On the modular classes of Poisson-Nijenhuis manifolds

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Abstract We prove a property of the Poisson-Nijenhuis manifolds which yields new proofs of the bihamiltonian properties of the hierarchy of modular vector fields defined by Damianou and Fernandes.

Introduction

In [2], Damianou and Fernandes defined the modular vector field and the modular class of a Poisson-Nijenhuis manifold, and they proved that the hierarchy generated by the modular vector field coincides with the canonical hierarchy of bihamiltonian vector fields already defined in [5]. A theorem of Beltrán and Monterde [1] states that, in a PN-manifold, the derived bracket (see e.g. [3]) of the interior products by $N$ and $P$ acting on forms is the interior product by the hamiltonian vector field with hamiltonian $-\frac{1}{2} \text{Tr} N$. In this Letter, we give an elementary proof of a particular case of this theorem, a simple consequence of which, stated in Corollary 1.1, enables us to give new proofs of the hamiltonian properties of the hierarchy of modular vector fields of PN-manifolds. These can be extended to the case of arbitrary PN-algebroids in a straightforward manner.

1 Poisson-Nijenhuis structures

There are many ways of expressing the compatibility of a pair $(P, N)$, where $N$ is a Nijenhuis tensor and $P$ is a Poisson bivector on a manifold $M$ satisfying the condition that $NP$ be skew symmetric, in order to ensure that $NP, N^2P, \ldots, N^kP, \ldots$ be a sequence of pairwise-compatible Poisson brackets. Let $d_N = [i_N, d]$ be the differential on forms associated with the deformed bracket of vector fields, $[,]_N$, and let $[,]_P$ be the graded bracket of forms defined by $P$. When no confusion is possible, we denote by $N$ both the Nijenhuis tensor and its transpose, and by $P$ both the Poisson bivector and the map from 1-forms to vectors it defines, with the convention $P\alpha = i_\alpha P$. Let $H_P^f = Pf$ be the hamiltonian vector field with hamiltonian $f \in C^\infty(M)$ in the Poisson structure $P$. The derived bracket $[[i_N, d], i_P] = [d_N, i_P]$ is denoted by $[i_N, i_P]_d$. 
Proposition 1.1. The following conditions on \( N \) and \( P \) are equivalent:

- (i) \( NP = PN \) and (ii) \( C(P, N) = 0 \), where, for all \( \alpha, \beta \in \Gamma(T^*M) \),
  \[
  C(P, N)(\alpha, \beta) = [\alpha, \beta]_N - ([N\alpha, \beta]_P + [\alpha, N\beta]_P - N[\alpha, \beta]_P).
  \]

- \( d_N \) is a derivation of bracket \([,]_P\).
- \( d_P = [P, \cdot] \) is a derivation of the deformed bracket \([,]_N\).
- Let \( \{,\}_NP \) be the Poisson bracket of functions with respect to \( NP \).
  (i) \( NP = PN \) and (ii) \( d\{f, g\}_NP = L_{H_P}d_Ng - L_{H_P}d_Nf - d_N(H_P(f)) \),
  for all \( f, g \in C^\infty(M) \).

This last condition follows from \( C(P, N)(df, dg) = 0 \), for all functions \( f, g \in C^\infty(M) \), using the relation \([\alpha, df]_P = -i_{H_P}d\alpha\).

Definition 1.1. When any one of the above conditions is satisfied, \( N \) and \( P \) are called compatible. The pair \((P, N)\) is a Poisson-Nijenhuis structure (or PN-structure) if \( N \) and \( P \) are compatible. A manifold with a Poisson-Nijenhuis structure is called a Poisson-Nijenhuis manifold (or PN-manifold).

The compatibility of \( P \) and \( N \) can also be stated in terms of the morphism properties of maps \( P, N^kP, N^k \) and \((tN)^k, k \geq 1\), relating the various Lie algebroid structures on \( TM \) and \( T^*M \).

Proposition 1.2. Let \( P \) be a Poisson bivector and \( N \) a Nijenhuis tensor on \( M \) such that \( PN = NP \). Then, for all \( \alpha \in \Gamma(T^*M) \),

\[
\frac{1}{2} Tr(C(P, N)\alpha) = \frac{1}{2} < PdTr N, \alpha > + [i_N, i_P]d\alpha ,
\]

where \([,]_d\) denotes the derived bracket.

Proof. We shall use the expression of the components of \( C(P, N) \) in local coordinates \([4]\),

\[
C^{kj}_m = P^{lj}\partial_lN^k_m + P^{kl}\partial_lN^j_m - N^l_m\partial_lP^{kj} + N^l_m\partial_mP^{kl} - P^{lj}\partial_mN^k_l ,
\]

whence

\[
C^k_j = P^{lj}\partial_lN^k_j + P^{jl}\partial_lN^j_k - N^l_j\partial_lP^{kj} + N^l_j\partial_kP^{kl} - P^{lj}\partial_kN^k_j .
\]

From the assumption \( NP = PN \), i.e., \( P^{lj}N^k_l + P^{lk}N^j_l = 0 \), we obtain

\[
N^l_j\partial_mP^{lj} + P^{lj}\partial_mN^k_l + N^l_j\partial_mP^{lk} + P^{lk}\partial_mN^j_l = 0 ,
\]

whence

\[
N^l_j\partial_kP^{lj} + P^{lj}\partial_kN^k_l + N^l_j\partial_kP^{lk} + P^{lk}\partial_kN^j_l = 0 .
\]
This identity implies that
\[ \frac{1}{2} C_{k}^{kj} = \frac{1}{2} P_{lj}^{k} \partial_{l} N_{k}^{j} + P_{lk}^{j} \partial_{k} N_{l}^{j}. \]
Thus, for any 1-form \( \alpha \),
\[ \frac{1}{2} \text{Tr}(C(P, N) \alpha) = \frac{1}{2} P_{lj}^{k} \partial_{l} N_{k}^{j} \alpha_{j} + P_{lk}^{j} \partial_{k} N_{l}^{j} \alpha_{j} \]
\[ = -\frac{1}{2} < PdTrN, \alpha > + i Pd i N \alpha - i NPd \alpha. \]
Since \( i NP = i PN = i Pi N \),
\[ (i Pd i N - i NP) \alpha = [i P, [d, i N]] \alpha = [[i N, d], i P] \alpha = [i N, i P] d \alpha . \]
These equalities imply (1.1).

The following corollary, a consequence of the compatibility, will be used in Section 2.

**Corollary 1.1.** Let \( (P, N) \) be a Poisson-Nijenhuis structure on a manifold. For all \( f \in C^\infty(M) \),
\[ i P (d N df) = -\frac{1}{2} H_{I_{1}}^{P}(f), \]
where \( H_{I_{1}}^{P} = PdTrN \) is the hamiltonian vector field with hamiltonian \( I_{1} = \text{Tr} N \) in the Poisson structure \( P \).

**Proof.** When \( C(P, N) = 0 \), formula (1.1) for \( \alpha = df \) yields (1.2).

**Remark 1.1.** When \( P \) and \( N \) are compatible, the derived bracket \([i N, i P]_{d}\) is a derivation of degree \(-1\) of the algebra of forms equal to the interior product by the vector field \(-\frac{1}{2} PdTrN\). A proof of this property can be found in [1].

2. The hierarchy of modular classes of a Poisson-Nijenhuis manifold

2.1 The modular class of \((TM, N, [\cdot, \cdot]_{N})\).

Let \( N \) be a Nijenhuis tensor on manifold \( M \). Given \( \lambda \otimes \mu \), where \( \lambda \) is a nowhere vanishing multivector of top degree and \( \mu \) a volume element on \( M \), the modular class of the Lie algebroid \((TM, N, [\cdot, \cdot]_{N})\) is the class in the \( d N \)-cohomology of the 1-form \( \xi^{(N)} \) such that, for all \( X \in \Gamma(TM) \),
\[ < \xi^{(N)}, X > \lambda \otimes \mu = [X, \lambda]_{N} \otimes \mu + \lambda \otimes L_{NX} \mu . \]
If \( e_1, \ldots, e_n \) is a local basis of \( \Gamma(TM) \) such that \( \lambda = e_1 \wedge \ldots \wedge e_n \), then 
\[
[X, \lambda]_N = \sum_{j=1}^{n} (-1)^j [X, e_j]_N e_1 \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_n .
\]
Since \([X, Y]_N = [NX, Y] + (L_X N) Y\), we obtain 
\[
[X, \lambda]_N = L_{NX} \lambda + \sum_{j=1}^{n} (L_{X N})^j e_1 \wedge \ldots \wedge \hat{e}_j \wedge \ldots \wedge e_n .
\]
Choosing \( \lambda \) and \( \mu \) such that \( < \lambda, \mu > = 1 \) which implies that \( L_{NX} \lambda \otimes \mu + \lambda \otimes L_{NX} \mu = 0 \), and using the relation \( \sum_{j=1}^{n} (L_{X N})^j = \sum_{j=1}^{n} L_{X} (N_j) \), we obtain 
\[
< \xi^{(N)}, X > \lambda \otimes \mu = i_X (d \text{Tr} N) \lambda \otimes \mu .
\]
Thus we have recovered the result of [2]:

**Proposition 2.1.** The modular class in the \( d_N \)-cohomology of the Lie algebroid \((TM, N, [\; , \; ]_N)\) is the class of the 1-form \( d \text{Tr} N \).

The \( d_N \)-cocycle \( \xi^{(N)} = d \text{Tr} N \) is in fact independent of the choice of \( \lambda \otimes \mu \). The class it defines can also be considered to be the class of the morphism of Lie algebroids \( N : (TM, N, [\; , \; ]_N) \rightarrow (TM, \text{id}, [\; , \; ]) \).

Similarly, the modular classes associated to the Nijenhuis tensors \( N_k \), \( k \in \mathbb{N}, k \geq 2 \), are the \( d_{N^k} \)-classes of the 1-forms \( d \text{Tr}(N^k) \).

### 2.2 The modular class of a Poisson-Nijenhuis manifold

We shall now consider the case of a manifold \( M \) with a PN-structure. Let \( P_0 = P \) and \( P_1 = NP, \ldots, P_k = N^k P, \ldots \).

For each Poisson structure \( P_k \) on \( M \), \( k \geq 0 \), the modular vector field associated to a volume form \( \mu \) on \( M \) is, by definition, the \( d_{P_k} \)-cocycle \( X^k_{\mu} \), satisfying
\[
< X^k_{\mu}, df > \mu = L_{H^k f_{\mu}} \mu , \quad (2.1)
\]
for all \( f \in C^\infty(M) \), that is \( < X^k_{\mu}, df > \mu = di_{P_k f_{\mu}} df \mu \). It follows that, for all 1-forms \( \alpha \),
\[
< X^k_{\mu}, \alpha > \mu = di_{P_k \alpha_{\mu}} (\mu - (i_{P_k} d\alpha)\mu) . \quad (2.2)
\]

We now consider the vector fields
\[
X^{(k)} = X^k_{\mu} - N X^{k-1}_{\mu} , \quad (2.3)
\]
for \( k \geq 1 \), and we list their basic properties:

- For each \( k \), \( X^{(k)} \) is independent of \( \mu \). It is called the \( k \)-th modular vector field of \((M, P, N)\).
• $X^{(k)}$ is a $d_P$-cocycle. Its class is called the $k$-th modular class of the PN-manifold. In particular, the $d_{NP}$-class of $X^{(1)}$ is called the modular class of $(M, P, N)$.

• The $k$-th modular class of $(M, P, N)$ is one-half the relative modular class of the morphism of Lie algebroids $\mathcal{N} : (T^*M, P_k, [, ]_{P_k}) \to (T^*M, P_{k-1}, [, ]_{P_{k-1}})$.

### 2.3 Properties of the hierarchy of modular vector fields

**Proposition 2.2.** The modular vector fields of a PN-manifold $(M, P, N)$ satisfy

$$X^{(k)} = -\frac{1}{2}H^P_{I_k}, \quad k \geq 1,$$

(2.4)

where $I_k = \text{Tr}_{N^k}^P$, $k \geq 1$, is the sequence of fundamental functions in involution.

**Proof.** For clarity, we first prove the case $k = 1$. It follows from formula (2.2) and Corollary 1.1 that, for all $f \in C^\infty(M)$,

$$< NX^0_\mu, df > \mu = < X^0_\mu, Ndf > \mu$$

$$= di_{NPdf} \mu - (ipdNdf) \mu = di_{NPdf} \mu + \frac{1}{2} < PdTrN, df > \mu ,$$

while

$$< X^1_\mu, df > \mu = di_{NPdf} \mu .$$

Therefore $X^{(1)} = X^1_\mu - N X^0_\mu = -\frac{1}{2} PdTrN = -\frac{1}{2} H^P_1$.

The case $k \geq 2$ is similar. Applying Corollary 1.1 to the compatible pair $(N^{k-1}P, N)$, we obtain

$$< X^{(k)} df > = i_{N^{k-1}PdNdf} = i_{N^{k-1}pdNdf} = -\frac{1}{2} < N^{k-1}PdTrN, df > .$$

The result follows from $N^{k-1}PdTrN = PN^{k-1}dTrN = PdTrN^k$.

**Remark 2.1.** The sequence of modular vector fields $X^{(k)}$, $k \geq 1$, coincides with the well-known sequence [5] of bihamiltonian vector fields of a PN-manifold. It follows that $X^{(k)} = NX^{(k-1)}$.

**Remark 2.2.** The sequence of modular vector fields of a Poisson-Nijenhuis manifold introduced by Damianou and Fernandes in [2] is $X_k$, $k \geq 1$, defined by the recursion $X_1 = X_N = X^1_\mu - N X^0_\mu$ and $X_k = N X_{k-1}$, for $k \geq 2$.

They proved that $X_k = -\frac{1}{2} PdTrN^k$, for $k \geq 1$. Though the definition of the hierarchy $X^{(k)}$ that we have considered differs from theirs, the two hierarchies still coincide.
If we denote the modular vector field of the PN-structure \((N, P)\) by \(X_{N, P}\), then \(X^{(k)} = X_{N,N^k-1P}\), while \(X_k = N^{k-1}X_{N,P}\). The vector fields \(X_{N, P}\) satisfy
\[
X_{N,N} + NX_{N,P} = X_{N2,P},
\]
and, more generally,
\[
X_{N,N^k} + NX_{N,N^{k-1}P} = X_{N2,N^{k-1}P}.
\]

This relation is immediate from the definition. Each term is a hamiltonian vector field with respect to \(N^kP\); each of the terms on the left-hand side is equal to \(-\frac{1}{2}PN^{k-1}d\text{Tr}N^2 = -PNKd\text{Tr}N\).

**Remark 2.3.** It follows from the morphism properties of \(P, NP\) and \(t^N\) that the relative modular classes of \(P: (T^*M, P, [\cdot, ]_P) \rightarrow (TM, \text{Id}, [\cdot, ]_P)\), \(NP: (T^*M, NP, [\cdot, ]_{NP}) \rightarrow (TM, \text{Id}, [\cdot, ]),\) and \(t^N: (T^*M, NP, [\cdot, ]_{NP}) \rightarrow (T^*M, P, [\cdot, ]_P)\) are defined and satisfy
\[
\text{Mod}^{NP} - N\text{Mod}^P = \text{Mod}^{t^N}.
\]

A representative of this \(dNP\)-cohomology class is \(-Pd\text{Tr}N = 2X^{(1)}\).

More generally, a representative of the modular class of the morphism \(t^N_k\) from \((T^*M, P_k, [\cdot, ]_{P_k})\) to \((T^*M, P, [\cdot, ]_P)\) is \(-Pd\text{Tr}N^k = 2kX^{(k)}\).

**Remark 2.4.** The modular classes of the morphisms \(N: (TM, N, [\cdot, ]_N) \rightarrow (TM, \text{Id}, [\cdot, ]),\) and \(t^N: (T^*M, NP, [\cdot, ]_{NP}) \rightarrow (T^*M, P, [\cdot, ]_P)\) are related by
\[
\text{Mod}^{t^N} = -P\text{Mod}^N.
\]

Relation (2.6) can be generalized in two ways.

**Proposition 2.3.** (i) The modular classes of the morphisms
\[
N^k: (TM, N^k, [\cdot, ]_{N^k}) \rightarrow (TM, \text{Id}, [\cdot, ])
\]
and
\[
t^N_k: (T^*M, P_k, [\cdot, ]_{P_k}) \rightarrow (T^*M, P, [\cdot, ]_P)
\]
are related by
\[
\text{Mod}^{t^N_k} = -P\text{Mod}^{N^k}.
\]

(ii) The modular classes of the morphisms
\[
N^{[k]}: (TM, N^k, [\cdot, ]_{N^k}) \rightarrow (TM, N^{k-1}, [\cdot, ]_{N^{k-1}})
\]
and
\[
t^N^{[k]}: (T^*M, P_k, [\cdot, ]_{P_k}) \rightarrow (T^*M, P_{k-1}, [\cdot, ]_{P_{k-1}})
\]
are related by
\[
\text{Mod}^{t^N^{[k]}} = -P\text{Mod}^{N^{[k]}},
\]
and a representative of the modular class of the morphism \(t^N^{[k]}\) is \(2X^{(k)}\).
Proof. (i) follows from Proposition 2.1 and Remark 2.3. To prove (ii), we compute a representative of the modular class of $N^{[k]}$, 
\[ d \text{Tr}N^k - tN \text{dTr}N^{k-1} = d \text{Tr} \frac{N^k}{k}, \]
and a representative of the modular class of $tN^{[k]}$, 
\[ 2(X^k - NX^{k-1}) = 2X^{(k)} = -Pd\text{Tr} \frac{N^k}{k}. \]

Remark 2.5. Computations of a representative of $\text{Mod}^{tN^k}$ either from the equality $2(X^k - NX^{k-1}) = 2\sum_{\ell=1}^{k} N^{k-\ell}X^{(\ell)}$ or from the equality $\text{Mod}^{tN^k} = \sum_{\ell=1}^{k} N^{k-\ell} \text{Mod}^{tN^{[\ell]}}$ both recover the fact, stated in Remark 2.3, that a representative of $\text{Mod}^{tN^k}$ is equal to $-\sum_{\ell=1}^{k} N^{k-\ell}Pd\text{Tr} \frac{N^\ell}{\ell} = -Pd\text{Tr}N^k = 2kX^{(k)}$.

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References