NONLINEAR REGULARIZING EFFECT FOR HYPERBOLIC PARTIAL DIFFERENTIAL EQUATIONS

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The Tartar-DiPerna compensated compactness method, used initially to construct global weak solutions of hyperbolic systems of conservation laws for large data, can be adapted in order to provide some regularity estimates on these solutions. This note treats two examples: (a) the case of scalar conservation laws with convex flux, and (b) the Euler system for a polytropic, compressible fluid, in space dimension one.

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1. Motivation

Hyperbolic PDEs such as the wave equation are known to propagate singularities, unlike parabolic (or elliptic) PDEs, whose solutions are more regular than the corresponding data. Besides, in the context of hyperbolic PDEs, nonlinearities are responsible for the build-up of finite time singularities in the form of shock waves. Therefore, the notion of a "nonlinear regularizing effect" for hyperbolic PDEs may seem somewhat of a paradox.

Yet it has been known since the work of P. Lax [8, 9] that the evolution semigroup defined by the entropy solution $u \equiv u(t,x) \in \mathbf{R}$ of a scalar conservation law

$$\partial_t u + \partial_x f(u) = 0\,, \quad (t,x) \in \mathbf{R}_+^* \times \mathbf{R}$$

with strictly convex flux f is compact in $L^1(\mathbf{R})$ for each $t > 0$. On the other hand, if the flux f is linear, solving the equation above explicitly by the method of characteristics shows that whichever singularities exist in the initial data $u\big|_{t=0}$ are propagated and persist in the solution $u(t,x)$ for all $t > 0$. This simple example suggests that some type of nonlinearities may indeed induce limited regularization effects in hyperbolic PDEs. The purpose of the present note is to investigate that question on two examples: (a) the case of a scalar conservation law with convex flux as above, and (b) the case of the Euler system for the dynamics of a polytropic, compressible fluid.

2. Regularizing effect for scalar conservation laws

Consider the Cauchy problem

$$\left\{\begin{array}{l}\partial_t u + \partial_x f(u) = 0\,, \quad x \in \mathbf{R}\,,\ t > 0\,,\\ u\big|_{t=0} = u^{in}\,,\end{array}\right. \tag{1}$$

with unknown $u \equiv u(t,x) \in \mathbf{R}$ and flux $f \in C^2(\mathbf{R};\mathbf{R})$, and assume without loss of generality that $f(0) = f'(0) = 0$.

One of the methods for constructing entropy solutions of Eq. (1) is based on the compensated compactness method proposed by Tartar [13]. A striking feature in Tartar's argument is that he obtains the compactness of some approximating sequence converging to the entropy solution of (1) *without* using any variant of the Ascoli-Arzelà theorem[a] based on Sobolev (or Besov) regularity estimates.

Our main purpose in the present note is to present a method for obtaining nonlinear regularization effects in the context of hyperbolic PDEs that is inspired from Tartar's compensated compactness argument, and follows it very closely.

Theorem 2.1. *Let $a, R > 0$ and assume that $f''(v) \geq a$ for all $v \in \mathbf{R}$, while the initial data $u^{in} \in L^\infty(\mathbf{R})$ satisfies $u^{in}(x) = 0$ for a.e. $|x| \geq R$. Then, the entropy solution u belongs to the Besov space $B^{1/4,4}_{\infty,loc}(\mathbf{R}^*_+ \times \mathbf{R})$; in other words*

$$\int_0^\infty \int_{\mathbf{R}} \chi(t,x)^2 |u(t,x) - u(t+s,x+y)|^4 dxdt = O(|s| + |y|)$$

as $|s| + |y| \to 0$, for each compactly supported $\chi \in C^1(\mathbf{R}^*_+ \times \mathbf{R})$.

Before giving the proof of this estimate, let us compare it with earlier results in the literature.

As is well known, the optimal regularity result for Eq. (1) was obtained by Lax [9], who proved that the entropy solution $u \in BV_{loc}(\mathbf{R}^*_+ \times \mathbf{R})$, as a consequence of the Lax-Oleinik one-sided inequality

$$\partial_x u(t,x) \leq \frac{1}{at}\,, \qquad (t,x) \in \mathbf{R}^*_+ \times \mathbf{R}\,.$$

Unfortunately, this inequality is specific to the case of scalar conservation laws in space dimension 1 with nondegenerate convex flux.

More recently, Lions-Perthame-Tadmor [10] and Jabin-Perthame [7] obtained a Sobolev regularity estimate, by using a "kinetic formulation" of the scalar conservation law (1), together with some appropriate "velocity averaging" result. Specifically, they proved that $u \in W^{s,p}_{loc}(\mathbf{R}^*_+ \times \mathbf{R})$ for all $s < \frac{1}{3}$ and $1 \leq p < \frac{3}{2}$.

On the other hand, a very interesting contribution of DeLellis and Westdickenberg [2] shows that one cannot obtain better regularity in the scale of Besov spaces than $B^{1/r,r}_\infty$ for $r \geq 3$ or $B^{1/3,r}_r$ for $1 \leq r < 3$, by using *only* that the entropy production is a bounded Radon measure, without using that it is a *positive* Radon measure.

[a]The same is true of the argument used by Lax in Ref. [8].

Our result in Theorem 2.1, like the one of Lions-Perthame-Tadmor or of Jabin-Perthame, does not use the positivity of the entropy production, and therefore belongs to the DeLellis-Westdickenberg optimality class.

Sketch of the proof. The proof is split in two steps. We henceforth denote $\mathbf{D}_{s,y}\phi(t,x) := \phi(t+s,x+y) - \phi(t,x)$.

Step 1: Let u be the entropy solution of Eq. (1), and consider the two vector fields $B := (u, f(u))$ and $E := (f(u), g(u))$, where $g'(v) = f'(v)^2$ for each $v \in \mathbf{R}$. That u is the entropy solution of Eq. (1) entails the two following equalities:

$$\mathrm{div}_{t,x} B = 0\,, \quad \text{ and } \quad \mathrm{div}_{t,x} E = -\mu\,,$$

where μ is a bounded Radon measure on $\mathbf{R}_+^* \times \mathbf{R}$. A variant of the Murat-Tartar div-curl lemma [12] leads to the inequality

$$\int_0^\infty \int_{\mathbf{R}} \chi^2 \mathbf{D}_{s,y} E \cdot J \mathbf{D}_{s,y} B dt dx \le C(|s| + |y|)\,,$$

where J denotes the rotation of an angle $\pi/2$, the function χ is C^1 with compact support in $\mathbf{R}_+^* \times \mathbf{R}$ and $C = C(\|u\|_\infty, \|\mu\|_1) > 0$. (The notations $\|u\|_{L^\infty}$ and $\|\mu\|_1$ designate respectively the sup norm of u and the total mass of μ.)

Step 2: The integrand in the l.h.s. of the inequality above is of the form

$$\begin{aligned}(w-v)(g(w)-g(v)) &- (f(w)-f(v))^2 \\ &= \int_v^w d\lambda \int_v^w f'(\lambda)^2 d\lambda - \left(\int_v^w f'(\lambda) d\lambda\right)^2 \ge 0\end{aligned}$$

by the Cauchy-Schwarz inequality, as observed by Tartar [13]. In fact, the r.h.s. of the identity above can be written as the double integral:

$$\begin{aligned}(w-v)(g(w)-g(v)) - (f(w)-f(v))^2 &= \int_v^w \int_v^w (f'(\zeta) - f'(\xi)) f'(\zeta) d\xi d\zeta \\ = \tfrac{1}{2} \int_v^w \int_v^w (f'(\zeta) - f'(\xi))^2 d\xi d\zeta &\ge \tfrac{1}{2} \int_v^w \int_v^w a^2 (\zeta - \xi)^2 d\xi d\zeta = \tfrac{a^2}{12} |w-v|^4\,,\end{aligned}$$

and substituting this lower bound in the inequality obtained at the end of Step 1 above entails the claimed $B^{1/4,4}_{\infty,loc}$ estimate. □

Remark 2.1. The same method also works for degenerate convex fluxes, for which $f''(v) \ge 0$ for all $v \in \mathbf{R}$, but may have finitely many zeros $v_1, \ldots, v_n$ of finite order — meaning that $f''(v) = O((v - v_k)^{2\beta_k})$ as $v \to v_k$ for some positive integer β_k. See Ref. [5].

Remark 2.2. The proof sketched above uses only the entropy condition with f as the entropy density. By using a family of entropies (e.g. all Kruzhkov's entropies) one can improve the argument above and obtain a Besov regularity estimate in $B^{1/3,3}_\infty$, known to be optimal according to DeLellis-Westdickenberg [2]. See Ref. [6].

3. The Euler system for polytropic compressible fluids

The Euler system governs the evolution of the density $\rho \equiv \rho(t,x) \geq 0$ and velocity field $u \equiv u(t,x) \in \mathbf{R}$ of a polytropic compressible fluid:

$$\begin{cases} \partial_t \rho + \partial_x(\rho u) = 0\,, & \rho\big|_{t=0} = \rho^{in}\,, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2 + \kappa\rho^\gamma\right) = 0\,, & u\big|_{t=0} = u^{in}\,. \end{cases} \tag{2}$$

We assume that this Cauchy problem is posed for all $x \in \mathbf{R}$ and $t > 0$. The pressure is $p(\rho) = \kappa\rho^\gamma$, and, by a convenient choice of units, one can assume that $\kappa = (\gamma-1)^2/4\gamma$. This system is known to be hyperbolic with characteristic speeds

$$\lambda_\pm = u \pm \theta\rho^\theta\,, \qquad \text{where } \theta = \frac{\gamma-1}{2}\,.$$

Besides, along C^1 solutions (ρ, u), Euler's system assumes the diagonal form

$$\begin{cases} \partial_t w_+ + \lambda_+ \partial_x w_+ = 0\,, \\ \partial_t w_- + \lambda_- \partial_x w_- = 0\,, \end{cases}$$

where $w_\pm \equiv w_\pm(\rho, u)$ are the Riemann invariants

$$w_+ := u + \rho^\theta > u - \rho^\theta =: w_-\,.$$

In 1983, DiPerna [3, 4] managed to extend Tartar's method to a certain class of nonlinear hyperbolic systems with two equations in space dimension one including Euler's system (2). He proved that, given $\bar\rho > 0$ and assuming that $\rho^{in} - \bar\rho$ and u are of class C^2 and compactly supported on $\mathbf{R}$, and that $\rho^{in} > 0$ on $\mathbf{R}$, there exists a least one entropy solution of (2) defined for all $t \geq 0$ and $x \in \mathbf{R}$. DiPerna's original proof could handle only exponents of the form $\gamma = 1 + 1/(2n+1)$ for each $n \in \mathbf{N}$. The case of an arbitrary $\gamma \in (1,3]$ was subsequently settled by Chen [1] and Lions-Perthame-Souganidis [11].

Definition 3.1. Given an open set $\mathcal{O} \subset \mathbf{R}_+^* \times \mathbf{R}$, an entropy solution $(\rho, \rho u)$ of Eq. (2) is called admissible on $\mathcal{O}$ if and only if there exist constants $C > 0$, $u^* > 0$ and $0 < \rho_* < \rho^*$ such that $\rho_* \leq \rho \leq \rho^*$ and $|u| \leq u^*$ on $\mathcal{O}$, and, for each smooth entropy-entropy flux pair (ϕ, ψ) for the system Eq. (2),

$$\iint_{\mathcal{O}} |\partial_t \phi(\rho, \rho u) + \partial_x \psi(\rho, \rho u)| \leq C\|D^2\phi\|_{L^\infty([\rho_*,\rho^*]\times[-\rho^* u^*, \rho^* u^*])}\,.$$

Any DiPerna solution whose artificial viscous approximation with viscosity $\epsilon > 0$ satisfies $\rho_\epsilon \geq \rho_*$ uniformly on $\mathcal{O}$ as $\epsilon \to 0$ is admissible on $\mathcal{O}$. Yet, the global existence of admissible solutions for initial data of arbitrary size remains an open problem at the time of this writing.

Theorem 3.1. *Assume that $\gamma \in (1,3)$ and let $\mathcal{O} \subset \mathbf{R}_+^* \times \mathbf{R}$ be open. Any admissible solution of Euler's system (2) on $\mathcal{O}$ satisfies*

$$\iint_{\mathcal{O}} |(\rho,u)(t+s, x+y) - (\rho,u)(t,x)|^2 dxdt = O(\ln(|s|+|y|)^{-2})$$

as $|s| + |y| \to 0$.

The only regularity result for large data known prior to this one is due to Lions-Perthame-Tadmor [10] and Jabin-Perthame [7], for the only case $\gamma = 3$. Using a kinetic formulation of Eqs. (2) and some appropriate velocity averaging argument, they proved that ρ and $\rho u \in W^{s,p}_{loc}(\mathbf{R}_+ \times \mathbf{R})$ for all $s < \frac{1}{4}$ and $1 \le p \le \frac{8}{5}$. Unfortunately, the structure of the compressible Euler system (2) prevents any obvious extension of their method to the case $\gamma \in (1, 3)$. While we doubt that the regularity obtained in Theorem 3.1 is optimal, some depletion of nonlinear interactions may occur when $\gamma = 3$, since the Euler system in Riemann invariants coordinates is then decoupled into two independent Hopf (i.e. inviscid Burgers) equations. This could account for the better regularity obtained when $\gamma = 3$.

The proof of Theorem 3.1 (see Ref. [5]) is again inspired from the compensated compactness method in Ref. [3] for hyperbolic systems. It uses two special features of Eq. (2). First, the characteristic speeds are linear in terms of the Riemann invariants: $(\lambda_+, \lambda_-) = (w_+, w_-)\mathcal{A}$, where the matrix $\mathcal{A}$ is symmetric. Moreover $\mathcal{A}$ is definite positive for $\gamma > 1$, and, whenever $\gamma \in (1, 3)$, satisfies the stronger coercivity property

$$(\sinh X, \sinh Y)\mathcal{A}(X, Y)^T \ge \tfrac{\gamma-1}{2}(X \sinh X + Y \sinh Y)\,, \qquad X, Y \in \mathbf{R}\,.$$

The second property of the Euler system (2) used in the proof is that the vector field $(w_+, w_-) \mapsto (\partial_{w_-}\lambda_+/(\lambda_+ - \lambda_-), \partial_{w_+}\lambda_-/(\lambda_- - \lambda_+))$ is a gradient.

4. Final remarks

Thus the Tartar-DiPerna compensated compactness method can be used to establish new regularizing effects in the context of hyperbolic systems of conservation laws. Open questions include (a) the case of scalar conservation laws in space dimension larger than one, (b) the case of more general pressure laws in the Euler system, and (c) the case of solutions of the Euler system with vanishing density.

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