LODAY ALGEBRAS (LEIBNIZ ALGEBRAS)

Loday algebras were introduced under the name *Leibniz algebras* by J.-L. Loday [10] [11] as noncommutative analogues of Lie algebras. They are defined by a bilinear bracket which is no longer skew-symmetric. See [12] for motivations, an overview and additional references. The term "Leibniz algebra" was used in all articles prior to 1996, and in many posterior ones. It had been chosen because, in the generalization of Lie algebras to Loday algebras, it is the derivation property of the adjoint maps, analogous to the Leibniz rule in elementary calculus, that is preserved, while the skew-symmetry of the bracket is not. However, it has been shown [1] [8] that in many instances it is necessary to consider both a bracket and an associative multiplication defined on the same space, and to impose a "Leibniz rule" relating both operations, stating that the adjoint maps are derivations of the associative multiplication. For this reason, it is preferable to adopt the term "Loday algebra" rather than "Leibniz algebra" when referring to the derivation property of the bracket alone.

A left *Loday algebra* over the field k is a vector space over k with a k-bilinear map $[\ ,\ ] : A \times A \to A$ satisfying

$$[a,[b,c]] = [[a,b],c] + [b,[a,c]] ,$$

for all $a, b, c \in A$. This property means that, for each a in A, the adjoint endomorphism of A, $\mathrm{ad}_a = [a, .]$, is a derivation of $(A, [\ ,\ ])$.

Similarly, by definition, in a right Loday algebra, for each $a \in A$, the map $x \in A \mapsto [x,a] \in A$ is a derivation of $(A, [\ ,\ ])$.

A left or right Loday algebra in which bracket $[\ ,\ ]$ is skew-symmetric (alternating, if k is of characteristic 2) is a **Lie algebra**.

Loday algebra structures on a vector space V can be defined as elements of square 0 with respect to a graded Lie bracket on the vector space of V-valued multilinear forms on V [3].

A graded version of a left (or right) Loday algebra has been introduced by F. Akman [1] and further studied in [8]. The graded Loday algebras generalize the **graded Lie algebras**.

Examples. The tensor module, $\bar{T}V = \oplus_{k\geq 1} V^{\otimes k}$ of any vector space V, can be turned into a Loday algebra such that $[w,v] = w \otimes v$, for $w \in \bar{T}V, v \in V$. This is the free Loday algebra over V.

Given any differential Lie algebra, more generally, any differential left (resp., right) Loday algebra, $(A, [\ ,\ ], d)$, define $[x,y]_d = [dx, y]$ (resp., $[x,y]_d = [x, dy]$. Then $[\ ,\ ]_d$ is a left (resp., right) Loday bracket, called the *derived bracket* [8]. There is a generalization of this construction to the graded case, and the derived brackets on differential graded Lie algebras, which are graded Loday brackets, have applications in differential and Poisson geometry.

Operads. The operad associated to the notion of Loday algebra is a Koszul **operad** [6]. There is a dual notion, the *dual-Loday algebras*, which are algebras over the dual operad.

Loday (Leibniz) homology. It is the homology of the complex $(\bar{T}V, d)$ with $d(x_1 \otimes x_2 \otimes ... \otimes x_n) = \sum_{1\leq i<j\leq n} (-1)^j x_1 \otimes \ldots \otimes x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \ldots \otimes \hat{x_j} \otimes \ldots \otimes x_n$, where $\hat{x_j}$ denotes that x_j is omitted. The homology complex of a Loday algebra is a coalgebra in the category of dual-Loday algebras.

The Loday homology of the algebra of matrices over an associative algebra A, over a field of characteristic zero, is isomorphic to the tensor module of the **Hochschild homology** of A, as a group in the category of dual-Loday algebras [4] [13] [16]. This is the analogue of the Loday-Quillen-Tsygan theorem relating the **Lie-algebra homology** of matrices to the graded symmetric algebra over the **cyclic homology** of A.

Loday (Leibniz) cohomology can be defined dually. The n-cochains on a Loday algebra A, with coefficients in a representation M of A (*see* [10] [13]), are the n-linear maps on A with values in M, to which the differential of the Chevalley-Eilenberg complex can be lifted. If M is the base field k with the trivial representation, the differential $d\alpha$ of an n-cochain α is defined by

$$d\alpha(x_0, x_1, \ldots, x_n) = \sum_{0\leq i<j\leq n} (-1)^j \alpha(x_0, \ldots, x_{i-1}, [x_i, x_j], x_{i+1}, \ldots, \hat{x_j}, \ldots, x_n) .$$

Dialgebras. A dialgebra is an algebra with two associative operations satisying additional axioms [12]. A dialgebra is a noncommutative analogue of an associative algebra, and any dialgebra structure on a vector space V gives rise to a Loday algebra structure on V. The universal enveloping algebra of a Loday algebra [13] has the structure of a *dialgebra*.

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