Spherical character of a supercuspidal representation as weighted orbital integral

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Abstract

Let E/F be an unramified quadratic extension of local non archimedean fields of characteristic 0. Let $\underline{H}$ be an algebraic reductive group, defined and split over F. We assume that the split connected component of the center of $\underline{H}$ is trivial. Let (τ, V) be a $\underline{H}(\mathrm{F})$-distinguished supercuspidal representation of $\underline{H}(\mathrm{E})$. Using the recent results of C. Zhang [Z], and the geometric side of a local relative trace formula obtained by P. Delorme, P. Harinck and S. Souaifi [DHS], we describe spherical characters associated to $\underline{H}(\mathrm{F})$-invariant linear forms on V in terms of weighted orbital integrals of matrix coefficients of τ.

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1 Introduction

Let E/F be an unramified quadratic extension of local non archimedean fields of characteristic 0. Let $\underline{H}$ be an algebraic reductive group, defined and split over F. We denote by $\underline{G} := \mathrm{Res}_{\mathrm{E/F}}\underline{H}_{/\mathrm{E}}$ the restriction of scalars of $\underline{H}_{/\mathrm{E}}$. Then $G := \underline{G}(\mathrm{F})$ is isomorphic to $\underline{H}(\mathrm{E})$. We set $H := \underline{H}(\mathrm{F})$. We denote by σ the involution of $\underline{G}$ induced by the nontrivial element of the Galois group of E/F.

An unitary irreducible admissible representation (π, V) of G is H-distinguished if the space $V^{*H} = \mathrm{Hom}_H(\pi, \mathbb{C})$ of H-invariant linear forms on V is nonzero. In that case, a distribution $m_{\xi,\xi'}$, called spherical character, can be associated to two H-invariant linear forms ξ, ξ' on V (cf. (2.1)). By ([Ha] Theorem 1), spherical characters are locally integrable functions on G, which are H biinvariant and smooth on the set $G^{\sigma-reg}$ of elements g, called σ-regular points, such that g is semisimple and $g^{-1}\sigma(g)$ is regular in G in the usual sense.

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We assume that the split component of the center of H is trivial. Let (τ, V) be a H-distinguished supercuspidal representation of G.

The aim of this note is to give the value of a spherical character $m_{\xi,\xi'}(g)$, when $g \in G$ is a regular point for the symmetric space $H\backslash G$ and $\xi, \xi' \in V^{*H}$, in terms of weighted orbital integrals of a matrix coefficient of τ (cf. Theorem 3.1). This result is analogous to that of J. Arthur in the group case ([Ar1]). Notice that this result of J. Arthur can be deduced from his local trace formula ([Ar2]) which was obtained later.

We use the recent work of C. Zhang [Z], which describes the space of H-invariant linear forms of supercuspidal representations, and the geometric side of a local relative trace formula obtained by P. Delorme, P. Harinck and S. Souaifi [DHS].

2 Spherical characters

We denote by $C_c^\infty(G)$ the space of compactly smooth functions on G.
We fix a H-distinguished supercuspidal representation (τ, V) of G. We denote by $d(\tau)$ its formal degree.

Let $(\cdot,\cdot)$ be a G-invariant hermitian inner product on V. Since τ is unitary, it induces an isomorphism $\iota : v \mapsto (\cdot, v)$ from the conjugate complex vector space $\overline{V}$ of V and the smooth dual $\check{V}$ of V, which intertwines the complex conjugate of τ and its contragredient $\check{\tau}$. If ξ is a linear form on V, we define the linear form $\overline{\xi}$ on $\overline{V}$ by $\bar{\xi}(u) := \overline{\xi(u)}$.
For ξ_1 and ξ_2 two H-invariant linear forms on V, we associate the spherical character m_{ξ_1,ξ_2} defined to be the distribution on G given by

$$m_{\xi_1,\xi_2}(f) := \sum_{u\in\mathcal{B}} \xi_1\big(\tau(f)u\big)\overline{\xi_2(u)}, \tag{2.1}$$

where $\mathcal{B}$ is an orthonormal basis of V. Since $\tau(f)$ is of finite rank, this sum is finite. Moreover, this sum does not depend on the choice of $\mathcal{B}$. Indeed, let (τ^*, V^*) be the dual representation of τ. For $f \in C_c^\infty(G)$, we set $\check{f}(g) := f(g^{-1})$. By ([R] Théorème III.3.4 and I.1.2), the linear form $\tau^*(\check{f})\xi$ belongs to $\check{V}$. Hence we can write $\iota^{-1}(\tau^*(\check{f})\xi) = \sum_{v\in\mathcal{B}} \big(\tau^*(\check{f})\xi\big)(v) \cdot v$ where $(\lambda, v) \mapsto \lambda \cdot v$ is the action of $\mathbb{C}$ on $\overline{V}$. Therefore we deduce easily that one has

$$m_{\xi_1,\xi_2}(f) = \overline{\xi}_2\big(\iota^{-1}(\tau^*(\check{f})\xi_1)\big). \tag{2.2}$$

Since τ is a supercuspidal representation, we can define the $H \times H$-invariant pairing $\mathcal{L}$ on $V \times \overline{V}$ by

$$\mathcal{L}(u, v) := \int_H (\tau(h)u, v)dh.$$

By ([Z] Theorem 1.5),

$$\text{the map } v \mapsto \xi_v : u \mapsto \mathcal{L}(u, v) \text{ is a surjective linear map from } \overline{V} \text{ onto } V^{*H}. \tag{2.3}$$

For $v, w \in V$, we denote by $c_{v,w}$ the corresponding matrix coefficient defined by $c_{v,w}(g) := (\tau(g)v, w)$ for $g \in G$.

2.1 Lemma. *Let $\xi_1, \xi_2 \in V^{*H}$ and $v, w \in V$. Then we have*

$$m_{\xi_1,\xi_2}(\check{c}_{v,w}) = d(\tau)^{-1}\xi_1(v)\overline{\xi_2(w)}.$$

Proof :

By (2.3), there exist v_1 and v_2 in V such that $\xi_j = \xi_{v_j}$ for $j = 1, 2$. By definition of the spherical character, for $f \in C_c^\infty(G)$ and $\mathcal{B}$ an orthonormal basis of V, one has

$$\begin{aligned} m_{\xi_1,\xi_2}(f) &= \sum_{u\in\mathcal{B}} \int_H (\tau(h)\tau(f)u, v_1)dh \int_H \overline{(\tau(h)u, v_2)}dh \\ &= \sum_{u\in\mathcal{B}} \int_{H\times H} (u, \tau(\check{f})\tau(h_1)v_1)(\tau(h_2)v_2, u)dh_1dh_2 \\ &= \int_{H\times H} (\tau(h_2)v_2, \tau(\check{f})\tau(h_1)v_1)dh_1dh_2 \end{aligned}$$

Hence we obtain

$$m_{\xi_1,\xi_2}(f) = \int_{H\times H}\int_G f(g)(\tau(h_1gh_2)v_2, v_1)dgdh_1dh_2. \tag{2.4}$$

Let $f(g) := \check{c}_{v,w}(g) = \overline{(\tau(g)w, v)}$. By the orthogonality relation of Schur, for $h_1, h_2 \in H$, one has

$$\int_G (\tau(g)\tau(h_2)v_2, \tau(h_1)v_1)\overline{(\tau(g)w, v)}dg = d(\tau)^{-1}(\tau(h_2)v_2, w)(v, \tau(h_1)v_1).$$

Thus, we deduce that

$$m_{\xi_1,\xi_2}(f) = d(\tau)^{-1}\xi_w(v_2)\xi_{v_1}(v) = d(\tau)^{-1}\xi_1(v)\overline{\xi_2(w)}. \qquad \square$$

3 Main result

We first recall some notations of [DHS] to introduce weighted orbital integrals.

We refer the reader to ([RR] §3) and ([DHS] §1.2 and 1.3) for the notations below and more details on σ-regular points. Let D_G be the usual Weyl discriminant function of G. By ([RR] Lemma 3.2 and Lemma 3.3), an element $g \in G$ is σ-regular if and only if $D_G(g^{-1}\sigma(g)) \neq 0$. The set $G^{\sigma-reg}$ of σ-regular points of G is decribed as follows. Let $\underline{S}$ be a maximal torus of $\underline{H}$. We denote by $\underline{S}_\sigma$ the connected component of the set of points $\gamma \in \mathrm{Res}_{\mathrm{E/F}}\underline{S}_{/\mathrm{E}}$ such that $\sigma(\gamma) = \gamma^{-1}$. We set $S_\sigma := \underline{S}_\sigma(\mathrm{F})$. By Galois cohomology, there exists a finite set $\kappa_S \subset G$ such that $\underline{HS}_\sigma \cap G = \cup_{x\in\kappa_S} HxS_\sigma$.
By ([RR] Theorem 3.4) and ([DHS] (1.30)), if $g \in G^{\sigma-reg}$, there exist a unique maximal torus $\underline{S}$ of $\underline{H}$ defined over F and 2 unique points $x \in \kappa_S$ and $\gamma \in S_\sigma$ such that $g = x\gamma$. We denote by M the centralizer of the split connected component of $S := \underline{S}(\mathrm{F})$. Then M is

Levi subgroup, that is the Levi component of a parabolic subgroup of H. We define the weight function w_M on $H \times H$ by

$$w_M(y_1, y_2) := \tilde{v}_M(1, y_1, 1, y_2),$$

where $\tilde{v}_M$ is the weight function defined in ([DHS] Lemma 2.10) and 1 is the neutral element of H.
For $x \in \kappa_S$, we set $d_{M,S,x} := c_M c_{S,x}$ where the constants c_M and $c_{S,x}$ are defined in ([DHS] (1.33)).

For $f \in C_c^\infty(G)$, we define the weighted orbital integral of f on $G^{\sigma-reg}$ as follows. Let $g \in G^{\sigma-reg}$. We keep the above notations and we write $g = x\gamma$ with $x \in \kappa_S$ and $\gamma \in S_\sigma$. We set

$$\mathcal{WM}(f)(g) := \frac{1}{d_{M,S,x}} |D_G(g^{-1}\sigma(g))|^{1/2} \int_{H\times H} f(y_1 g y_2) w_M(y_1, y_2) dy_1 dy_2.$$

3.1 Theorem. *For $v, w \in V$, we have*

$$\mathcal{WM}(c_{v,w})(g) = m_{\xi_w,\xi_v}(g), \quad g \in G^{\sigma-reg}.$$

Proof :
Let f_1 be a matrix coefficient of τ and $f_2 \in C_c^\infty(G)$. We set $f := f_1 \otimes f_2$. Let R be the regular representation of $G \times G$ on $L^2(G)$ given by $[R(x_1, x_2)\Psi](g) = \Psi(x_1^{-1} g x_2)$. Then $R(f)$ is an integral operator with smooth kernel K_f given by $K_f(x, y) = \int_G f_1(xu) f_2(uy) du$. As in ([DHS] §2.2), we introduce the truncated kernel

$$K^T(f) := \int_{H\times H} K_f(x, y) u(x, T) u(y, T) dx dy$$

where $u(x, T)$ is the truncated function of J. Arhur on H (cf. [DHS] (2.7)). It is the characteristic function of a compact subset of H, depending on a parameter T in a finite dimensional vector space, which converges to the function equal to 1 when $\|T\|$ approaches $+\infty$. We will give the spectral asymptotic expansion of $K^T(f)$.
For $x \in G$, we define

$$h(g) := \int_G f_1(xu) f_2(ugx) du,$$

so that

$$K_f(x, y) = \big[\rho(yx^{-1})h\big](e),$$

where ρ is the right regular representation of G.
If π is a unitary irreducible admissible representation of G, one has

$$\pi\big(\rho(yx^{-1})h\big) = \int_{G\times G} f_1(xu) f_2(ugy) \pi(g) du dg$$

$$= \int_{G\times G} f_1(xu) f_2(u_2) \pi(u^{-1} u_2 y^{-1}) du du_2 = \int_{G\times G} f_1(u_1^{-1}) f_2(u_2) \pi(u_1 x u_2 y^{-1}) du_1 du_2$$

$$= \pi(\check{f}_1)\pi(x)\pi(f_2)\pi(y^{-1}).$$

Since τ is supercuspidal and f_1 is a matrix coefficient of τ, we deduce that $\pi\big(\rho(yx^{-1})h\big)$ is equal to 0 if π is not equivalent to τ. Therefore, applying the Plancherel formula ([W2] Théorème VIII.1.1.) to $\big[\rho(yx^{-1})h\big]$, we obtain

$$K_f(x,y) = d(\tau)\mathrm{tr}\big(\tau(\check{f}_1)\tau(x)\tau(f_2)\tau(y^{-1})\big).$$

We identify $\check{V}\otimes V$ with a subspace of Hilbert-Schmidt operators on V. Taking an orthonormal basis $\mathcal{B}_{HS}(V)$ of $\check{V}\otimes V$ for the scalar product $(S,S') := \mathrm{tr}(SS'^*)$, one obtains

$$\begin{aligned} K_f(x,y) &= d(\tau)\mathrm{tr}\Big(\tau(\check{f}_1)\tau(x)\tau(f_2)\tau(y)^*\Big) = d(\tau)(\tau(\check{f}_1)\tau(x)\tau(f_2),\tau(y)) \\ &= d(\tau)\sum_{S\in\mathcal{B}_{HS}(V)} (\tau(\check{f}_1)\tau(x)\tau(f_2),S^*)\overline{(\tau(y),S^*)} \\ &= d(\tau)\sum_{S\in\mathcal{B}_{HS}(V)} \mathrm{tr}\big(\tau(x)\tau(f_2)S\tau(\check{f}_1)\big)\mathrm{tr}\overline{\big(\tau(y)S\big)}, \end{aligned}$$

where the sums over S are finite since $\tau(f_2)$ and $\tau(\check{f}_1)$ are of finite rank. Therefore, the truncated kernel is equal to

$$K^T(f) = d(\tau)\sum_{S\in\mathcal{B}_{HS}(V)} P_\tau^T(\check{\tau}\otimes\tau(f)S)\overline{P_\tau^T(S)}$$

where

$$P_\tau^T(S) = \int_H \mathrm{tr}\big(\tau(h)S\big)u(h,T)dh, \quad S\in\check{V}\otimes V.$$

For $\check{v}\otimes v\in\check{V}\otimes V$, one has $\mathrm{tr}\big(\tau(h)(\check{v}\otimes v)\big) = c_{\check{v},v}(h)$. Since $c_{\check{v},v}$ is compactly supported, the truncated local period $P_\tau^T(S)$ converges when $\|T\|$ approaches infinity to

$$P_\tau(S) = \int_H \mathrm{tr}\big(\tau(h)S\big)dh.$$

Therefore, we obtain

$$\lim_{\|T\|\to+\infty} K^T(f) = d(\tau)m_{P_\tau,P_\tau}(f), \tag{3.1}$$

where m_{P_τ,P_τ} is the spherical character of the representation $\check{\tau}\otimes\tau$ associated to the $H\times H$-invariant linear form P_τ on $\check{V}\otimes V$.

By ([DHS] Theorem 2.15), the truncated kernel $K^T(f)$ is asymptotic to a distribution $J^T(f)$ as $\|T\|$ approaches $+\infty$ and the constant term $\tilde{J}(f)$ of $J^T(f)$ is explicitly given in ([DHS] Corollary 2.11). Therefore, we deduce that

$$d(\tau)m_{P_\tau,P_\tau}(f) = \tilde{J}(f). \tag{3.2}$$

We now express m_{P_τ,P_τ} in terms of H-invariant linear forms on V. Let V_H be the orthogonal of V^{*H} in V. Since $\xi_u(v) = \overline{\xi_v(u)}$ for $u,v\in V$, the space $\overline{V}_H$ is the kernel of $v\mapsto\xi_v$. Let W be a complementary subspace of V_H in V. Then, the map $v\mapsto\xi_v$ is an isomorphism from $\overline{W}$ to V^{*H} and $(u,v)\mapsto\xi_v(u)$ is a nondegenerate hermitian form on

W. Let $(e_1, \ldots, e_n)$ be an orthogonal basis of W for this hermitian form. We set $\xi_i := \xi_{e_i}$ for $i = 1, \ldots, n$. Thus we have $\xi_i(e_i) \neq 0$.
We identify $\overline{V}$ and $\check{V}$ by the isomorphism ι. We claim that

$$P_\tau = \sum_{i=1}^{n} \frac{1}{\xi_i(e_i)} \overline{\xi_i} \otimes \xi_i \tag{3.3}$$

Indeed, we have $P_\tau(v \otimes u) = \xi_v(u) = \overline{\xi_u(v)}$. Hence, the two sides are equal to 0 on $\overline{V} \otimes V_H + \overline{V}_H \otimes V + \overline{V}_H \otimes V_H$ and take the same value $\xi_k(e_l)$ on $e_k \otimes e_l$ for $k, l \in \{1, \ldots n\}$. Hence, by definition of spherical characters, we deduce that

$$m_{P_\tau, P_\tau}(f_1 \otimes f_2) = \sum_{u \otimes v \in \, o.b.(\bar{V} \otimes V)} P_\tau\Big(\bar{\tau}(f_1) \otimes \tau(f_2)(u \otimes v)\Big)\overline{P_\tau(u \otimes v)}$$

$$= \sum_{u \otimes v \in \, o.b.(\bar{V} \otimes V)} \sum_{i,j=1}^{n} \frac{1}{\xi_i(e_i)\xi_j(e_j)} \overline{\xi_i}(\bar{\tau}(f_1)u)\xi_i(\tau(f_2)v)\overline{\overline{\xi_j}(u)\xi_j(v)},$$

where $o.b.(\bar{V} \otimes V)$ is an orthonormal basis of $\bar{V} \otimes V$. By definition of $\bar{\xi}$ for $\xi \in V^{*H}$, one has $\bar{\xi}(\bar{\tau}(f_1)u) = \overline{\xi(\tau(\bar{f_1}))}$. Therefore, we obtain

$$m_{P_\tau, P_\tau}(f_1 \otimes f_2) = \sum_{i,j=1}^{n} \frac{1}{\xi_i(e_i)\xi_j(e_j)} \overline{m_{\xi_i, \xi_j}(\bar{f_1})} m_{\xi_i, \xi_j}(f_2). \tag{3.4}$$

Let v and w in V. Let $f_1 := c_{v,w}$ so that $\bar{f_1} = \check{c}_{v,w}$. . If $v \in V_H$ or $w \in V_H$, it follows from Lemma 2.1 that $m_{\xi_i, \xi_j}(\bar{f_1}) = 0$ for $i, j \in \{1, \ldots, n\}$, hence $m_{P_\tau, P_\tau}(f_1 \otimes f_2) = 0$. Thus, we deduce from (3.2) that

$$\tilde{J}(c_{v,w} \otimes f_2) = 0, \quad v \in V_H \text{ or } w \in V_H. \tag{3.5}$$

Let $k, l \in \{1, \ldots, n\}$. We set $f_1 := c_{e_k, e_l}$, hence $\bar{f_1} = \check{c}_{e_l, e_k}$. By Lemma 2.1, one has $m_{\xi_i, \xi_j}(\bar{f_1}) = d(\tau)^{-1}\xi_i(e_l)\xi_j(e_k)$. Therefore, by (3.2) and (3.4) we obtain

$$\tilde{J}(c_{e_k, e_l} \otimes f_2) = m_{\xi_l, \xi_k}(f_2). \tag{3.6}$$

By sesquilinearity, ones deduces from (3.5) and (3.6) that one has

$$\tilde{J}(c_{v,w} \otimes f_2) = m_{\xi_w, \xi_v}(f_2) \quad v, w \in V. \tag{3.7}$$

Let $g \in G^{\sigma-reg}$. Let $(J_n)_n$ be a sequence of compact open sugroups whose intersection is equal to the neutral element of G. The characteristic function ϕ_n of $J_n g J_n$ approaches the Dirac measure at g as n approaches $+\infty$. Thus, if $v, w \in V$ then $m_{\xi_w, \xi_v}(\phi_n)$ converges to $m_{\xi_w, \xi_v}(g)$. By ([DHS] Corollary 2.11) the constant term $\tilde{J}(c_{v,w} \otimes \phi_n)$ converges to $\mathcal{WM}(c_{v,w})(g)$. We deduce the Theorem from (3.7). □

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