Poisson algebra - An algebra, usually over the field of real or complex numbers, equipped with a bilinear mapping satisfying the properties of the usual Poisson bracket of functions. Let $A$ be an associative commutative algebra over a commutative ring, $R$. A Poisson algebra structure on $A$ is defined by an $R$-bilinear, skew-symmetric mapping, $\{\ , \ \}$ : $A \times A \rightarrow A$ such that (i) $(A, \{\ , \ \})$ is a Lie algebra over $R$, (ii) the Leibniz rule is satisfied, namely, $\{a, bc\} = \{a, b\}c + b\{a, c\}$, for all $a, b, c \in A$. The element $\{a, b\}$ is called the Poisson bracket of $a$ and $b$. The main example is that of the algebra of smooth functions on a Poisson manifold.

On a Poisson algebra, one can define [12] a skew-symmetric, $A$-bilinear map, $P$, which generalizes the Poisson bivector on Poisson manifolds, mapping a pair of Kähler (or formal) differentials on $A$ to the algebra $A$ itself. There exists a unique $R$-bilinear bracket, $[\ , \ ]_P$ on the $A$-module, $\Omega^1(A)$, of Kähler differentials satisfying $[da, db]_P = d\{a, b\}$ and lending it the structure of a Lie-Rinehart algebra, $[da, fdb]_P = f[da, db]_P + P^\#(da)(f)db$, for all $a, b, f$ in $A$. (Here $P^\#$ is the adjoint of $P$, mapping the Kähler differentials into the derivations of $A$.) The Poisson cohomology of $A$ is then defined and, when $\Omega^1(A)$ is projective as an $A$-module, is equal to the cohomology of the complex of alternating $A$-linear maps on $\Omega^1(A)$ with values in $A$, with the differential [1] defined by the Lie-Rinehart algebra structure. In the case of the algebra of functions on a smooth manifold, the Poisson cohomology coincides with the cohomology of the complex of multivectors, with differential $dp = [P, \cdot]$, where $P$ is the Poisson bivector and $[\ , \ ]$ is the Schouten bracket.

In a canonical ring [4], the Poisson bracket is defined by a given mapping $P^\#$, Dirac structures [13] on complexes over Lie algebras are a generalization of the Poisson algebras, adapted to the theory of infinite-dimensional Hamiltonian systems, where the ring of functions is replaced by the vector space of functionals.

In the category of $\mathbb{Z}$-graded algebras, there are even and odd Poisson algebras, called graded Poisson algebras and Gerstenhaber algebras, respectively. Let $A = \oplus A^i$ be an associative, graded commutative algebra. A graded Poisson (resp., Gerstenhaber) algebra structure on $A$ is a graded Lie algebra structure $\{\ , \ \}$ (resp., where the grading is shifted by 1), such that a graded version of the Leibniz rule holds: for each $a \in A^i$, $\{a, \cdot\}$ is a derivation of degree $i$ (resp., $i+1$) of the graded commutative algebra $A = \oplus A^i$. Examples of Gerstenhaber algebras are: the Hochschild cohomology of an associative algebra [2], in particular, the Schouten algebra of multivectors on a smooth manifold [3], the exterior algebra of a Lie algebra, the algebra of differential forms on a Poisson manifold [9], the space of sections of the exterior algebra of a Lie algebroid, the algebra of functions on an odd Poisson supermanifold of type $(n|n)$ [7]. Batalin-Vilkovisky or BV-algebras are exact Gerstenhaber algebras, i.e., their Lie bracket is a coboundary in the graded Hochschild cohomology of the algebra. Such structures arise on the BRST cohomology of topological field theories [14].

References


Y. Kosmann-Schwarzbach

AMS 1991 Mathematics Subject Classification: 17B60, 17B70, 16W55, 17B56, 17B81, 58A10.