LIE ALGEBROID

Lie algebroids were first introduced and studied by J. Pradines [11], following work by C. Ehresmann and P. Libermann on *differentiable groupoids* (later called *Lie groupoids*). Just as Lie algebras are the infinitesimal objects of Lie groups, Lie algebroids are the infinitesimal objects of **Lie groupoids**. They are generalizations of both **Lie algebras** and **tangent vector bundles**. For a comprehensive treatment and lists of references, see [8], [9]. See also [1], [4], [6], [13], [14].

A real *Lie algebroid* $(A, [\ ,\ ]_A, q_A)$, is a smooth real vector bundle A over base M, with a real Lie algebra structure $[\ ,\ ]_A$ on the vector space $\Gamma(A)$ of the smooth global sections of A, and a morphism of vector bundles $q_A : A \to TM$, where TM is the tangent bundle of M, called the *anchor*, such that

- $[X, fY]_A = f[X, Y]_A + (q_A(X).f)\ Y$, for all $X, Y \in \Gamma(A)$ and $f \in C^\infty(M)$,
- q_A defines a Lie algebra homomorphism from the Lie algebra of sections of A, with Lie bracket $[\ ,\ ]_A$, into the Lie algebra of vector fields on M.

Complex Lie algebroid structures [1] on complex vector bundles over real bases can be defined similarly, replacing the tangent bundle of the base by the complexified tangent bundle.

The space of sections of a Lie algebroid is a **Lie-Rinehart algebra**, also called a Lie d-ring or a Lie pseudoalgebra. (See [4], [6], [9].) More precisely, it is an $(R, \mathcal{A})$-Lie algebra, where R is the field of real (or complex) numbers, and $\mathcal{A}$ is the algebra of functions on the base manifold. In fact, the Lie-Rinehart algebras are the algebraic counterparts of the Lie algebroids, just as the modules over a ring are the algebraic counterparts of the vector bundles.

Examples

1. A Lie algebroid over a one-point set, with the zero anchor, is a Lie algebra.

2. The tangent bundle, TM, of a manifold M, with bracket the Lie bracket of vector fields and with anchor the identity of TM, is a Lie algebroid over M. Any integrable sub-bundle of TM, in particular the tangent bundle along the leaves of a **foliation**, is also a Lie algebroid.

3. A vector bundle with a smoothly varying Lie algebra structure on the fibers (in particular, a Lie-algebra bundle [8]) is a Lie algebroid, with pointwise bracket of sections and zero anchor.

4. If M is a **Poisson manifold** then the cotangent bundle T^*M of M is, in a natural way, a Lie algebroid over M. The anchor is the map $P^\sharp : T^*M \to TM$ defined by the Poisson bivector P. The Lie bracket $[\ ,\ ]_P$ of differential 1-forms satisifes $[df, dg]_P = d\{f, g\}_P$, for any functions $f, g \in C^\infty(M)$, where $\{f, g\}_P = P(df, dg)$ is the Poisson bracket of functions, defined by P. When P is nondegenerate, M is a symplectic manifold and this Lie algebra structure of $\Gamma(T^*M)$ is isomorphic to that of $\Gamma(TM)$. For references to the early occurrences of this bracket, which seems to have first appeared in [3], see [4], [6] and [13]. It was shown in [2] that $[\ ,\ ]_P$ is a Lie algebroid bracket on T^*M.

5. The Lie algebroid of a Lie groupoid $(\mathcal{G}, \alpha, \beta)$, where α is the source map and β is the target map [11] [8] [13]. It is defined as the normal bundle along the base of the groupoid, whose sections can be identified with the right-invariant, α-vertical vector fields. The bracket is induced by the Lie bracket of vector fields on the groupoid, and the anchor is $T\beta$.

6. Atiyah sequence. If P is a principal bundle with structure group G, base M and projection p, the G-invariant vector fields on P are the sections of a vector bundle with base M, denoted TP/G, and sometimes called the *Atiyah bundle* of the principal bundle P. This vector bundle is a Lie algebroid, with bracket induced by the Lie bracket of vector fields on P, and with surjective anchor induced by Tp. The kernel of the anchor is the adjoint bundle, $(P \times \mathfrak{g})/G$. Splittings of the anchor are **connections** on P. The Atiyah bundle of P is the Lie algebroid of the Ehresmann gauge groupoid $(P \times P)/G$. If P is the frame bundle of a vector bundle E, then the sections of the Atiyah bundle of P are the covariant differential operators on E, in the sense of [8].

7. Other examples are the trivial Lie algebroids $TM \times \mathfrak{g}$, the transformation Lie algebroids $M \times \mathfrak{g} \to M$, where Lie algebra $\mathfrak{g}$ acts on manifold M, the deformation Lie algebroid $A \times \mathbb{R}$ of a Lie algebroid A,

where $A \times \{\hbar\}$, for $\hbar \neq 0$, is isomorphic to A, and $A \times \{0\}$ is isomorphic to vector bundle A with the abelian Lie algebroid structure (zero bracket and zero anchor), the prolongation Lie algebroids of a Lie algebroid, etc.

de Rham differential. Given any Lie algebroid A, a differential d_A is defined on the graded algebra of sections of the exterior algebra of the dual vector bundle, $\Gamma(\bigwedge A^*)$, called the *de Rham differential* of A. Then $\Gamma(\bigwedge A^*)$ can be considered as the algebra of functions on a **supermanifold**, d_A being an odd vector field with square zero [12].

If A is a Lie algebra $\mathfrak{g}$, then d_A is the Chevalley-Eilenberg cohomology operator on $\bigwedge(\mathfrak{g}^*)$.

If $A = TM$, then d_A is the usual de Rham differential on forms.

If $A = T^*M$ is the cotangent bundle of a Poisson manifold, then d_A is the Lichnerowicz-Poisson differential $[P, \ .\ ]_A$ on fields of multivectors on M.

Schouten algebra. Given any Lie algebroid A, on the graded algebra of sections of the exterior algebra of vector bundle A, $\Gamma(\bigwedge A)$, there is a *Gerstenhaber algebra* structure (see **Poisson algebra**), denoted by $[\ ,\ ]_A$. With this graded Lie bracket, $\Gamma(\bigwedge A)$ is called the *Schouten algebra* of A.

If A is a Lie algebra $\mathfrak{g}$, then $[\ ,\ ]_A$ is the *algebraic Schouten bracket* on $\bigwedge \mathfrak{g}$.

If $A = TM$, then $[\ ,\ ]_A$ is the usual Schouten bracket of fields of multivectors on M.

If $A = T^*M$ is the cotangent bundle of a Poisson manifold, then $[\ ,\ ]_A$ is the Koszul bracket [7] [13] [5] of differential forms.

Morphims of Lie algebroids and the linear Poisson structure on the dual. A base-preserving morphism from Lie algebroid A_1 to Lie algebroid A_2, over the same base M, is a base-preserving vector-bundle morphism, $\mu : A_1 \to A_2$, such that $q_{A_2} \circ \mu = q_{A_1}$, inducing a Lie-algebra morphism from $\Gamma(A_1)$ to $\Gamma(A_2)$.

If A is a Lie algebroid, the dual vector bundle A^* is a *Poisson vector bundle*. This means that the total space of A^* has a Poisson structure such that the Poisson bracket of two functions which are linear on the fibers is linear on the fibers. A base-preserving morphism from vector bundle A_1 to vector bundle A_2 is a morphism of Lie algebroids if and only if its transpose is a Poisson morphism.

Lie bialgebroids [10] [5] are pairs of Lie algebroids (A, A^*) in duality satisfying the compatibility condition that d_{A^*} be a derivation of the graded Lie bracket $[\ ,\ ]_A$. They generalize the Lie bialgebras in the sense of V. G. Drinfel'd (see **quantum groups** and **Poisson Lie groups**) and also the pair (TM, T^*M), where M is a Poisson manifold.

There is no analogue to Lie's third theorem in the case of Lie algebroids, since not every Lie algebroid can be integrated to a global Lie groupoid, although there are local versions of this result. (See [8], [1].)

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