be checked étale locally, so we can assume that Z is snc. Finally, if Z is snc then Z_i is snc by Lemma 4.2.9 below.

(ii) Permissible towers are very common in embedded desingularization because they do not destroy regularity of the ambient scheme and the normal crossings (or snc) property of the boundary (or accumulated exceptional divisor). Even when one starts with an empty boundary, a non-trivial boundary appears after the first step, and this restricts the choice of further centers. Actually, any known self-contained proof of embedded desingularization constructs a permissible resolution tower.

4.1.5. G-permissible towers. — In addition, we will only blow up G-equivariant centers V. So, $f \colon X' = Bl_V(X) \to X$ is G-equivariant and the exceptional divisor $E = f^{-1}(V)$ is regular and G-equivariant and hence G-strict. Such a blow up (or their tower) will be called G-**permissible**. It follows by induction that the exceptional divisor of such a tower is G-strict.

4.1.6. Step 2. Making Z snc and G-strict. — Consider the stratification of Z by multiplicity: a point $z \in Z$ is in Z^n if it has exactly n preimages in the normalization of Z. Note that $\{Z^n\}$ is precisely the log stratification as defined in 3.3.1. By **depth** of the stratification we mean the maximal d such that $Z^d \neq \varnothing$. In particular, Z^d is the only closed stratum. Step 2 proceeds as follows: $X_{i+1} \to X_i$ is the blow up along the closed stratum of Z_i^{st}.

Remark 4.1.7. — What we use above is the standard algorithm that achieves the following two things: Z' is snc and $Z^{st} = \varnothing$ (see, for example, [**de Jong, 1996**, 7.2]). Even when Z is snc, the second property is often used in the embedded desingularization algorithms to get rid of the old components of the boundary.

4.1.8. Justification of Step 2. — Since the construction is well known, we just sketch the argument. First, observe that Z^d has normal crossings with Z, that is, $X' = Bl_{Z^d}(X) \to X$ is permissible. Thus, Z' is normal crossings and hence Z^{st} is also normal crossings. A simple computation with blow up charts shows that the depth of Z^{st} is $d-1$ (for example, one can work étale locally, and then this follows from Lemma 4.2.8 below). It follows by induction that the tower produced by Step 2 is permissible, of length d and with $Z_d^{st} = \varnothing$. So, $Z_d = E_d$ is snc by Remark 4.1.4 and G-strict by 4.1.5.

4.1.9. Step 3. Making the inertia groups abelian and the action of G on $X \setminus Z$ free. — Recall (exp. **VI**, 4.1) that the inertia strata are of the form $X_H = X^H \setminus \bigcup_{H \subsetneq H'} X^{H'}$. Step 3 runs analogously to Step 2, but this time we will work with the inertia stratification of X instead of the log stratification, and will have to apply the same operation a few times. Let us first describe the blow up algorithm used in this step; its justification will be given in §4.2.

Let $f_{\{X^H\}} \colon X' \dashrightarrow X$ denote the following blow up tower. First we blow up the union of all closed strata X_H. In other words, V_0 is the union of all non-empty minimal X^H, i.e. non-empty X^H that do not contain X^K with $\varnothing \subsetneq X^K \subsetneq X^H$. Next, we consider the family of all strict transforms of X^H and blow up the union of the non-empty minimal ones, etc. Obviously, the construction is G-equivariant. We will prove in Proposition 4.2.11 that $f_{\{X^H\}}$ is permissible of length bounded by the length of the maximal chains of subgroups. Also, we will show in 4.2.13 that $f_{\{X^H\}}$ decreases all non-abelian inertia groups, so the algorithm of Step 3 goes as follows: until all inertia groups become abelian, apply $f_{\{X^H\}} \colon X' \dashrightarrow X$ (i.e. replace (X, Z) with (X', Z')).

4.2. Justification of Step 3. —

4.2.1. Weakly snc families. — Assume that X is regular, $Z \hookrightarrow X$ is an snc divisor, and $\{X_i\}_{i \in I}$ is a finite collection of closed subschemes of X. For any $J \subset I$ we denote by X_J the scheme-theoretic intersection $\bigcap_{j \in J} X_j$. The family $\{X_i\}$ is called **weakly snc** if each X_i is nowhere dense and X_J is regular. The family $\{X_i\}$ is called **weakly Z-snc** if it is weakly snc and each X_J has normal crossings with Z. In particular, $\{X_i\}_{i \in I}$ is weakly snc (resp. weakly Z-snc) if and only if the family $\{X_J\}_{\varnothing \neq J \subset I}$ is weakly snc (resp. weakly Z-snc).

Remark 4.2.2. — (i) Here is a standard criterion of being an snc divisor, which is often taken as a definition. Let $D \hookrightarrow X$ be a divisor with irreducible components $\{D_i\}_{i\in I}$. Then D is snc if and only if the set of its irreducible components is weakly snc and each irreducible component of D_J is of codimension $|J|$ in X.

(ii) The condition on the codimension is essential. For example, $xy(x+y) = 0$ defines a weakly snc but not snc family of irreducible divisors in $\mathbf{A}^2_k = \mathrm{Spec}(k[x,y])$.

(iii) The criterion from (i) implies that if X is qe then the snc locus of D is open—it is the complement of the union of singular loci of D_J's and the loci where D_J is of codimension smaller than $|J|$. (Note that this makes sense for all points of X because D is snc at a point $x \in X \setminus D$ if and only if $X = D_\varnothing$ is regular at x.)

Lemma 4.2.3. — *Let* (X, Z) *and* G *be as achieved in Step 2. Then the family* $\{X^H\}_{1\neq H\subset G}$ *is weakly* Z*-snc.*

Proof. — Recall that for any subgroup $H \subset G$ the fixed point subscheme X^H is regular by Proposition exp. **VI**, 4.2, and X^H is nowhere dense for $H \neq 1$ by generic freeness of the action of G. Since for any pair of subgroups $K, H \subset G$ we have that $X^H \times_X X^K = X^{KH}$, the family is weakly snc. It remains to show that $Y = X^H$ has normal crossings with Z. Note that it is enough to consider the case when X is local with closed point x and $G = H$. The cotangent spaces at x will be denoted $T^*X = m_{X,x}/m^2_{X,x}$, T^*Y, etc. Their dual spaces will be called the tangent spaces, and denoted TX, TY, etc. Let $\phi^* : T^*X \twoheadrightarrow T^*Y$ denote the natural map and let $\phi : TY \hookrightarrow TX$ denote its dual. We will systematically use without mention that $|G|$ is coprime to char $k(x)$, in particular, the action of G on T^*X is semi-simple.

The proof of exp. **VI**, 4.2 also shows that for any point $x \in X^H$, the tangent space $T_x(X^H)$ is isomorphic to $(T_xX)^H$. In particular, $TY \xrightarrow{\sim} (TX)^G$ and hence ϕ^* maps $(T^*X)^G \subset T^*X$ isomorphically onto T^*Y. Therefore, $U = \mathrm{Ker}(\phi^*)$ is the G-orthogonal complement to $(T^*X)^G$, i.e. the only G-invariant subspace such that $(T^*X)^G \oplus U \xrightarrow{\sim} T^*X$. Let $Z_i = V(t_i)$, $1 \leq i \leq n$ be the components of Z and let $dt_i \in T^*X$ denote the image of t_i. By Z-strictness of G, each line $L_i = \mathrm{Span}(dt_i)$ is G-invariant, so G acts on dt_i by a character χ_i. Without restriction of generality, $\chi_1, \ldots, \chi_l$ for some $0 \leq l \leq n$ are the only trivial characters. In particular, $L = \mathrm{Span}(dt_1, \ldots, dt_n)$ is the direct sum of $L^G = \mathrm{Span}(dt_1, \ldots, dt_l)$ and its G-orthogonal complement $L \cap U$, which (by uniqueness of the complement) coincides with $\mathrm{Span}(dt_{l+1}, \ldots, dt_n)$. Complete the basis of L to a basis $\{dt_1, \ldots, dt_n, e_1, \ldots, e_m\}$ of T^*X such that $\{dt_{l+1}, \ldots, dt_n, e_1, \ldots, e_r\}$ for some $r \leq m$ is a basis of U and choose functions $s_1, \ldots, s_m$ on X so that $ds_j = e_j$ and $s_1, \ldots, s_r$ vanish on Y. Clearly, $t_1, \ldots, t_n, s_1, \ldots, s_m$ is a regular family of parameters of $\mathcal{O}_{X,x}$, so the lemma would follow if we prove that $Y = V(t_{l+1}, \ldots, t_n, s_1, \ldots, s_r)$.

Since Y is regular and $\mathrm{Ker}(\phi^*)$ is spanned by the images of $t_{l+1}, \ldots, t_n, s_1, \ldots, s_r$, we should only check that these functions vanish on Y. The s_j's vanish on Y by the construction, so we should check that t_i vanishes on Y whenever $l < i \leq n$. Using the functorial definition from exp. **VI**, 4.1 of the subscheme of fixed points, we obtain that $Z_i^G = Z_i \times_X X^G$, hence $Z_i \times_X Y$ is regular by exp. **VI**, 4.2. However, TY is contained in TZ_i, which is the vanishing space of dt_i, hence we necessarily have that $Y \hookrightarrow Z_i$. □

4.2.4. Snc families. — A family of nowhere dense closed subschemes $\{X_i\}_{i\in I}$ is called **snc** (resp. **Z-snc**) at a point x if X is regular at x and there exists a regular family of parameters $t_j \in \mathcal{O}_{X,x}$ such that in a neighborhood of x each X_i (resp. and each irreducible component Z_k of Z) passing through x is given by the vanishing of a subfamily $t_{j_1}, \ldots, t_{j_l}$. Note that the family $\{X_i\}_{i\in I}$ is Z-snc if and only if the union $\{X_i\} \cup \{Z_k\}$ is snc. A family is **snc** if it is so at any point of X (in particular, X is regular).

Remark 4.2.5. — (i) It is easy to see that the family $\{X^H\}_{H\subset G}$ is snc whenever G is abelian. Indeed, it suffices to show that for any point x there exists a basis of T_xX such that each $T_x(X^H)$ is given by vanishing of some of the coordinates. But this is so because the action of G on T_xX is (geometrically) diagonalizable and $T_x(X^H) = (T_xX)^H$. In general, the family $\{X^H\}_{H\subset G}$ does not have to be snc, as the example of a dihedral group D_n with $n \geq 3$ acting on the plane shows.

(ii) If $Z \hookrightarrow X$ is an snc divisor with components Z_i and $V \hookrightarrow X$ is a closed subscheme then the family $\{Z_i, V\}$ is snc if and only if V has normal crossings with Z.

The transversal case of the following lemma can be deduced from [**ÉGA** IV$_4$ §19.1], but we could not find the general case in the literature (although it seems very probable that it should have appeared somewhere).

Lemma 4.2.6. — *Any weakly snc family with* $|I| = 2$ *is snc.*

Proof. — We should prove that if $X = \mathrm{Spec}(A)$ is a regular local scheme and Y, Z are regular closed subschemes such that $T = Y \times_X Z$ is regular then there exists a regular family of parameters $t_1, \ldots, t_n \in A$ such that Y and Z are given by vanishing of some set of these parameters. Let m be the maximal ideal of A, and let I, J and $K = I + J$ be the ideals defining Y, Z and T, respectively. By $T^*X = m/m^2$, T^*Y, etc., we denote the cotangent spaces at the closed point of X. Note that $I/\mathfrak{m}I \xrightarrow{\sim} \mathrm{Ker}(T^*X \to T^*Y)$, and similar formulas hold for $J/\mathfrak{m}J$ and $K/\mathfrak{m}K$. Indeed, we can choose the parameters so that $Y = V(t_1, \dots, t_l)$ and then the images of $t_1, \dots, t_l$ form a basis both of $I/\mathfrak{m}I$ and $\mathrm{Ker}(T^*X \to T^*Y)$.

Now, let us prove the lemma. Assume first that Y and Z are transversal, i.e. $T^*X \hookrightarrow T^*Y \oplus T^*Z$. Choose elements $t_1, \dots, t_{l+k}$ such that $Y = V(t_1, \dots, t_l)$, $Z = V(t_{l+1}, \dots, t_{l+k})$, $l = \mathrm{codim}(Y)$ and $k = \mathrm{codim}(Z)$. Then the images $dt_i \in T^*X$ of t_i are linearly independent because $dt_1, \dots, dt_l$ span $\mathrm{Ker}(T^*X \to T^*Y)$ and $dt_{l+1}, \dots, dt_{l+k}$ span $\mathrm{Ker}(T^*X \to T^*Z)$. Hence we can complete t_i's to a regular family of parameters by choosing $t_{l+k+1}, \dots, t_n$ such that $dt_1, \dots, dt_n$ is a basis of T^*X. This proves the transversal case, and to establish the general case it now suffices to show that if Y and Z are not transversal then there exists an element $t_1 \in \mathfrak{m} \setminus \mathfrak{m}^2$ which vanishes both on Y and Z. (The we can replace X with $X_1 = V(t_1)$ and repeat this process until Y and Z are transversal in $X_a = V(t_1, \dots, t_a)$.) Tensoring the exact sequence $0 \to I \cap J \to I \oplus J \to K \to 0$ with $A/\mathfrak{m}$ we obtain an exact sequence

$$(I \cap J)/\mathfrak{m}(I \cap J) \to I/\mathfrak{m}I \oplus J/\mathfrak{m}J \xrightarrow{\phi} K/\mathfrak{m}K \to 0$$

The failure of transversality is equivalent to non-injectivity of ϕ, hence there exists an element $f \in I \cap J$ with a non-zero image in $I/\mathfrak{m}I \oplus J/\mathfrak{m}J$. Thus, $f \in \mathfrak{m} \setminus \mathfrak{m}^2$ and we are done. □

4.2.7. Blowing up the minimal strata of a weakly snc family. — Given a weakly snc family $\{X_i\}_{i \in I}$, we say that a scheme X_J with $J \subset I$ is **minimal** if it is non-empty and any $X_{J'} \subsetneq X_J$ is empty. Also, we will need the following notation: if $Z \hookrightarrow X$ is a closed subscheme and $D \hookrightarrow X$ is a Cartier divisor with the corresponding ideals $\mathscr{I}_Z, \mathscr{I}_D \subset \mathscr{O}_X$, then $Z + D$ is the closed subscheme defined by the ideal $\mathscr{I}_Z \mathscr{I}_D$.

Proposition 4.2.8. — *Assume that X is regular, $Z \hookrightarrow X$ is an snc divisor with irreducible components $Z_1, \dots, Z_l$, and $\{X_i\}_{i \in I}$ is a Z-snc (resp. weakly Z-snc) family of subschemes. Let V be the union of all non-empty minimal subschemes X_J, $f\colon X' = \mathrm{Bl}_V(X) \to X$, $X'_i = f^{\mathrm{st}}(X_i)$ and $Z' = f^{-1}(Z \cup V)$. Then*

(i) X' is regular and Z' is snc.

(ii) The family $\{X'_i\}_{i \in I}$ is Z'-snc (resp. weakly Z'-snc).

(iii) For any $J \subset I$, the scheme-theoretical intersection $X'_J = \bigcap_{j \in J} X'_j$ coincides with $f^{\mathrm{st}}(X_J)$.

(iv) For any $J \subset I$ the total transform $X_J \times_X X'$ is of the form $X'_J + D'_J$ where D'_J is the divisor consisting of all connected components of $E' = f^{-1}(V)$ contained in $f^{-1}(X_J)$.

Proof. — We start with the following lemma.

Lemma 4.2.9. — *Assume that X is regular, Z is an snc divisor and $V \hookrightarrow Y$ are closed subschemes having normal crossings with Z. Let $f\colon X' \to X$ be the blow up along V, $Y' = f^{\mathrm{st}}(Y)$, $Z' = f^{-1}(Z \cup V)$, and $E' = f^{-1}(V)$. Then Z' is snc, Y' has normal crossings with Z' and $Y \times_X X' = Y' + E'$.*

Proof. — The proof is a usual local computation with charts. Take any point $u \in V$ and choose a regular family of parameters $t_1, \dots, t_n$ at u such that Y (resp. V, resp. Z) are given by the vanishing of $t_1, \dots, t_m$ (resp. $t_1, \dots, t_l$, resp. $\prod_{i \in I} t_i$), where $0 \le m \le l \le n$ and $I \subset \{1, \dots, n\}$. Locally over u the blow up is covered by l charts, and the local coordinates on the i-th chart are t'_j such that $t'_j = t_j$ for $j > l$ or $j = i$ and $t'_j = \frac{t_j}{t_i}$ otherwise. On this chart, $Y \times_X X'$ (resp. Y', resp. E', resp. Z') is given by the vanishing of $t_1, \dots, t_m$ (resp. $t'_1, \dots, t'_m$, resp. t'_i, resp. $\prod_{j \in I \cup \{i\}} t'_j$), hence the lemma follows. □

The lemma implies (i). In addition, it follows from the lemma that $f^{\mathrm{st}}(X_J)$ has normal crossings with Z' and $X_J \times_X X' = f^{\mathrm{st}}(X_J) + D'_J$. Thus, (iii) implies (iv), and (iii) implies (ii) in the case when the family $\{X_i\}_{i \in I}$ is weakly Z-snc. Note also that if this family is Z-snc then locally at any point $x \in X$ there exists a family of regular parameters $\underline{t} = \{t_1, \dots, t_n\}$ such that each X_J and each component of Z is given by the vanishing of a subfamily of $\underline{t}$ locally in a neighborhood of x. Then the same local computation as was used in the proof of Lemma 4.2.9 proves also claims (ii) and (iii) of the proposition. So, it remains to prove (iii) when the family is weakly snc. It suffices to prove that if (iii) holds for X_J and X_K then it holds for $X_{J \cup K}$. Moreover, (iii) does not involve the boundary so we can assume that $Z = \varnothing$. It remains to note that $\{X_J, X_S\}$ is an snc family by Lemma 4.2.6, hence our claim follows from the snc case. □

4.2.10. Blow up tower of a weakly Z-snc family. — Let X be a regular scheme, Z be an snc divisor and $\{X_i\}_{i \in I}$ be a weakly Z-snc family. By the blow up tower $f_{\{X_i\}}$ of $\{X_i\}$ we mean the following tower: the first blow up $h_1 : X_1 \to X_0 = X$ is along the union of all non-empty minimal schemes of the form X_J for $\varnothing \neq J \subset I$, the second blow up is along the union of all non-empty minimal schemes of the form $h_1^{\mathrm{st}}(X_J)$ for $\varnothing \neq J \subset I$, etc.

Proposition 4.2.11. — *Keep the above notation. Then the tower $f_{\{X_i\}}$ is permissible with respect to Z and its length equals to the maximal length of chains $\varnothing \neq X_{J_1} \subsetneq \cdots \subsetneq X_{J_d}$ with $\varnothing \neq J_d \subsetneq \cdots \subsetneq J_1 \subset I$. Furthermore, the strict transform of any scheme X_J is empty and the total transform of X_J is a Cartier divisor.*

Proof. — The claim about the length is obvious. Let $f_{\{X_i\}} : X_d \dashrightarrow X_0 = X$ and let $h_n : X_n \dashrightarrow X_0 = X$ be its n-th truncation. For each $i \in I$ set $X_{n,i} = h_n^{st}(X_i)$ and for each $J \subset I$ set $X_{n,J} = \bigcap_{i \in J} X_{n,i}$. Using Proposition 4.2.8 and straightforward induction on length, we obtain that the family $\{X_{n,i}\}_{i \in I}$ is weakly Z_n-snc, $X_{n,J} = h_n^{st}(X_J)$, the blow up $X_{n+1} \to X_n$ is along the union of non-empty minimal $X_{n,J}$'s, and $X_J \times_X X_n = X_{n,J} + D_{J,n}$, where $D_{J,n}$ is a divisor. So, the tower is permissible, and since $X_{d,J} = \varnothing$ we also have that $X_J \times_X X_d$ is a divisor. □

Remark 4.2.12. — We will not need this, but it is easy to deduce from the proposition that on the level of morphisms the modification $X_d \to X$ is isomorphic to the blow up along $\prod_{\varnothing \neq J \subset I} \mathscr{I}_{X_J}$.

4.2.13. Justification of Step 3. — The blow up tower $f_{\{X^H\}} \colon X' \dashrightarrow X$ from Step 3 is G-equivariant in an obvious way, and it is permissible by Proposition 4.2.11. In addition, $Z' = f^{-1}(Z) \cup f^{-1}(\bigcup_{1 \neq H \subset G} X^H)$ and, since G acts freely on $X' \setminus f^{-1}(\bigcup_{1 \neq H \subset G} X^H) \widetilde{\to} X \setminus \bigcup_{1 \neq H \subset G} X^H$, it also acts freely on $X' \setminus Z'$. It remains to show that applying $f_{\{X^H\}}$ we decrease all non-abelian inertia groups. Namely, for any $x' \in X'$ mapped to $x \in X$ we want to show that either $G_{\overline{x}}$ is abelian or the inclusion $G_{\overline{x}'} \subsetneq G_{\overline{x}}$ is strict.

Let $H \subset G$ be any non-abelian subgroup with commutator $K = [H, H]$. Since $X^K \times_X X'$ is a divisor by Proposition 4.2.11, the universal property of blow ups implies that $X' \to X$ factors through $Y = Bl_{X^K}(X)$. On the other hand, it is proved in exp. **VI**, 4.8 that $Y \to X$ is an H-equivariant blow up, and if a geometric point $\overline{x} \to X$ with $G_{\overline{x}} = H$ lifts to a geometric point $\overline{y} \to Y$ then $G_{\overline{y}} \subsetneq H$. Therefore, the same is true for the G-equivariant modification $X' \to X$, and we are done.

5. Proof of Theorem 1.1 — abelian inertia

5.1. Conventions. — Throughout §5 we assume that (X, Z) and G satisfy all conditions achieved at Steps 1, 2, 3, and our aim is to construct a modification $f_{(G,X,Z)} \colon X' \to X$ as in Theorem 1.1. Unless specially mentioned, we do not assume that X is qe. This is done in order to isolate the only place where this assumption is needed (existence of rigidifications).

5.2. Outline of our method and other approaches. —

5.2.1. Combinatorial nature of the problem. — On the intuitive level it is natural to expect that "everything relevant to our problem" should be determined by the following "combinatorial" data: the log structure of X, the inertia stratification of X by $X_H := X^H \setminus \bigcup_{H' \subsetneq H} X^{H'}$ and the representations of the inertia groups on the tangent spaces (which are essentially constant along X_H). This combinatorial nature is manifested in both approaches to the problem that we describe below.

5.2.2. Combinatorial algorithm. — A natural approach is to seek for a "combinatorial algorithm" that iteratively blows up disjoint unions of a few closures of connected components of **log-inertia strata** (i.e. intersections of a log stratum with an inertia stratum). For example, our (very simple) algorithms in §4 were of this type. The choice of the centers should be governed by the following combinatorial data: the number of components of Z through a point x and the history of their appearance (similarly to the desingularization algorithms), and the representation of $G_{\overline{x}}$ on T_x plus the history of representations (i.e. the list of representations for all predecessors of x in the blow up tower).

It is natural to expect that building such an algorithm would lead to a relatively simple proof of Theorem 1.1. In particular, it would be non-sensitive to quasi-excellence issues. Unfortunately, despite partial positive results, we could not construct such an algorithm. Thus, the question whether it exists remains open.[ii]

5.2.3. Our method. — A general plan of our method is as follows. In §5.3 we will show that such a modification f exists étale locally on the base if X is qe. A priori, our construction will be canonical up to an auxiliary choice, but then we will prove in §5.5 that actually it is independent of the choice and hence descends to a modification f as required. To prove independence we will show in §5.4 that the construction is functorial with respect to strict **inert** morphisms (i.e. morphisms that preserve both the log and the inertia structures, see 5.3.6). The latter is a manifestation of the "combinatorial nature" of our algorithm.

5.3. Local construction. —

[ii] F. Pop told the second author that he has a plan for constructing such a combinatorial algorithm.

5.3.1. Very tame action and Zariski topology. — Note that the log structure on (X, Z) is Zariski since Z is snc. Thus, it will be convenient to describe very tame action in terms of Zariski topology. We say that the action of G is **very tame at** a point $x \in X$ if for any geometric point $\overline{x}$ over x the action is very tame at $\overline{x}$. Clearly, the $G_{\overline{x}}$-equivariant log scheme $\mathrm{Spec}(\mathcal{O}_{X,\overline{x}})$ is independent of the choice of $\overline{x}$ up to an isomorphism. In particular, the action is very tame at x if and only if it is very tame at a single geometric point $\overline{x}$ above x.

Lemma 5.3.2. — *Assume that (G, X, Z) is as in §5.1, $x \in X$ is a point, and T is the set of points of X at which the action is very tame. Then*

(i) T is open,

(ii) $x \in T$ if and only if the local log stratification of X (see Remark 3.3.2) is finer than the inertia stratification in a neighborhood of x.

Proof. — The first assertion follows from exp. **VI**, 3.9. To prove the second claim we note that the conditions (i) and (ii) from exp. **VI**, 3.1 are automatically satisfied at $\overline{x}$. Indeed, (i) is satisfied because the action is tame by assumption of 1.1 and (ii) is satisfied because Z is snc and G-strict. Since log and inertia stratifications are compatible with the strict henselization morphism, condition (iii) from exp. **VI**, 3.1 is satisfied at $\overline{x}$ if and only if the local log stratification is finer than the inertia stratification at x. This proves (ii). □

5.3.3. Admissibility. — In the sequel, by saying that X is **admissible** we mean that the action of G on X is admissible in the sense of [**SGA 1** V 1.7] (e.g. X is affine). This is needed to ensure that X/G exists as a scheme. An alternative would be to allow X/G to be an algebraic space (and $(X/G, Z/G)$ to be a log algebraic space).

5.3.4. Rigidification. — By a **rigidification** of X we mean a G-equivariant normal crossings divisor $\overline{Z}$ that contains Z and such that the action of G on the log regular log scheme $\overline{X} = (X, \overline{Z})$ is very tame. If $\overline{Z}$ is snc then we say that the rigidification is **strict**. Sometimes, by a rigidification of $X = (X, Z)$ we will mean the log scheme $\overline{X}$ itself. Our construction of a modification f uses a rigidification, so let us first establish local results on existence of the latter.

In the sequel, we say that μ_N is **split** over a scheme X if it is isomorphic to the discrete group $(\mathbf{Z}/N\mathbf{Z})^*$ over X. This happens if and only if X admits a morphism to $\mathrm{Spec}(\mathbf{Z}[\frac{1}{N}, \mu_N])$, where we use the notation $\mathbf{Z}[\frac{1}{N}, \mu_N] = \mathbf{Z}[\frac{1}{N}, x]/(x^n - 1)$.

Lemma 5.3.5. — *Let X, Z, G be as in §5.1. Assume that $X = \mathrm{Spec}(A)$ is a local scheme with closed point x, and μ_N is split over X, where N is the order of $G = G_{\overline{x}}$. Then X possesses a strict rigidification.*

Proof. — Choose $t_1, \dots, t_n \in A$ such that $Z_i = (t_i)$ are the components of Z. By G-strictness of Z, for any $g \in G$ we have that $Z_i = (gt_i)$ locally at x, in particular, the tangent space to each Z_i at x is G-invariant. Now, we can use averaging by the G-action to make the parameters G-equivariant. Namely, G acts on dt_i by a character χ_i and replacing t_i with $\frac{1}{|G|}\sum_{g\in G}\frac{gt_i}{\chi_i(g)}$ we do not change dt_i and achieve that $gt_i = \chi_i(g)t_i$.

The action of G on the cotangent space at x is diagonalizable because G is abelian and μ_N is split over $k(x)$. In particular, we can complete the family $dt_1, \dots, dt_n$ to a basis $dt_1, \dots, dt_l$ such that $t_i \in \mathcal{O}_{X,x}$ and G acts on each dt_i by a character χ_i. Then $t_1, \dots, t_l$ is a regular family of parameters of $\mathcal{O}_{X,x}$ and using the same averaging procedure as above we can make them G-equivariant. Take now $\overline{Z}$ to be the union of all divisors (t_i) with $1 \le i \le l$. The action of G on $(X, \overline{Z})$ is very tame at x because it is the only point of its log stratum and so Lemma 5.3.2 applies. □

5.3.6. Inert morphisms. — Let (X, Z) and G be as in §5.1. Our next aim is to find an étale cover $f: (Y, T) \to (X, Z)$ which "preserves" the log-inertia structure of (X, Z) and such that (Y, T) admits a rigidification. The condition on the log structure is obvious: we want f to be strict, i.e. $f^{-1}(Z) = T$. Let us introduce a restriction related to the inertia groups.

Assume that (Y, T) with an action of H is another such triple, and let $\lambda : H \to G$ be a homomorphism. A λ-equivariant morphism $f\colon (Y, T) \to (X, Z)$ will be called **inert** if for any point $y \in Y$ with $x \in X$ the induced homomorphism of inertia groups $H_{\overline{y}} \to G_{\overline{x}}$ is an isomorphism. In particular, the inertia stratification of Y is the preimage of the inertia stratification of X.

Remark 5.3.7. — When λ is an identity, inert morphisms are usually called "fixed point reflecting". We prefer to change the terminology since "inert" is brief and adequate.

Lemma 5.3.8. — *Let X, Z and G be as above, and assume that X is qe. Then there exists a G-equivariant surjective étale inert strict morphism $h\colon (Y, T) \to (X, Z)$ such that Y is affine and (Y, T) possesses a strict rigidification.*

Proof. — First, we note that the problem is local on X. Namely, it suffices for any point $x \in X$ to find a $G_{\overline{x}}$-equivariant étale inert strict morphism $h\colon (Y,T) \to (X,Z)$ such that Y is affine, $x \in h(Y)$, any point $x' \in h(Y)$ satisfies $G_{\overline{x}'} \subset G_{\overline{x}}$, and (Y,T) admits a strict rigidification. Indeed, h can be extended to a G-equivariant morphism $Y \times_X (X \times G/G_{\overline{x}}) = \coprod_{g\in G/G_{\overline{x}}} Y_g \to X$, where each Y_g is isomorphic to Y and the morphism $Y_g \to X$ is obtained by composing $Y \to X$ with $g\colon X \to X$. Clearly, the latter morphism is étale, inert, and strict, and by quasi-compactness of X we can combine finitely many such morphism to obtain a required cover of (X,Z).

Now, fix $x \in X$ and consider the $G_{\overline{x}}$-invariant neighborhood $X' = X \setminus \bigcup_{H \not\subseteq G_{\overline{x}}} X^H$ of x. We will work over X', so the condition $G_{\overline{x}'} \subset G_{\overline{x}}$ will be automatic. Let N be the order of $G_{\overline{x}}$, and consider the $G_{\overline{x}}$-equivariant morphism $f\colon Y = X' \times \mathrm{Spec}(\mathbf{Z}[\frac{1}{N}, \mu_N]) \to X$ (with $G_{\overline{x}}$ acting trivially on the second factor). Let $T = f^{-1}(Z)$ and let y be any lift of x. It suffices to show that (Y,T) admits a rigidification in an affine $G_{\overline{x}}$-invariant neighborhood of y (such neighborhoods form a fundamental family of neighborhoods of y). By Lemma 5.3.5, the localization $Y_y = \mathrm{Spec}(\mathscr{O}_{Y,y})$ with the restriction T_y of T possesses a strict rigidification $\overline{T}_y$. Clearly, $\overline{T}_y$ extends to a divisor $\overline{T}$ with $T \hookrightarrow \overline{T} \hookrightarrow Y$ and we claim that it is a rigidification in a neighborhood of y. Indeed, $\overline{T}$ is snc at y, hence it is snc in a neighborhood of y by Remark 4.2.2, and it remains to use Lemma 5.3.2. □

5.3.9. Main construction. — Assume, now, that $X = (X,Z)$ is admissible and admits a rigidification $\overline{Z}$. We are going to construct a G-equivariant modification

$$f_{(G,X,Z,\overline{Z})}\colon (X',Z') \to (X,Z)$$

such that G acts very tamely on the target and $f_{(G,X,Z,\overline{Z})}$ is independent of the rigidification. The latter is a subtle property (missing in the obvious modification $(X,\overline{Z}) \to (X,Z)$), and it will take us a couple of pages to establish it.

The quotient log scheme $\overline{Y} = (Y,\overline{T}) = (X/G, \overline{Z}/G)$ is log regular by Theorem exp. **VI**, 3.2, hence by Theorem 3.3.16 there exists a functorial saturated log blow up tower $\overline{h} = \widetilde{\mathscr{F}}^{\log}(\overline{Y})\colon \overline{Y}' = (Y',\overline{T}') \to \overline{Y}$ with a regular and log regular source. Let $\overline{f}\colon \overline{X}' = (X',\overline{Z}') \to \overline{X}$ be the pullback of $\overline{h}$ (as a saturated log blow up tower, see 3.3.11), then $\alpha'\colon \overline{X}' \to \overline{Y}'$ is a Kummer étale G-cover because $\alpha\colon \overline{X} \to \overline{Y}$ is so by exp. **VI**, 3.2 as the square

$$\begin{array}{ccc} (X',\overline{Z}') & \xrightarrow{\overline{f}} & (X,\overline{Z}) \\ \downarrow{\scriptstyle \alpha'} & & \downarrow{\scriptstyle \alpha} \\ (Y',\overline{T}') & \xrightarrow{\overline{h}} & (Y,\overline{T}) \end{array}$$

is cartesian in the category of fs log schemes.

Since G acts freely on $U = X - Z$, $V = U/G$ is regular and $\overline{T}|_V$ is snc. In particular, $\overline{h}$ is an isomorphism over V and hence $\overline{f}$ is an isomorphism over U. We claim that the Weil divisor $T = Z/G$ of Y is $\mathbf{Q}$-Cartier (it does not have to be Cartier, as the orbifold case with $X = \mathbf{A}^2$, $Z = \mathbf{A}^1$ and $G = \{\pm 1\}$ shows). Indeed, it suffices to check this étale locally at a point $y \in Y$. In particular, we can assume that μ_N is split over $\mathscr{O}_{Y,y}$, where $N = |G|$. Then, as we showed in the proof of Lemma 5.3.5, Z can be locally defined by equivariant parameters, in particular, $Z = V(f)$ where G acts on f by characters. Therefore, f^N is G-fixed, and we obtain that T is the reduction of the Cartier divisor C of Y given by $f^N = 0$. (The same argument applied to $T \times_Y X$ shows that NT is Cartier, so T is $\mathbf{Q}$-Cartier.) So, $C' = C \times_Y Y'$ is a Cartier divisor whose reduction is $T' = \overline{h}^{-1}(T)$. Since Y' is regular and T' lies in the snc divisor $\overline{T}'$, we obtain that T' is itself an snc divisor.

Let Z' denote the divisor $\alpha'^{-1}(T') = \overline{f}^{-1}(Z)$. Since $X' \to Y'$ is étale over $V = Y' - T'$, the morphism of log schemes $(X',Z') \to (Y',T')$ is a Kummer étale G-cover. This follows from a variant of the classical Abhyankar's lemma (exp. **IX**, 2.1), which is independent of the results of the present exposé.

In particular, $X' = (X',Z')$ is log regular and it follows that the action of G on X' is very tame. We define $f_{(G,X,Z,\overline{Z})}$ to be the modification $X' \to X$.

Remark 5.3.10. — (i) Note that $X' \to X$ satisfies all conditions of Theorem 1.1 because the action is very tame and G acts freely on $X' \setminus f^{-1}(Z)$. So, we completed the proof in the case when (X,Z) admits a rigidification $\overline{Z}$. Our last task will be to get rid of the rigidification.

(ii) The only dependence of our construction on the rigidification is when we construct the resolution of $(Y,\overline{T})$. Conjecturally, it depends only on the scheme Y, and then (Y',T'), and hence also (X',Z'), would depend only on (X,Z). Recall that we established in Theorem 3.4.15 the particular case of this conjecture when all maximal points of the log strata of $(Y,\overline{T})$ are of characteristic zero. Hence independence of the rigidification is unconditional in this case, and, fortunately, this will suffice.

5.3.11. Finer structure of $f_{(G,X,Z,\overline{Z})}$. — Obviously, the saturated log blow up tower $\overline{f}\colon (X',\overline{Z}') \to (X,\overline{Z})$ depends on the rigidification, and this is the reason why we prefer to consider the modification $f_{(G,X,Z,\overline{Z})}\colon (X',Z') \to (X,Z)$ instead. However, there is an additional structure on $f_{(G,X,Z,\overline{Z})}$ that has a chance to be independent of $\overline{Z}$, and which should be taken into account. By §3.4.8, the modification of schemes $f\colon X' \to X$ has a natural structure of a normalized blow up tower $X_\bullet$ with $X = X_0$ and $X' = X_n$. Note also that the tower contains no empty blow ups because this is true for $\widetilde{\mathscr{F}}^{\log}(X/G,\overline{Z}/G)$ and $f_{(G,X,Z,\overline{Z})}$ is its strict transform with respect to the surjective morphism $X \to X/G$.

Note also that the log structure on (X',Z') is reconstructed uniquely from f because $Z' = f^{-1}(Z)$ and (X',Z') is saturated and log regular. Therefore, it is safe from now on to view $f_{(G,X,Z,\overline{Z})}$ as a normalized blow up tower of X, but the modification of log schemes $(X',Z') \to (X,Z)$ will also be denoted as $f_{(G,X,Z,\overline{Z})}$.

Remark 5.3.12. — Although we do not assume that X is qe, all normalizations in the tower $f_{(G,X,Z,\overline{Z})}$ are finite. This happens because they underly saturations of fine log schemes, which are always finite morphisms.

5.4. Functoriality. — Clearly, the construction of f depends canonically on $(G,X,Z,\overline{Z})$, i.e. is compatible with any automorphism of such quadruple. Our next aim is to establish functoriality with respect to strict inert λ-equivariant morphisms $\phi\colon (H,Y,T,\overline{T}) \to (G,X,Z,\overline{Z})$ (i.e. morphisms that "preserve the combinatorial structure"). For this one has first to study the quotient morphism of log schemes $\widetilde{\phi}\colon (Y/H,\overline{T}/H) \to (X/G,\overline{Z}/G)$.

5.4.1. Log structure of the quotients. — Recall the following facts from Proposition exp. **VI**, 3.4(b) and its proof. Assume that $X = (X, M_X)$ is an fs log scheme provided with a very tame action of a group G. After replacing X with its strict localization at a geometric point $\overline{x}$, it admits an equivariant chart $X \to \mathrm{Spec}(\Lambda[Q])$, where $\Lambda = \mathbf{Z}[1/N, \mu_N]$ for the order N of $G_{\overline{x}}$, Q is an fs monoid and the action of $\overline{G}_x$ is via a pairing $\chi\colon G_{\overline{x}} \otimes Q \to \mu_N$. Moreover, if $P \subset Q$ is the maximal submonoid with $\chi(G_{\overline{x}} \otimes P) = 1$ then $\mathrm{Spec}(\Lambda[Q]) \to \mathrm{Spec}(\Lambda[P])$ is a chart of $X \to X/G_{\overline{x}}$. Now, let us apply this description to the study of $\widetilde{\phi}$.

Proposition 5.4.2. — *Assume that fs log schemes X, Y are provided with admissible very tame actions of groups G and H, respectively, $\lambda\colon H \to G$ is a homomorphism, and $\phi\colon Y \to X$ is a strict inert λ-equivariant morphism. Then the quotient morphism $\widetilde{\phi}\colon Y/H \to X/G$ is strict.*

Proof. — Fix a geometric point $\overline{y}$ of Y and let $\overline{x}$ be its image in X. It suffices to show that $\widetilde{\phi}$ is strict at the image of $\overline{y}$ in Y/H. The morphism $Y/H_{\overline{y}} \to Y/H$ is strict (and étale) over the image of $\overline{y}$, and the same is true for X. Therefore we can replace H and G with $H_{\overline{y}} \xrightarrow{\sim} G_{\overline{x}}$, and then we can also replace X and Y with their strict localizations at $\overline{x}$ and $\overline{y}$. Now, the morphism $X \to \widetilde{X} = X/G_{\overline{x}}$ admits an equivariant chart $h\colon \mathrm{Spec}(\Lambda[Q]) \to \mathrm{Spec}(\Lambda[P])$ as explained before the proposition. Since ϕ is strict, the induced morphism $Y \to \mathrm{Spec}(\Lambda[Q])$ is also a chart and hence h is also a chart of $Y \to Y/G_{\overline{y}}$. Thus, $\widetilde{\phi}$ is strict. □

5.4.3. An application to functoriality of $f_\bullet$. — Assume that $(G,X,Z,\overline{Z})$ is as earlier, and let $(H,Y,T,\overline{T})$ be another such quadruple (i.e. (Y,T) with the action of H satisfies conditions of Steps 1, 2, 3 and $(Y,\overline{T})$ is its rigidification).

Corollary 5.4.4. — *Assume that $\lambda\colon H \to G$ is a homomorphism and $\phi\colon Y \to X$ is a λ-equivariant inert morphism underlying strict morphisms of log schemes $\psi\colon (Y,T) \to (X,Z)$ and $\overline{\psi}\colon (Y,\overline{T}) \to (X,\overline{Z})$. Then $f_\bullet$ is compatible with ϕ in the sense that $f_{(H,Y,T,\overline{T})}$ is the contraction of $\phi^{\mathrm{st}}(f_{(G,X,Z,\overline{Z})})$. In addition, $\phi^{\mathrm{st}}(f_{(G,X,Z,\overline{Z})}) = f_{(G,X,Z,\overline{Z})} \times_X Y$.*

Proof. — Since $\overline{\psi}$ is strict its quotient is strict by Proposition 5.4.2, and by functoriality of saturated monoidal desingularization we obtain that $\widetilde{\mathscr{F}}^{\log}(Y/H,\overline{T}/H)$ is the contracted pullback of $\widetilde{\mathscr{F}}^{\log}(X/G,\overline{Z}/G)$. So, both $f_{(H,Y,T,\overline{T})}$ and $f_{(G,X,Z,\overline{Z})}$ are obtained as the contraction of the strict transform of $\widetilde{\mathscr{F}}^{\log}(X/G,\overline{Z}/G)$. The first claim of the Corollary follows.

Furthermore, $f_{(G,X,Z,\overline{Z})}$ underlies a log blow up tower of $(X,\overline{Z})$ which is the strict transform of $\widetilde{\mathscr{F}}^{\log}(X/G,\overline{Z}/G)$, and the same is true for $f_{(H,Y,T,\overline{T})}$. Since $\overline{\psi}$ is strict it follows from Lemma 3.4.6(ii) that the strict transform is a pullback, i.e. $\phi^{\mathrm{st}}(f_{(G,X,Z,\overline{Z})}) = f_{(G,X,Z,\overline{Z})} \times_X Y$. □

5.4.5. Localizations and completions. — In particular, it follows that the construction of $f_\bullet$ is compatible with localizations and completions. Namely, if $x \in X$ is a point, $X_x = \mathrm{Spec}(\mathscr{O}_{X,x})$, $Z_x = Z \times_X X_x$ and $\overline{Z}_z = \overline{Z} \times_X X_x$, then $f_{(G_{\overline{x}},X_x,Z_z,\overline{Z}_x)}$ is the contraction of $f_{(G,X,Z,\overline{Z})} \times_X X_x$. Similarly, if $\widehat{X}_x = \mathrm{Spec}(\widehat{\mathscr{O}}_{X,x})$, $\widehat{Z}_x = Z \times_X \widehat{X}_x$ and $\widehat{\overline{Z}}_x = \overline{Z} \times_X \widehat{X}_x$, then $f_{(G_{\overline{x}},\widehat{X}_x,\widehat{Z}_x,\widehat{\overline{Z}}_x)}$ is the contraction of $f_{(G,X,Z,\overline{Z})} \times_X \widehat{X}_x$.

5.5. Globalization. — To complete the proof of Theorem 1.1 it suffices to show that $f_{(G,X,Z,\overline{Z})}$ is independent of $\overline{Z}$, and hence the local constructions glue to a global normalized blow up tower. The main idea is to simultaneously lift two rigidifications to characteristic zero and apply Theorem 3.4.15.

5.5.1. Independence of rigidification. — We start with the case of complete local rings. Then the problem is solved by lifting to characteristic zero and referencing to 3.4.15. The general case will follow rather easily.

Lemma 5.5.2. — *Keep assumptions on (X,Z) and G as in §5.1 and assume, in addition, that $X = \coprod_{i=1}^m \mathrm{Spec}(A_i)$ where each A_i is a complete noetherian regular local ring with a separably closed residue field. Then for any pair of rigidifications $\overline{Z}$ and $\overline{Z}'$ the equality $f_{(G,X,Z,\overline{Z})} = f_{(G,X,Z,\overline{Z}')}$ holds.*

Proof. — Almost the whole argument runs independently on each irreducible component, so assume first that $X = \mathrm{Spec}(A)$ is irreducible. Set $k = A/m_A$. By Remark 5.3.10(ii), it suffices to consider the case when $\mathrm{char}(k) = p > 0$, so let $C(k)$ be a Cohen ring of k. Note that we can work with $H = G_{\overline{x}}$ instead of G because $f_{(H,X,Z,\overline{Z})} = f_{(G,X,Z,\overline{Z})}$ by Corollary 5.4.4. Since H acts trivially on k, for any element $t \in A$ its H-averaging is an element of A^H with the same image in the residue field. Hence k is the residue field of A^H and the usual theory of Cohen rings provides a homomorphism $C(k) \to A^H$ that lifts $C(k) \to A^H/m_{A^H}$. Note that $\overline{Z}$ and $\overline{Z}'$ are snc because each A_i is strictly henselian. Using averaging on the action of H again, we can find regular families of H-equivariant parameters $\underline{z} = (z_1, \dots, z_d)$ and $\underline{z}' = (z'_1, \dots, z'_d)$ such that $Z = V(\prod_{i=1}^l z_i)$, $z'_i = z_i$ for $1 \le i \le l$, $\overline{Z} = V(\prod_{i=1}^n z_i)$ and $\overline{Z}' = V(\prod_{i=1}^{n'} z'_i)$. Explicitly, the action on z_i (resp. z'_i) is by a character $\chi_i \colon H \to \mu_N$ (resp. $\chi'_i \colon H \to \mu_N$).

Since the image of $\underline{z}$ is a basis of the cotangent space at x, we obtain a surjective homomorphism $f\colon B = C(k)[[t_1, \dots, t_d]] \to A$ taking t_i to z_i. Provide B with the action of H which is trivial on $C(k)$ and acts on t_i via χ_i, in particular, f is H-equivariant. Let us also lift each z'_i to an H-equivariant parameter $t'_i \in B$. For $i \le l$ we take $t'_i = t_i$, and for $i > l$ we first choose any lift and then replace it with its χ_i-weighted H-averaging. Consider the regular scheme $Y = \mathrm{Spec}(B)$ with H-equivariant snc divisors $T = V(\prod_{i=1}^l t_i)$, $\overline{T} = V(\prod_{i=1}^n t_i)$ and $\overline{T}' = V(\prod_{i=1}^{n'} t'_i)$.

Since H acts vary tamely on $(X, \overline{Z})$, it acts trivially on $V(z_1, \dots, z_n) = \mathrm{Spec}(k[[z_{n+1}, \dots, z_d]])$ and we obtain that $\chi_i = 1$ for $i > n$. Therefore, H also acts trivially on $\mathrm{Spec}(B/(t_1, \dots, t_n)) = \mathrm{Spec}(C(k)[[t_{n+1}, \dots, t_d]])$ and we obtain that the action on $(Y, \overline{T})$ is very tame. Since the closed immersion $j : X \to Y$ is H-equivariant and strict, and $\overline{Z} = \overline{T} \times_Y X$, Corollary 5.4.4 implies that $f_\bullet$ is compatible with j, i.e., $f_{(H,X,Z,\overline{Z})}$ is the contracted strict transform of $f_{(H,Y,T,\overline{T})}$. The same argument applies to the rigidifications $\overline{Z}'$ and $\overline{T}'$, so it now suffices to show that $f_{(H,Y,T,\overline{T})} = f_{(H,Y,T,\overline{T}')}$. For this we observe that maximal points of log strata of the log schemes $(Y/H, \overline{T}/H)$ and $(Y/H, \overline{T}'/H)$ are of characteristic zero, hence the latter equality holds by Theorem 3.4.15.

Finally, let us explain how one deals with the case of $m > 1$. First one finds an H-equivariant strict closed immersion $i : X \to Y$ such that $\overline{Z}$ and $\overline{Z}'$ extend to rigidifications $\overline{T}$ and $\overline{T}'$ of (Y,T), and the maximal points of the log strata of $(Y/H, \overline{T}/H)$ and $(Y/H, \overline{T}'/H)$ are of characteristic zero. For this we apply independently the above construction to the connected components of X. Once i is constructed, the same reference to 3.4.15 shows that $f_{(H,Y,T,\overline{T})} = f_{(H,Y,T,\overline{T}')}$ and hence $f_{(H,X,Z,\overline{Z})} = f_{(H,X,Z,\overline{Z}')}$ □

Corollary 5.5.3. — *Let (X,Z) and G be as in §5.1 and assume that the action is admissible. Then for any choice of rigidifications $\overline{Z}$ and $\overline{Z}'$ we have that $f_{(G,X,Z,\overline{Z})} = f_{(G,X,Z,\overline{Z}')}$.*

Proof. — For a point $x \in X$ let $\widehat{\mathcal{O}^{sh}_{X,x}}$ denote the completion of the strict henselization of $\mathcal{O}_{X,x}$. It suffices to check that for any point x the normalized blow up towers $f_{(G,X,Z,\overline{Z})}$ and $f_{(G,X,Z,\overline{Z}')}$ pull back to the same normalized blow up towers of $\widehat{X}^{sh}_x = \mathrm{Spec}(\widehat{\mathcal{O}^{sh}_{X,x}})$ (with respect to the morphism $\widehat{X}^{sh}_x \to X$). Indeed, any normalized blow up tower $\mathcal{X} = (X_\bullet, V_\bullet)$ is uniquely determined by its centers V_i. For each i the morphism $Y_i = \coprod_{x \in X} X_i \times_X \widehat{X}^{sh}_x \to X_i$ is faithfully flat, hence V_i is uniquely determined by $V_i \times_{X_i} Y_i$, which is the center of $\mathcal{X} \times_X \coprod_{x\in X} \widehat{X}^{sh}_x$.

By 5.4.5, $f_{(G_{\overline{x}}, \widehat{X}^{sh}_x, Z\times_X \widehat{X}^{sh}_x, \overline{Z}\times_X \widehat{X}^{sh}_x)}$ is the contracted pullback of $f_{(G,X,Z,\overline{Z})}$, and an analogous result is true for $f_{(G,X,Z,\overline{Z}')}$. Thus the contracted pullbacks are equal by Lemma 5.5.2. We have, however, to worry also for the synchronization, i.e. to establish equality of non-contracted pullbacks. For this we will use the following trick. Consider the finite set $S = \mathrm{Ass}(f_{(G,X,Z,\overline{Z})}) \cup \mathrm{Ass}(f_{(G,X,Z,\overline{Z}')})$ (see 2.2.11). Set $\widehat{X}^{sh}_S = \coprod_{s\in S} \widehat{X}^{sh}_s$, then the pullbacks of $f_{(G,X,Z,\overline{Z})}$ and $f_{(G,X,Z,\overline{Z}')}$ to $\widehat{X}^{sh}_S$ are already contracted. Now, in order to compare the pullbacks to $\widehat{X}^{sh}_x$, consider the pullbacks to $\widehat{X}^{sh}_x \coprod \widehat{X}^{sh}_S$. They are contracted, so Lemma 5.5.2 (which covers disjoint unions) implies that these pullbacks are equal. Restricting them onto $\widehat{X}^{sh}_x$ we obtain equality of non-contracted pullbacks to $\widehat{X}^{sh}_x$. □

Remark 5.5.4. — (i) The above corollary implies that the modification $f_{(G,X,Z,\overline{Z})}$ depends only on (G, X, Z), so it will be denoted $f_{(G,X,Z)}$ in the sequel. At this stage, $f_{(G,X,Z)}$ is defined only when X is admissible and (X, Z) admits a rigidification.

(ii) Corollaries 5.4.4 and 5.5.3 imply that $f_{(G,X,Z)}$ is functorial with respect to equivariant strict inert morphisms.

5.5.5. Theorem 1.1 — end of proof. — Let $X = (X, Z)$ be as assumed in §5.1, and suppose that X is qe. By Lemma 5.3.8 there exists a surjective étale inert strict morphism $h\colon X_0 \to X$ such that X_0 is affine and possesses a rigidification. Then $X_1 = X_0 \times_X X_0$ is affine and also admits a rigidification (e.g. the preimage of that of X_0 by one of the canonical projections). By Remark 5.5.4(i), X_0 and X_1 possess normalized blow up towers $f_{(G,X_0,Z_0)}$ and $f_{(G,X_1,Z_1)}$, which are compatible with both projections $X_1 \to X_0$ by Remark 5.5.4(ii). It follows that $f_{(G,X_0,Z_0)}$ is induced from a unique normalized blow up tower of X that we denote as $f_{(G,X,Z)}$. This modification satisfies all assertions of Theorem 1.1 because $f_{(G,X_0,Z_0)}$ does so by Remark 5.3.10(i).

5.6. Additional properties of $f_{(G,X,Z)}$. — Finally, let us formulate an addendum to Theorem 1.1 where we summarize additional properties of the constructed modification of (X, Z). At this stage we drop any assumptions on (X, Z) beyond the assumptions of 1.1. By $f_{(G,X,Z)}$ we denote below the entire modification from Theorem 1.1 that also involves the modifications of Steps 1, 2, 3.

Theorem 5.6.1. — *Keep assumptions of Theorem 1.1. In addition to assertions of the theorem, the modifications $f_{(G,X,Z)}$ can be constructed uniformly for all triples (G, X, Z) such that the following properties are satisfied:*

(i) Each $f_{(G,X,Z)}$ is provided with a structure of a normalized blow up tower and its centers are contained in the preimages of $Z \cup T$.

(ii) For any homomorphism $\lambda : H \to G$ the construction is functorial with respect to λ-equivariant inert strict regular morphisms $(Y, T) \to (X, Z)$.

Proof. — The total modification $f_{(G,X,Z)}$ is obtained by composing four modifications f_1, f_2, f_3 and f_4: the modifications from Steps 1, 2, 3 and the modification we have constructed in §5. Recall that f_1 and f_4 are constructed as normalized blow up towers. Modifications f_2 and f_3 are permissible blow up towers, hence they are also normalized blow up towers with the same centers. This establishes the first part of (i).

Concerning claim (ii), recall that normalized blow ups are compatible with regular morphisms by Lemma 2.2.9, hence we should check that the centers of $f_{(H,Y,T)}$ are the pullbacks of the centers of $f_{(G,X,Z)}$. For f_1, f_2 and f_3 this is clear, so it remains to prove that if (G, X, Z) is as in §5.1 and $(Y, T) \to (X, Z)$ is λ-equivariant, inert, strict and regular then $f_{(H,Y,T)}$ is the pullback of $f_{(G,X,Z)}$. We can work étale locally on X (replacing Y with the base change). Thus, using Lemma 5.3.8 we can assume that X possesses a rigidification $\overline{Z}$. Since $Y \to X$ is regular and inert, the preimage $\overline{T}$ of $\overline{Z}$ is a rigidification of Y. Thus, $f_{(H,Y,T)} = f_{(H,Y,T,\overline{T})}$ and $f_{(G,X,Z)} = f_{(G,X,Z,\overline{Z})}$ and it remains to use that $f_{(H,Y,T,\overline{T})}$ is the pullback of $f_{(G,X,Z,\overline{Z})}$ by Lemma 5.4.4.

To prove the second part of (i) we use (ii) to restrict $f_{(G,X,Z)}$ onto $U = X \setminus Z \cup T$. Then U is a regular scheme with a trivial log structure which is acted freely by G. It follows from the definitions of f_1, f_2, f_3 and f_4 that they are trivial for such U. So, $f_{(G,U,\varnothing)}$ is the trivial tower, and hence all centers of $f_{(G,X,Z)}$ are disjoint from the preimage of U. □

Remark 5.6.2. — One may wonder if the functoriality in Theorem 5.6.1(ii) holds for λ-equivariant strict inert morphisms which are not necessarily regular. In this case, the normalized blow up tower of X does not have to pullback to Y, but one can hope that $f_{(H,Y,T)}$ is the strict transform of $f_{(G,X,Z)}$. We indicate without proofs what can be done in this direction.

A careful examination of the argument shows that f_1 and f_2 are functorial with respect to all strict morphisms, while f_3 is functorial with respect to all schematically inert morphisms as defined below. A λ-equivariant morphism of separated schemes $Y_1 \to X_1$ is called **schematically inert** if for any subgroup $G' \subset G$ the pullback $X_1^{G'} \times_{X_1} Y_1$ of $X_1^{G'}$ is a disjoint union of $Y_1^{H'_i}$ with $\lambda(H'_i) = G'$. For example, if μ_2 acts on the coordinate x of $X_1 = \operatorname{Spec} \mathbf{Q}[x]$ by the non-trivial character then the μ_2-equivariant morphism $X_1 \to X_1$ sending x to x^3 is inert but not schematically inert.

It remains to examine the construction of f_4. Note that we only used the regularity of $h\colon Y \to X$ to construct compatible rigidifications of X and Y, since Lemma 5.4.4 holds for arbitrary strict inert morphisms. In fact, one can formulate a necessary and sufficient criterion guaranteeing that étale locally on X there exist compatible rigidifications $\overline{Z}$ and $\overline{T}$. Note that if $\overline{Z}$ and $\overline{T}$ exist then the inertia strata of X and Y are (set-theoretic) disjoint unions of connected components of log-strata of $\overline{Z}$ and $\overline{T}$. Since $\overline{T}^i = \overline{Z}^i \times_X Y$, this implies that h is schematically inert.

In addition, $\overline{Z}^i$ and $\overline{T}^i$ are of the same codimension hence the following condition holds: $(*)$ for any connected component V of $X^{G'}$ with non-empty preimage in Y, the codimension of V in X equals to the codimension of $V \times_X Y$ in Y. This motivates a further strengthening of the notion of inertness: a schematically inert morphism $h_1 \colon Y_1 \to X_1$ is called **derived inert** if the morphisms h_1 and $X_1^{G'} \to X_1$ are Tor-independent for any subgroup $G' \subset G$. One can show that in the case of regular X_1 and Y_1 this happens if and only if $(*)$ is satisfied. For example, if μ_n acts on the coordinates of $X = \operatorname{Spec} \mathbf{Q}[x, y]$ by characters then the μ_n-equivariant closed immersion $\operatorname{Spec} \mathbf{Q}[x] \hookrightarrow X$ is always schematically inert, and it is derived inert if and only if the action on y is through the trivial character.

As we saw, existence of compatible rigidifications guarantees that $h \colon Y \to X$ is derived inert, and strengthening Lemma 5.3.8 one can show that, conversely, if h is derived inert then compatible rigidifications exist étale locally. In particular, functoriality holds for arbitrary morphisms which are equivariant, strict and derived inert.

References

[Bierstone & Milman, 1997] Bierstone, E. & Milman, P. (1997). Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. math.*, 128(2), 207–302. ↑ 7

[Bierstone et al., 2011] Bierstone, E., Milman, P., & Temkin, M. (2011). **Q**-universal desingularization. *Asian Journal of Mathematics*, 15, 229–250. ↑ 7, 8

[Bravo et al., 2005] Bravo, A. M., Encinas, S., & Villamayor U., O. (2005). A simplified proof of desingularization and applications. *Rev. Mat. Iberoamericana*, 21(2), 349–458. ↑ 7

[Conrad, 2007] Conrad, B. (2007). Deligne's notes on Nagata compactification. *J. Ramanujan Math. Soc.*, 22, 205–257. ↑ 2

[de Jong, 1996] de Jong, A. J. (1996). Smoothness, semistability and alterations. *Publications mathématiques de l'IHÉS*, 83, 51–93. ↑ 25

[Deitmar, 2005] Deitmar, A. (2005). Schemes over $\mathbf{F}_1$. In *Number fields and function fields—two parallel worlds*, volume 239 des *Progress in Mathematics* (pp. 87–100). Birkhäuser. ↑ 10, 13

[Fulton, 1993] Fulton, W. (1993). *Introduction to toric varieties*, volume 131 des *Annals of Mathematics Studies*. Princeton University Press. ↑ 15

[Illusie, 1972] Illusie, L. (1971-1972). *Complexe cotangent et déformations,* volume 239 & 283. Springer-Verlag. Lecture Notes in Mathematics. ↑ 9, 10

[Kato, 1994] Kato, K. (1994). Toric singularities. *Amer. J. Math.*, 116(5), 1073–1099. ↑ 10, 11, 13, 14, 18, 19, 20

[Kollár, 2007] Kollár, J. (2007). *Lectures on resolution of singularities*, volume 166 des *Annals of Mathematics Studies*. Princeton University Press. ↑ 7

[Matsumura, 1980] Matsumura, H. (1980). *Commutative algebra*, volume 56 des *Mathematics Lecture Note Series*. The Benjamin/Cummings Publishing Company, seconde édition. ↑ 5, 15, 21

[Niziol, 2006] Niziol, W. (2006). Toric singularities: log-blow-ups and global resolutions. *J. Algebraic Geom.*, 15(1), 1–29. ↑ 14, 17, 18, 19, 20, 21

[Ogus, 2013] Ogus, A. (2013). Lectures on Logarithmic Algebraic Geometry. Projet de livre, non publié. ↑ 10, 12, 13

[Raynaud, 1967] Raynaud, M. (1967). Un critère d'effectivité de descente. In *Séminaire Samuel. Algèbre commutative*, volume 2, exp. n°5 (pp. 1–22). ↑ 3, 4

[Swan, 1998] Swan, R. G. (1998). Néron-Popescu desingularization. In *Algebra and geometry (Taipei, 1995)*, volume 2 des *Lectures in Algebra and Geometry* (pp. 135–192). International Press of Boston. ↑ 10

[Temkin, 2012] Temkin, M. (2012). Functorial desingularization of quasi-excellent schemes in characteristic zero: the nonembedded case. *Duke Math. J.*, 161(11), 2207–2254. ↑ 3, 6, 7, 8

[Włodarczyk, 2005] Włodarczyk, J. (2005). Simple Hironaka resolution in characteristic zero. *J. Amer. Math. Soc.*, 18(4), 779–822. ↑ 7