A local relative trace formula for $PGL(2)$

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Abstract

Following a scheme inspired by the recent results of B. Feigon, where she obtains what she called a local relative trace formula for PGL_2 and a local Kutznetsov trace formula for $U(2)$, we describe the spectral side of a local relative trace formula for $G := PGL(2, \mathrm{E})$ relative to the symmetric subgroup $H := PGL(2, \mathrm{F})$ where E/F is an unramified quadratic extension of local non archimedean fields of characteristic 0. This spectral side is given in terms of regularized normalized periods and normalized C-functions of Harish-Chandra. Using the geometric side of such local relative trace formula obtained in a more general setting by P. Delorme, P. Harinck and S. Souaifi, we deduce a local relative trace formula for G relative to H. We apply our result to invert some orbital integrals.

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1 Introduction

Let E/F be an unramified quadratic extension of local non archimedean fields of characteristic 0. In this paper, we prove a local relative trace formula for $G := PGL(2, \mathrm{E})$ relative to the symmetric subgroup $H := PGL(2, \mathrm{F})$ following a scheme inspired by B. Feigon [F].

As in [Ar], the way to establish a local relative trace formula is to describe two asymptotic expansions of a truncated kernel associated to the regular representation of $G \times G$ on $L^2(G)$, the first one in terms of weighted orbital integrals (called the geometric expansion), and the second one in terms of irreducible representations of G (called the spectral expansion). The truncated kernel we consider is defined as follows. The regular representation R of $G \times G$ on $L^2(G)$ is given by $(R(g_1, g_2)\psi)(x) = \psi(g_2^{-1} x g_1)$. For $f = f_1 \otimes f_2$, where f_1 and f_2 are two smooth compactly supported functions on G, the corresponding operator $R(f)$ is an integral operator on $L^2(G)$ with smooth kernel

$$K_f(x, y) = \int_G f_1(gy) f_2(xg) dg = \int_G f_1(x^{-1} gy) f_2(g) dg.$$

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We define the truncated kernel $K^n(f)$ by

$$K^n(f) := \int_{H\times H} K_f(x,y)u(x,n)u(y,n)dxdy,$$

where the truncated function $u(\cdot,n)$ is the characteristic function of a large compact subset in H depending on a positive integer n as in [Ar] or [DHSo].

In [DHSo], we study such a truncated kernel in the more general setting where H is the group of F-points of a reductive algebraic group $\underline{H}$ defined and split over F and G is the group of F-points of the restriction of scalars $\underline{G} := \mathrm{Res}_{\mathrm{E/F}}\underline{H}$ from E to F and we obtain an asymptotic geometric expansion of this truncated kernel in terms of weighted orbital integrals.

It is considerably more difficult to obtain a spectral asymptotic expansion of the truncated kernel and the main part of this paper is devoted to give it for $\underline{H} = PGL(2)$.

First, we express the kernel K_f in terms of normalized Eisenstein integrals using the Plancherel formula for G (cf. section 3) . Then the truncated kernel can be written as a finite linear combination, depending on unitary irreducible representations of G, of terms involving scalar product of truncated periods (cf. Corollary 4.2). The difficulty appears in the terms depending on principal series of G.

Let M (resp., P) be the image in G of the group of diagonal (resp., upper triangular) matrices of $GL(2,\mathrm{E})$ and let $\bar{P}$ be the parabolic subgroup opposite to P. As M is isomorphic to $\mathrm{E}^\times$, we identify characters on M and on $\mathrm{E}^\times$. The group of unramified characters of M is isomorphic to $\mathbb{C}^*$ by a map $z \to \chi_z$. Let δ be a unitary character of $\mathrm{E}^\times$, which is trivial on a fixed uniformizer of $\mathrm{F}^\times$. For $z \in \mathbb{C}^*$, we set $\delta_z := \delta \otimes \chi_z$. We denote by $(i_P^G\delta_z, i_P^G\mathbb{C}_{\delta_z})$ the normalized induced representation and by $(i_P^G\check{\delta}_z, i_P^G\check{\mathbb{C}}_{\delta_z})$ its contragredient. Then, the normalized truncated period is defined by

$$P^n_{\delta_z}(S) := \int_H E^0(P,\delta_z,S)(h)u(h,n)dh, \quad S \in i_P^G\mathbb{C}_{\delta_z} \otimes i_{\bar{P}}^G\check{\mathbb{C}}_{\delta_z},$$

where $E^0(P,\delta_z,\cdot)$ is the normalized Eisenstein integral associated to $i_P^G\delta_z$ (cf. (3.6)). The contribution of $i_P^G\delta_z$ in $K^n(f)$ is a finite linear combination of integrals

$$I^n_\delta(S,S') := \int_{\mathcal{O}} P^n_{\delta_z}(S)\overline{P^n_{\delta_z}(S')}\frac{dz}{z}, \quad S,S' \in i_P^G\mathbb{C}_{\delta_z} \otimes i_{\bar{P}}^G\check{\mathbb{C}}_{\delta_z}$$

where $\mathcal{O}$ is the torus of complex numbers of modulus equal to 1.

To establish the asymptotic expansion of this integral, we recall the notion of normalized regularized period introduced by B. Feigon (cf. section 4). This period, denoted by

$$P_{\delta_z}(S) := \int_H^* E^0(P,\delta_z,S)(h)dh$$

is meromorphic in a neighborhood $\mathcal{V}$ of $\mathcal{O}$ with at most a simple pole at $z = 1$ and defines a $H \times H$ invariant linear form on $i_P^G\mathbb{C}_{\delta_z} \otimes i_{\bar{P}}^G\check{\mathbb{C}}_{\delta_z}$. Moreover, the difference $P_{\delta_z}(S) - P^n_{\delta_z}(S)$ is a rational function in z on $\mathcal{V}$ with at most a simple pole at $z = 1$ which depends on the normalized C-functions of Harish-Chandra. As normalized Eisenstein integrals and normalized C-functions are holomorphic in a neighborhood of $\mathcal{O}$, we can deduce an asymptotic behavior of the integrals $I_\delta^n(S,S')$ in terms of normalized regularized periods and normalized C-functions (cf. Proposition 7.1).

Our first result (cf. Theorem 7.3) asserts that $K^n(f)$ is asymptotic to a polynomial function in n of degree 1 whose coefficients are described in terms of generalized matrix coefficients $m_{\xi,\xi'}$ associated to unitary irreducible representations (π, V_π) of G where ξ and ξ' are linear forms on V_π. When (π, V_π) is a normalized induced representation, these linear forms are defined from the regularized normalized periods, its residues, and the normalized C-functions of Harish-Chandra.

We make precise the geometric asymptotic expansion of $K^n(f)$ obtained in [DHSo] for $\underline{H} := PGL(2)$. Therefore, comparing the two asymptotic expansions of $K^n(f)$, we deduce our relative local trace formula and a relation between orbital integrals on elliptic regular points in $H\backslash G$ and some generalized matrix coefficients of induced representations (Theorem 8.1).

As corollary of these results, we give an inversion formula for orbital integrals on regular elliptic points of $H\backslash G$ and for orbital integrals of a matrix coefficient associated to a cuspidal representation of G.

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2 Notation

Let F be a non archimedean local field of characteristic 0 and odd residual characteristic q. Let E be an unramified quadratic extension of F. Let $\mathcal{O}_{\mathrm{F}}$ (resp., $\mathcal{O}_{\mathrm{E}}$) denote the ring of integers in F (resp., in E). We fix a uniformizer ϖ in the maximal ideal of $\mathcal{O}_{\mathrm{F}}$. Thus ϖ is also a uniformizer of E. We denote by $v(\cdot)$ the valuation of F, extended to E. Let $|\cdot|_{\mathrm{F}}$ (resp., $|\cdot|_{\mathrm{E}}$) denote the normalized valuation on F (resp., on E). Thus for $a \in \mathrm{F}^\times$, one has $|a|_{\mathrm{F}} = |a|_{\mathrm{E}}^2$.

Let $N_{\mathrm{E/F}}$ be the norm map from $\mathrm{E}^\times$ to $\mathrm{F}^\times$. We denote by E^1 the set of elements in $\mathrm{E}^\times$ whose norm is equal to 1.

Let $\underline{H} := PGL(2)$ defined over F and let $\underline{G} := \mathrm{Res}_{\mathrm{E/F}}(\underline{H} \times_{\mathrm{F}} \mathrm{E})$ be the restriction of scalars of $\underline{H}$ from E to F. We set $H := \underline{H}(\mathrm{F}) = PGL(2,\mathrm{F})$ and $G := \underline{G}(\mathrm{F}) = PGL(2,\mathrm{E})$. Let $K := \underline{G}(\mathcal{O}_F) = PGL(2, \mathcal{O}_E)$.

We denote by $C^\infty(G)$ the space of smooth functions on G and by $C_c^\infty(G)$ the subspace of compactly supported functions in $C^\infty(G)$. If V is a vector space of valued functions on G which is invariant by right (resp., left) translations, we will denote by ρ (resp., λ) the right (resp., left) regular representation of G in V.

If V is a vector space, V' will denote its dual. If V is real, $V_{\mathbb{C}}$ will denote its complexification.

Let p be the canonical projection of $GL(2,\mathrm{E})$ onto G. We denote by M and N the image by p of the subgroups of diagonal matrices and upper triangular unipotent matrices of $GL(2,\mathrm{E})$ respectively. We set $P := MN$ and we denote by $\bar{P}$ the parabolic subgroup opposite to P. Let δ_P be the modular function of P. We denote by 1 and w the representatives in K of the Weyl group W^G of M in G.

For $J = K, M$ or P, we set $J_H := J \cap H$.

For a, b in $\mathrm{E}^\times$, we denote by $diag_G(a,b)$ the image by p of the diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in GL(2,E)$. The natural map $(a,b) \mapsto diag_G(a,b)$ induces an isomorphism from $E^\times \times E^\times / diag(E^\times) \simeq E^\times$ to M where $diag(E^\times)$ is the diagonal of $E^\times \times E^\times$.

Hence, each character χ of $E^\times$ defines a character of M given by $diag_G(a,b) \mapsto \chi(ab^{-1})$, which we will denote by the same letter. (2.1)

We define the map $h_M : M \to \mathbb{R}$ by

$$q^{-h_M(m)} = |ab^{-1}|_{\mathrm{E}} \quad \text{for } m = diag_G(a,b). \tag{2.2}$$

We define similarly h_{M_H} on M_H by $q^{-h_{M_H}(diag_G(a,b))} = |ab^{-1}|_{\mathrm{F}}$ for $a, b \in F^\times$. Then for $m \in M_H$, one has $\delta_P(m) = \delta_{P_H}(m)^2 = q^{-2h_{M_H}(m)}$.

We normalize the Haar measure dx on F so that $\mathrm{vol}(\mathcal{O}_{\mathrm{F}}) = 1$. We define the measure $d^\times x$ on $\mathrm{F}^\times$ by $d^\times x = \dfrac{1}{1-q^{-1}} \dfrac{1}{|x|_{\mathrm{F}}} dx$. Thus, we have $\mathrm{vol}(\mathcal{O}_{\mathrm{F}}^\times) = 1$. We let M and M_H have the measure induced by $d^\times x$. We normalize the Haar measure on K so that $\mathrm{vol}(K) = 1$. Let dn be the Haar measure on N such that

$$\int_N \delta_{\bar{P}}(m_{\bar{P}}(n)) dn = 1.$$

Let dg be the Haar measure on G such that

$$\int_G f(g) dg = \int_M \int_N \int_K f(mnk) dk\, dn\, dm.$$

We define dh on H similarly.

The Cartan decomposition of H is given by

$$H = K_H M_H^+ K_H \text{ where } M_H^+ := \{diag_G(a,b); a, b \in \mathrm{F}^\times, |ab^{-1}|_{\mathrm{F}} \leqslant 1\}, \tag{2.3}$$

and for any integrable function f on H, we have the standard integration formula

$$\int_H f(x) dx = \int_{K_H} \int_{K_H} \int_{M_H} D_{P_H}(m) f(k_1 m k_2) dm dk_2 dk_1, \tag{2.4}$$

where

$$D_{P_H}(m) = \begin{cases} \delta_{P_H}(m)^{-1}(1+q^{-1}) & \text{if } m \in M_H^+, \\ 0 & \text{otherwise.} \end{cases}$$

For $h \in H$, we denote by $\mathcal{M}(h)$ an element of M_H^+ such that $h \in K_H \mathcal{M}(h) K_H$. The element $h_{M_H}(\mathcal{M}(h))$ is independent of this choice. We thank E. Lapid who suggests us the proof of the following Lemma.

2.1 Lemma. *Let Ω be a compact subset of H. There is $N_0 > 0$ sastisfying the following property:*
for any $h \in \Omega$, there exists $X_h \in \mathbb{R}$ such that, for all $m \in M_H^+$ satisfying $h_{M_H}(m) \geqslant N_0$, one has

$$h_{M_H}(\mathcal{M}(mh)) = h_{M_H}(m) + X_h.$$

Proof :
For a matrix $x = (x_{i,j})_{i,j}$ of $GL(2, \mathrm{F})$, we set

$$F(x) := \log \max_{i,j} \Big(\frac{|x_{i,j}|_{\mathrm{F}}^2}{|detx|_{\mathrm{F}}} \Big).$$

The function F is clearly invariant under the action of the center of $GL(2, \mathrm{F})$, hence it defines a function on H which we denote by the same letter.

Since $| \cdot |_{\mathrm{F}}$ is ultrametric, for $k \in K_H$ and $h \in H$, we have $F(kh) \leqslant F(h)$, hence, $F(k^{-1}kh) \leqslant F(kh)$. Using the same argument on the right, we deduce that F is right and left invariant by K_H.

If $m = diag_G(\omega^{n_1}, \omega^{n_2})$ with $n_1 - n_2 \geqslant 0$ then $F(m) = \log \max \Big(\frac{q^{-2n_1}}{q^{-n_1-n_2}}, \frac{q^{-2n_2}}{q^{-n_1-n_2}} \Big) = (n_1 - n_2) \log q = h_{M_H}(m) \log q$. Thus, we deduce that

$$F(h) = h_{M_H}(\mathcal{M}(h)) \log q, \quad h \in H.$$

If $h = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $m = diag_G(\omega^{n_1}, \omega^{n_2})$, then

$$F(mh) = \log \max \Big(|a|_{\mathrm{F}}^2 q^{n_2-n_1}, |b|_{\mathrm{F}}^2 q^{n_2-n_1}, |c|_{\mathrm{F}}^2 q^{n_1-n_2}, |d|_{\mathrm{F}}^2 q^{n_1-n_2} \Big) - \log |ad - bc|_{\mathrm{F}}.$$

Therefore, we can choose $N_0 > 0$ such that, for any $h \in \Omega$ and $m \in M_H^+$ with $h_M(m) > N_0$, we have

$$\begin{aligned} F(mh) &= \log \max \Big(|c|_{\mathrm{F}}^2 q^{n_1-n_2}, |d|_{\mathrm{F}}^2 q^{n_1-n_2} \Big) - \log |ad - bc|_{\mathrm{F}} \\ &= (n_1 - n_2) \log q + \log \max(|c|_{\mathrm{F}}^2, |d|_{\mathrm{F}}^2) - \log |ad - bc|_{\mathrm{F}}. \end{aligned}$$

Hence, we obtain the Lemma. □

3 Normalized Eisenstein integrals and Plancherel formula

We denote by $\widehat{M_2}$ the set of unitary characters of $E^\times$ which are trivial on ω.

Let $X(M)$ be the complex torus of unramified characters of M and $X(M)_u$ be the compact subtorus of unitary unramified characters of M. For $z \in \mathbb{C}^*$, we denote by χ_z the unramified character of $\mathrm{E}^\times$ defined by $\chi_z(\omega) = z$. By definition of h_M, we have $\chi_z(m) = z^{h_M(m)/2}$. Each element of $X(M)$ is of the form χ_z for some $z \in \mathbb{C}^*$ and $X(M)_u$ identifies with the group $\mathcal{O}$ of complex numbers of modulus equal to 1.
For $\delta \in \widehat{M_2}$ and $z \in \mathbb{C}^*$, we set $\delta_z := \delta \otimes \chi_z$. We will denote by $\mathbb{C}_{\delta_z}$ the space of δ_z.

Let $Q = MU$ be equal to P or to $\bar{P}$. Let $\delta \in \widehat{M_2}$ and $z \in \mathbb{C}^*$. We denote by $i_Q^G \delta_z$ the right representation of G in the space $i_Q^G \mathbb{C}_{\delta_z}$ of maps v from G to $\mathbb{C}$, right invariant by a compact open subgroup of G and such that $v(mug) = \delta_Q(m)^{1/2}\delta_z(m)f(g)$ for all $m \in M, u \in U$ and $g \in G$.

One denotes by $(\bar{i}_Q^G\delta_z, i_{K\cap Q}^K\mathbb{C})$ the compact realization of $(i_Q^G\delta_z, i_Q^G\mathbb{C}_{\delta_z})$ obtained by restriction of functions. If $v \in i_{Q\cap K}^K\mathbb{C}$, one denotes by v_z the element of $i_Q^G\mathbb{C}_{\delta_z}$ whose restriction to K is equal to v.

One defines a scalar product on $i_{Q\cap K}^K\mathbb{C}$ by

$$(v, v') = \int_K v(k)\overline{v'(k)}dk, \quad v, v' \in i_{Q\cap K}^K\mathbb{C}. \tag{3.1}$$

If $z \in \mathcal{O}$ (hence δ_z is unitary), the representation $\bar{i}_Q^G(\delta_z)$ is unitary. Therefore, by "transport de structure", $i_Q^G(\delta_z)$ is also unitary.
Let $(\check{\delta_z}, \check{\mathbb{C}_{\delta_z}})$ be the contragredient representation of $(\delta_z, \mathbb{C}_{\delta_z})$. We can and will identify $(i_Q^G\check{\delta_z}, i_Q^G\check{\mathbb{C}_{\delta_z}})$ with the contragredient representation of $(i_Q^G\delta_z, i_Q^G\mathbb{C}_{\delta_z})$ and $i_Q^G\mathbb{C}_{\delta_z} \otimes i_Q^G\check{\mathbb{C}_{\delta_z}}$ with a subspace of $\mathrm{End}_G(i_Q^G\mathbb{C}_{\delta_z})$ ([W], I.3).

Using the isomorphism between $i_Q^G\mathbb{C}_{\delta_z}$ and $i_{Q\cap K}^K\mathbb{C}$, we can define the notion of rational or polynomial map from $X(M)$ to a space depending on $i_Q^G\mathbb{C}_{\delta_z}$ as in ([W] IV.1 and VI.1).

We denote by $A(\bar{Q}, Q, \delta_z) : i_Q^G\mathbb{C}_{\delta_z} \to i_{\bar{Q}}^G\mathbb{C}_{\delta_z}$ the standard intertwining operator. By ([W], IV. 1. and Proposition IV.2.2), the map $z \in \mathbb{C}^* \mapsto A(\bar{Q}, Q, \delta_z) \in \mathrm{Hom}_G(i_Q^G\mathbb{C}_{\delta_z}, i_{\bar{Q}}^G\mathbb{C}_{\delta_z})$ is a rational function on $\mathbb{C}^*$. Moreover, there exists a rational complex valued function $j(\delta_z)$ depending only on M such that $A(Q, \bar{Q}, \delta_z) \circ A(\bar{Q}, Q, \delta_z)$ is the dilation of scale $j(\delta_z)$. We set

$$\mu(\delta_z) := j(\delta_z)^{-1}. \tag{3.2}$$

By ([W] Lemme V.2.1), the map $z \mapsto \mu(\delta_z)$ is rational on $\mathbb{C}^*$ and regular on $\mathcal{O}$.
The Eistenstein integral $E(Q, \delta_z)$ is the map from $i_Q^G\mathbb{C}_{\delta_z} \otimes i_Q^G\check{\mathbb{C}_{\delta_z}}$ to $C^\infty(G)$ defined by

$$E(Q, \delta_z, v \otimes \check{v})(g) = \langle (i_Q^G\delta_z)(g)v, \check{v}\rangle, \quad v \in i_Q^G\mathbb{C}_{\delta_z}, \check{v} \in i_Q^G\check{\mathbb{C}_{\delta_z}}. \tag{3.3}$$

If $\psi \in i_Q^G\mathbb{C}_{\delta_z} \otimes i_Q^G\check{\mathbb{C}_{\delta_z}}$ is identified with an endomorphism of $i_Q^G\mathbb{C}_{\delta_z}$, we have

$$E(Q, \delta_z, \psi)(g) = \mathrm{tr}(i_Q^G\delta_z(g)\psi). \tag{3.4}$$

We introduce the operator $C_{P,P}(1, \delta_z) := Id \otimes A(\bar{P}, P, \check{\delta_z})$ from $i_P^G\mathbb{C}_{\delta_z} \otimes i_P^G\check{\mathbb{C}_{\delta_z}}$ to $i_P^G\mathbb{C}_{\delta_z} \otimes i_{\bar{P}}^G\check{\mathbb{C}_{\delta_z}}$. By ([W], Lemme V.2.2), one has

$$\text{the operator } \mu(\delta_z)^{1/2}C_{P,P}(1, \delta_z) \text{ is unitary and regular on } \mathcal{O}. \tag{3.5}$$

We define the normalized Eisenstein integral $E^0(P, \delta_z) : i_P^G\mathbb{C}_{\delta_z} \otimes i_{\bar{P}}^G\check{\mathbb{C}_{\delta_z}} \to C^\infty(G)$ by

$$E^0(P, \delta_z, \Psi) = E(P, \delta_z, C_{P|P}(1, \delta_z)^{-1}\Psi). \tag{3.6}$$

By ([S], §5.3.5), we have

$$E^0(P, \delta_z, \Psi) \text{ is regular on } \mathcal{O}. \tag{3.7}$$

For $f \in C_c^\infty(G)$, we denote by $\check{f}$ the function defined by $\check{f}(g) := f(g^{-1})$. Then, the operator $i_P^G\delta_z(\check{f})$ belongs to $i_P^G\mathbb{C}_{\delta_z} \otimes i_P^G\check{\mathbb{C}}_{\delta_z} \subset \mathrm{End}_G(i_P^G\mathbb{C}_{\delta_z})$. We define the Fourier transform $\mathcal{F}(P, \delta_z, f) \in i_P^G\mathbb{C}_{\delta_z} \otimes i_P^G\check{\mathbb{C}}_{\delta_z}$ of f by

$$\mathcal{F}(P, \delta_z, f) = i_P^G\delta_z(\check{f}).$$

The G-invariant scalar product on $i_P^G\mathbb{C}_{\delta_z}$ defined in (3.1) induces a G-invariant scalar product on $i_P^G\mathbb{C}_{\delta_z} \otimes i_P^G\check{\mathbb{C}}_{\delta_z}$ given by

$$(v_1 \otimes \check{v}_1, v_2 \otimes \check{v}_2) = (v_1, v_2)(\check{v}_1, \check{v}_2).$$

Notice that by the inclusion $i_P^G\mathbb{C}_{\delta_z} \otimes i_P^G\check{\mathbb{C}}_{\delta_z} \subset \mathrm{End}(i_P^G\mathbb{C}_{\delta_z})$, this scalar product coincides with the Hilbert-Schmidt scalar product on the space of Hilbert-Schmidt operators on $i_P^G\mathbb{C}_{\delta_z}$ defined by

$$(S, S') = \mathrm{tr}(SS'^*), \tag{3.8}$$

where $\mathrm{tr}(SS'^*) = \sum_{o.n.b.}\langle SS'^*u_i, u_i\rangle$ and this sum converges absolutely and does not depend on the basis.

Then, the Fourier transform is the unique element of $i_P^G\mathbb{C}_{\delta_z} \otimes i_P^G\check{\mathbb{C}}_{\delta_z}$ such that

$$(E(P, \delta_z, \Psi), f)_G = (\Psi, \mathcal{F}(P, \delta_z, f)). \tag{3.9}$$

Moreover, we have ([W] Lemme VII.1.1)

$$E(P, \delta_z, \mathcal{F}(P, \delta_z, f))(g) = \mathrm{tr}\big[(i_P^G\delta_z)(\lambda(g)\check{f})\big]. \tag{3.10}$$

We define the normalized Fourier transform $\mathcal{F}^0(P, \delta_z, f)$ of $f \in C_c^\infty(G)$ as the unique element of $i_P^G\mathbb{C}_{\delta_z} \otimes i_{\bar{P}}^G\check{\mathbb{C}}_{\delta_z}$ such that

$$(\Psi, \mathcal{F}^0(P, \delta_z, f)) = (E^0(P, \delta_z, \Psi), f)_G, \qquad \Psi \in i_P^G\mathbb{C}_{\delta_z} \otimes i_{\bar{P}}^G\check{\mathbb{C}}_{\delta_z}.$$

It follows easily from (3.9) and (3.5) that

$$\mathcal{F}^0(P, \delta_z, f) = \mu(\delta_z)C_{P|P}(1, \delta_z)\mathcal{F}(P, \delta_z, f),$$

thus we deduce that

$$E^0(P, \delta_z, \mathcal{F}^0(P, \delta_z, f)) = \mu(\delta_z)E(P, \delta_z, \mathcal{F}(P, \delta_z, f)). \tag{3.11}$$

Therefore, we can describe the spectral decomposition of the regular representation $R := \rho \otimes \lambda$ of $G \times G$ on $L^2(G)$ of ([W] Théorème VIII.1.1) in terms of normalized Eisenstein integrals as follows. Let $\mathcal{E}_2(G)$ be the set of classes of irreducible admissible representations of G whose matrix coefficients are square-integrable. We will denote by $d(\tau)$ the formal degree of $\tau \in \mathcal{E}_2(G)$. Then we have

$$f(g) = \sum_{\tau \in \mathcal{E}_2(G)} d(\tau)\mathrm{tr}(\tau(\lambda(g)\check{f})) + \frac{1}{4i\pi}\sum_{\delta \in \widehat{M}_2}\int_{\mathcal{O}} E^0(P, \delta_z, \mathcal{F}^0(P, \delta_z, f))(g)\frac{dz}{z}. \tag{3.12}$$

4 The truncated kernel

Let $f \in C_c^\infty(G \times G)$ be of the form $f(y_1, y_2) = f_1(y_1)f_2(y_2)$ with $f_j \in C_c^\infty(G)$. Then the operator $R(f)$ (where $R := \rho \otimes \lambda$) is an integral operator with smooth kernel

$$K_f(x,y) = \int_G f_1(gy)f_2(xg)dg = \int_G f_1(x^{-1}gy)f_2(g)dg.$$

Notice that the kernel studied in [Ar], [F] or [DHSo] corresponds to the kernel of the representation $\lambda \times \rho$ which coincides with $K_{f_2 \otimes f_1}(x,y) = K_{f_1 \otimes f_2}(x^{-1}, y^{-1})$.

The aim of this part is to give a spectral expansion of the truncated kernel obtained by integrating K_f against a truncated function on $H \times H$ as in [Ar].

4.1 Lemma. *For $(\tau, V_\tau) \in \mathcal{E}_2(G)$, we fix an orthonormal basis $\mathcal{B}_\tau$ of the space of Hilbert-Schmidt operators on V_τ. For $\delta \in \widehat{M}_2$ and $z \in \mathcal{O}$, we fix an orthonormal basis $\mathcal{B}_{\bar{P},P}(\mathbb{C})$ of $i_{P\cap K}^K \mathbb{C} \otimes i_{\bar{P}\cap K}^K \check{\mathbb{C}}$. Using the isomorphism $S \mapsto S_z$ between $i_{P\cap K}^K \mathbb{C} \otimes i_{\bar{P}\cap K}^K \check{\mathbb{C}}$ and $i_P^G \mathbb{C}_{\delta_z} \otimes i_{\bar{P}}^G \check{\mathbb{C}}_{\delta_z}$, we have*

$$K_f(x,y) = \sum_{\tau \in \mathcal{E}_2(G)} \sum_{S \in \mathcal{B}_\tau} d(\tau) tr(\tau(x)\tau(f_1)S\tau(\check{f}_2))\overline{tr(\tau(y)S)}$$

$$+\frac{1}{4i\pi} \sum_{\delta \in \widehat{M}_2} \sum_{S \in \mathcal{B}_{\bar{P},P}(\mathbb{C})} \int_{\mathcal{O}} E^0(P, \delta_z, \Pi_{\delta_z}(f)S_z)(x)\overline{E^0(P, \delta_z, S_z)(y)}\frac{dz}{z},$$

where $\Pi_{\delta_z}(f)S_z := (i_P^G\delta_z \otimes i_{\bar{P}}^G\check{\delta}_z)(f)S_z = (i_P\delta_z)(f_1)S_z(i_{\bar{P}}\delta_z)(\check{f}_2)$ and the sums over S are all finite.

Proof :

For $x \in G$, we set

$$h(v) := \int_G f_1(uvx)f_2(xu)du,$$

so that

$$K_f(x,y) = \big[\rho(yx^{-1})h\big](e). \tag{4.1}$$

If π is a representation of G, one has

$$\pi\big(\rho(yx^{-1})h\big) = \int_{G\times G} f_1(ugy)f_2(xu)\pi(g)dudg = \int_{G\times G} f_1(u_1)f_2(xu)\pi(u^{-1}u_1y^{-1})dudu_1$$

$$= \int_{G\times G} f_1(u_1)f_2(u_2)\pi(u_2^{-1}xu_1y^{-1})du_1du_2 = \pi(\check{f}_2)\pi(x)\pi(f_1)\pi(y^{-1}).$$

Therefore, using the Hilbert-Schmidt scalar product (3.8), one obtains for $\tau \in \mathcal{E}_2(G)$,

$$\begin{aligned} &\text{tr } \tau\big(\rho(yx^{-1})h\big) = \text{tr } \tau(\check{f}_2)\tau(x)\tau(f_1)\tau(y)^* = (\tau(\check{f}_2)\tau(x)\tau(f_1), \tau(y)) \\ &= \sum_{S \in \mathcal{B}_\tau} (\tau(\check{f}_2)\tau(x)\tau(f_1), S^*)\overline{(\tau(y), S^*)} = \sum_{S \in \mathcal{B}_\tau} \text{tr } (\tau(x)\tau(f_1)S\tau(\check{f}_2))\overline{tr(\tau(y)S)}, \end{aligned} \tag{4.2}$$

where the sum over S in $\mathcal{B}_\tau$ is finite.

We consider now $\pi := i_P^G \delta_z$ with $\delta \in \widehat{M}_2$ and $z \in \mathcal{O}$. By (3.10) and (3.11), we have

$$E^0(P, \delta_z, \mathcal{F}^0(P, \delta_z, [\rho(yx^{-1})h]\check{\ })(e) = \mu(\delta_z)\mathrm{tr}\ \pi(\rho(yx^{-1})h). \tag{4.3}$$

Let $\mathcal{B}_{P,P}(\mathbb{C}_{\delta_z})$ be an orthonormal basis of $i_P^G \mathbb{C}_{\delta_z} \otimes i_P^G \check{\mathbb{C}}_{\delta_z}$. Since $f_1, f_2 \in C_c^\infty(G)$, the operators $\pi(f_1)$ and $\pi(\check{f}_2)$ are of finite rank. Therefore, we deduce as above that

$$\mathrm{tr}\ \pi\big(\rho(yx^{-1})h\big) = \mathrm{tr}\big(\pi(\check{f}_2)\pi(x)\pi(f_1)\pi(y)^{-1}\big) = \sum_{S \in \mathcal{B}_{P,P}(\mathbb{C}_{\delta_z})} tr(\pi(x)\pi(f_1)S\pi(\check{f}_2))\overline{tr(\pi(y)S)},$$

where the sum over S in $\mathcal{B}_{P,P}(\mathbb{C}_{\delta_z})$ is finite.

In what follows, the sums over elements of an orthonormal basis will be always finite. Hence, by (3.4), we deduce that

$$\mathrm{tr}\ \pi(\rho(yx^{-1})h) = \sum_{S \in \mathcal{B}_{P,P}(\mathbb{C}_{\delta_z})} E(P, \delta_z, \pi(f_1)S\pi(\check{f}_2))(x)\overline{E(P, \delta_z, S)(y)}. \tag{4.4}$$

Recall that we fix an orthonormal basis $\mathcal{B}_{\bar{P},P}(\mathbb{C})$ of the space $i_{P\cap K}^K \mathbb{C} \otimes i_{\bar{P}\cap K}^K \check{\mathbb{C}}$ which is isomorphic to $i_P^G \mathbb{C}_{\delta_z} \otimes i_{\bar{P}}^G \check{\mathbb{C}}_{\delta_z}$ by the map $S \mapsto S_z$. By (3.5), the family $\tilde{S}(\delta_z) := \mu(\delta_z)^{-1/2} C_{P,P}(1, \delta_z)^{-1} S_z$ for $S \in \mathcal{B}_{\bar{P},P}(\mathbb{C})$ is an orthonormal basis of $i_P^G \mathbb{C}_{\delta_z} \otimes i_P^G \check{\mathbb{C}}_{\delta_z}$.

Moreover, using the inclusion $i_P^G \mathbb{C}_{\delta_z} \otimes i_{\bar{P}}^G \check{\mathbb{C}}_{\delta_z} \subset \mathrm{Hom}_G(i_{\bar{P}}^G \mathbb{C}_{\delta_z}, i_P^G \mathbb{C}_{\delta_z})$, and the adjonction property of the intertwining operator ([W], IV.1. (11)), we have $C_{P,P}(1, \delta_z)^{-1} S = S \circ A(P, \bar{P}, \delta_z)^{-1}$, for all $S \in i_P^G \mathbb{C}_{\delta_z} \otimes i_{\bar{P}}^G \check{\mathbb{C}}_{\delta_z}$. Since $A(P, \bar{P}, \delta_z)^{-1} \circ i_P^G(\delta_z) = i_{\bar{P}}^G(\delta_z) \circ A(P, \bar{P}, \delta_z)^{-1}$, writing (4.4) for the basis $\tilde{S}(\delta_z)$, we obtain

$$\begin{aligned}
&\mathrm{tr}\ \pi(\rho(yx^{-1})h)\\
&= \mu(\delta_z)^{-1} \sum_{S \in \mathcal{B}_{\bar{P},P}(\mathbb{C})} E(P, \delta_z, \pi(f_1)C_{P,P}(1, \delta_z)^{-1}(S_z)\pi(\check{f}_2))(x)\overline{E(P, \delta_z, C_{P,P}(1, \delta_z)^{-1}S_z)(y)}\\
&= \mu(\delta_z)^{-1} \sum_{S \in \mathcal{B}_{\bar{P},P}(\mathbb{C})} E(P, \delta_z, C_{P,P}(1, \delta_z)^{-1}[(i_P^G \delta_z)(f_1)S_z(i_{\bar{P}}^G \delta_z)(\check{f}_2)])(x)\overline{E(P, \delta_z, C_{P,P}(1, \delta_z)^{-1}S_z)(y)}\\
&= \mu(\delta_z)^{-1} \sum_{S \in \mathcal{B}_{\bar{P},P}(\mathbb{C})} E^0(P, \delta_z, (i_P^G \delta_z)(f_1)S_z(i_{\bar{P}}^G \delta_z)(\check{f}_2))(x)\overline{E^0(P, \delta_z, S_z)(y)}.
\end{aligned}$$

We set $\Pi_{\delta_z} := i_P^G \delta_z \otimes i_{\bar{P}}^G \check{\delta}_z$. Then we have

$$\Pi_{\delta_z}(f)S_z = (i_P^G \delta_z)(f_1)S_z(i_{\bar{P}}^G \delta_z)(\check{f}_2). \tag{4.5}$$

By (4.3), we obtain

$$E^0(P, \delta_z, \mathcal{F}^0(P, \delta_z, [\rho(yx^{-1})h]\check{\ }))(e) = \sum_{S \in \mathcal{B}_{\bar{P},P}(\mathbb{C})} E^0(P, \delta_z, \Pi_{\delta_z}(f)S_z)(x)\overline{E^0(P, \delta_z, S_z)(y)}.$$

The Lemma follows from (3.12), (4.1), (4.2) and the above result. □

To integrate the kernel K_f on $H \times H$, we introduce truncation as in [Ar]. Let n be a positive integer. Let $u(\cdot, n)$ be the truncated function defined on H by

$$u(h,n) = \begin{cases} 1 & \text{if } h = k_1 m k_2 \text{ with } k_1, k_2 \in K_H, m \in H \text{ such that } 0 \leqslant |h_{M_H}(m)| \leqslant n \\ 0 & \text{otherwise} \end{cases}$$

We define the truncated kernel by

$$K^n(f) := \int_{H \times H} K_f(x,y) u(x,n) u(y,n) dx dy. \tag{4.6}$$

Since $K_f(x^{-1}, y^{-1})$ coincides with the kernel studied in ([DHSo] 2.2) and $u(x,n) = u(x^{-1}, n)$, this definition of the truncated kernel coincides with that of [DHSo].
We defined truncated periods by

$$P^n_\tau(S) := \int_H \mathrm{tr}(\tau(y)S) u(y,n) dy, \quad (\tau, V_\tau) \in \mathcal{E}_2(G), S \in \mathrm{End}_{fin.rk}(V_\tau), \tag{4.7}$$

where $\mathrm{End}_{fin.rk}(V_\tau)$ is the space of finite rank operators in $\mathrm{End}(V_\tau)$, and

$$P^n_{\delta_z}(S) := \int_H E^0(P, \delta_z, S_z)(y) u(y,n) dy, \quad \delta \in \widehat{M}_2, z \in \mathcal{O}, S \in i^K_{P \cap K}\mathbb{C} \otimes i^K_{\bar{P} \cap K}\check{\mathbb{C}}. \tag{4.8}$$

4.2 Corollary. *With notation of Lemma 4.1, one has*

$$K^n(f) = \sum_{\tau \in \mathcal{E}_2(G)} \sum_{S \in \mathcal{B}_\tau} d(\tau) P^n_\tau(\tau \otimes \check{\tau}(f) S) \overline{P^n_\tau(S)}$$

$$+ \frac{1}{4i\pi} \sum_{\delta \in \widehat{M}_2} \sum_{S \in \mathcal{B}_{\bar{P},P}(E)} \int_{\mathcal{O}} P^n_{\delta_z}(\overline{\Pi}_{\delta_z}(f) S) \overline{P^n_{\delta_z}(S)} \frac{dz}{z},$$

where the sums over S are all finite and $\overline{\Pi}_{\delta_z} := \bar{i}^G_P \delta_z \otimes \bar{i}^G_{\bar{P}} \check{\delta}_z$.

Proof :
For $\tau \in \mathcal{E}_2(G)$ and $S \in \mathcal{B}_\tau$, one has $\tau(f_1) S \tau(\check{f}_2) = \tau \otimes \check{\tau}(f) S$. Therefore, since the functions we integrate are compactly supported, the assertion follows from Lemma 4.1. □

5 Regularized normalized periods

To determine the asymptotic expansion of the truncated kernel, we recall the notion of regularized period introduced in ([F]). It is defined by meromorphic continuation.
Let $z_0 \in \mathbb{C}^*$. Then, for $z \in \mathbb{C}^*$ such that $|zz_0| < 1$, the integral

$$\int_{M_H^+} \chi_{z_0}(m) \chi_z(m)(1 - u(m, n_0)) dm = \sum_{n > n_0} (zz_0)^n = \frac{(zz_0)^{n_0+1}}{1 - zz_0}$$

is well defined and has a meromorphic continuation at $z = 1$. Morever this meromorphic continuation is holomorphic on $\mathcal{V} - \{1\}$ with a simple pole at $z_0 = 1$.

Let $\delta \in \widehat{M_2}$. We consider now an holomorphic function $z \mapsto \varphi_z \in C^\infty(G)$ defined in a neighborhood $\mathcal{V}$ of $\mathcal{O}$ in $\mathbb{C}^*$ such that

$$\begin{gathered}\text{there exist a positive integer } n_0 \text{ and two holomorphic functions } z \in \mathcal{V} \mapsto \phi_z^i \in C^\infty(K_H \times K_H), i = 1,2 \text{ such that, for } k_1, k_2 \in K_H, \text{ and } m \in M_H^+ \text{ satisfying } h_{M_H}(m) > n_0, \text{ we have}\\ \delta_P(m)^{-1/2}\varphi_z(k_1 m k_2) = \delta_z(m)\phi_z^1(k_1,k_2) + \delta_{z^{-1}}(m)\phi_z^2(k_1,k_2).\end{gathered} \tag{5.1}$$

Recall that $\mathcal{M}(h)$ for $h \in H$ is an element in M_H^+ such that $h \in K_H\mathcal{M}(h)K_H$. By the integral formula (2.4), we deduce that for $|z| < \min(|z_0|, |z_0|^{-1})$, the integral

$$\int_H \varphi_{z_0}(h)\chi_z(\mathcal{M}(h))(1 - u(h,n_0))dh$$

$$= (1+q^{-1})\Big(\int_{K_H\times K_H}\phi_{z_0}^1(k_1,k_2)dk_1dk_2\Big)\int_{M_H^+}\delta(m)\chi_{z_0 z}(m)(1-u(m,n_0))dm$$

$$+(1+q^{-1})\Big(\int_{K_H\times K_H}\phi_{z_0}^2(k_1,k_2)dk_1dk_2\Big)\int_{M_H^+}\delta(m)\chi_{z_0^{-1}z}(m)(1-u(m,n_0))dm$$

is also well defined and has a meromorphic continuation at $z = 1$. Morever this meromorphic continuation is holomorphic on $\mathcal{V} - \{1\}$ with at most a simple pole at $z_0 = 1$. As $u(\cdot, n_0)$ is compactly supported, we deduce that the integral

$$\int_H \varphi_{z_0}(h)\chi_z(\mathcal{M}(h))dh = \int_H \varphi_{z_0}(h)\chi_z(\mathcal{M}(h))u(h,n_0)dh + \int_H \varphi_{z_0}(h)\chi_z(\mathcal{M}(h))(1-u(h,n_0))dh.$$

has a meromorphic continuation at $z = 1$ which we denote by

$$\int_H^* \varphi_{z_0}(h)dh.$$

The above discussion implies that $\int_H^* \varphi_{z_0}(h)dh$ is holomorphic on $\mathcal{V} - \{1\}$ with at most a simple pole at $z_0 = 1$.

The next result is established in ([F] Proposition 4.6), but we think that the proof is not complete. We thank E. Lapid who suggests us the proof below.

5.1 Proposition. *(H-invariance) For $x \in H$, we have*

$$\int_H^* \varphi_{z_0}(hx)dh = \int_H^* \varphi_{z_0}(h)dh.$$

Proof :

We fix $x \in H$. For z, z' in $\mathbb{C}^*$, we set $F(\varphi_{z_0}, z, z')(h) := \varphi_{z_0}(h)\chi_z(\mathcal{M}(h))\chi_{z'}(\mathcal{M}(hx^{-1}))$. By (5.1), for $k_1, k_2 \in K_H$, and $m \in M_H^+$ with $h_{M_H}(m) > n_0$, we have

$$\delta_P(m)^{-1/2}F(\varphi_{z_0}, z, z')(k_1 m k_2) = \phi_{z_0}^1(k_1,k_2)\delta(m)(z_0 z)^{h_{M_H}(m)} z'^{h_{M_H}(\mathcal{M}(k_1 m k_2 x^{-1}))}$$

$$+\phi^2_{z_0}(k_1,k_2)\delta(m)(z_0^{-1}z)^{h_{M_H}(m)}z'^{h_{M_H}(\mathcal{M}(k_1mk_2x^{-1}))}.$$

We can choose n_0 such that Lemma 2.1 is satisfied. Thus, for any $k_2 \in K_H$, there exists $X_{k_2x^{-1}} \in \mathbb{R}$ such that, for any $m \in M_H^+$ satisfying $1-u(m,n_0) \neq 0$, we have $h_{M_H}(\mathcal{M}(k_1mk_2x^{-1})) = h_{M_H}(m) + X_{k_2x^{-1}}$. We deduce that

$$\delta_P(m)^{-1/2}F(\varphi_{z_0},z,z')(k_1mk_2)(1-u(m,n_0)) = \phi^1_{z_0}(k_1,k_2)\delta(m)(z_0zz')^{h_{M_H}(m)}z'^{X_{k_2x^{-1}}}$$

$$+\phi^2_{z_0}(k_1,k_2)\delta(m)(z_0^{-1}zz')^{h_{M_H}(m)}z'^{X_{k_2x^{-1}}}.$$

Therefore, by Hartogs' Theorem and the same argument as above, the function

$$(z_0,z,z') \mapsto \int_H \varphi_{z_0}(h)\chi_z(\mathcal{M}(h))\chi_{z'}(\mathcal{M}(hx^{-1}))dh$$

is well defined for $|z_0zz'| < 1$, and has a meromorphic continuation on $\mathcal{V} \times (\mathbb{C}^*)^2$. We denote by $I(\varphi_{z_0},z,z')$ this meromorphic continuation. Moreover, for $z_0 \neq 1$, the function $(z,z') \mapsto I(\varphi_{z_0},z,z')$ is holomorphic in a neighborhood of $(1,1)$.
For $|z_0z| < 1$, we have $I(\varphi_{z_0},z,1) = \int_H \varphi_{z_0}(h)\chi_z(\mathcal{M}(h))dh$. Hence we deduce that

$$I(\varphi_{z_0},1,1) = \int_H^* \varphi_{z_0}(h)dh.$$

On the other hand, we have $I(\varphi_{z_0},1,z') = \int_H \varphi_{z_0}(hx)\chi_{z'}(\mathcal{M}(h))dh$ for $|z_0z'| < 1$, therefore, one obtains

$$I(\varphi_{z_0},1,1) = \int_H^* \varphi_{z_0}(hx)dh.$$

This finishes the proof of the proposition. □

We will apply this to normalized Eisenstein integrals. Let $\delta \in \widehat{M_2}$ and $z \in \mathbb{C}^*$. Recall that we have defined the operator $C_{P,P}(1,\delta_z)$ by

$$C_{P,P}(1,\delta_z) := Id \otimes A(\bar{P},P,\check{\delta}_z) \in \mathrm{Hom}_G\big(i_P^G\mathbb{C}_{\delta_z}\otimes i_P^G\check{\mathbb{C}}_{\delta_z}, i_P^G\mathbb{C}_{\delta_z}\otimes i_{\bar{P}}^G\check{\mathbb{C}}_{\delta_z}\big).$$

We set

$$C_{P,P}(w,\delta_z) := A(P,\bar{P},w\delta_z)\lambda(w)\otimes\lambda(w) \in \mathrm{Hom}_G\big(i_P^G\mathbb{C}_{\delta_z}\otimes i_P^G\check{\mathbb{C}}_{\delta_z}, i_P^G\mathbb{C}_{w\delta_z}\otimes i_{\bar{P}}^G\check{\mathbb{C}}_{w\delta_z}\big).$$

where $\lambda(w)$ is the left translation by w which induces an isomorphism from $i_P^G\mathbb{C}_{\delta_z}$ to $i_{\bar{P}}^G\mathbb{C}_{w\delta_z}$. For $s \in W^G$, we define

$$C^0_{P,P}(s,\delta_z) := C_{P,P}(s,\delta_z)\circ C_{P,P}(1,\delta_z)^{-1} \in \mathrm{Hom}_G\big(i_P^G\mathbb{C}_{\delta_z}\otimes i_{\bar{P}}^G\check{\mathbb{C}}_{\delta_z}, i_P^G\mathbb{C}_{s\delta_z}\otimes i_{\bar{P}}^G\check{\mathbb{C}}_{s\delta_z}\big). \quad (5.2)$$

In particular, $C^0_{P,P}(1,\delta_z)$ is the identity map of $i_P^G\mathbb{C}_{\delta_z}\otimes i_{\bar{P}}^G\check{\mathbb{C}}_{\delta_z}$. By arguments analogous to those of ([W] Lemme V.3.1.), we obtain that

$$\text{for } s \in W^G, \text{ the rational operator } C^0_{P|P}(s, \delta_z) \text{ is regular on } \mathcal{O}. \tag{5.3}$$

Let $S \in i^K_{P\cap K}\mathbb{C} \otimes i^K_{\bar{P}\cap K}\check{\mathbb{C}}$. By (3.7), the normalized Eisenstein integral $E^0(P, \delta_z, S_z)$ is holomorphic in a neighborhood $\mathcal{V}$ of $\mathcal{O}$. We may and will assume that $\mathcal{V}$ is invariant by the map $z \mapsto z^{-1}$. By ([He] Theorem 1.3.1) applied to $\lambda(k_1^{-1})\rho(k_2)E^0(P, \delta_z, S_z), k_1, k_2 \in K_H$, there exists a positive integer n_0 such that, for $k_1, k_2 \in K_H$, and $m \in M_H^+$ satisfying $h_{M_H}(m) > n_0$, we have

$$\delta_P(m)^{-1/2}E^0(P, \delta_z, S_z)(k_1 m k_2)$$
$$= \delta(m)\Big(\chi_z(m)\mathrm{tr}\big(\big[C^0_{P,P}(1, \delta_z)S_z\big](k_1, k_2)\big) + \chi_{z^{-1}}(m)\mathrm{tr}\big(\big[C^0_{P,P}(w, \delta_z)S_z\big](k_1, k_2)\big)\Big).$$

Therefore, the normalized Eisenstein integral satisfies (5.1). Hence, we can define the normalized regularized period by

$$P_{\delta_z}(S) := \int_H^* E^0(P, \delta_z, S_z)(h)dh, \quad S \in i^K_{P\cap K}\mathbb{C} \otimes i^K_{\bar{P}\cap K}\check{\mathbb{C}}. \tag{5.4}$$

The above discussion implies that $P_{\delta_z}(S)$ is a meromorphic function on the neighborhood $\mathcal{V}$ of $\mathcal{O}$ which is holomorphic on $\mathcal{V} - \{1\}$.

For $s \in W^G$ and $S \in i^K_{P\cap K}\mathbb{C} \otimes i^K_{\bar{P}\cap K}\check{\mathbb{C}}$, we set

$$C(s, \delta_z)(S) := (1 + q^{-1}) \int_{K_H \times K_H} \mathrm{tr}\big(\big[C^0_{P,P}(s, \delta_z)S_z\big](k_1, k_2)\big)dk_1 dk_2. \tag{5.5}$$

By the same argument as in ([F] Proposition 4.7), we have the following relations between the truncated period and the normalized regularized period.

$$\text{If } \delta_{|\mathrm{F}^\times} \neq 1 \text{ then, for } n \text{ large enough, we have } P_{\delta_z}(S) = P^n_{\delta_z}(S), \tag{5.6}$$

If $\delta_{|\mathrm{F}^\times} = 1$ then, for n large enough, we have

$$P_{\delta_z}(S) = P^n_{\delta_z}(S) + \frac{z^{n+1}}{1-z}C(1, \delta_z)(S) + \frac{z^{-(n+1)}}{1-z^{-1}}C(w, \delta_z)(S). \tag{5.7}$$

The following Lemma is analoguous to ([F] Lemma 4.8).

5.2 Lemma. *Let $z \in \mathbb{C}^*$ and $S \in i^K_{P\cap K}\mathbb{C} \otimes i^K_{\bar{P}\cap K}\check{\mathbb{C}}$.*

1. *If $\delta_{|F^\times} \neq 1$ and $\delta_{|E^1} \neq 1$ then, for n large enough, we have*

$$P_{\delta_z}(S) = P^n_{\delta_z}(S) = 0.$$

2. *If $\delta_{|F^\times} \neq 1$ and $\delta_{|E^1} = 1$ then, for n large enough, we have*

$$P_{\delta_z}(S) = P^n_{\delta_z}(S).$$

3. *If $\delta_{|F^\times} = 1$ and $\delta_{|E^1} \neq 1$ then $P_{\delta_z}(S) = 0$ whenever it is defined, and*

$$C(1, \delta_1)(S) = C(w, \delta_1)(S).$$

4. *If $\delta_{|F^\times} = 1$ and $\delta_{|E^1} = 1$ then $\delta^2 = 1$. We have $C(1, \delta_1)(S) = -C(w, \delta_1)(S)$ and the regularized normalized period $P_{\delta_z}(S)$ is meromorphic with a unique pole at $z = 1$ which is simple.*

Proof :

Case 2 follows from (5.6). By ([JLR] Proposition 22), if $\delta_{|E^1} \neq 1$ and $z \neq 1$ then the representation $i_P^G \delta_z$ admits no nontrivial H-invariant linear form. Thus in that case, Proposition 5.1 implies $P_{\delta_z}(S) = 0$ whenever it is defined. We deduce case 1 from (5.6) and in case 3, it follows from (5.7) that

$$P^n_{\delta_z}(S) = -\Big(\frac{z^{n+1}}{1-z} C(1, \delta_z)(S) + \frac{z^{-(n+1)}}{1-z^{-1}} C(w, \delta_z)(S)\Big).$$

Since $P^n_{\delta_z}(S)$ and $C(s, \delta_z)(S)$ for $s \in W^G$ are holomorphic functions at $z = 1$, and

$$\begin{aligned} \mathrm{Res}(\frac{z^{n+1}}{1-z} C(1, \delta_z)(S), z = 1) &= -C(1, \delta_1)(S), \\ \mathrm{Res}(\frac{z^{-(n+1)}}{1-z^{-1}} C(w, \delta_z)(S), z = 1) &= C(w, \delta_1)(S), \end{aligned} \tag{5.8}$$

we deduce the result in the case 3.

In case 4, we obtain easily $\delta^2 = 1$. By ([W] Corollaire IV.1.2.), the intertwining operator $A(\bar{P}, P, \delta_z)$ has a simple pole at $z = 1$. Thus the function $\mu(\delta_z)$ has a zero of order 2 at $z = 1$. In that case, by ([S], proof of Theorem 5.4.2.1), the operators $C_{P|P}(s, \delta_z)$ for $s \in W^G$ have a simple pole at $z = 1$ and

$$\mathrm{Res}(C_{P|P}(1, \delta_z), z = 1) = -\mathrm{Res}(C_{P|P}(w, \delta_z), z = 1).$$

Therefore, if we set $T_z := (z-1)C_{P|P}(1, \delta_z)$ and $U_z := (z-1)C_{P|P}(w, \delta_z)$, then U_z and T_z^{-1} are holomorphic near $z = 1$ and $T_1 = -U_1$ as $\delta^2 = 1$. By definition (cf. (5.2)), we have $C^0_{P|P}(w, \delta_z) = U_z T_z^{-1}$. Hence, one deduces that $C^0_{P|P}(w, \delta_1) = -Id = -C^0_{P|P}(1, \delta_1)$, where Id is the identity map of $i_P^G \mathbb{C}_{\delta_1} \otimes i_P^G \check{\mathbb{C}}_{\delta_1}$. We deduce the first assertion in case 4 from the definition of $C(s, \delta_z)(S)$ (cf.(5.5)).

Since $P^n_{\delta_z}(S)$ and $C(s, \delta_z)(S)$ for $s \in W^G$ are holomorphic functions at $z = 1$, the last assertion follows from (5.7), (5.8) and the above result. This finishes the proof of the Lemma. □

6 Preliminary Lemma

In this part, we prove a preliminary lemma which will allow us to get the asymptotic expansion of the truncated kernel in terms of regularized normalized periods.

Let $\mathcal{V}$ be a neighborhood of $\mathcal{O}$ in $\mathbb{C}^*$. We assume that $\mathcal{V}$ is invariant by the map $z \mapsto \bar{z}^{-1}$. Let f be a meromorphic function on $\mathcal{V}$. We assume that f has at most a pole at $z = 1$ in $\mathcal{V}$.
For $r < 1$ (resp. $r > 1$) such that f is defined on the set of complex numbers of modulus r, then the integral $\displaystyle\int_{|z|=r} f(z)dz$ does not depend of the choice of r. We set

$$\int_{\mathcal{O}^-} f(z)dz := \int_{|z|=r} f(z)dz, \quad r < 1. \tag{6.1}$$

and

$$\int_{\mathcal{O}^+} f(z)dz := \int_{|z|=r} f(z)dz, \quad r > 1. \tag{6.2}$$

Notice that we have

$$\int_{\mathcal{O}^+} f(z)dz - \int_{\mathcal{O}^-} f(z)dz = 2i\pi\mathrm{Res}(f(z), z = 1). \tag{6.3}$$

The two following properties are easily consequences of the definitions:

$$\lim_{n\to+\infty} \int_{\mathcal{O}^-} z^n f(z)dz = 0, \quad \text{and} \quad \lim_{n\to+\infty} \int_{\mathcal{O}^+} z^{-n} f(z)dz = 0 \tag{6.4}$$

We have assumed that $\mathcal{V}$ is invariant by the map $z \to \bar{z}^{-1}$. Then, the function $\tilde{f}(z) := \overline{f(\bar{z}^{-1})}$ is also a meromorphic function on $\mathcal{V}$ with at most a pole at $z = 1$ and it satisfies $\tilde{f}(z) = \overline{f(z)}$ for $z \in \mathcal{O}$.

Let $c(s, z)$ and $c'(s, z)$, for $s \in W^G$ be holomorphic functions on $\mathcal{V}$ such that $c(s, 1) \neq 0$ and $c'(s, 1) \neq 0$. Let p and p' be two meromorphic functions on $\mathcal{V}$ with at most a pole at $z = 1$. We set

$$\begin{aligned} p_n(z) &:= p(z) - \Big[\frac{z^{n+1}}{1-z}c(1, z) + \frac{z^{-(n+1)}}{1-z^{-1}}c(w, z)\Big] \\ &\text{and} \\ p'_n(z) &:= p'(z) - \Big[\frac{z^{n+1}}{1-z}c'(1, z) + \frac{z^{-(n+1)}}{1-z^{-1}}c'(w, z)\Big]. \end{aligned} \tag{6.5}$$

6.1 Lemma. *We assume that p_n and p'_n are holomorphic on $\mathcal{V}$ and that either p and p' are vanishing functions or $c(1, 1) = -c(w, 1)$ and $c'(1, 1) = -c'(w, 1)$. Then, the integral*

$$\int_{\mathcal{O}} p_n(z)\overline{p'_n(z)}\frac{dz}{z}$$

is asymptotic as n approaches $+\infty$ to the sum of

$$\int_{\mathcal{O}^-} \Big(p(z)\tilde{p}'(z) + \frac{c(1, z)\tilde{c}'(1, z)}{(1-z)(1-z^{-1})} + \frac{c(w, z)\tilde{c}'(w, z)}{(1-z)(1-z^{-1})}\Big)\frac{dz}{z}, \tag{6.6}$$

$$-2i\pi\Big[\frac{d}{dz}\Big(c(w, z)\tilde{c}'(1, z)\Big)\Big]_{z=1} + 2i\pi\Big[\frac{d}{dz}\Big(c(w, z)(z-1)\tilde{p}'(z) + \tilde{c}'(1, z)(z-1)p(z)\Big)\Big]_{z=1}, \tag{6.7}$$

and

$$2i\pi(2n + 1)c(w, 1)\tilde{c}'(1, 1) - 2i\pi(n + 1)\big(c(w, 1)Res(\tilde{p}', z = 1) + \tilde{c}'(1, 1)Res(p, z = 1)\big). \tag{6.8}$$

Proof :

Since p_n and $\tilde{p'}_n$ are holomorphic functions on $\mathcal{V}$, we have

$$\int_{\mathcal{O}} p_n(z)\overline{p'_n(z)}\frac{dz}{z} = \int_{\mathcal{O}^-} p_n(z)\tilde{p'_n}(z)\frac{dz}{z}$$

$$= \int_{\mathcal{O}^-} \Big(p(z) - \frac{z^{n+1}}{1-z}c(1,z) - \frac{z^{-(n+1)}}{1-z^{-1}}c(w,z)\Big)\Big(\tilde{p'}(z) - \frac{z^{-(n+1)}}{1-z^{-1}}\tilde{c'}(1,z) - \frac{z^{n+1}}{1-z}\tilde{c'}(w,z)\Big)\frac{dz}{z}$$

$$= \int_{\mathcal{O}^-} \Big(p(z)\tilde{p'}(z) + \frac{c(1,z)\tilde{c'}(1,z)}{(1-z)(1-z^{-1})} + \frac{c(w,z)\tilde{c'}(w,z)}{(1-z)(1-z^{-1})}\Big)\frac{dz}{z}$$

$$+ \int_{\mathcal{O}^-} z^{2(n+1)}\frac{c(1,z)\tilde{c'}(w,z)}{(1-z)^2}\frac{dz}{z} - \int_{\mathcal{O}^-} z^{n+1}\Big(\frac{c(1,z)\tilde{p'}(z) + p(z)\tilde{c'}(w,z)}{1-z}\Big)\frac{dz}{z}$$

$$+ \int_{\mathcal{O}^-} z^{-2(n+1)}\frac{c(w,z)\tilde{c'}(1,z)}{(1-z^{-1})^2}\frac{dz}{z} - \int_{\mathcal{O}^-} z^{-(n+1)}\Big(\frac{c(w,z)\tilde{p'}(z) + p(z)\tilde{c'}(1,z)}{1-z^{-1}}\Big)\frac{dz}{z}.$$

By (6.4), the second and third terms of the right hand side converge to 0 as n approaches $+\infty$.

By (6.3), one has

$$\int_{\mathcal{O}^-} z^{-2(n+1)}\frac{c(w,z)\tilde{c'}(1,z)}{(1-z^{-1})^2}\frac{dz}{z} = \int_{\mathcal{O}^+} z^{-2(n+1)}\frac{c(w,z)\tilde{c'}(1,z)}{(1-z^{-1})^2}\frac{dz}{z} - 2i\pi \mathrm{Res}(z^{-2(n+1)}\frac{c(w,z)\tilde{c'}(1,z)}{z(1-z^{-1})^2}, z=1).$$

Let $\phi(z) := z^{-2(n+1)}\frac{c(w,z)\tilde{c'}(1,z)}{z(1-z^{-1})^2} = z^{-(2n+1)}\frac{c(w,z)\tilde{c'}(1,z)}{(z-1)^2}$. Since $c(w,z)$ and $\tilde{c'}(1,z)$ are holomorphic functions on $\mathcal{V}$, the function ϕ has a unique pole of order 2 at $z=1$. Thus, we obtain

$$\mathrm{Res}(\phi, z=1) = \Big[\frac{d}{dz}\Big((z-1)^2\phi(z)\Big)\Big]_{z=1} = -(2n+1)c(w,1)\tilde{c'}(1,1) + \Big[\frac{d}{dz}\Big(c(w,z)\tilde{c'}(1,z)\Big)\Big]_{z=1}.$$

We deduce from (6.4) that

$$\int_{\mathcal{O}^-} z^{-2(n+1)}\frac{c(w,z)\tilde{c'}(1,z)}{(1-z^{-1})^2}\frac{dz}{z} = 2i\pi(2n+1)c(w,1)\tilde{c'}(1,1) - 2i\pi\Big[\frac{d}{dz}\Big(c(w,z)\tilde{c'}(1,z)\Big)\Big]_{z=1} + \epsilon_1(n), \tag{6.9}$$

where $\lim_{n\to+\infty}\epsilon_1(n) = 0$.

When p and p' are vanishing functions, we obtain the result of the Lemma. Otherwise, by (6.5) and our assumptions, the function $\frac{c(w,z)\tilde{p'}(z) + p(z)\tilde{c'}(1,z)}{1-z^{-1}}$ is a meromorphic function with a unique pole of order 2 at $z=1$. Applying the same argument as above, we obtain

$$\int_{\mathcal{O}^-} z^{-(n+1)}\Big(\frac{c(w,z)\tilde{p'}(z) + p(z)\tilde{c'}(1,z)}{1-z^{-1}}\Big)\frac{dz}{z}$$

$$= \int_{\mathcal{O}^+} z^{-(n+1)}\Big(\frac{c(w,z)\tilde{p'}(z) + p(z)\tilde{c'}(1,z)}{1-z^{-1}}\Big)\frac{dz}{z} - 2i\pi\Big[\frac{d}{dz}\Big(z^{-(n+1)}(z-1)(c(w,z)\tilde{p'}(z) + p(z)\tilde{c'}(1,z))\Big)\Big]_{z=1}$$

$$= 2i\pi(n+1)\big(c(w,1)\mathrm{Res}(\tilde{p'}, z=1) + \mathrm{Res}(p, z=1)\tilde{c'}(1,1)\big)$$
$$-2i\pi\Big[\frac{d}{dz}\Big(c(w,z)(z-1)\tilde{p'}(z) + (z-1)p(z)\tilde{c'}(1,z)\Big)\Big]_{z=1} + \epsilon_2(n),$$
where $\lim_{n\to+\infty} \epsilon_2(n) = 0$.

Therefore, we obtain the Lemma by (6.9) and the above result. □

7 Spectral side of a local relative trace formula

We recall the spectral expression of the truncated kernel obtained in Corollary 4.2:

$$K^n(f) = \sum_{\tau\in\mathcal{E}_2(G)} \sum_{S\in\mathcal{B}_\tau} d(\tau)P^n_\tau(\tau\otimes\check{\tau}(f)S)\overline{P^n_\tau(S)}$$
$$+\frac{1}{4i\pi}\sum_{\delta\in\widehat{M}_2}\sum_{S\in\mathcal{B}_{\bar{P},P}(E)}\int_{\mathcal{O}} P^n_{\delta_z}(\overline{\Pi}_{\delta_z}(f)S)\overline{P^n_{\delta_z}(S)}\frac{dz}{z},$$

where the sums over S are all finite and $\overline{\Pi}_{\delta_z} := \bar{i}^G_P\delta_z \otimes \bar{i}^G_{\bar{P}}\check{\delta}_z$.
By ([F] Lemma 4.10), if $(\tau, V_\tau) \in \mathcal{E}_2(G)$ and $S \in \mathrm{End}_{fin.rk}(V_\tau)$, then

$$\lim_{n\to+\infty} P^n_\tau(S) = \int_H \mathrm{tr}(\tau(h)S)dh. \quad (7.1)$$

We consider now the second term of the above expression of $K^n(f)$. Let $\delta \in \widehat{M}_2$ and $S \in i^K_{P\cap K}\mathbb{C} \otimes i^K_{\bar{P}\cap K}\check{\mathbb{C}}$. We keep notation of the previous section. In particular, for $z \in \mathbb{C}^*$, we have $\tilde{C}(s,\delta_z)(S) = \overline{C(s,\delta_{\bar{z}^{-1}})(S)}$ and $\tilde{P}_{\delta_z}(S) = \overline{P_{\bar{z}^{-1}}(S)}$. By definition of δ_z, we have $\delta_1 = \delta$.

7.1 Proposition. *Let $S \in i^K_{P\cap K}\mathbb{C} \otimes i^K_{\bar{P}\cap K}\check{\mathbb{C}}$. We set $S'_z := \overline{\Pi}_{\delta_z}(f)S$.*

1. *If $\delta_{|\mathrm{F}^\times} \neq 1$ and $\delta_{|\mathrm{E}^1} \neq 1$ then, for $n \in \mathbb{N}$ large enough, one has*
$$\int_{\mathcal{O}} P^n_{\delta_z}(S'_z)\overline{P^n_{\delta_z}(S)}\frac{dz}{z} = 0.$$

2. *If $\delta_{|F^\times} \neq 1$ and $\delta_{|E^1} = 1$ then*
$$\lim_{n\to+\infty}\int_{\mathcal{O}} P^n_{\delta_z}(S'_z)\overline{P^n_{\delta_z}(S)}\frac{dz}{z} = \int_{\mathcal{O}} P_{\delta_z}(S'_z)\overline{P_{\delta_z}(S)}\frac{dz}{z}.$$

3. *Assume that $\delta_{|F^\times} = 1$ and $\delta_{|E^1} \neq 1$. Then*
$$\int_{\mathcal{O}} P^n_{\delta_z}(S'_z)\overline{P^n_{\delta_z}(S)}\frac{dz}{z}$$
is asymptotic when n approaches $+\infty$ to
$$2i\pi(2n+1)C(1,\delta)(S'_1)\overline{C(1,\delta)(S)}$$

$$+\int_{\mathcal{O}^-}\Big(\frac{C(1,\delta_z)(S'_z)\tilde{C}(1,\delta_z)(S)}{(1-z)(1-z^{-1})}+\frac{C(w,\delta_z)(S'_z)\tilde{C}(w,\delta_z)(S)}{(1-z)(1-z^{-1})}\Big)\frac{dz}{z}$$

$$-2i\pi\frac{d}{dz}\Big[C(w,\delta_z)(S'_z)\tilde{C}(1,\delta_z)(S)\Big]_{z=1}.$$

4. Assume that $\delta_{|F^\times}=1$ *and* $\delta_{|E^1}=1$*. Then*

$$\int_{\mathcal{O}} P^n_{\delta_z}(S'_z)\overline{P^n_{\delta_z}(S)}\frac{dz}{z}$$

is asymptotic when n *approaches* $+\infty$ *to*

$$2i\pi(2n+3)C(1,\delta)(S'_1)\overline{C(1,\delta)(S)}$$

$$+\int_{\mathcal{O}^-}\Big(P_{\delta_z}(S'_z)\overline{P_{\delta_z}(S)}+\frac{C(1,\delta_z)(S'_z)\tilde{C}(1,\delta_z)(S)}{(1-z)(1-z^{-1})}+\frac{C(w,\delta_z)(S'_z)\tilde{C}(w,\delta_z)(S)}{(1-z)(1-z^{-1})}\Big)\frac{dz}{z}$$

$$-2i\pi\frac{d}{dz}\Big[C(w,\delta_z)(S'_z)\tilde{C}(1,\delta_z)(S)\Big]_{z=1}$$

$$+2i\pi\Big[\frac{d}{dz}\Big((z-1)P_{\delta_z}(S'_z)\tilde{C}(1,\delta_z)(S)+C(w,\delta_z)(S'_z)(z-1)\tilde{P}_{\delta_z}(S)\Big)\Big]_{z=1}.$$

Proof. The two first assertions are immediate consequences of Lemma 5.2. To prove *3.* and 4., we set:

$$p_n(z):=P^n_{\delta_z}(S'_z(f)),\quad p'_n(z):=P^n_{\delta_z}(S),\qquad p(z):=P_{\delta_z}(S'_z(f)),\quad p'(z):=P_{\delta_z}(S)$$

and $c(s,z):=C(s,\delta_z)(S'_z(f))$, $\;c'(s,z):=C(s,\delta_z)(S)$ for $s\in W^G$.

By (5.7) and Lemma 5.2, these functions satisfy (6.5) and we can apply Lemma 6.1. The result in case *3* follows immediately since $p(z)=p'(z)=0$ by Lemma 5.2.

In case *4*, we have $c(1,1)=-c(w,1)$ and $c'(1,1)=-c'(w,1)$ by Lemma 5.2. Moreover, the relations (6.5) give $\mathrm{Res}(p,z=1)=-c(1,1)+c(w,1)$ and $\mathrm{Res}(\tilde{p'},z=1)=c'(1,1)-c'(w,1)$. Hence, we obtain

$$2i\pi(2n+1)c(w,1)\tilde{c'}(1,1)-2i\pi(n+1)\big(c(w,1)\mathrm{Res}(\tilde{p'},z=1)+\tilde{c'}(1,1)\mathrm{Res}(p,z=1)\big).$$

$$=2i\pi(2n+3)c(1,1)\tilde{c'}(1,1),$$

and the result in that case follows from Lemma 6.1. □

To describe the spectral side of our local relative trace formula, we introduce generalized matrix coefficients.

Let (π,V) be a smooth unitary representation of G. We denote by (π',V') its dual representation. Let ξ and ξ' be two linear forms on V. For $f\in C_c^\infty(G)$, the linear form $\pi'(\check{f})\xi$ belongs to the smooth dual $\check{V}$ of V ([R] Théorème III.3.4 and I.1.2). The scalar product on V induces an isomorphism $j: v\mapsto(\cdot,v)$ from the conjugate complex vector space $\overline{V}$ of V and $\check{V}$, which intertwines the complex conjugate of π and $\check{\pi}$ as π is unitary. One has

$$\check{v}(v) = (v, j^{-1}(\check{v})), \quad v \in V, \check{v} \in \check{V}.$$

Therefore, for $v \in V$, we have

$$\big(\pi'(\check{f})\xi\big)(v) = \xi\big(\pi(f)v\big) = (v, j^{-1}\big(\pi'(\check{f})\xi\big)).$$

As $\pi(f)$ is an operator of finite rank, we have for any orthonormal basis $\mathcal{B}$ of V

$$j^{-1}\big(\pi'(\check{f})\xi\big) = \sum_{v \in \mathcal{B}} (\pi'(\check{f})\xi)(v) \cdot v \tag{7.2}$$

where the sum over v is finite, and $(\lambda, v) \mapsto \lambda \cdot v$ is the action of $\mathbb{C}$ on $\overline{V}$.
Let $\overline{\xi'}$ be the linear form on $\overline{V}$ defined by $\overline{\xi'}(u) = \overline{\xi'(u)}$. We define the generalized matrix coefficient $m_{\xi,\xi'}$ by

$$m_{\xi,\xi'}(f) = \overline{\xi'}\Big(j^{-1}\big(\pi'(\check{f})\xi\big)\Big).$$

Then, by (7.2), we obtain

$$m_{\xi,\xi'}(f) = \sum_{v \in \mathcal{B}} \xi(\pi(f)v)\overline{\xi'}(v). \tag{7.3}$$

Hence, this sum is independent of the choice of the basis $\mathcal{B}$.

Let $z \in \mathbb{C}^*$. We set $(\Pi_z, V_z) := (i_P^G \delta_z \otimes i_{\bar{P}}^G \check{\delta_z}, i_P^G \mathbb{C}_{\delta_z} \otimes i_{\bar{P}}^G \mathbb{C}_{\check{\delta_z}})$. We denote by $(\overline{\Pi}_z, V)$ its compact realization. We define meromorphic linear forms on V_z using the isomorphism $V_z \simeq V$.

7.2 Lemma. *Let ξ_z and ξ'_z be two linear forms on V which are meromorphic in z on a neighborhood $\mathcal{V}$ of $\mathcal{O}$. Let $\mathcal{B}$ be an orthonormal basis of V. Then, for $f \in C_c^\infty(G \times G)$, the sum*

$$\sum_{S \in \mathcal{B}} \xi_z(\overline{\Pi}_z(f)S)\overline{\xi_{\bar{z}^{-1}}(S)}$$

is a finite sum over S which is independent of the choice of the basis $\mathcal{B}$.

Proof :

For $z \in \mathcal{O}$, the representation Π_z is unitary. Hence (7.3) gives the Lemma in that case. Since the linear forms ξ_z and ξ'_z are meromorphic on $\mathcal{V}$, we deduce the result of the Lemma for any z in $\mathcal{V}$ by meromorphic continuation. □

With notation of the Lemma, we define, for $z \in \mathcal{V}$, the generalized matrix coefficient $m_{\xi_z,\xi'_{\bar{z}^{-1}}}$ associated to (ξ_z, ξ'_z) by

$$m_{\xi_z,\xi'_{\bar{z}^{-1}}}(f) := \sum_{S \in \mathcal{B}} \xi_z(\overline{\Pi}_z(f)S)\overline{\xi_{\bar{z}^{-1}}(S)}.$$

Therefore, using Proposition 7.1, we can deduce the asymptotic behavior of the truncated kernel in terms of generalized matrix coefficients.

7.3 Theorem. *As n approaches $+\infty$, the truncated kernel $K^n(f)$ is asymptotic to*

$$n \sum_{\delta \in \widehat{M}_2, \delta_{|\mathrm{F}^\times}=1} m_{C(1,\delta),C(1,\delta)}(f) + \sum_{\tau \in \mathcal{E}_2(G)} d(\tau) m_{P_\tau,P_\tau}(f) + \frac{1}{4i\pi} \sum_{\delta \in \widehat{M}_2, \delta_{|\mathrm{F}^\times} \neq 1, \delta_{|\mathrm{E}^1}=1} \int_{\mathcal{O}} m_{P_{\delta_z},P_{\delta_z}}(f) \frac{dz}{z}$$

$$+ \frac{1}{4i\pi} \sum_{\delta \in \widehat{M}_2, \delta_{|\mathrm{F}^\times}=1, \delta_{|E^1} \neq 1} R_\delta(f) + \int_{\mathcal{O}^-} \frac{m_{C(1,\delta_z),C(1,\delta_{\bar{z}^{-1}})}(f) + m_{C(w,\delta_z),C(w,\delta_{\bar{z}^{-1}})}(f)}{(1-z)(1-z^{-1})} \frac{dz}{z}$$

$$+ \frac{1}{4i\pi} \sum_{\delta \in \widehat{M}_2, \delta_{|\mathrm{F}^\times}=\delta_{|E^1}=1} \tilde{R}_\delta(f) + \int_{\mathcal{O}^-} \frac{m_{C(1,\delta_z),C(1,\delta_{\bar{z}^{-1}})}(f) + m_{C(w,\delta_z),C(w,\delta_{\bar{z}^{-1}})}(f)}{(1-z)(1-z^{-1})} \frac{dz}{z}$$

$$+ \int_{\mathcal{O}^-} m_{P_{\delta_z},P_{\delta_{\bar{z}^{-1}}}}(f) \frac{dz}{z}.$$

where

$$R_\delta(f) := 2i\pi \Big(m_{C(1,\delta),C(1,\delta)}(f) - \Big[\frac{d}{dz} m_{C(w,\delta_z),C(1,\delta_{\bar{z}^{-1}})}(f) \Big]_{z=1} \Big),$$

$$\tilde{R}_\delta(f) = 2i\pi \Big(3\, m_{C(1,\delta),C(1,\delta)}(f) - \Big[\frac{d}{dz} m_{C(w,\delta_z),C(1,\delta_{\bar{z}^{-1}})}(f) \Big]_{z=1}$$

$$+ \Big[\frac{d}{dz}(z-1)\Big(m_{P_{\delta_z},C(1,\delta_{\bar{z}^{-1}})}(f) + m_{C(w,\delta_z),P_{\bar{z}^{-1}}}(f) \Big) \Big]_{z=1} \Big),$$

$$P_\tau(S) = \int_H tr(\tau(h)S)dh, \quad S \in End_{fin.rk}(V_\tau),$$

$$P_{\delta_z}(S) = \int_H^* E^0(P, \delta_z, S_z)(h)dh, \quad S \in i_{P\cap K}^K \mathbb{C} \otimes i_{P \cap K}^K \check{\mathbb{C}}$$

and

$$C(s,\delta_z)(S) := (1+q^{-1}) \int_{K_H \times K_H} tr\big(\big[C^0_{P,P}(s,\delta_z) S_z \big](k_1,k_2) \big) dk_1 dk_2, \quad s \in W^G$$

8 A local relative trace formula for $PGL(2)$

We make precise the geometric expansion of the truncated kernel obtained in ([DHSo] Theorem 2.3) for $\underline{H} := PGL(2)$. This geometric expansion depends on orbital integrals of f_1 and f_2, and on a weight function v_L where $L = H$ or M. To recall the definition of these objects, we need to introduce some notation.

If $\underline{J}$ is an algebraic group defined over F, we denote by J its group of points over F and we identify $\underline{J}$ with the group of points of $\underline{J}$ over an algebraic closure of F. Let $\underline{J}_H$ be an algebraic subgroup of $\underline{H}$ defined over F. We denote by $\underline{J} := \mathrm{Res}_{\mathrm{E/F}}(\underline{J}_H \times_{\mathrm{F}} \mathrm{E})$ the restriction of scalars of $\underline{J}_H$ from E to F. Then, the group $J := \underline{J}(\mathrm{F})$ is isomorphic to $\underline{J}_H(\mathrm{E})$.

The nontrivial element of the Galois group of E/F induces an involution σ of $\underline{G}$ defined over F.

We denote by $\underline{\mathcal{P}}$ the connected component of 1 in the set of x in $\underline{G}$ such that $\sigma(x) = x^{-1}$. A torus $\underline{A}$ of $\underline{G}$ is called a σ-torus if $\underline{A}$ is a torus defined over F contained in $\underline{\mathcal{P}}$. Let $\underline{S}_H$ be a maximal torus of $\underline{H}$. We denote by $\underline{S}_\sigma$ the connected component of $\underline{S} \cap \underline{\mathcal{P}}$. Then $\underline{S}_\sigma$ is a maximal σ-torus defined over F and the map $S_H \mapsto S_\sigma$ is a bijective correspondence between H-conjugacy classes of maximal tori of H and H-conjugacy classes of maximal σ-tori of G. (cf. [DHSo] 1.2).

Each maximal torus of H is either anisotropic or H-conjugate to M. We fix $\mathcal{T}_H$ a set of representatives for the H-conjugacy classes of maximal anisotropic torus in H.

By ([DHSo] (1.28)), for each maximal torus S_H of H, we can fix a finite set of representatives $\kappa_S = \{x_m\}$ of the (H, S_σ)-double cosets in $\underline{H}\underline{S}_\sigma \cap G$ such that each element x_m may be written $x_m = h_m a_m^{-1}$ where $h_m \in \underline{H}$ centralizes the split component A_S of S_H and $a_m \in \underline{S}_\sigma$.

The orbital integral of a compactly supported smooth function is defined on the set $G^{\sigma-reg}$ of σ-regular points of G, that is the set of point x in G such that $\underline{H}x\underline{H}$ is Zariski closed and of maximal dimension. The set $G^{\sigma-reg}$ can be described in terms of maximal σ-tori as follows. If $\underline{S}_H$ is a maximal torus of $\underline{H}$, we denote by $\underline{\mathfrak{s}}$ the Lie algebra of $\underline{S}$ and we set $\mathfrak{s} := \underline{\mathfrak{s}}(\mathrm{F})$. We set

$$\Delta_\sigma(g) = \det(1 - \mathrm{Ad}(g^{-1}\sigma(g))_{\mathfrak{g}/\mathfrak{s}}), \quad g \in G.$$

By ([DHSo] (1.30)), if $x \in G^{\sigma-reg}$ then there exists a maximal torus S_H of H such that $\Delta_\sigma(x) \neq 0$. Morever, there are two elements $x_m \in \kappa_S$ and $\gamma \in S_\sigma$ such that $x = x_m\gamma$. We define the orbital integral $\mathcal{M}(f)$ of a function $f \in C_c^\infty(G)$ on $G^{\sigma-reg}$ as follows. Let S_H be a maximal torus of H. For $x_m \in \kappa_S$ and $\gamma \in S_\sigma$ with $\Delta_\sigma(x_m\gamma) \neq 0$, we set

$$\mathcal{M}(f)(x_m\gamma) := |\Delta_\sigma(x_m\gamma)|_{\mathrm{F}}^{1/4} \int_{diag(A_S)\backslash(H\times H)} f(h^{-1}x_m\gamma l)d\overline{(h,l)} \tag{8.1}$$

where $diag(A_S)$ is the diagonal of $A_S \times A_S$.

We now give an explicit expression of the truncated function $v_L(\cdot, n)$ defined in ([DHSo] (2.12)), where n is a positive integer and L is equal to H or M. Let n be a positive integer. It follows immediately from the definition ([DHSo] (2.12)) that we have

$$v_H(x_1, y_1, x_2, y_2, n) = 1, \quad x_1, y_1, x_2, y_2 \in H. \tag{8.2}$$

We will describe v_M using ([DHSo] (2.63)). Since $H = P_H K_H$, each $x \in H$ can be written $x = m_{P_H}(x)n_{P_H}(x)k_{P_H}(x)$ with $m_{P_H}(x) \in M_H, n_{P_H}(x) \in N_H$ and $k_{P_H}(x) \in K_H$. We take similar notation if we consider $\bar{P}$ instead of P. For $Q = P$ or $\bar{P}$, we set

$$h_{Q_H}(x) := h_{M_H}(m_{Q_H}(x)).$$

With our definition of h_{M_H} (2.2), the map $M_H \to \mathbb{R}$ given in ([DHSo] (1.2)) coincides with $-(\log q)h_{M_H}$.

For x_1, y_1, x_2 and y_2 in H, we set

$$z_P(x_1, y_1, x_2, y_2) := \inf\big(h_{\bar{P}_H}(x_1) - h_{P_H}(y_1), h_{\bar{P}_H}(x_2) - h_{P_H}(y_2)\big),$$

and

$$z_{\bar{P}}(x_1, y_1, x_2, y_2) := -\inf\big(h_{\bar{P}_H}(y_1) - h_{P_H}(x_1), h_{\bar{P}_H}(y_2) - h_{P_H}(x_2)\big).$$

We omit x_1, y_1, x_2 and y_2 in this notation if there is no confusion. Hence the elements Z_P^0 and $Z_{\bar{P}}^0$ of ([DHSo] (2.55)) coincide with $(\log q) z_P$ and $(\log q) z_{\bar{P}}$ respectively. Therefore, the relation ([DHSo] (2.63)) gives

$$\begin{aligned} v_M(x_1, y_1, x_2, y_2, n) &= \lim_{\lambda \to 0} \Big(\frac{q^{\lambda(n+z_P)}}{1 - q^{-2\lambda}}(1 + q^{-\lambda}) + \frac{q^{\lambda(-n+z_{\bar{P}})}}{1 - q^{2\lambda}}(1 + q^{\lambda}) \Big) \\ &= \lim_{\lambda \to 0} \Big(\frac{q^{\lambda(n+z_P)}}{1 - q^{-\lambda}} + \frac{q^{-\lambda(n-z_{\bar{P}})}}{1 - q^{\lambda}} \Big) = \lim_{\lambda \to 0} \frac{q^{\lambda(n+z_P)} - q^{-\lambda(n-z_{\bar{P}}+1)}}{1 - q^{-\lambda}} \\ &= 2n + 1 + z_P - z_{\bar{P}}. \end{aligned}$$

We set

$$v_M^0(x_1, y_1, x_2, y_2) := z_P - z_{\bar{P}}$$

$$= \inf\big(h_{\bar{P}_H}(x_1) - h_{P_H}(y_1), h_{\bar{P}_H}(x_2) - h_{P_H}(y_2)\big) + \inf\big(h_{\bar{P}_H}(y_1) - h_{P_H}(x_1), h_{\bar{P}_H}(y_2) - h_{P_H}(x_2)\big).$$

Therefore, ([DHSo] Theorem 2.3) gives:

As n approaches to $+\infty$, the truncated kernel $K^n(f)$ is asymptotic to

$$\begin{aligned} &2n \sum_{x_m \in \kappa_M} c_{M,x_m}^0 \int_{M_\sigma} \mathcal{M}(f_1)(x_m\gamma)\mathcal{M}(f_2)(x_m\gamma) d\gamma \\ &+ \sum_{S_H \in \mathcal{T}_H \cup \{M_H\}} \sum_{x_m \in \kappa_S} c_{S,x_m}^0 \int_{S_\sigma} \mathcal{M}(f_1)(x_m\gamma)\mathcal{M}(f_2)(x_m\gamma) d\gamma + \sum_{x_m \in \kappa_M} c_{M,x_m}^0 \int_{M_\sigma} \mathcal{WM}(f)(x_m\gamma) d\gamma, \end{aligned} \tag{8.3}$$

where the constants c_{M,x_m}^0 are defined in ([RR] Theorem 3.4) and $\mathcal{WM}(f)$ is the weighted integral orbital given by

$$\Delta_\sigma(x_m\gamma)^{-1/2} \mathcal{WM}(f)(x_m\gamma)$$

$$= \int_{diag(M_H)\backslash H\times H} \int_{diag(M_H)\backslash H\times H} f_1(x_1^{-1} x_m \gamma x_2) f_2(y_1^{-1} x_m \gamma y_2) v_M^0(x_1, y_1, x_2, y_2) d\overline{(x_1, x_2)} d\overline{(y_1, y_2)}.$$

Therefore, comparing asymptotic expansions of $K^n(f)$ in Theorem 7.3 and (8.3), we obtain:

8.1 Theorem. *For f_1 and f_2 in $C_c^\infty(G)$ then we have:*

1.

$$2 \sum_{x_m \in \kappa_M} c_{M,x_m}^0 \int_{M_\sigma} \mathcal{M}(f_1)(x_m\gamma)\mathcal{M}(f_2)(x_m\gamma) d\gamma = \sum_{\delta \in \widehat{M}_2, \delta_{|\mathrm{F}^\times} = 1} m_{C(1,\delta),C(1,\delta)}(f).$$

2. *(Local relative trace formula). The expression*

$$\sum_{S_H\in\mathcal{T}_H\cup\{M_H\}}\sum_{x_m\in\kappa_S} c^0_{S,x_m}\int_{S_\sigma}\mathcal{M}(f_1)(x_m\gamma)\mathcal{M}(f_2)(x_m\gamma)d\gamma+\sum_{x_m\in\kappa_M} c^0_{M,x_m}\int_{M_\sigma}\mathcal{WM}(f)(x_m\gamma)d\gamma$$

equals

$$\sum_{\tau\in\mathcal{E}_2(G)} d(\tau)m_{P_\tau,P_\tau}(f)+\frac{1}{4i\pi}\sum_{\delta\in\widehat{M}_2,\delta_{|\mathrm{F}^\times}\neq 1,\delta_{|\mathrm{E}^1}=1}\int_{\mathcal{O}} m_{P_{\delta_z},P_{\delta_z}}(f)\frac{dz}{z}$$

$$+\frac{1}{4i\pi}\sum_{\delta\in\widehat{M}_2,\delta_{|\mathrm{F}^\times}=1,\delta_{|E^1}\neq 1} R_\delta(f)+\int_{\mathcal{O}^-}\frac{m_{C(1,\delta_z),C(1,\delta_{\bar{z}^{-1}})}(f)+m_{C(w,\delta_z),C(w,\delta_{\bar{z}^{-1}})}(f)}{(1-z)(1-z^{-1})}\frac{dz}{z}$$

$$+\frac{1}{4i\pi}\sum_{\delta\in\widehat{M}_2,\delta_{|\mathrm{F}^\times}=\delta_{|E^1}=1}\tilde{R}_\delta(f)+\int_{\mathcal{O}^-}\frac{m_{C(1,\delta_z),C(1,\delta_{\bar{z}^{-1}})}(f)+m_{C(w,\delta_z),C(w,\delta_{\bar{z}^{-1}})}(f)}{(1-z)(1-z^{-1})}\frac{dz}{z}$$

$$+\int_{\mathcal{O}^-} m_{P_{\delta_z},P_{\delta_{\bar{z}^{-1}}}}(f)\frac{dz}{z}.$$

where

$$R_\delta(f):=2i\pi\Big(m_{C(1,\delta),C(1,\delta)}(f)-\Big[\frac{d}{dz}m_{C(w,\delta_z),C(1,\delta_{\bar{z}^{-1}})}(f)\Big]_{z=1}\Big),$$

$$\tilde{R}_\delta(f)=2i\pi\Big(3\,m_{C(1,\delta),C(1,\delta)}(f)-\Big[\frac{d}{dz}m_{C(w,\delta_z),C(1,\delta_{\bar{z}^{-1}})}(f)\Big]_{z=1}$$
$$+\Big[\frac{d}{dz}(z-1)\Big(m_{P_{\delta_z},C(1,\delta_{\bar{z}^{-1}})}(f)+m_{C(w,\delta_z),P_{\bar{z}^{-1}}}(f)\Big)\Big]_{z=1}\Big),$$

$$P_\tau(S)=\int_H tr(\tau(h)S)dh,\quad S\in End(V_\tau),$$

$$P_{\delta_z}(S)=\int_H^* E^0(P,\delta_z,S_z)(h)dh,\quad S\in i^K_{P\cap K}\mathbb{C}\otimes i^K_{\bar{P}\cap K}\check{\mathbb{C}}$$

and

$$C(s,\delta_z)(S):=(1+q^{-1})\int_{K_H\times K_H} tr\big(\big[C^0_{P,P}(s,\delta_z)S_z\big](k_1,k_2)\big)dk_1dk_2,\quad s\in W^G.$$

As an application of this Theorem, we will invert orbital integrals on the anisotropic σ-torus M_σ of G.

Let $\delta\in\widehat{M}_2$. As the operator $C^0_{P,P}(1,\delta)$ is the identity operator of $i^G_P\mathbb{C}_{\delta_z}\otimes i^G_{\bar{P}}\check{\mathbb{C}_{\delta_z}}$, one has

$$C(1,\delta)(v\otimes\check{w})=(1+q^{-1})\int_{K_H\times K_H} v(k_1)\check{w}(k_2)dk_1dk_2,\quad v\otimes\check{w}\in i^K_{P\cap K}\mathbb{C}\otimes i^K_{\bar{P}\cap K}\check{\mathbb{C}}.$$

Hence, we have $C(1,\delta)=(1+q^{-1})\xi_\delta\otimes\xi_{\check{\delta}}$ where ξ_δ and $\xi_{\check{\delta}}$ are the H-invariant linear forms on $i^K_{P\cap K}\mathbb{C}$ and $i^K_{\bar{P}\cap K}\check{\mathbb{C}}$ respectively given by the integration over K_H. Therefore, one deduces that

$$m_{C(1,\delta),C(1,\delta)}(f_1\otimes f_2)=m_{\xi_\delta,\xi_\delta}(f_1)m_{\xi_{\check{\delta}},\xi_{\check{\delta}}}(f_2).$$

Moreover, by ([AGS] Corollary 5.6.3), the distribution $f\mapsto m_{\xi_{\check{\delta}},\xi_{\check{\delta}}}(f)$ is smooth in a neighborhood of any σ-regular point of G.

8.2 Corollary. *Let $f \in C_c^\infty(G)$. Let $x_m \in \kappa_M$ and $\gamma \in M_\sigma$ such that $x_m\gamma$ is σ-regular. Then we have*

$$c^0_{M,x_m}|\Delta_\sigma(x_m\gamma)|^{1/4}\mathcal{M}(f)(x_m\gamma) = \sum_{\delta\in\widehat{M_2},\delta_{|\mathrm{F}^\times}=1} m_{\xi_\delta,\xi_\delta}(f)m_{\xi_{\check\delta},\xi_{\check\delta}}(x_m\gamma).$$

Proof :

Let $(J_n)_n$ be a sequence of compact open sugroups whose intersection is equal to the neutral element of G. Then the characteristic function g_n of $J_nx_m\gamma J_n$ approaches the Dirac measure at $x_m\gamma$. Therefore, taking $f_1 := f$ and $f_2 := g_n$ in Theorem 8.1 *1.*, we obtain the result. □

Remark. Let (τ, V_τ) be a supercuspidal representation of G and f be a matrix coefficient of τ. Then we deduce from the corollary that the orbital integral of f on σ-regular points of M_σ is equal to 0.

Moreover, by ([Fli], Proposition 11) we have *dim* $V'^H_\tau = 1$. Let ξ be a nonzero H-invariant linear form on V_τ. Let S_H be an anisotropic torus of H and $x_m \in \kappa_S$. Then, applying our local relative trace formula to $f_1 := f$ and f_2 approaching the Dirac measure at a σ-regular point $x_m\gamma$ with $\gamma \in S_\sigma$, we obtain

$$|\Delta_\sigma(x_m\gamma)|^{1/4}\mathcal{M}(f)(x_m\gamma) = cm_{\xi,\xi}(f)m_{\xi,\xi}(x_m\gamma),$$

where c is some nonzero constant.

J. Hakim obtained these results by other methods ([Ha] Proposition 8.1 and Lemma 8.1).

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