Poisson algebra - An algebra, usually over the field of real or complex numbers, equipped with a bilinear mapping satisfying the properties of the usual Poisson bracket of functions. Let A be an associative commutative algebra over a commutative ring, R. A *Poisson algebra structure* on A is defined by an R-bilinear, skew-symmetric mapping, $\{\ ,\ \} : A \times A \to A$ such that (i) $(A, \{\ ,\ \})$ is a Lie algebra over R, (ii) the Leibniz rule is satisfied, namely, $\{a, bc\} = \{a, b\}c + b\{a, c\}$, for all $a, b, c \in A$. The element $\{a, b\}$ is called the *Poisson bracket* of a and b. The main example is that of the algebra of smooth functions on a *Poisson manifold* [5].

On a Poisson algebra, one can define [12] a skew-symmetric, A-bilinear map, P, which generalizes the Poisson bivector on Poisson manifolds, mapping a pair of Kähler (or formal) differentials on A to the algebra A itself. There exists a unique R-bilinear bracket, $[\ ,\ ]_P$ on the A-module, $\Omega^1(A)$, of Kähler differentials satisfying $[da, db]_P = d\{a, b\}$ and lending it the structure of a Lie-Rinehart algebra, $[da, fdb]_P = f[da, db]_P + P^\sharp(da)(f)db$, for all a, b, f in A. (Here $P^\sharp$ is the adjoint of P, mapping the Kähler differentials into the derivations of A.) The *Poisson cohomology* of A is then defined and, when $\Omega^1(A)$ is projective as an A-module, is equal to the cohomology of the complex of alternating A-linear maps on $\Omega^1(A)$ with values in A, with the differential [1] defined by the Lie-Rinehart algebra structure. In the case of the algebra of functions on a smooth manifold, the Poisson cohomology coincides with the cohomology of the complex of multivectors, with differential $d_P = [P, .]$, where P is the Poisson bivector and $[\ ,\ ]$ is the Schouten bracket.

In a *canonical ring* [4], the Poisson bracket is defined by a given mapping $P^\sharp$. *Dirac structures* [13] on complexes over Lie algebras are a generalization of the Poisson algebras, adapted to the theory of infinite-dimensional Hamiltonian systems, where the ring of functions is replaced by the vector space of functionals.

In the category of $\mathbf{Z}$-graded algebras, there are even and odd Poisson algebras, called *graded Poisson algebras* and *Gerstenhaber algebras*, respectively. Let $A = \oplus A^i$ be an associative, graded commutative algebra. A *graded Poisson* (resp., *Gerstenhaber*) *algebra structure* on A is a graded Lie algebra structure $\{\ ,\ \}$ (resp., where the grading is shifted by 1), such that a graded version of the Leibniz rule holds: for each $a \in A^i$, $\{a, .\}$ is a derivation of degree i (resp., $i+1$) of the graded commutative algebra $A = \oplus A^i$. Examples of Gerstenhaber algebras are: the *Hochschild cohomology* of an associative algebra [2], in particular, the Schouten algebra of multivectors on a smooth manifold [3], the exterior algebra of a Lie algebra, the algebra of differential forms on a Poisson manifold [9], the space of sections of the exterior algebra of a *Lie algebroid*, the algebra of functions on an *odd Poisson supermanifold* of type $(n|n)$ [7]. *Batalin-Vilkovisky* or *BV-algebras* are exact Gerstenhaber algebras, *i. e.*, their Lie bracket is a coboundary in the graded Hochschild cohomology of the algebra. Such structures arise on the BRST cohomology of topological field theories [14].

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