CHAPTER 3

The Boltzmann Equation and Its Hydrodynamic Limits

François Golse

Institut Universitaire de France, and Laboratoire Jacques-Louis Lions, Université Paris 7,
Boîte Courrier 187, F75252 Paris Cedex 05, France
*E-mail: golse@math.jussieu.fr*

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1. Introduction

The classical models of fluid dynamics, such as the Euler or Navier–Stokes equations, were first established by applying Newton's second law of motion to each infinitesimal volume element of the fluid considered, see, for instance, Chapter 1 of [75]. While this method has the advantage of being universal – indeed, all hydrodynamic models can be obtained in this way – it has one major drawback: equations of state and transport coefficients (such as the viscosity or heat conductivity) are given as phenomenological or experimental data, and are not related to microscopic data (essentially, to the laws governing molecular interactions). As a matter of fact, a microscopic theory of liquids is most likely too complex to be of any use in deriving the macroscopic models of fluid mechanics. In the case of gases or plasmas, however, molecular interactions are on principle much more elementary, so that one can hope to express thermodynamic functions and transport coefficients in terms of purely mechanical data concerning collisions between gas molecules.

In fact, the subject of hydrodynamic limits goes back to the work of the founders J. Clerk Maxwell and L. Boltzmann, of the kinetic theory of gases. Both checked the consistency of their new – and, at the time, controversial – theory with the well-established laws of fluid mechanics. Interestingly, while the very existence of atoms was subject to heated debates, kinetic theory would provide estimates on the size of a gas molecule from macroscopic data such as the viscosity of the gas.

Much later, D. Hilbert formulated the question of hydrodynamic limits as a mathematical problem, as an example in his 6th problem on the axiomatization of physics [68]. In Hilbert's own words "[...] Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes [...] which lead from the atomistic view to the laws of motion of continua". Some years later, Hilbert himself attacked the problem in [69], as an application of his own fundamental work on integral equations.

There is an ambiguity in Hilbert's formulation. Indeed, what is meant by "the atomistic view" could designate two very different theories. One is molecular dynamics (i.e., the N-body problem of classical mechanics with elastic collisions, assuming for simplicity all bodies to be spherical and of equal mass). The other possibility is to start from the kinetic theory of gases, and more precisely from the Boltzmann equation, which is what Hilbert himself did in [69]. However, one should be aware that the Boltzmann equation is not itself a "first principle" of physics, but a low density limit of molecular dynamics. In the days of Maxwell and Boltzmann, and maybe even at the time of Hilbert's own papers on the subject, this may not have been so clear to everyone. In particular, much of the controversy on irreversibility could perhaps have been avoided with a clear understanding of the relations between molecular gas dynamics and the kinetic theory of gases.

In any case, the problem of hydrodynamic limits is to obtain rigorous derivations of macroscopic models such as the fundamental partial differential equations (PDEs) of fluid mechanics from a microscopic description of matter, be it molecular dynamics or the kinetic theory of gases. The situation can be illustrated by the following diagram.

$$\boxed{\text{MOLECULAR DYNAMICS}} \longrightarrow \boxed{\text{KINETIC THEORY}}$$

$$\searrow \qquad \downarrow$$

$$\boxed{\text{HYDRODYNAMICS}}$$
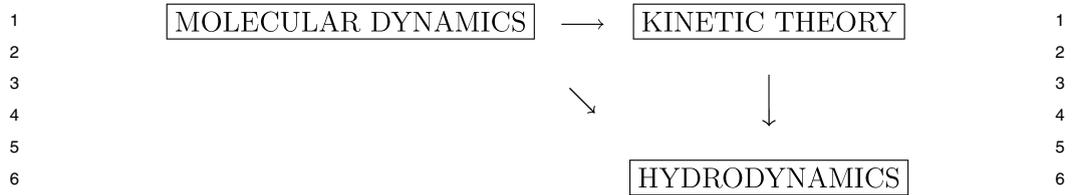
Throughout the present chapter, we are concerned with only the vertical arrow in the diagram above. As a matter of fact, this is perhaps the part of the subject that is best understood so far, at least according to the mathematical standards of rigor.

The other arrows in this diagram correspond with situations that are only partially understood, and where certain issues are still clouded with mystery. Before starting our discussion of the hydrodynamic limits of the kinetic theory of gases, let us say a few words on these other limits and direct the interested reader at the related literature.

Although beyond the scope of this chapter, the horizontal arrow is of considerable interest to our discussion, being a justification of the kinetic theory of gases on the basis of the molecular gas dynamics (viewed as a first principle of classical, nonrelativistic physics). A rigorous derivation of the Boltzmann equation from molecular dynamics on short time intervals was obtained by Lanford [77]; see also the very nice rendition of Lanford's work in the book [28]. Hence, although not a first principle itself, the Boltzmann equation is rigorously derived from first principles and therefore has more physical legitimacy than phenomenological models (such as lattice gases or stochastic Hamiltonian models). Besides, the Boltzmann equation is currently used by engineers in aerospace industry, in vacuum technology, in nuclear engineering, as well as several other applied fields, a more complete list of those being available in the Proceedings of the Rarefied Gas Dynamics Symposia.

On the other hand, "formal" derivations of the Euler system for compressible fluids from molecular dynamics were proposed by Morrey [98]. Later on, S.R.S. Varadhan and his collaborators studied the same limit, however with a different method. Instead of taking molecular dynamics as their starting point, they modified slightly the N-body Hamiltonian by adding an arbitrarily small noise term to the kinetic energy; they also cut off high velocities at a threshold compatible with the maximum speed observed on the macroscopic system. Starting from this stochastic variant of molecular gas dynamics, they derived the Euler system of compressible fluids for short times (before the onset of singularities such as shock waves); see for instance [120] and the references therein, notably [104], see also [33], and the more recent reference [44]. The role of the extra noise term in their derivation is to guarantee some form of the ergodic principle, i.e., that the only invariant measure for the Hamiltonian in the limit of infinitely many particles is a local Gibbs state (parametrized by macroscopic quantities). At the time of this writing, deriving the Euler system of compressible fluids from molecular gas dynamics without additional noise terms as in [104] and for all positive times seems beyond reach.

For these reasons, we have limited our discussion to only the derivation of hydrodynamic models from the kinetic theory of gases, i.e., from the Boltzmann equation. For a

more general view of the subject of hydrodynamic limits, the reader is advised to read the excellent survey article by Esposito and Pulvirenti [41], whose selection of topics is quite different from ours.

This chapter is organized as follows: in Section 2 we review the classical models of fluid mechanics. Section 3 introduces the Boltzmann equation and discusses its structure and main formal properties. In Section 4 we discuss the dimensionless form of the Boltzmann equation and introduce its main scaling parameters. Sections 5 and 6 explain in detail the formal derivation of the most classical PDEs of fluid mechanics from the Boltzmann equation by several different methods. Section 7 recalls the known mathematical results on the Cauchy problem for the PDEs of fluid mechanics. In Section 8 we review the state of the art on the existence theory for the Boltzmann equation. Sections 9–11 sketch the mathematical proofs of the formal derivations described in Sections 6 and 7; here again, we present three different methods for establishing these hydrodynamic limits and discuss their respective merits.

We have chosen to emphasize compactness methods, leading to global results, and especially the derivation of global weak solutions of the incompressible Navier–Stokes equations from renormalized solutions of the Boltzmann equation. There is more than a simple matter of taste in this choice. Indeed, it is a nontrivial question to decide whether these hydrodynamic limits are intrinsic properties of the microscopic versus macroscopic models governing the dynamics of gases, or simply an illustration of more or less standard techniques in asymptotic analysis. The second viewpoint leads to derivations of hydrodynamic models that fall short of describing any singular behavior beyond isolated shock waves in compressible gas dynamics. The first viewpoint uses the specific structure of the Boltzmann equation to design convergence proofs that are based on only the a priori estimates on this equation that have an intrinsic physical meaning; these convergence proofs are insensitive to whether singularities appear in finite time on the limiting hydrodynamic model.

2. Fluid dynamics: A presentation of models

Usually, one thinks of a fluid – more generally, a continuous medium – as a set of material points which, at any given time t, fill a smooth domain in the Euclidean space $\mathbb{R}^N$, where $N = 1, 2, 3$ are the dimensions of physical interest.

The purpose of fluid dynamics is to describe the state of the fluid at any instant of time with a *small* number of fields – such as the velocity or temperature fields – defined on the domain filled by the fluid.

These fields are governed by several partial differential equations that share a common structure which we briefly recall below.

Consider the motion of a continuous medium, and denote by

$$X(t, s; a) \in \mathbb{R}^N$$

the position at time t of the material point which occupied position a at time s.

The kinematics of such a medium is based on the *parallel transport* along the family of curves $t \mapsto X(t, s; a)$ indexed by a (s, being the origin of times, is kept fixed). The infinitesimal description of this parallel transport involves the first-order differential operator

$$\frac{\mathrm{D}}{\mathrm{D}t} = \partial_t + u(t, x) \cdot \nabla_x = \partial_t + \sum_{j=1}^{N} u_j(t, x) \, \partial_{x_j},$$

where the velocity field $u(t, x)$ is defined in terms of the particle paths $X(t, s; a)$ by the formula

$$\frac{\mathrm{d}}{\mathrm{d}t} X(t, s; a) = u\big(t, X(t, s; a)\big).$$

The operator $\frac{\mathrm{D}}{\mathrm{D}t}$ is usually called *the material derivative*, and using it allows one to eliminate the trajectories $X(t, s; a)$. In other words, instead of following the motion of each material point, one looks at any fixed point in the Euclidean space $\mathbb{R}^N$, say x, and observes, at any given time t, the velocity $u(t, x)$ of the material point that is located at the position x at time t. This is called the *Eulerian description* of a continuous medium, whereas the description in terms of $X(t, s; a)$ is called the *Lagrangian description*. Interestingly, the connections between the kinetic theory of gases (or plasmas) and fluid dynamics are always formulated in terms of the Eulerian, instead of the Lagrangian description, although the latter may seem more natural when dealing with the motion of a gas at the atomic or molecular level.

Fluid dynamics rests on three fundamental laws – or equations:
- the continuity equation,
- the motion equation, and
- the energy balance equation.

The continuity equation states that the density ρ of the fluid is transported by the flow, i.e., that the measure $\rho(t, x) \, \mathrm{d}x$ is the image of the measure $\rho(s, a) \, \mathrm{d}a$ under the map $a \mapsto X(t, s; a)$. The infinitesimal formulation of this fact is

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho \, \mathrm{div}_x \, u. \tag{2.1}$$

The motion equation states that each portion of the fluid obeys Newton's second law of motion (i.e., $\frac{\mathrm{d}}{\mathrm{d}t}$(momentum) = force). The acceleration is computed in terms of the material derivative, and the infinitesimal formulation of the motion equation is

$$\rho \frac{\mathrm{D}u}{\mathrm{D}t} = \mathrm{div}_x \, S + \rho \mathbf{f}, \tag{2.2}$$

where $\mathbf{f}$ is the external force field (e.g., gravity, Lorentz force in the case of a plasma . . .) and S is the stress tensor. The meaning of S is as follows: at any given time t, isolate a smooth domain Ω in the fluid, denote by $\partial \Omega$ its boundary, by n_x the unit normal field

on $\partial\Omega$ pointing toward the outside of Ω, and by $d\sigma(x)$ the surface element on $\partial\Omega$. Then, the force exerted by the fluid outside Ω on the fluid inside Ω is

$$\int_{\partial\Omega} S(t,x)n_x \, d\sigma(x).$$

Finally, the energy balance equation involves the internal energy of the fluid per unit of mass E; the total energy per unit of mass is $\frac{1}{2}|u|^2 + E$ (the sum of the kinetic energy and the internal energy). It states that the material derivative of the total energy of any portion of fluid is the sum of the works of the stresses and of the external force $\mathbf{f}$, minus the heat flux lost by that portion of fluid. Its infinitesimal formulation is

$$\rho \frac{D}{Dt}\left(\frac{1}{2}|u|^2 + E\right) = -\text{div}_x Q + \text{div}_x(Su) + \rho \mathbf{f} \cdot u, \tag{2.3}$$

where Q is the heat flux.

In the motion and energy balance equations, $\mathbf{f}$ is a given vector field, while the density ρ, the velocity field u, the internal energy E, the stress tensor S and the heat flux Q are unknown. However, these quantities are usually not independent, but are related by *equations of state* that depend on the fluid considered.

Equations (2.1)–(2.3) are *Galilean invariant*. Specifically, let $v \in \mathbb{R}^3$; define the Galilean transformation

$$x' = x + vt, \qquad u'(t,x') = u(t,x) + v, \qquad \phi'(t,x') = \phi(t,x)$$

for $\phi = \rho, S, \mathbf{f}, E, Q$. Then, setting

$$\frac{D'}{Dt} = \partial_t + u' \cdot \nabla_{x'}$$

one deduces from (2.1)–(2.3) that

$$\frac{D'}{Dt}\rho' = -\rho' \, \text{div}_{x'} u',$$

$$\rho' \frac{D'}{Dt}u' = \text{div}_{x'} S' + \rho' \mathbf{f}',$$

$$\rho' \frac{D'}{Dt}\left(\frac{1}{2}|u'|^2 + E'\right) = -\text{div}_{x'} Q' + \text{div}_{x'}(S'u') + \rho' \mathbf{f}' \cdot u'.$$

2.1. *The compressible Euler system*

An *ideal fluid* is one where the effects of viscosity and thermal conductivity can be neglected. In this case, $Q = 0$ and the stress tensor is of the form

$$S = -pI,$$

where the scalar p is the pressure. Hence the system consisting of the continuity equation the motion equation and the energy balance equation becomes

$$\partial_t \rho + \operatorname{div}_x(\rho u) = 0,$$

$$\rho\big(\partial_t u + (u \cdot \nabla_x)u\big) = -\nabla_x p + \rho\mathbf{f}, \qquad (2.4)$$

$$\rho\big(\partial_t E + (u \cdot \nabla_x)E\big) = -p\operatorname{div}_x u + \rho\mathbf{f} \cdot u.$$

Thus, the unknowns are the density ρ, the velocity field u, the pressure p and the internal energy E. However, the quantities ρ, p and E are not independent, but are related by *equations of state*.

Choosing the density ρ and the temperature θ as independent thermodynamic variables, these equations of state are relations that express the pressure p and the internal energy E in terms of ρ and θ

$$p \equiv p(\rho, \theta), \qquad E \equiv E(\rho, \theta). \qquad (2.5)$$

Hence (2.4) is a system of $N+2$ partial differential equations for the unknowns ρ, u and θ; notice that there are in fact $N+2$ scalar unknowns, ρ and θ, plus the N components of the vector field u.

The case of a perfect gas is of particular importance for the rest of this chapter. In this case, the equations of state are

$$p(\rho, \theta) = k\rho\theta, \qquad e(\rho, \theta) = \frac{k\theta}{\gamma - 1}, \qquad (2.6)$$

where k is the Boltzmann constant ($k = 1.38 \cdot 10^{-23}$ J K^{-1}) and $\gamma > 1$ is a constant called the adiabatic exponent. For a perfect gas whose molecules have n degrees of freedom

$$\gamma = 1 + \frac{2}{n}.$$

For instance, in the case of a perfect monatomic gas, each molecule has 3 degrees of freedom (the coordinates of its center of mass); hence $\gamma = 5/3$. In the case of a diatomic gas, each molecule has 5 degrees of freedom (the coordinates of its center of mass and the direction of the line passing through the centers of both atoms); hence $\gamma = 7/5$.

From now on, we choose a temperature scale such that $k = 1$.

Adding the continuity equation to the motion equation and to the energy balance equation, one can recast (2.4) in the form

$$\partial_t \rho + \operatorname{div}_x(\rho u) = 0,$$

$$\partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x(\rho\theta) = \rho\mathbf{f}, \qquad (2.7)$$

$$\partial_t\left(\rho\left(\frac{1}{2}|u|^2 + \frac{1}{\gamma - 1}\theta\right)\right) + \operatorname{div}_x\left(\rho u\left(\frac{1}{2}|u|^2 + \frac{\gamma}{\gamma - 1}\theta\right)\right) = \rho\mathbf{f} \cdot u.$$

In the absence of external force, i.e., when $\mathbf{f} = 0$, (2.7) is a hyperbolic system of conservation laws.

2.2. *The compressible Navier–Stokes system*

If the fluid considered is not ideal, the viscous forces and heat conduction must be taken into account.

In the case of moderate temperature gradients in the fluid, heat conduction is usually modeled with Fourier's law: the heat flux Q is proportional to the temperature gradient, i.e.,

$$Q = -\kappa \nabla_x \theta,$$

where the coefficient κ is called *the heat conductivity*. Usually, κ is a function of the pressure and the temperature. Because of the equation of state for the pressure, one has equivalently $\kappa \equiv \kappa(\rho, \theta) > 0$.

The viscous forces are modeled by adding a correction term to the pressure in the stress tensor S. In the case where the gradient of the velocity field is not too large, this correction term is linear in the gradient of the velocity field – by analogy with Fourier's law. Usually, the fluid under consideration is isotropic, and this implies that this correcting term is a linear combination of the scalar tensor $(\operatorname{div}_x u) I$ and of the traceless part of the symmetrized gradient of the velocity field

$$D(u) = \nabla_x u + \nabla_x u^{\mathrm{T}} - \frac{2}{N} (\operatorname{div}_x u) I.$$

In other words, the stress tensor takes the form

$$S = -pI + \mu(\operatorname{div}_x u)I + \lambda D(u),$$

where λ and μ are two positive scalar quantities referred to as the viscosity coefficients. Again, λ and μ are functions of the pressure and temperature, which, by the equation of state for the pressure, can be transformed into $\lambda \equiv \lambda(\rho, \theta)$ and $\mu \equiv \mu(\rho, \theta)$.

Inserting this form of the stress tensor in the motion and energy balance equation, one finds the system of Navier–Stokes equations for compressible fluids

$$\partial_t \rho + \operatorname{div}_x(\rho u) = 0,$$
$$\rho\big(\partial_t u + (u \cdot \nabla_x)u\big) = -\nabla_x p(\rho, \theta)$$
$$\qquad + \rho\mathbf{f} + \operatorname{div}_x\big(\lambda(\rho, \theta)D(u)\big) + \nabla_x\big(\mu(\rho, \theta)\operatorname{div}_x(u)\big), \quad (2.8)$$
$$\rho\big(\partial_t E(\rho, \theta) + (u \cdot \nabla_x)E(\rho, \theta)\big) = -p(\rho, \theta)\operatorname{div}_x u + \operatorname{div}_x\big(\kappa(\theta)\nabla_x\theta\big)$$
$$\qquad + \frac{1}{2}\lambda(\rho, \theta)D(u) : D(u) + \mu(\rho, \theta)(\operatorname{div}_x u)^2.$$

This is a degenerate parabolic system of partial differential equations in the unknowns ρ, u and θ. Observe that there is no diffusion term in the first equation, which is clear on physical grounds. Indeed, the meaning of the continuity equation is purely geometric – namely, the fact that the measure $\rho \, dx$ is transported by the fluid flow – and cannot be affected by physical assumptions on the fluid (such as whether the fluid is ideal or not).

2.3. *The acoustic system*

The acoustic waves in an ideal fluid are small amplitude disturbances of a constant equilibrium state. Therefore the propagation of acoustic waves is governed by the linearization at a constant state $(\bar{\rho}, \bar{u}, \bar{\theta})$ of the compressible Euler system. Without loss of generality, one can assume by Galilean invariance that $\bar{u} = 0$. The density, velocity and temperature fields are written as

$$\rho = \bar{\rho} + \tilde{\rho}, \qquad u = \tilde{u}, \qquad \theta = \bar{\theta} + \tilde{\theta},$$

where the letters adorned with tildes designate small disturbances of the background equilibrium state $(\bar{\rho}, 0, \bar{\theta})$. In the case of a perfect gas, and in the absence of external force (i.e., for $\mathbf{f} = 0$), the acoustic system takes the form

$$\partial_t \tilde{\rho} + \bar{\rho} \operatorname{div}_x \tilde{u} = 0,$$

$$\partial_t \tilde{u} + \frac{\bar{\theta}}{\bar{\rho}} \nabla_x \tilde{\rho} + \nabla_x \tilde{\theta} = 0, \tag{2.9}$$

$$\frac{1}{\gamma - 1} \partial_t \tilde{\theta} + \bar{\theta} \operatorname{div}_x \tilde{u} = 0.$$

By combining the first and the last equation in the system above, one can put it in the form

$$\partial_t \left(\frac{\tilde{\rho}}{\bar{\rho}} + \frac{\tilde{\theta}}{\bar{\theta}} \right) + \gamma \operatorname{div}_x \tilde{u} = 0,$$

$$\partial_t \tilde{u} + \bar{\theta} \nabla_x \left(\frac{\tilde{\rho}}{\bar{\rho}} + \frac{\tilde{\theta}}{\bar{\theta}} \right) = 0. \tag{2.10}$$

Splitting the fluctuation of velocity field $\tilde{u}$ as the sum of a gradient field and of a *solenoidal* (i.e., divergence-free) field

$$\tilde{u} = -\nabla_x \varphi + \tilde{u}^{\mathrm{s}}, \qquad \operatorname{div}_x u^{\mathrm{s}} = 0,$$

one deduces from the system (2.10) – together with boundary conditions, or conditions at infinity, or else conditions on the mean value of the fields, whose detailed description does

not belong here – that

$$\left(\partial_{tt} - \gamma\bar{\theta}\Delta_x\right)\left(\frac{\tilde{\rho}}{\bar{\rho}} + \frac{\tilde{\theta}}{\bar{\theta}}\right) = 0,$$

$$\left(\partial_{tt} - \gamma\bar{\theta}\Delta_x\right)\varphi = 0, \tag{2.11}$$

$$\partial_t u^{\mathrm{s}} = 0.$$

In other words, the acoustic system can be reduced to two independent wave equations for $(\tilde{\rho}/\bar{\rho} + \tilde{\theta}/\bar{\theta})$ (the relative pressure fluctuation) and φ (the fluctuating stream function), while the solenoidal part of the velocity fluctuation u^{s} is a constant of motion.

2.4. *The incompressible Euler equations*

Consider next the case of an incompressible, homogeneous ideal fluid. The evolution of such a fluid is governed by the system (2.4) with $\rho = $ const. The continuity and motion equations in (2.4) reduce to

$$\mathrm{div}_x\, u = 0,$$

$$\partial_t u + (u \cdot \nabla_x)u = -\nabla_x \pi + \mathbf{f}, \tag{2.12}$$

where $\pi = p/\rho$. At variance with the compressible Euler system, there is no need of an equation of state to determine π. Indeed, taking the divergence of both sides of the motion equation leads to

$$-\Delta_x \pi = \mathrm{div}_x(u \cdot \nabla_x u) - \mathrm{div}_x\, \mathbf{f} = \mathrm{trace}\big((\nabla_x u)^2\big) - \mathrm{div}_x\, \mathbf{f},$$

so that π can be expressed in terms of u by solving the Laplace equation. In other words, π must be thought of as the Lagrange multiplier associated to the constraint $\mathrm{div}_x\, u = 0$.

The incompressible Euler equations arise in a different context, namely in the description of *incompressible flows* of compressible fluids (such as perfect gases, for instance).

The dimensionless number that monitors the compressibility is the *Mach number*, i.e., the ratio of the length of the velocity field to the speed of sound. In the case of a perfect gas with adiabatic exponent γ, our discussion of the acoustic system above shows that the speed of sound in the gas at a temperature θ is $c = \sqrt{\gamma\theta}$, so that the Mach number in that case is

$$Ma = \frac{|u|}{\sqrt{\gamma\theta}}. \tag{2.13}$$

With this definition, the Mach number is a local quantity, since u and θ are in general functions of x and t. But one can replace $|u|$ and θ in the definition above by constant quantities of the same order of magnitude, for instance by averages of $|u|$ and θ over large spatial and temporal domains.

Flows of perfect gases are incompressible in the small Mach number limit. Setting $\varepsilon = \sqrt{Ma} \ll 1$, consider the rescaled density, velocity and temperature fields defined by

$$\rho_\varepsilon(t, x) = \rho\left(\frac{t}{\varepsilon}, x\right),$$

$$u_\varepsilon(t, x) = \frac{1}{\varepsilon} u\left(\frac{t}{\varepsilon}, x\right), \tag{2.14}$$

$$\theta_\varepsilon(t, x) = \theta\left(\frac{t}{\varepsilon}, x\right),$$

assuming (ρ, u, θ) is a solution of the compressible Euler system (2.7), with $\mathbf{f} \equiv 0$ for simplicity. Hence $(\rho_\varepsilon, u_\varepsilon, \theta_\varepsilon)$ satisfies

$$\partial_t \rho_\varepsilon + \operatorname{div}_x(\rho_\varepsilon u_\varepsilon) = 0,$$

$$\rho_\varepsilon\big(\partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x)u_\varepsilon\big) + \frac{1}{\varepsilon^2}\nabla_x(\rho_\varepsilon \theta_\varepsilon) = 0, \tag{2.15}$$

$$\partial_t \theta_\varepsilon + u_\varepsilon \cdot \nabla_x \theta_\varepsilon + (\gamma - 1)\theta_\varepsilon \operatorname{div}_x u_\varepsilon = 0.$$

The leading-order term in the momentum equation is the gradient of the pressure field, which suggests that, in the limit as $\varepsilon \to 0$, $\rho_\varepsilon \theta_\varepsilon \simeq C(t)$; then, combining the continuity and temperature equations above leads to

$$\gamma \operatorname{div}_x u_\varepsilon = -\partial_t \ln(\rho_\varepsilon \theta_\varepsilon) - u_\varepsilon \cdot \nabla_x \ln(\rho_\varepsilon \theta_\varepsilon) \simeq \frac{\mathrm{d}}{\mathrm{d}t}\big(\ln C(t)\big).$$

In many situations – for instance, if the spatial domain is a periodic box, or in the case of a bounded domain Ω with the usual boundary condition $u_\varepsilon \cdot n_x = 0$ on $\partial\Omega$ – integrating in x both sides of this equality leads to the incompressibility condition

$$\operatorname{div}_x u_\varepsilon \simeq 0 \quad \text{in the limit as } \varepsilon \to 0.$$

Hence the continuity equation reduces to

$$\partial_t \rho_\varepsilon + u_\varepsilon \cdot \nabla_x \rho_\varepsilon \simeq 0$$

so that, if the initial data for ρ_ε is a constant $\bar{\rho}$, then

$$\rho_\varepsilon(t, x) \simeq \bar{\rho} \quad \text{in the limit as } \varepsilon \to 0.$$

Then, the momentum equation reduces to

$$\partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x)u_\varepsilon \simeq -\frac{1}{\varepsilon^2}\nabla_x \theta_\varepsilon = \text{gradient field}$$

in the limit as $\varepsilon \to 0$. This discussion suggests that, in the small Mach number limit, flows of a compressible fluid such as a perfect gas are well described by the incompressible Euler equations.

2.5. *The incompressible Navier–Stokes equations*

Next, we start from the compressible Navier–Stokes system (2.8), and assume that the density ρ is a constant. As above, the continuity equation in (2.8) reduces to the incompressibility condition $\operatorname{div}_x u = 0$. Moreover, assuming that the viscosity λ is a constant, we find that the momentum equation reduces to

$$\rho\big(\partial_t u + (u \cdot \nabla_x)u\big) + \nabla_x p = \rho \mathbf{f} + \lambda \Delta_x u.$$

Defining the kinematic viscosity to be

$$\nu = \frac{\lambda}{\rho}$$

and setting $\pi = p/\rho$, we arrive at the incompressible Navier–Stokes equations

$$\operatorname{div}_x u = 0,$$
$$\partial_t u + (u \cdot \nabla_x)u + \nabla_x \pi = \mathbf{f} + \nu \Delta_x u. \tag{2.16}$$

We leave it to the reader to verify that the incompressible Navier–Stokes equations can be viewed as the small Mach number limit of the compressible Navier–Stokes system, as was done in the case of the incompressible Euler system. The scaling law is slightly different from the Euler case: for $\varepsilon = \sqrt{Ma}$, set

$$\rho_\varepsilon(t, x) = \rho\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right),$$
$$u_\varepsilon(t, x) = \frac{1}{\varepsilon}u\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right), \tag{2.17}$$
$$\theta_\varepsilon(t, x) = \theta\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right),$$

where (ρ, u, θ) is a solution to the Navier–Stokes system (2.8), with $\mathbf{f} \equiv 0$. Then, to leading order as $\varepsilon \to 0$, u_ε satisfies (2.16) with $\mathbf{f} \equiv 0$.

So far, we have said nothing about the temperature field in incompressible flows; this will be the subject matter of the next subsection.

2.6. *The temperature equation for incompressible flows*

Going back to the Navier–Stokes system (2.8) for a perfect gas with adiabatic exponent γ, we see that, in the incompressible case where $\rho = \text{const}$, the third equation reduces to

$$\frac{1}{\gamma - 1}\rho(\partial_t\theta + u \cdot \nabla_x\theta) = \text{div}_x\big(\kappa(\theta)\nabla_x\theta\big) + \frac{1}{2}\lambda D(u) : D(u). \tag{2.18}$$

On the right-hand side of (2.18), the first term represents the divergence of the heat flux due to thermal conduction, as described by Fourier's law, while the second term represents the production of heat by intermolecular friction and is called the *viscous heating* term.

In some models that can be found in the literature, the viscous heating term is absent from the temperature equation. Whether the viscous heating term should be taken into account or not depends in fact on the relative size of the fluctuations of velocity field about its average value, and of the fluctuations of temperature field about its average values.

If the fluctuations of velocity field are of a smaller order than the square-root of the temperature fluctuations, then a straightforward scaling argument shows that the viscous heating term can indeed be neglected in (2.18). If however, the fluctuations of velocity field are at least of the same order of magnitude as the square-root of the temperature fluctuations, then the viscous heating term cannot be neglected in (2.18). We shall discuss this alternative further in the description of the incompressible hydrodynamic limits of the Boltzmann equation.

2.7. *Coupling of the velocity and temperature fields by conservative forces*

In our discussion of the incompressible flows as low Mach number limits, we have neglected so far the external force $\mathbf{f}$. Split it as the sum of a gradient field (i.e., of a conservative force) and of a solenoidal field

$$\mathbf{f} = -\nabla_x\phi + \mathbf{f}^s, \qquad \text{div}_x\,\mathbf{f}^s = 0.$$

Scale ϕ and $\mathbf{f}^s$ as

$$\phi_\varepsilon(t,x) = \frac{1}{\varepsilon}\phi\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right), \qquad \mathbf{f}^s_\varepsilon(t,x) = \frac{1}{\varepsilon^3}\mathbf{f}^s\left(\frac{t}{\varepsilon^2}, \frac{x}{\varepsilon}\right), \tag{2.19}$$

and assume that ρ_ε and θ_ε have fluctuations of order ε about their constant average values

$$\rho_\varepsilon = \bar\rho + \varepsilon\tilde\rho_\varepsilon, \qquad \theta_\varepsilon = \bar\theta + \varepsilon\tilde\theta_\varepsilon.$$

In that case, the leading order in ε of the momentum equation in the Navier–Stokes system (2.8) reduces to

$$\nabla_x\big(\bar\rho\tilde\theta_\varepsilon + \bar\theta\tilde\rho_\varepsilon\big) + \bar\rho\nabla_x\phi_\varepsilon \simeq 0,$$

or in other words,

$$\frac{\tilde{\rho}_\varepsilon}{\bar{\rho}} + \frac{\tilde{\theta}_\varepsilon}{\bar{\theta}} + \frac{\phi_\varepsilon}{\bar{\theta}} \simeq C(t), \tag{2.20}$$

an equality known as *Boussinesq's relation*. In many cases, the boundary conditions (or decay at infinity, or else periodicity conditions) entail that $C(t) = 0$.

The next order in ε of the momentum equation in the Navier–Stokes system is

$$\bar{\rho}\big(\partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x)u_\varepsilon\big) + \tilde{\rho}_\varepsilon \nabla_x \phi_\varepsilon \simeq \lambda \Delta_x u_\varepsilon + \bar{\rho}\mathbf{f}_\varepsilon + \text{gradient field}$$

and one expresses the action of the conservative force $\tilde{\rho}_\varepsilon \nabla_x \phi_\varepsilon$ as

$$\tilde{\rho}_\varepsilon \nabla_x \phi_\varepsilon \simeq -\frac{\bar{\rho}}{\bar{\theta}}\left(\tilde{\theta}_\varepsilon \nabla_x \phi_\varepsilon + \frac{1}{2}\nabla_x\big(\phi_\varepsilon^2\big)\right)$$

so that the momentum equation reduces to

$$\partial_t u_\varepsilon + (u_\varepsilon \cdot \nabla_x)u_\varepsilon - \frac{\theta_\varepsilon}{\bar{\theta}}\nabla_x \phi_\varepsilon \simeq \nu \Delta_x u_\varepsilon + \mathbf{f}_\varepsilon + \text{gradient field.} \tag{2.21}$$

As for the temperature equation, one should refrain from using directly (2.18). Indeed, this equation has been derived from (2.8) in the purely incompressible case where $\rho = \text{const}$, while in the present case $\rho = \text{const}$ modulo terms of order ε.

In the present case, we must go back to the Navier–Stokes system (2.8) and write the continuity and energy equation in terms of the fluctuations of density and temperature

$$\varepsilon(\partial_t \tilde{\rho}_\varepsilon + u_\varepsilon \cdot \nabla_x \tilde{\rho}_\varepsilon) + \bar{\rho}\,\text{div}_x\, u_\varepsilon = \text{o}(\varepsilon),$$

$$\frac{1}{\gamma - 1}\varepsilon\big(\partial_t \tilde{\theta}_\varepsilon + u_\varepsilon \cdot \nabla_x \tilde{\theta}_\varepsilon\big) + \bar{\theta}\,\text{div}_x\, u_\varepsilon = \varepsilon\,\text{div}_x\left(\frac{\kappa(\bar{\theta})}{\bar{\rho}}\nabla_x \tilde{\theta}_\varepsilon\right) + \text{o}(\varepsilon).$$

Next we must eliminate $\text{div}_x\, u_\varepsilon$ between both equations above; indeed, we only know that $\text{div}_x\, u_\varepsilon \simeq 0$ to leading order in ε, and it may not be true that $\text{div}_x\, u_\varepsilon = \text{o}(\varepsilon)$. Dividing the first equation above by $\bar{\rho}$ and the second by $\bar{\theta}$, one arrives at

$$\varepsilon(\partial_t + u_\varepsilon \cdot \nabla_x)\left(\frac{1}{\gamma - 1}\frac{\tilde{\theta}_\varepsilon}{\bar{\theta}} - \frac{\tilde{\rho}_\varepsilon}{\bar{\rho}}\right) = \varepsilon\frac{\kappa(\bar{\theta})}{\bar{\rho}}\Delta_x\left(\frac{\tilde{\theta}_\varepsilon}{\bar{\theta}}\right) + \text{o}(\varepsilon).$$

We further eliminate the fluctuation of density by Boussinesq's relation, and eventually arrive at

$$\frac{\gamma}{\gamma - 1}\bar{\rho}(\partial_t \theta_\varepsilon + u_\varepsilon \cdot \nabla_x \theta_\varepsilon) + \bar{\rho}u_\varepsilon \cdot \nabla_x \phi_\varepsilon \simeq \kappa\big(\bar{\theta}\big)\Delta_x \theta_\varepsilon. \tag{2.22}$$

Collecting both equations (2.21) and (2.22), we arrive at the coupled system in the small ε limit

$$\partial_t u + u \cdot \nabla_x u + \frac{\theta}{\bar{\theta}} \nabla_x \phi + \nabla_x \pi = \nu \Delta_x u, \qquad \text{div}_x u = 0,$$

$$\partial_t \theta + u \cdot \nabla_x \theta + \frac{\gamma - 1}{\gamma} u \cdot \nabla_x \phi = \bar{\kappa} \Delta_x \theta, \tag{2.23}$$

where

$$\bar{\kappa} = \frac{\gamma - 1}{\gamma} \frac{\kappa(\bar{\theta})}{\bar{\rho}}.$$

It is interesting to compare the temperature equation in (2.23) with (2.18); notice that the heat conductivity in (2.23) is $1/\gamma$ that in (2.18). Besides there is no viscous heating term in the temperature equation in (2.23), at variance with (2.18). This, however, is a consequence of the scaling considered in the discussion above: indeed, the fluctuations of velocity and temperature fields are of the same order of magnitude, so that viscous heating is a lower-order effect. On the contrary, if one sets the temperature fluctuations to be of the order of the squared fluctuations of velocity field, one recovers a viscous heating term in (2.23).

The material in this section is fairly classical and can be found in most textbooks on fluid mechanics; more information can be gathered from the excellent introductory section of [86]; see also the classical treatise [75].

3. The Boltzmann equation and its formal properties

The Boltzmann equation is the model that governs the evolution of perfect gases in kinetic theory. While fluid dynamics describes the state of a fluid with a few scalar or vector fields defined on the domain filled by the fluid, such as the temperature or velocity fields, kinetic theory describes the state of a gas with the *number density* (also called the *distribution function*) $F \equiv F(t, x, v) \geqslant 0$ that is the density of gas molecules which, at time $t \geqslant 0$, are located at the position $x \in \mathbb{R}^3$ and have velocity $v \in \mathbb{R}^3$. Put in other words, in any infinitesimal volume $\mathrm{d}x \, \mathrm{d}v$ centered at the point $(x, v) \in \mathbb{R}^3 \times \mathbb{R}^3$ of the single particle phase space, one can find approximately $F(t, x, v) \, \mathrm{d}x \, \mathrm{d}v$ like particles at time t. In the classical kinetic theory of gases, the molecular radius is neglected, except in the collision cross-section: this has important consequences, as will be seen later.
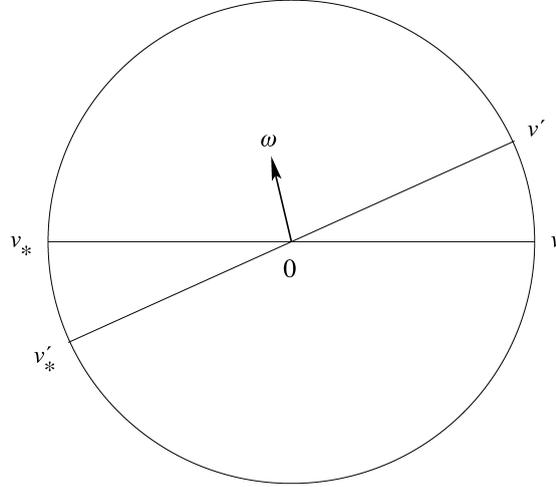
The Boltzmann equation takes the form

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \tag{3.1}$$

where $\mathcal{B}(F, F)$ is the collision integral. This collision integral $\mathcal{B}(F, F)$ is a quadratic integral operator acting only on the v-argument of the number density F, and takes the form

$$\mathcal{B}(F, F)(t, x, v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \left(F' F'_* - F F_* \right) b(v - v_*, \omega) \, \mathrm{d}\omega \, \mathrm{d}v_*, \tag{3.2}$$

Fig. 1. The pre- and post-collision velocities in the reference frame of the center of mass of the particle pair.

where the notations F, F_*, F' and F'_* designate respectively the values $F(t, x, v)$, $F(t, x, v_*)$, $F(t, x, v')$ and $F(t, x, v'_*)$, with $v' \equiv v'(v, v_*, \omega)$ and $v'_* \equiv v'_*(v, v_*, \omega)$ given in terms of v, v_* and ω by the formulas

$$v' = v - (v - v_*) \cdot \omega\omega, \qquad v'_* = v_* + (v - v_*) \cdot \omega\omega, \tag{3.3}$$

where $\omega \in \mathbb{S}^2$ is an arbitrary unit vector. These formulas represent all the solutions $(v', v'_*) \in \mathbb{R}^3 \times \mathbb{R}^3$ of the system of equations

$$v' + v'_* = v + v_*, \qquad |v'|^2 + |v'_*|^2 = |v|^2 + |v_*|^2, \tag{3.4}$$

where $(v, v_*) \in \mathbb{R}^3 \times \mathbb{R}^3$ is given. If (v, v_*) are the velocities of a pair of like particles before collision, and (v', v'_*) are the velocities of the same pair of particles after collision – or vice versa, equalities (3.4) express the conservation of momentum and kinetic energy during the collision.

Only the binary collisions are accounted for in Boltzmann's equation. Indeed, since the molecular radius is neglected in the kinetic theory of gases, one can show that collisions involving more than two particles are events that occur with probability zero, and therefore can be neglected for all practical purposes.

Moreover, kinetic energy is the only form of energy conserved during collisions. In fact, the Boltzmann collision integral (3.2) applies only to monatomic gases. Polyatomic gases can also be treated by the methods of kinetic theory; however this require using complicated variants of Boltzmann's original collision integral that involve vibrational and rotational energies in addition to the kinetic energy of the center of mass of each molecule; besides, these additional energy variables are quantized in certain applications. While such considerations are important for understanding some real gas effects, they lead to heavy

technicalities which do not belong to an expository article such as the present one. For these reasons, we shall implicitly restrict our attention to monatomic gases and to the collision integral (3.2) in the sequel.

The function $b \equiv b(V, \omega)$ is the *collision kernel*, an a.e. positive function that is of the form

$$b(V, \omega) = |V| \Sigma\big(|V|, \big|\cos(\widehat{V, \omega})\big|\big), \tag{3.5}$$

where Σ is the *scattering cross-section* (see Section 3.5 for a precise definition of this notion).

We shall discuss later the physical meaning of the function Σ, together with the usual mathematical assumptions on the collision kernel b. For the moment, assume that b is locally integrable on $\mathbb{R}^3 \times \mathbb{S}^2$, and consider a number density $F \equiv F(t, x, v)$ which, at any arbitrary instant of time t and location x, is continuous with compact support in the velocity variable v. Then, the collision integral $\mathcal{B}(F, F)(t, x, v)$ can be split as

$$\mathcal{B}(F, F)(t, x, v) = \mathcal{B}_+(F, F)(t, x, v) - \mathcal{B}_-(F, F)(t, x, v), \tag{3.6}$$

where

$$\mathcal{B}_+(F, F)(t, x, v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} F' F'_* \, b(v - v_*, \omega) \, d\omega \, dv_*,$$

$$\mathcal{B}_-(F, F)(t, x, v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} F F_* \, b(v - v_*, \omega) \, d\omega \, dv_* \tag{3.7}$$

are called respectively the *gain term* and the *loss term* in the collision integral $\mathcal{B}(F, F)$. The physical meaning of both the gain and loss terms – and that of the collision integral itself – can be explained in the following manner:

- $\mathcal{B}_-(F, F)(t, x, v) \, dv$ is the number of particles located at x at time t that exit the volume element dv centered at v in the velocity space by colliding with another particle with an arbitrary velocity v_* located at the same position x at the same time t, and
- $\mathcal{B}_+(F, F)(t, x, v) \, dv$ is the number of particles located at x at time t that enter the volume element dv centered at v in the velocity space as the result of a collision involving two particles with pre-collisional velocities v' and v'_* at the same time t and the same position x.

Notice that, in this model, collisions are purely local and instantaneous, which is another consequence of having neglected the molecular radius. Moreover, it is assumed that the joint distribution of any pair of particles located at the same position x at the same time t with velocities v and v_* *and that are about to collide* is the product

$$F(t, x, v) F(t, x, v_*).$$

In other words, such particles are assumed to be statistically uncorrelated; however, this assumption, which is crucial in the physical derivation of the Boltzmann equation, is needed

only for particle pairs about to collide – and is obviously false for a pair of particles having just collided.

Going back to the Boltzmann equation (3.1) in the form

$$\partial_t F = -v \cdot \nabla_x F + \mathcal{B}(F, F),$$

it follows from the above discussion that the first term on the right-hand side represents the net number of particles entering the infinitesimal phase-space volume $dx\,dv$ centered at (x, v) as the result of inertial motion of particles between collisions, while the second term represents the net number of particles entering that same volume as the result of instantaneous and purely local collisions.

3.1. *Conservation laws*

Throughout this subsection, it is assumed that the collision kernel b is locally integrable,

$$b \in L^1_{\text{loc}}\big(\mathbb{R}^3 \times \mathbb{S}^2\big). \tag{3.8}$$

The first major result about the Boltzmann collision integral is the following proposition.

PROPOSITION 3.1. *Let* $F \equiv F(v) \in C_c(\mathbb{R}^3)$ *and* $\phi \in C(\mathbb{R}^3)$. *Then*

$$\int_{\mathbb{R}^3} \mathcal{B}(F, F)(v)\phi(v)\,dv$$

$$= \frac{1}{4} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \big(F'F'_* - FF_*\big)\big(\phi + \phi_* - \phi' - \phi'_*\big)b(v - v_*, \omega)\,dv\,dv_*\,d\omega.$$

This result is essential to understanding the Boltzmann equation and especially its relations to hydrodynamics. For this reason, we shall give a complete proof of it.

PROOF OF PROPOSITION 3.1. The second relation in (3.4) and the fact that F is compactly supported shows that the support of $(v, v_*, \omega) \mapsto F'F'_* - FF_*$ is compact in $\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$. Hence both integrals

$$\frac{1}{4} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \big(F'F'_* - FF_*\big)\big(\phi + \phi_* - \phi' - \phi'_*\big)b(v - v_*, \omega)\,dv\,dv_*\,d\omega$$

and

$$\int_{\mathbb{R}^3} \mathcal{B}(F, F)(v)\phi(v)\,dv$$

$$= \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \big(F'F'_* - FF_*\big)\phi b(v - v_*, \omega)\,dv\,dv_*\,d\omega$$

are well defined since F and ϕ are continuous and b satisfies (3.8). In the latter integral, apply the change of variables $(v, v_*) \mapsto (v_*, v)$, while keeping ω fixed: (3.3) show that (v', v_*') is changed into (v_*', v'), so that the expression $(F'F_*' - FF_*)$ is invariant, while (3.5) shows that the collision kernel satisfies $b(v_* - v, \omega) = b(v - v_*, \omega)$. Hence

$$\iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F'F_*' - FF_*\right)\phi b(v - v_*, \omega)\, dv\, dv_*\, d\omega$$

$$= \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F'F_*' - FF_*\right)\phi_* b(v - v_*, \omega)\, dv\, dv_*\, d\omega$$

$$= \frac{1}{2} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F'F_*' - FF_*\right)(\phi + \phi_*) b(v - v_*, \omega)\, dv\, dv_*\, d\omega.$$

Now in the latter integral, for a.e. fixed $\omega \in \mathbb{S}^2$, apply the change of variables $(v, v_*) \mapsto (v', v_*')$ defined by (3.3). It is easily seen that this transformation is an involution of $\mathbb{R}^3 \times \mathbb{R}^3$, so that this change of variables maps (v', v_*') onto (v, v_*): hence $F'F_*' - FF_*$ is transformed into its opposite $FF_* - F'F_*'$. Formulas (3.3) also show that

$$\left|v' - v_*'\right| = v - v_* \quad \text{and} \quad \left(v' - v_*'\right) \cdot \omega = -(v - v_*) \cdot \omega$$

so that, by (3.5), one has $b(v' - v_*', \omega) = b(v - v_*, \omega)$. Finally, this change of variables is an isometry of $\mathbb{R}^3 \times \mathbb{R}^3$ by the second relation of (3.4), and therefore preserves the Lebesgue measure. Eventually, we have proved that

$$\iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F'F_*' - FF_*\right)(\phi + \phi_*) b(v - v_*, \omega)\, dv\, dv_*\, d\omega$$

$$= -\iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F'F_*' - FF_*\right)\left(\phi' + \phi_*'\right) b(v - v_*, \omega)\, dv\, dv_*\, d\omega$$

$$= \frac{1}{2} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F'F_*' - FF_*\right)(\phi + \phi_* - \phi' - \phi_*') b(v - v_*, \omega)\, dv\, dv_*\, d\omega$$

and this entails the announced formula. $\square$

Notice that it may not be necessary to assume that F has compact support in Proposition 3.1. For instance, the same result holds for all F and $\phi \in C(\mathbb{R}^3)$ if there exists $m > 0$ such that

$$\left|\phi(v)\right| + \int_{\mathbb{S}^2} b(v, \omega)\, d\omega = \mathrm{O}\left(|v|^m\right) \quad \text{while} \quad F(v) = \mathrm{O}\left(|v|^{-n}\right)$$

$$\text{as } |v| \to +\infty, \text{ with } n > 2m + 3. \tag{3.9}$$

An important consequence of this proposition is the following corollary.

COROLLARY 3.2. *Under the same assumptions as in Proposition* 3.1 *or* (3.9), *one has*

$$\int_{\mathbb{R}^3} \mathcal{B}(F, F)(v)\, \mathrm{d}v = 0 \qquad (conservation\ of\ mass),$$

$$\int_{\mathbb{R}^3} v_k \mathcal{B}(F, F)(v)\, \mathrm{d}v = 0 \qquad (conservation\ of\ momentum),$$

$$\int_{\mathbb{R}^3} \frac{1}{2}|v|^2 \mathcal{B}(F, F)(v)\, \mathrm{d}v = 0 \quad (conservation\ of\ energy),$$

for $k = 1, 2, 3$.

When applied to a solution of the Boltzmann equation $F \equiv F(t, x, v)$, these five relations are the net conservation of mass – equivalently, of the total number of particles – momentum and energy in each phase-space cylinder $\mathrm{d}x \times \mathbb{R}_v^3$, where $\mathrm{d}x$ is any infinitesimal phase-space element in the space of positions $\mathbb{R}_x^3$.

PROOF OF COROLLARY 3.2. Assuming that $\phi(v)$ is one of the functions 1, v_k for $k = 1, 2, 3$, and $\frac{1}{2}|v|^2$, one has

$$\phi(v) + \phi(v_*) - \phi\big(v'\big) - \phi\big(v_*'\big) = 0$$

for each $(v, v_*, \omega) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2$, because of (3.4). Applying Proposition 3.1 shows the five relations stated in Corollary 3.2.   □

Let $F \equiv F(t, x, v)$ be a solution of the Boltzmann equation; assume that $F(t, x, \cdot)$ is continuous with compact support on $\mathbb{R}_v^3$ a.e. in $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, or satisfies (3.9). Then Corollary 3.2 implies that

$$\partial_t \int_{\mathbb{R}^3} F\, \mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v F\, \mathrm{d}v = 0,$$

$$\partial_t \int_{\mathbb{R}^3} v F\, \mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v \otimes v F\, \mathrm{d}v = 0, \qquad (3.10)$$

$$\partial_t \int_{\mathbb{R}^3} \frac{1}{2}|v|^2 F\, \mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v \frac{1}{2}|v|^2 F\, \mathrm{d}v = 0.$$

These equalities are the local conservation laws of mass, momentum and energy in space–time divergence form.

Assume further, for simplicity, that for a.e. $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$,

$$\int_{\mathbb{R}^3} F(t, x, v)\, \mathrm{d}v > 0;$$

define then

$$\rho(t, x) = \int_{\mathbb{R}^3} F(t, x, v)\, dv \qquad \text{(macroscopic density)},$$

$$u(t, x) = \frac{1}{\rho(t, x)} \int_{\mathbb{R}^3} v F(t, x, v)\, dv \quad \text{(bulk velocity)}, \qquad (3.11)$$

$$\theta(t, x) = \frac{1}{\rho(t, x)} \int_{\mathbb{R}^3} \frac{1}{3} |v - u(t, x)|^2 F(t, x, v)\, dv \quad \text{(temperature)}.$$

With these definitions, the local conservation laws (3.10) take the form

$$\partial_t \rho + \mathrm{div}_x (\rho u) = 0,$$

$$\partial_t (\rho u) + \mathrm{div}_x (\rho u \otimes u) + \nabla_x (\rho \theta) = -\mathrm{div}_x \int_{\mathbb{R}^3} A(v - u) F\, dv,$$

$$\partial_t \left(\rho \left(\frac{1}{2} |u|^2 + \frac{3}{2} \theta \right) \right) + \mathrm{div}_x \left(\rho u \left(\frac{1}{2} |u|^2 + \frac{5}{2} \theta \right) \right) \qquad (3.12)$$

$$= -\mathrm{div}_x \int_{\mathbb{R}^3} B(v - u) F\, dv - \mathrm{div}_x \int_{\mathbb{R}^3} A(v - u) \cdot u F\, dv,$$

where

$$A(z) = z \otimes z - \frac{1}{3} |z|^2, \qquad B(z) = \frac{1}{2} \big(|z|^2 - 5 \big) z.$$

The left-hand side of the equalities above coincides with that of the compressible Euler system (2.7) with $\gamma = 5/3$ (the adiabatic exponent for point particles, i.e., for particles with 3 degrees of freedom). The right-hand side, on the contrary, depends on the solution of the Boltzmann equation F and is in general not determined by the macroscopic variables ρ, u and θ.

However, in some limit, it may be possible to approximate the right-hand side of (3.12) by appropriate functions of ρ, u and θ, thereby arriving at a system in closed form with unknown (ρ, u, θ).

For instance, deriving the compressible Euler system (2.7) as some asymptotic limit of the Boltzmann equation would consist in proving that the right-hand side of the second and third equations in (3.12) vanishes in that limit. Deriving the Navier–Stokes system (2.8) from the Boltzmann equation would consist in finding some (other) asymptotic limit such that

$$\int_{\mathbb{R}^3} A(v - u) F\, dv \simeq -\lambda D(u) \quad \text{and} \quad \int_{\mathbb{R}^3} B(v - u) F\, dv = -\kappa \nabla_x \theta,$$

and so on.

The problem of finding such closure relations is the key to all the derivations of hydrodynamic models from the Boltzmann equation.

3.2. *Boltzmann's H-theorem*

We have seen in the last subsection how the symmetries of the Boltzmann collision integral entail the local conservation of mass, momentum and energy.

Another important feature of these symmetries is that they also entail a variant of the second principle of thermodynamics, as we shall now explain.

PROPOSITION 3.3 (Boltzmann's *H*-theorem). *Assume that the collision kernel b satisfies* (3.8), *that* $F \in C(\mathbb{R}^3)$ *is positive and rapidly decaying at infinity, and that, for some* $m > 0$, *one has*

$$\int_{\mathbb{S}^2} b(v, \omega)\, d\omega + \left| \ln F(v) \right| = \mathrm{O}\big(|v|^m\big) \quad \text{as } |v| \to +\infty.$$

Then

$$\int_{\mathbb{R}^3} \mathcal{B}(F, F) \ln F\, dv$$

$$= -\frac{1}{4} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \big(F'F'_* - FF_*\big) \ln\bigg(\frac{F'F'_*}{FF_*}\bigg) b(v - v_*, \omega)\, dv\, dv_*\, d\omega \leqslant 0.$$

Moreover, the following conditions are equivalent

  (i) $\int_{\mathbb{R}^3} \mathcal{B}(F, F) \ln F\, dv = 0$,

  (ii) $\mathcal{B}(F, F)(v) = 0$ *for all* $v \in \mathbb{R}^3$,

  (iii) *F is a Maxwellian distribution, i.e., there exists* $\rho, \theta > 0$ *and* $u \in \mathbb{R}^3$ *such that* $F = \mathcal{M}_{(\rho, u, \theta)}$, *where*

$$\mathcal{M}_{(\rho, u, \theta)}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} \mathrm{e}^{-|v-u|^2/(2\theta)} \quad \text{for each } v \in \mathbb{R}^3. \tag{3.13}$$

As was already the case of Proposition 3.1, Boltzmann's *H*-theorem is so essential in deriving hydrodynamic equations from the Boltzmann equation that we give a complete proof of it.

PROOF OF PROPOSITION 3.3. The assumptions on F and b are such that F, b and $\phi = \ln F$ satisfy the assumption (3.9). Applying Proposition 3.1 implies that

$$\int_{\mathbb{R}^3} \mathcal{B}(F, F) \ln F\, dv$$

$$= -\frac{1}{4} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \big(F'F'_* - FF_*\big) \ln\bigg(\frac{F'F'_*}{FF_*}\bigg) b(v - v_*, \omega)\, dv\, dv_*\, d\omega.$$

$$\tag{3.14}$$

Since the logarithm is an increasing function, one has

$$(f - g)\ln\left(\frac{f}{g}\right) = (f - g)(\ln f - \ln g) \geqslant 0 \quad \text{for each } f, g > 0,$$

so that the expression on the right-hand side of (3.14) is nonpositive.

If that expression is equal to zero, the integrand must vanish a.e., meaning that

$$F'F'_* = FF_* \quad \text{for a.e. } (v, v_*, \omega) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2,$$

since the collision kernel b is a.e. positive. This accounts for the equivalence between conditions (i) and (ii). That (iii) implies (i) is proved by inspection; for instance, one can observe that, if F is a Maxwellian distribution, $\ln F$ is a linear combination of 1, v_1, v_2, v_3 and $|v|^2$, so that

$$\ln F' + \ln F'_* - \ln F - \ln F_* = 0 \quad \text{for all } (v, v_*, \omega) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2,$$

because of the microscopic conservation laws (3.4). Finally, (i) implies (iii), as shown by the next lemma, and this concludes the proof of Boltzmann's H-theorem. $\qquad\square$

LEMMA 3.4. *Let $\phi \geqslant 0$ a.e. be such that $(1 + |v|^2)\phi \in L^1(\mathbb{R}^3)$. If*

$$\phi'\phi'_* = \phi\phi_* \quad \text{for a.e. } (v, v_*, \omega) \in \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2,$$

then ϕ is either a.e. 0 or a Maxwellian (i.e., is of the form (3.13)).

The following proof is due to Perthame [105]; Boltzmann's original argument can be found in Section 18 of [16].

PROOF OF LEMMA 3.4. After translation and multiplication by a constant, one can always assume that

$$\int_{\mathbb{R}^3} \phi(v)\,dv = 1, \qquad \int_{\mathbb{R}^3} v\phi(v)\,dv = 0 \tag{3.15}$$

unless $\phi = 0$ a.e. Denoting by $\hat{\phi}$ the Fourier transform of ϕ, our assumptions implies that, for a.e. $\omega \in \mathbb{S}^2$, one has

$$\hat{\phi}(\xi)\hat{\phi}(\xi_*) = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \phi(v')\phi(v'_*)e^{-i\xi \cdot v - i\xi_* \cdot v_*}\,dv\,dv_*$$

$$= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \phi(v)\phi(v_*)e^{-i\xi \cdot v' - i\xi \cdot v'_*}\,dv\,dv_*$$

$$= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \phi(v)\phi(v_*)e^{-i\xi \cdot v - i\xi_* \cdot v_*}e^{i((\xi - \xi_*) \cdot \omega)((v - v_*) \cdot \omega)}\,dv\,dv_*.$$

(Notice that the second equality follows from the same change of variables $(v, v_*) \mapsto (v', v'_*)$ as in the proof of Proposition 3.1.) In fact, this relation holds for all $\omega \in \mathbb{S}^2$ since both sides of the equality above are continuous in ω.

Since the left-hand side of the equality above is independent of ω, one can differentiate in ω to obtain that

$$0 = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \phi(v)\phi(v_*)\mathrm{e}^{-\mathrm{i}\xi \cdot v - \mathrm{i}\xi_* \cdot v_*}(v - v_*) \cdot \omega_0 \, \mathrm{d}v \, \mathrm{d}v_*$$

for any $\xi \neq \xi_* \in \mathbb{R}^3$ and $\omega_0 \in \mathbb{S}^2$ such that $\omega_0 \perp (\xi - \xi_*)$. In other words,

$$\omega_0 \perp (\xi - \xi_*) \quad \Longrightarrow \quad (\nabla_\xi - \nabla_{\xi_*})\hat{\phi}(\xi)\hat{\phi}(\xi_*) \perp \omega_0.$$

This implies that, for all $\xi \neq \xi_* \in \mathbb{R}^3$, one has

$$(\nabla_\xi - \nabla_{\xi_*})\hat{\phi}(\xi)\hat{\phi}(\xi_*) \parallel (\xi - \xi_*). \tag{3.16}$$

Applying this with $\xi_* = 0$ leads to

$$\nabla_\xi \hat{\phi}(\xi) \parallel \xi,$$

on account of the normalization condition (3.15). Hence $\hat{\phi}$ is of the form

$$\hat{\phi}(\xi) = \psi(|\xi|^2), \quad \xi \in \mathbb{R}^3.$$

Writing (3.16) with this form of $\hat{\phi}$, one finds that

$$\xi \psi'(|\xi|^2)\psi(|\xi_*|^2) - \xi_* \psi(|\xi|^2)\psi'(|\xi_*|^2) \parallel (\xi - \xi_*).$$

Whenever ξ and ξ_* are not colinear, i.e., for a dense subset of all $\xi, \xi_* \in \mathbb{R}^3$, this implies that

$$\psi'(|\xi|^2)\psi(|\xi_*|^2) = \psi(|\xi|^2)\psi'(|\xi_*|^2).$$

Since $\phi \in L^1((1 + |v|^2)\,\mathrm{d}v)$, $\hat{\phi} \in C^2(\mathbb{R})$ and the normalization conditions (3.15) imply that $\hat{\phi}(0) = 1$ and $\hat{\phi}'(0) = 0$; hence $\psi \in C^1(\mathbb{R}^2)$ and the relation above holds for each $(\xi, \xi_*) \in \mathbb{R}^3 \times \mathbb{R}^3$. This relation implies in turn that ψ is of the form

$$\psi(r) = \mathrm{e}^{-\theta r/2}.$$

Hence $\hat{\phi}$ is of the form

$$\hat{\phi}(\xi) = \mathrm{e}^{-\theta |\xi|^2/2},$$

so that ϕ is of the form $\phi = \mathcal{M}_{(1,0,\theta)}$. $\qquad\square$

At this point, it is natural to introduce the notion of *collision invariant*.

DEFINITION 3.5. A collision invariant is a measurable function ϕ defined a.e. on $\mathbb{R}^3$ that satisfies

$$\phi(v) + \phi(v_*) - \phi(v') - \phi(v'_*) = 0 \quad \text{a.e. in } (v, v_*, \omega),$$

where $v' \equiv v'(v, v_*, \omega)$ and $v'_* \equiv v'_*(v, v_*, \omega)$ are defined by (3.3).

For instance, in the proof of Boltzmann's H-theorem, Maxwellian densities are characterized as the densities whose logarithms are collision invariants.

A variant of Lemma 3.4 characterizes collision invariants.

PROPOSITION 3.6. *A function ϕ is a collision invariant if and only if there exists five constants $a_0, a_1, a_2, a_3, a_4 \in \mathbb{R}$ such that*

$$\phi(v) = a_0 + a_1 v_1 + a_2 v_2 + a_3 v_3 + a_4 |v|^2 \quad \text{a.e. in } \mathbb{R}^3.$$

See Section 3.1 in [28] for a proof of the proposition above.

3.3. *H-theorem and a priori estimates*

We conclude with the main application of Boltzmann's H-theorem, i.e., getting a priori estimates on the solution of the Boltzmann equation. We shall discuss four different cases.

Case 1: *The periodic box.*   Consider the Cauchy problem

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3,$$

$$F|_{t=0} = F^{\text{in}}.$$

Let F be a solution of the Boltzmann equation such that, for a.e. $(t, x) \in \mathbb{R}_+ \times \mathbb{T}^3$, $F(t, x, \cdot)$ satisfies the assumptions of Proposition 3.3. Then the number density F satisfies the local entropy inequality (3.26). Integrating this differential inequality on $[0, t] \times \mathbb{T}^3$, one arrives at

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} F \ln F(t, x, v) \, dx \, dv$$

$$+ \frac{1}{4} \int_0^t \int_{\mathbb{T}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} (F'F'_* - FF_*) \ln\left(\frac{F'F'_*}{FF_*}\right) b \, dv \, dv_* \, d\omega \, dx \, ds$$

$$= \iint_{\mathbb{T}^3 \times \mathbb{R}^3} F^{\text{in}} \ln F^{\text{in}}(x, v) \, dx \, dv \tag{3.17}$$

for each $t \geqslant 0$.

The following definition explains the name "H-theorem".

DEFINITION 3.7. Let $F \geqslant 0$ a.e. be an element of $L^1(\mathbb{T}^3 \times \mathbb{R}^3)$ such that

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \big| F \ln F(x, v) \big| \, dx \, dv < +\infty.$$

One denotes by $H(F)$ the quantity

$$H(F) = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} F \ln F(x, v) \, dx \, dv.$$

Whenever there is no risk of ambiguity, we use the notation $H(t)$ to designate $H(F(t, \cdot, \cdot))$, when F is a solution of the Boltzmann equation. Equality (3.17) implies that $H(F)$ is a nonincreasing function of time; it was this property that Boltzmann called "the H-theorem". Moreover, $H(F)$ is stationary only if F is a Maxwellian (see Section 3.4.2). Hence, from the physical viewpoint, it is natural to think of $H(F(t, \cdot, \cdot))$ as *minus the entropy* of the system of particles distributed under $F(t, \cdot, \cdot)$.

In order to obtain a bound on the entropy production, it is convenient to introduce another (closely related) concept of entropy.

DEFINITION 3.8. Let $F \geqslant 0$ a.e. and $G > 0$ be two measurable functions on $\mathbb{T}^3 \times \mathbb{R}^3$; the relative entropy of F with respect to G is

$$H(F|G) = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \left(F \ln\left(\frac{F}{G}\right) - F + G \right) dx \, dv.$$

Notice that the integrand in the definition of $H(F|G)$ is an a.e. nonnegative measurable function, so that the relative entropy $H(F|G)$ is well defined as an element of $[0, +\infty]$.

Let $\rho, \theta > 0$ and $u \in \mathbb{R}^3$, then

$$H(F|\mathcal{M}_{(\rho,u,\theta)}) = H(t) + \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{|v - u|^2}{2\theta} F \, dx \, dv$$

$$- \left(1 + \ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right) \right) \iint_{\mathbb{T}^3 \times \mathbb{R}^3} F \, dx \, dv + \rho. \qquad (3.18)$$

Hence, if $F \in L^1(\mathbb{T}^3 \times \mathbb{R}^3; (1 + |v|^2) \, dv \, dx)$ and if $H(0)$ is finite, then $H(t)$ is finite for each $t \geqslant 0$, and

$$- \left(\left| \ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right) \right| + \frac{|u|^2 + 1}{\theta} \right) \iint_{\mathbb{T}^3 \times \mathbb{R}^3} (1 + |v|^2) F \, dv \, dx - \rho$$

$$\leqslant H(t) \leqslant H(0).$$

On the other hand, F also satisfies the local conservation of mass, momentum, and energy (3.10), so that, integrating these local conservation laws on $[0, t] \times \mathbb{T}^3$, one arrives at

the *global* variant of these conservation laws

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} F(t, x, v) \, dv \, dx = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} F(0, x, v) \, dv \, dx,$$

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} v F(t, x, v) \, dv \, dx = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} v F(0, x, v) \, dv \, dx, \qquad (3.19)$$

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{1}{2} |v|^2 F(t, x, v) \, dv \, dx = \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{1}{2} |v|^2 F(0, x, v) \, dv \, dx.$$

Since

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{|v - u|^2}{2\theta} F \, dx \, dv$$

$$= \frac{1}{2\theta} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \left(|v|^2 + |u|^2 \right) F \, dx \, dv - \frac{1}{\theta} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} u \cdot v F \, dx \, dv,$$

one has

$$H\big(F(t)\big|\mathcal{M}_{(\rho,u,\theta)}\big) = H(t) + \text{globally conserved quantities}$$

so that

$$H\big(F(t)\big|\mathcal{M}_{(\rho,u,\theta)}\big) - H\big(F(0)\big|\mathcal{M}_{(\rho,u,\theta)}\big) = H(t) - H(0).$$

Hence, the global entropy relation (3.17) is recast in terms of the relative entropy as

$$\frac{1}{4} \int_0^t \int_{\mathbb{T}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} (F' F_*' - F F_*) \ln\left(\frac{F' F_*'}{F F_*} \right) b \, dv \, dv_* \, d\omega \, dx \, ds$$

$$= H\big(F(0)\big|\mathcal{M}_{(\rho,u,\theta)}\big) - H\big(F(t)\big|\mathcal{M}_{(\rho,u,\theta)}\big) \qquad (3.20)$$

for each $\rho, \theta > 0$ and each $u \in \mathbb{R}^3$. This implies, in particular,

- the relative entropy bound

$$0 \leqslant H\big(F(t)\big|\mathcal{M}_{(\rho,u,\theta)}\big) \leqslant H\big(F(0)\big|\mathcal{M}_{(\rho,u,\theta)}\big), \quad t \geqslant 0;$$

- the following entropy control

$$-\left(\left| \ln\left(\frac{\rho}{(2\pi\theta)^{3/2}} \right) \right| + \frac{|u|^2 + 1}{\theta} \right) \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \left(1 + |v|^2 \right) F^{\text{in}} \, dv \, dx - \rho$$

$$\leqslant H(F)(t) \leqslant H(F)(0);$$

- the entropy production estimate

$$\frac{1}{4} \int_0^{+\infty} \int_{\mathbb{T}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} (F'F'_* - FF_*) \ln\left(\frac{F'F'_*}{FF_*}\right) b \, dv \, dv_* \, d\omega \, dx \, ds$$

$$\leqslant H\big(F(0)\big|\mathcal{M}_{(\rho,u,\theta)}\big).$$

Case 2: *A bounded domain with specular reflection on the boundary.*   The periodic box is a somewhat academic choice of a spatial domain for studying the Boltzmann equation. The next case that we consider now is very similar but more realistic. Let Ω be a smooth, bounded domain of $\mathbb{R}^3$. Starting from a given number density F^{in} at time $t = 0$, we consider the initial boundary value problem

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \quad (x, v) \in \Omega \times \mathbb{R}^3,$$

$$F(t, x, v) = F(t, x, \mathcal{R}_x v), \quad (x, v) \in \partial\Omega \times \mathbb{R}^3,$$

$$F|_{t=0} = F^{\mathrm{in}},$$

where $\mathcal{R}_x$ designates the specular reflection defined by the outward unit normal n_x at $x \in \partial\Omega$

$$\mathcal{R}_x v = v - 2(v \cdot n_x)n_x.$$

Assume that the initial boundary value problem above has a solution F satisfying the assumptions of Proposition 3.3. One multiplies the Boltzmann equation above by $\ln F + 1$ and integrates first in v to obtain the identity (3.26), and then integrates in x, which leads to

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Omega \times \mathbb{R}^3} F \ln F \, dx \, dv + \iint_{\partial\Omega \times \mathbb{R}^3} F \ln F v \cdot n_x \, d\sigma(x) \, dv$$

$$= -\frac{1}{4} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} (F'F'_* - FF_*) \ln\left(\frac{F'F'_*}{FF_*}\right) b(v - v_*, \omega) \, dv \, dv_* \, d\omega.$$

Changing the v variable in the boundary term by $v \mapsto w = \mathcal{R}_x v$, one sees that $w \cdot n_x = -v \cdot n_x$, while the specular reflection condition satisfied by F on $\partial\Omega$ implies that $F(t, x, v) = F(t, x, w)$; besides this change of variables preserves the Lebesgue measure dv since $\mathcal{R}_x$ is an isometry. Hence

$$\iint_{\partial\Omega \times \mathbb{R}^3} F \ln F v \cdot n_x \, d\sigma(x) \, dv = -\iint_{\partial\Omega \times \mathbb{R}^3} F \ln F w \cdot n_x \, d\sigma(x) \, dw = 0$$

and therefore

$$\frac{\mathrm{d}}{\mathrm{d}t} H(F)(t)$$

$$= -\frac{1}{4} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} (F'F'_* - FF_*) \ln\left(\frac{F'F'_*}{FF_*}\right) b(v - v_*, \omega) \, dv \, dv_* \, d\omega.$$

$$(3.21)$$

Proceeding similarly with the local conservation laws (3.10) shows that

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Omega \times \mathbb{R}^3} F \, \mathrm{d}x \, \mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Omega \times \mathbb{R}^3} \frac{1}{2}|v|^2 F \, \mathrm{d}x \, \mathrm{d}v = 0. \tag{3.22}$$

At this point, we apply the formula (3.18) in the case where $u = 0$ and deduce from (3.22) that

$$H(F|\mathcal{M}_{(\rho,0,\theta)}) = H(F) + \text{globally conserved quantities.}$$

Notice that, at variance with the case of the periodic box, the total momentum is not conserved, so that the formula above only holds with centered Maxwellians (i.e., Maxwellians with zero bulk velocity).

Therefore, as in the case of the periodic box, one has, for each $\rho, \theta > 0$,

$$\frac{1}{4} \int_0^t \int_\Omega \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F'F'_* - FF_* \right) \ln\left(\frac{F'F'_*}{FF_*} \right) b \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\omega \, \mathrm{d}x \, \mathrm{d}s$$

$$= H\left(F(0) \big| \mathcal{M}_{(\rho,0,\theta)} \right) - H\left(F(t) \big| \mathcal{M}_{(\rho,0,\theta)} \right) \tag{3.23}$$

for each $t \geqslant 0$. Again we obtain
- the relative entropy bound

$$0 \leqslant H\left(F(t) \big| \mathcal{M}_{(\rho,u,\theta)} \right) \leqslant H\left(F(0) \big| \mathcal{M}_{(\rho,u,\theta)} \right), \quad t \geqslant 0;$$

- the following entropy control

$$-\left(\left| \ln\left(\frac{\rho}{(2\pi\theta)^{3/2}} \right) \right| + \frac{1}{\theta} \right) \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \left(1 + |v|^2 \right) F^{\mathrm{in}} \, \mathrm{d}v \, \mathrm{d}x - \rho$$

$$\leqslant H(F)(t) \leqslant H(F)(0);$$

- the entropy production estimate

$$\frac{1}{4} \int_0^{+\infty} \int_\Omega \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F'F'_* - FF_* \right) \ln\left(\frac{F'F'_*}{FF_*} \right) b \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\omega \, \mathrm{d}x \, \mathrm{d}s$$

$$\leqslant H\left(F(0) \big| \mathcal{M}_{(\rho,u,\theta)} \right).$$

Case 3: *The Euclidean space with Maxwellian equilibrium at infinity.*   Next we study cases where the spatial domain is unbounded. The simplest of such cases is that of a cloud of gas which is in Maxwellian equilibrium at infinity. Therefore, we consider the Cauchy problem

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3,$$

$$F(t, x, v) \to \mathcal{M}_{(\rho,u,\theta)}, \qquad |x| \to +\infty,$$

$$F|_{t=0} = F^{\mathrm{in}}.$$

We shall assume that F converges to the Maxwellian state $\mathcal{M}_{(\rho,u,\theta)}$ rapidly enough so that the relative entropy

$$
H\big(F(t)\big|\mathcal{M}_{(\rho,u,\theta)}\big)
$$

$$
= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(F \ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)} \right) dx\, dv < +\infty
$$

for each $t \geqslant 0$. We claim that the same entropy relation as in the case of the three-torus also holds in the present situation

$$
\frac{1}{4} \int_0^t \int_{\mathbb{R}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F'F'_* - FF_*\right) \ln\left(\frac{F'F'_*}{FF_*}\right) b\, dv\, dv_*\, d\omega\, dx\, ds
$$

$$
= H\big(F(0)\big|\mathcal{M}_{(\rho,u,\theta)}\big) - H\big(F(t)\big|\mathcal{M}_{(\rho,u,\theta)}\big) \tag{3.24}
$$

for each $t \geqslant 0$. However, this equality is not obtained in the same way, since neither the globally conserved quantities nor the H-function itself are well-defined objects in this case (these quantities involve divergent integrals because of the Maxwellian condition at infinity).

Observe instead that, by the same argument as in the case of the three-torus, one has

$$
\int_{\mathbb{R}^3} \left(F \ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)} \right) dv
$$

$$
= \int_{\mathbb{R}^3} F \ln F\, dv + \frac{1}{2\theta} \int_{\mathbb{R}^3} \left(|v|^2 + |u|^2\right) F\, dv
$$

$$
- \frac{1}{\theta} \int_{\mathbb{R}^3} u \cdot v F\, dv + \left(1 + \ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right)\right) \int_{\mathbb{R}^3} F\, dv + \rho
$$

while

$$
\int_{\mathbb{R}^3} v \left(F \ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)} \right) dv
$$

$$
= \int_{\mathbb{R}^3} v F \ln F\, dv + \frac{1}{2\theta} \int_{\mathbb{R}^3} v\left(|v|^2 + |u|^2\right) F\, dv
$$

$$
- \frac{1}{\theta} \int_{\mathbb{R}^3} v u \cdot v F\, dv + \left(1 + \ln\left(\frac{\rho}{(2\pi\theta)^{3/2}}\right)\right) \int_{\mathbb{R}^3} v F\, dv.
$$

In other words,

$$
\int_{\mathbb{R}^3} \left(F \ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)} \right) dv
$$

$$
= \int_{\mathbb{R}^3} F \ln F\, dv + \text{locally conserved quantity}
$$

while

$$\int_{\mathbb{R}^3} v\left(F \ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)} \right) dv$$

$$= \int_{\mathbb{R}^3} v F \ln F \, dv + \text{flux of that locally conserved quantity}$$

so that

$$\partial_t \int_{\mathbb{R}^3} \left(F \ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)} \right) dv$$

$$+ \text{div}_x \int_{\mathbb{R}^3} v \left(F \ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)} \right) dv$$

$$= \partial_t \int_{\mathbb{R}^3} F \ln F \, dv + \text{div}_x \int_{\mathbb{R}^3} v F \ln F \, dv$$

so that

$$\partial_t \int_{\mathbb{R}^3} \left(F \ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)} \right) dv$$

$$+ \text{div}_x \int_{\mathbb{R}^3} v \left(F \ln\left(\frac{F}{\mathcal{M}_{(\rho,u,\theta)}}\right) - F + \mathcal{M}_{(\rho,u,\theta)} \right) dv$$

$$= -\frac{1}{4} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F' F'_* - F F_* \right) \ln\left(\frac{F' F'_*}{F F_*} \right) b \, dv \, dv_* \, d\omega.$$

Integrating further on $[0, t] \times \mathbb{R}^3$, one arrives at (3.24).

To summarize, we deduce from (3.24) that

• the relative entropy bound

$$0 \leqslant H\left(F(t) \big| \mathcal{M}_{(\rho,u,\theta)} \right) \leqslant H\left(F^{\text{in}} \big| \mathcal{M}_{(\rho,u,\theta)} \right) \quad \text{for each } t \geqslant 0;$$

• the entropy production estimate

$$\frac{1}{4} \int_0^{+\infty} \int_{\mathbb{R}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F' F'_* - F F_* \right) \ln\left(\frac{F' F'_*}{F F_*} \right) b \, dv \, dv_* \, d\omega \, dx \, ds$$

$$\leqslant H\left(F^{\text{in}} \big| \mathcal{M}_{(\rho,u,\theta)} \right).$$

Case 4: *The Euclidean space with vacuum at infinity.*   Finally, we consider the case of a cloud of gas expanding in the vacuum. As we shall see, this case is slightly different from

the previous ones. Consider the Cauchy problem

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3,$$

$$F(t, x, v) \to 0, \quad |x|, |v| \to +\infty,$$

$$F|_{t=0} = F^{\text{in}}.$$

We shall assume that F vanishes rapidly enough at infinity so that the relative entropy

$$H\big(F(t)\big|\mathcal{G}\big) < +\infty \quad \text{for each } t \geqslant 0,$$

where $\mathcal{G}$ is the centered reduced Gaussian

$$\mathcal{G}(x, v) = \frac{1}{(2\pi)^3} e^{-(|x|^2 + |v|^2)/2}.$$

Assume that

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} F^{\text{in}}\big(\big|\ln F^{\text{in}}\big| + |x|^2 + |v|^2 + 1\big)\, dx\, dv < +\infty.$$

We claim that, for each $\geqslant 0$,

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |x - tv|^2 F(t, x, v)\, dx\, dv = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |x|^2 F(0, x, v)\, dx\, dv. \tag{3.25}$$

Indeed

$$\frac{d}{dt} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |x - tv|^2 F(t, x, v)\, dx\, dv$$

$$= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \partial_t \big(|x - tv|^2 F(t, x, v)\big)\, dx\, dv$$

$$= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} (\partial_t + v \cdot \nabla_x)\big(|x - tv|^2 F(t, x, v)\big)\, dx\, dv$$

$$= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |x - tv|^2 (\partial_t + v \cdot \nabla_x) F(t, x, v)\, dx\, dv$$

$$= \int_{\mathbb{R}^3} \left(\int_{\mathbb{R}^3} |x - tv|^2 \mathcal{B}(F, F)(t, x, v)\, dv\right) dx = 0.$$

Observe that

$$H(F|\mathcal{G}) = H(F) + \frac{1}{2} \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \big(|x|^2 + |v|^2\big) F\, dx\, dv$$

$$+ \big(3 \ln(2\pi) - 1\big) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} F\, dx\, dv + 1.$$

Because of (3.25), one has

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |x|^2 F(t) \, dx \, dv$$

$$\leqslant 2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |x - tv|^2 F(t) \, dx \, dv + 2t^2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 F(t) \, dx \, dv$$

$$= 2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |x|^2 F(0) \, dx \, dv + 2t^2 \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 F(0) \, dx \, dv$$

so that

$$-\iint_{\mathbb{R}^3 \times \mathbb{R}^3} |x|^2 F^{\mathrm{in}} \, dx \, dv - \left(\frac{1}{2} + t^2\right) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} |v|^2 F^{\mathrm{in}} \, dx \, dv$$

$$- \left(3 \ln(2\pi) - 1\right) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} F^{\mathrm{in}} \, dx \, dv - 1 \leqslant H(t) \leqslant H(0).$$

Integrating on $[0, t] \times \mathbb{R}^3$ the local entropy equality (3.26), one arrives at the equality

$$H(0) - H(t)$$

$$= \frac{1}{4} \int_0^t \int_{\mathbb{R}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F' F'_* - F F_*\right) \ln\left(\frac{F' F'_*}{F F_*}\right) b \, dv \, dv_* \, d\omega \, dx \, ds$$

$$\leqslant C\left(1 + t^2\right) \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(1 + |x|^2 + |v|^2 + \left|\ln F^{\mathrm{in}}\right|\right) F^{\mathrm{in}} \, dv \, dx.$$

As we shall see below, Cases 1–3 are the most useful in the context of hydrodynamic limits. Case 4 is also interesting, although not for hydrodynamic limits: it provides one of the important estimates in the construction of global weak solutions to the Boltzmann equation by R. DiPerna and P.-L. Lions (see further).

Another case, which we did not discuss in spite of its obvious interest for applications, is that of a spatial domain that is the complement in $\mathbb{R}^3$ of a regular compact set, assuming specular reflection of the particles at the boundary of the domain. This case is handled by a straightforward adaptation of the arguments in Cases 2 and 3.

Let us now briefly discuss some of the main consequences of Boltzmann's H-theorem.

3.4. *Further remarks on the H-theorem*

3.4.1. *H-theorem and the second principle of thermodynamics.*   To begin with, let F be a solution of the Boltzmann equation such that, for a.e. $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, $F(t, x, \cdot)$ satisfies the assumptions of Proposition 3.3. Multiplying both sides of the Boltzmann equation by $\ln F + 1$ and applying Proposition 3.3 and Corollary 3.2 with $\phi \equiv 1$, one arrives at the

identity

$$\partial_t \int_{\mathbb{R}^3} F \ln F \, dv + \text{div}_x \int_{\mathbb{R}^3} v F \ln F \, dv$$

$$= -\frac{1}{4} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F'F'_* - FF_*\right) \ln\left(\frac{F'F'_*}{FF_*}\right) b(v - v_*, \omega) \, dv \, dv_* \, d\omega$$

$$\leqslant 0. \tag{3.26}$$

It is interesting to compare the equality (3.26) with the second principle of thermodynamics applied to any portion of a fluid in a smooth domain Ω. Denoting by n_x the outward unit normal field on $\partial\Omega$, and by s the entropy per unit of mass in the fluid, one has

$$\frac{d}{dt} \int_\Omega \rho s(t, x) \, dt \geqslant -\int_{\partial\Omega} \rho s u(t, x) \cdot n_x \, d\sigma(x) - \int_{\partial\Omega} \frac{q(t, x)}{\theta(t, x)} \cdot n_x \, d\sigma(x),$$

where ρ is the density of the fluid, u the velocity field, θ the temperature, q the heat flux and $d\sigma(x)$ the surface element on $\partial\Omega$. The infinitesimal version of this inequality is

$$\partial_t(\rho s) + \text{div}_x\left(\rho s u + \frac{q}{\theta}\right) \geqslant 0, \tag{3.27}$$

which is obviously analogous to (3.26). In particular,

$$\text{the quantity} \ -\int_{\mathbb{R}^3} F \ln F \, dv \ \text{is analogous to} \ \rho s$$

and

$$\text{the quantity} \ -\int_{\mathbb{R}^3} v F \ln F \, dv \ \text{is analogous to} \ \rho s u,$$

while the quantity

$$\frac{1}{4} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F'F'_* - FF_*\right) \ln\left(\frac{F'F'_*}{FF_*}\right) b(v - v_*, \omega) \, dv \, dv_* \, d\omega$$

is the local entropy production. Notice that fluid dynamics does not in general provide any expression of the entropy production in terms of ρ, u, θ, s and q. On the contrary, in the kinetic theory of gases, the entropy production is given in terms of the number density by the integral above.

3.4.2. *Relaxation towards equilibrium.* One application of the second principle of thermodynamics is the relaxation towards equilibrium for closed systems. Assume that a gas described by the Boltzmann equation is enclosed in some container Ω that is a smooth, bounded domain of $\mathbb{R}^3$. At the microscopic level, we assume that the gas molecules are

reflected on the surface of the container without exchanging heat. One model for this is the ideal situation where each molecule impinging on the boundary of the container is specularly reflected, this being Case 2 of Section 3.3.

Starting from a given number density F^{in} at time $t = 0$, we consider the initial boundary value problem

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \quad (x, v) \in \Omega \times \mathbb{R}^3,$$

$$F(t, x, v) = F(t, x, \mathcal{R}_x v), \quad (x, v) \in \partial\Omega \times \mathbb{R}^3,$$

$$F|_{t=0} = F^{\text{in}},$$

where $\mathcal{R}_x$ designates the specular reflection defined by the outward unit normal n_x at $x \in \partial\Omega$

$$\mathcal{R}_x v = v - 2v \cdot n_x n_x.$$

Now pick any sequence $t_n \to +\infty$ such that

$$F_n(t, x, v) := F(t + t_n, x, v) \to E(t, x, v) \tag{3.28}$$

in a weak topology that is compatible with the conservation laws (3.22). Then, by weak convergence and convexity, the bound on entropy production obtained in Case 2 of Section 3.3 implies that $E(t, x, v)$ is a local Maxwellian – meaning that the function $v \mapsto E(t, x, v)$ is a.e. a Maxwellian with parameters ρ, u, θ that are functions of t, x – which satisfies

$$(\partial_t E + v \cdot \nabla_x E)(t, x, v) = 0, \quad (x, v) \in \Omega \times \mathbb{R}^3,$$
$$E(t, x, v) = E(t, x, \mathcal{R}_x v), \quad (x, v) \in \partial\Omega \times \mathbb{R}^3. \tag{3.29}$$

Whenever Ω is not rotationally invariant with respect to some axis of symmetry, the only local Maxwellians that solve the system of equations (3.29) are the global Maxwellians of the form

$$E(t, x, v) = \mathcal{M}_{(\rho, 0, \theta)}(v) \quad \text{for some constant } \rho, \theta > 0. \tag{3.30}$$

Since we assumed that the conservation laws (3.22) are compatible with the topology in which the long time limit holds, the constants ρ and θ are given by

$$\iint_{\Omega \times \mathbb{R}^3} F^{\text{in}} \, dx \, dv = \rho |\Omega|, \qquad \iint_{\Omega \times \mathbb{R}^3} \frac{1}{2} F^{\text{in}} \, dx \, dv = \frac{3}{2} \rho\theta |\Omega|.$$

However, if Ω is rotationally invariant around some axis of symmetry, the system (3.29) has other solutions than the global Maxwellians (3.30), namely all functions of the form

$$E(t, x, v) = \mathcal{M}_{(\rho, 0, \theta)}(v) e^{\lambda(k \times (x - x_0)) \cdot v},$$

where $\lambda \neq 0$ is a constant, while the axis of rotational symmetry for Ω is the line of direction $k \in \mathbb{S}^2$ passing through x_0.

In fact, Lions proved in Section V of [84] that the convergence (3.28) is locally uniform in t with values in the strong topology of $L^1(\Omega \times \mathbb{R}^3)$.

Let us mention that, in spite of his apparent simplicity, the problem of relaxation toward equilibrium for the Boltzmann equation is still open, in spite of recent progress by Desvillettes and Villani [35]. The main issue is the lack of a tightness estimate in the v variable for $|v|^2 F(t, x, v)$ as $t \to +\infty$ that would apply to all initial data of finite mass, energy and entropy. Short of such an estimate, the part of the argument above identifying the temperature in terms of the initial data fails. The only cases where such estimates have been obtained correspond to initial data that are already close enough to some global Maxwellian, or that are independent of the space variable (i.e., the space homogeneous case).

However incomplete, this discussion shows the importance of Boltzmann's H-theorem whenever one seeks to estimate how close to the class of local Maxwellians a given solution of the Boltzmann equation may be. This particular point is of paramount importance for hydrodynamic limits.

3.5. *The collision kernel*

So far, our discussion of the Boltzmann equation – in fact, of the Boltzmann collision integral – did not use much of the properties of the collision kernel b. Indeed, we only took advantage of the symmetries of b in (3.5) and some additional bounds such as (3.8) or (3.9).

However, the derivation of hydrodynamic limits requires further properties of the collision integral, for which a more extensive discussion of the collision kernel becomes necessary.

First we recall some elementary facts concerning the two-body problem. Consider two points of unit mass subject to a repulsive interaction potential $U \equiv U(r)$, where r is the distance between these two points. In other words, assume that U satisfies the properties

$$U \in C^\infty\big(\mathbb{R}_+^*\big) \text{ is decreasing}, \qquad \lim_{r \to 0^+} U(r) = +\infty, \qquad \lim_{r \to +\infty} U(r) = 0^+.$$

It is well known that both points stay in the same plane for all times. Pick a Galilean frame where one of the points is at rest; then, the trajectory of the moving point is easily expressed in polar coordinates (r, θ) with the fixed point as origin. Choosing the origin of polar angles to be the line asymptotic to the trajectory of the moving particle in the past, the trajectory is determined as follows. Let v be the speed of the moving particle at infinity, and let h be the impact parameter defined by

$$vh = r^2(t)\dot{\theta}(t), \quad t \in \mathbb{R},$$

this quantity being a well-known first integral of the motion (see Figure 2). In other words, h is the distance between the line asymptotic to the trajectory of the moving particle in the
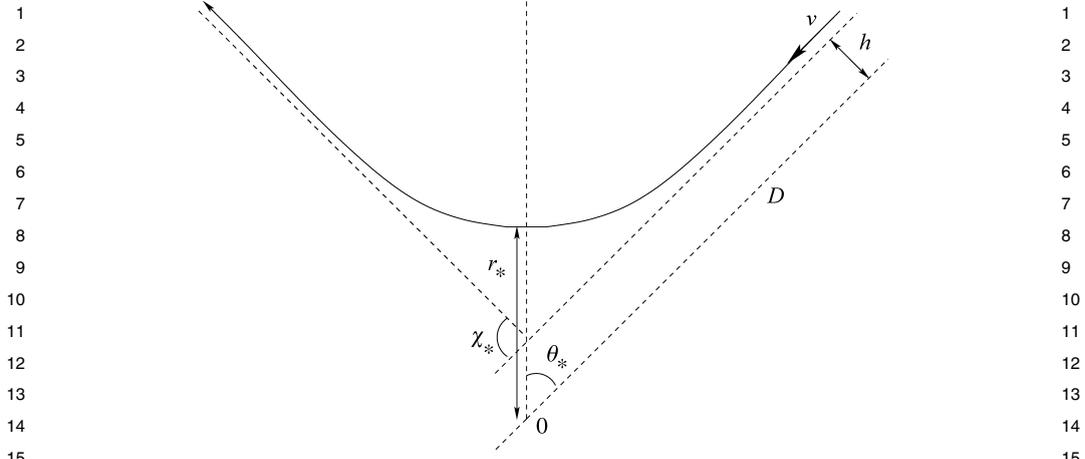
Fig. 2. Deflection of a particle subject to a radial repulsion potential from a particle at rest.

past and the parallel line going through the particle at rest (see Figure 2). Let $z_* > 0$ be the unique solution to

$$1 - z_*^2 - \frac{4}{v^2} U\left(\frac{h}{z_*}\right) = 0,$$

and set

$$r_* = \frac{h}{z_*} \quad \text{and} \quad \theta^* = \int_0^{z_*} \frac{dz}{\sqrt{1 - z^2 - 4/v^2 U(h/z)}}. \tag{3.31}$$

The point of polar coordinates (r_*, θ_*) is the apse of the trajectory, i.e., the closest to the particle at rest. Then the trajectory of the moving particles is given in polar coordinates by the equation

$$\theta = \begin{cases} \int_0^{h/r} \frac{dz}{\sqrt{1 - z^2 - 4/v^2 U(h/z)}} & \text{for } \theta \in (0, \theta_*] \text{ and } r > r_*, \\ 2\theta_* - \int_0^{h/r} \frac{dz}{\sqrt{1 - z^2 - 4/v^2 U(h/z)}} & \text{for } \theta \in [\theta_*, 2\theta_*) \text{ and } r > r_*. \end{cases}$$

Notice in particular that the moving particle is deflected of an angle $2\theta_*$.

Next we recall the notion of *scattering cross-section*. Pick an arbitrary relative speed v at infinity, and consider the deflection angle $\chi_* = \pi - 2\theta_*$ as a function of the impact parameter h. It is easily seen that

$$\chi_* \text{ is decreasing}, \qquad \lim_{h \to 0^+} \chi_*(h) = \pi^- \quad \text{and} \quad \lim_{h \to +\infty} \chi_*(h) = 0^+.$$

Because the two-body problem is invariant by any rotation around the line D passing through the particle at rest that is parallel to the asymptote in the past to the trajectory
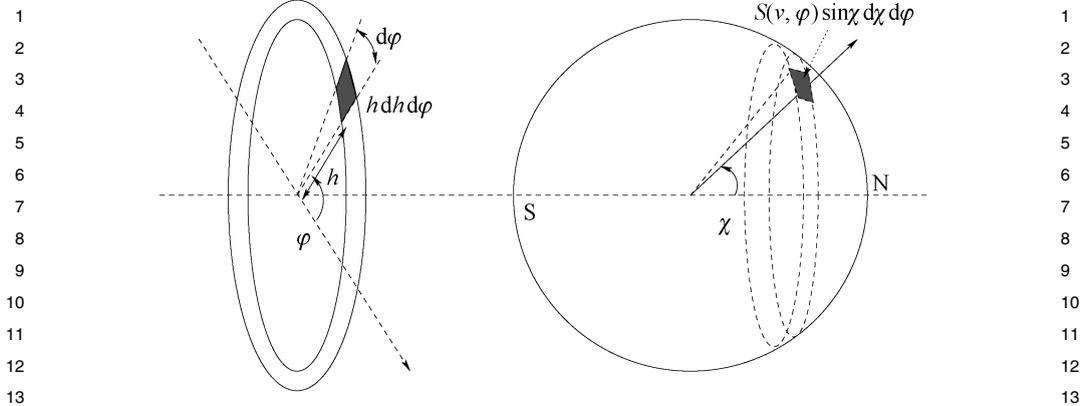
Fig. 3. The scattering cross-section corresponding to the relative speed v and in the direction χ corresponding to the impact parameter h.

of the moving particle, consider the map

$$\mathbb{R}^2 \setminus \{0\} \simeq \mathbb{R}_+^* \times [0, 2\pi) \to (0, \pi) \times [0, 2\pi) \simeq \mathbb{S}^2 \setminus \{N, S\},$$

$$(h, \phi) \mapsto (\chi_*(h), \phi), \tag{3.32}$$

where the first identification is through polar coordinates in the plane orthogonal to D with origin $h = 0$ (the intersection of D with that plane), while the second identification is through spherical coordinates, with D as the polar axis and $\mathbb{S}^2 \cap D = \{N, S\}$, see Figure 3. The image of the Lebesgue measure under this map is a surface measure on $\mathbb{S}^2$ of the form

$$S(v, \chi) \sin \chi \, d\chi \, d\phi,$$

and $S(v, \chi)$ is the scattering cross-section in the direction χ corresponding to the relative speed v. Because S is the density with respect to the Euclidean surface element on the unit sphere (which is dimensionless) of the image of the two-dimensional Lebesgue measure (which has the dimension of a surface) under the map (3.32), it has the dimension of a surface, which justifies the name "cross-section".

The scattering cross-section clearly depends upon the computation of the deflection angle in (3.31)

$$S(v, \chi) = \left. \frac{h}{\sin \chi \, |\chi_*'(h)|} \right|_{\chi_*(h) = \chi}.$$

Here are the scattering cross-sections for a few typical interactions:
- in the case of a hard sphere interaction,

$$U(r) = 0 \quad \text{if } r \geqslant d_0 \quad \text{and} \quad U(r) = +\infty \quad \text{if } 0 < r < d_0.$$

In this example, U is not decreasing but only nonincreasing and has finite range. Therefore, the definition of the scattering cross-section must be modified as follows. The map (3.32) is replaced with

$$B(0, d_0) \simeq (0, d_0) \times [0, 2\pi) \rightarrow [0, \pi) \times [0, 2\pi) \simeq \mathbb{S}^2 \setminus \{N, S\},$$

$$(h, \phi) \mapsto (\chi_*(h), \phi), \tag{3.33}$$

and $S(v, \chi) \sin \chi \, d\chi \, d\phi$ is the image under the above map of the restriction to the disk $B(0, d_0) \subset \mathbb{R}^2$ of the Lebesgue measure. With this slightly modified definition, it is found that

$$S(v, \chi) = \frac{1}{4} d_0^2;$$

- if $U(r) = kr^{-s}$ with $s > 0$, set

$$\vartheta(l) = \int_0^{\zeta(l)} \frac{dz}{\sqrt{1 - z^2 - 2(z/l)^s}},$$

where $l = (v^2/2k)^{1/s} h$, and where $\zeta(l)$ is the only positive root of the denominator of the integrand above; set $\vartheta \mapsto \lambda(\vartheta)$ to be the inverse of $l \mapsto \vartheta(l)$ so defined. Then

$$S(v, \chi) = (2k)^{2/s} v^{-4/s} \frac{\beta((\pi - \chi)/2)}{\sin \chi},$$

where $\beta(\vartheta) = \lambda(\vartheta)\lambda'(\vartheta)$. One finds that β is singular near $\theta = \pi/2$, which corresponds to $\chi = 0$ and $l \rightarrow +\infty$, i.e., to collisions with small deflection angles, or equivalently to the case of *grazing collisions*

$$\beta(\theta) \simeq C \left(\frac{\pi}{2} - \theta \right)^{-1-2/s} \quad \text{as } \theta \rightarrow \frac{\pi}{2}^-,$$

while $\beta(\theta) = O(\theta)$ for $\theta \rightarrow 0$.

Although the usual definition of the scattering cross-section involves the deflection angle χ, one might find it easier to use instead the angle $\theta = (\pi - \chi)/2$ (see Figure 2). It is easily seen that the scattering cross-section S as above can be expressed in terms of a function $\Sigma \equiv \Sigma(v, \mu)$ defined on $\mathbb{R}_+ \times [0, 1)$ by the formula

$$\Sigma(v, |\cos \theta|) = \frac{1}{2} S(v, \pi - 2\theta) |\cos \theta|.$$

This function Σ has the following geometric interpretation: the mapping

$$\mathbb{S}^2 \setminus \{N, S\} \simeq (0, \pi) \times [0, 2\pi) \rightarrow \mathbb{S}^2 \setminus \{N\},$$

$$\omega \simeq (\theta, \phi) \mapsto (-\cos(2\theta), \sin(2\theta) \cos \phi, \sin(2\theta) \sin \phi)$$
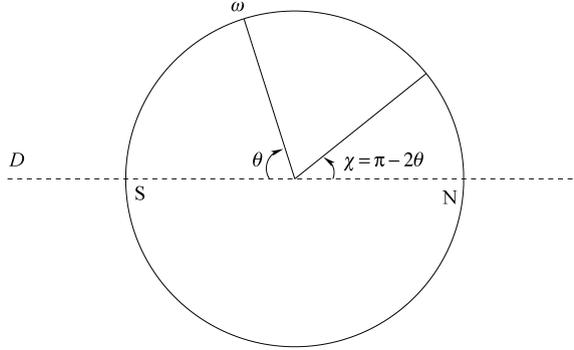
Fig. 4. The double cover $\theta \mapsto \chi$.

is a double cover (see Figure 4) and the image of the surface measure $\Sigma(v, |\cos\theta|) \times \sin\theta\, d\theta\, d\phi$ under the above mapping is $2S(v, \chi) \sin\chi\, d\chi\, d\phi$.

Then, the collision kernel b is given by

$$b(V, \omega) = |V| \Sigma\big(|V|, \big|\cos(\widehat{V, \omega})\big|\big).$$

With the formulas for the scattering cross-section given above, one sees that

- in the case of hard-spheres with radius r_0,

$$b(V, \omega) = \frac{1}{2} r_0^2 |V \cdot \omega|; \tag{3.34}$$

- in the case of an interaction potential $U(r) = kr^{-s}$ for $s > 0$,

$$b(V, \omega) = \frac{1}{4}(2k)^{2/s}|V|^{1-4/s}\frac{\beta(\theta)}{\sin\theta} \quad \text{with } \theta = (\widehat{V, \omega}) \tag{3.35}$$

  with $\beta(\theta) = O(\theta)$ as $\theta \to 0^+$ while $\beta(\theta) = O((\pi/2 - \theta)^{-1-2/s})$ as $\theta \to \pi^-/2$;
- whenever $s = 4$, the collision kernel b is independent of $|V|$; such potentials are usually referred to as *Maxwellian* potentials, and considerably facilitate the analysis of the collision integral;
- for $s = 1$, which is the case of a repulsive Coulomb potential, one has

$$b(V, \omega) = k^2|V|^{-3}\frac{\beta(\theta)}{\sin\theta} \quad \text{with } \beta(\theta) = O\bigg(\Big(\frac{\pi}{2} - \theta\Big)^{-3}\bigg) \text{ as } \theta \to \frac{\pi}{2}. \tag{3.36}$$

Observe that, for any inverse power-law potential $U(r) = kr^{-s}$,

$$\int_{\mathbb{S}^2} b(V, \omega)\, d\omega = +\infty \tag{3.37}$$

because of the singularity at $\theta = \pi/2$, so that our earlier assumption (3.8) is violated by such potentials. In particular, whenever the particle interaction is given by an inverse power-law potential, one cannot split the collision integral as

$$\mathcal{B}(F, F) = \mathcal{B}_+(F, F) - \mathcal{B}_-(F, F)$$

as was done earlier in this section. One way around this is to define $\mathcal{B}(F, F)$ in the sense of distributions, as follows

$$\langle \mathcal{B}(F, F), \phi \rangle$$
$$= \frac{1}{4} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} (F' F'_* - F F_*)(\phi + \phi_* - \phi' - \phi'_*) b(v - v_*, \omega) \, dv \, dv_* \, d\omega$$

for each $F \in C_c^1(\mathbb{R}^3)$ and $\phi \in C_c^1(\mathbb{R}^3)$. Then

$$(F' F'_* - F F_*)(\phi + \phi_* - \phi' - \phi'_*) = \mathrm{O}(|(v - v_*) \cdot \omega|^2)$$

so that the integrand in the right-hand side above is $\mathrm{O}((\pi/2 - \theta)^{1-2/s})$. This procedure can handle all inverse power-law potentials $U(r) = k r^{-s}$ for $s > 1$; however, the Coulomb case $s = 1$ remains excluded.

Observe that, in addition to the singularity in the deflection angle at $\theta = \pi/2$, the collision kernel of the Coulomb potential also has a $|v - v_*|^{-3}$ singularity in the velocity variable. Physicists deal with the latter singularity by introducing a further truncation near $V = 0$; the dependence upon this truncation parameter of the collision integral is only logarithmic, so that the result does not depend "too much" on the truncation parameter. In other words, the collision integral is computed modulo a scaling factor known as the *Coulomb logarithm* (see [76], Section 41).

Another way of avoiding the singularity of the collision kernel at $\theta = \pi/2$ consists in assuming that the interaction potential is truncated at large distance, in other words, that

$$U(r) = U(r_\mathrm{C}) \quad \text{whenever } r \geqslant r_\mathrm{C}. \tag{3.38}$$

In this case (as in the hard sphere case), U is not decreasing but only nonincreasing, and the definition of the scattering cross-section must be modified as in (3.33) with r_C in the place of d_0. Call b_C the collision kernel corresponding to the potential truncated as in (3.38), and $S_\mathrm{C}, \Sigma_\mathrm{C}$ the associated scattering cross-section. Then

$$\int_{\mathbb{S}^2} b_\mathrm{C}(V, \omega) \, d\omega = |V| \int_0^{2\pi} \int_0^\pi \Sigma_\mathrm{C}(|V|, |\cos\theta|) \sin\theta \, d\theta \, d\phi$$
$$= |V| \int_0^{2\pi} \int_0^\pi S_\mathrm{C}(|V|, \chi) \sin\chi \, d\chi \, d\phi$$
$$= |V| \int_0^{r_c} \int_0^{2\pi} h \, dh \, d\phi = |V| \pi r_\mathrm{C}^2$$

since $S_C(v, \chi) \sin \chi \, d\chi \, d\phi$ is the image under the map (3.33) of the Lebesgue measure on the two-dimensional disk of radius r_C. Hence the truncation (3.38) leads to a collision kernel whose integral over the angle variables is finite, thereby avoiding the divergence (3.37) that occurs for any infinite range, inverse power law potential.

Yet another way of avoiding the singularity of the collision kernel at $\theta = \pi/2$ was proposed by Grad [64]. He considered molecular force laws for which the collision kernel satisfies the condition

$$\frac{b(V, \omega)}{|\cos(V, \omega)|} \leqslant C\big(|V| + |V|^{1-\varepsilon}\big), \tag{3.39}$$

where $C > 0$ and $\varepsilon \in (0, 1)$. Comparing the case of a power law potential $U(r) = kr^{-s}$ with Grad's assumption above, we see that the collision kernel in (3.35) satisfies (3.39) provided that one modifies the function β near $\theta = \pi/2$ so that $\beta(\theta) = O(\pi/2 - \theta)$. One possibility is to replace β with

$$\tilde{\beta}(\theta) = \beta(\theta)\mathbb{1}_{\theta \leqslant \theta_0}, \tag{3.40}$$

where $\theta_0 \in (0, \frac{\pi}{2})$ is some arbitrary value. Then, the associated truncated collision kernel $\tilde{b}$ satisfies the bounds

$$\tilde{b}(V, \omega) \leqslant C_b\big(1 + |V|\big)^{1-4/s} \quad \text{and} \quad \int_{\mathbb{S}^2} b(V, \omega) \, d\omega \geqslant \frac{1}{C_b}\big(1 + |V|\big)^{1-4/s} \tag{3.41}$$

for some positive constant C_b. The potential U is called *a hard cut-off potential* if $s \geqslant 4$, and a *soft cut-off potential* if $s < 4$.

Grad defined more general classes of hard and soft cut-off potentials; specifically, a general hard cut-off potential corresponds to the condition

$$\int_{\mathbb{S}^2} b(V, \omega) \, d\omega \geqslant \frac{c|V|}{1 + |V|}$$

while a soft cut-off potential is defined by the condition

$$\int_{\mathbb{S}^2} b(V, \omega) \, d\omega \leqslant c\big(1 + |V|^{\varepsilon - 1}\big)$$

for some $c > 0$ and $\varepsilon \in (0, 1)$, in addition to the bound (3.39). In the sequel, we shall mostly restrict our attention to those hard cut-off potentials that satisfy the same bound (3.41) as in the inverse power law case.

The terminology "cut-off potential" attached to Grad's cut-off prescription, is somewhat unfelicitous. Indeed, it is not equivalent to truncating the potential at large intermolecular distances as in (3.38). Indeed, the angular truncation (3.40) prohibits grazing collisions

with deflection angle less than a threshold $\pi - 2\theta_0$ that is independent of the relative velocity V. On the contrary, with a potential truncated as in (3.38), there exist grazing collisions with deflection angle arbitrarily small for large enough relative velocity $|V|$. Hence, it would be more appropriate to refer to Grad's procedure as leading to a "cut-off scattering cross-section" rather than a "cut-off potential". Yet the latter terminology is commonly used in the literature, so that changing it would only cause confusion.

Let us conclude this subsection with a few words on the physical relevance of Grad's cut-off assumption. Grad observed that, in gases of neutral particles with short range interactions, grazing collisions are not statistically dominant as in the case of plasmas, where the long range effect of the Coulomb interaction must be accounted for. The latter case requires using a mean-field description of the long-range interaction potential, in addition to the Landau collision integral, an approximation of Boltzmann's collision integral in the regime of essentially grazing collisions.

That the Coulomb potential, probably the best known interaction in physics, is a singular point in the theory of the Boltzmann collision integral may seem highly regrettable. However, in view of the remark above, the reader should bear in mind that the Boltzmann equation is essentially meant to model collisional processes in neutral gases with short-range molecular interactions, and not in plasmas, so that the Coulomb potential is not really relevant in this context.

3.6. *The linearized collision integral*

Let ρ and $\theta > 0$, and pick $u \in \mathbb{R}^3$; the linearization at $\mathcal{M}_{(\rho,u,\theta)}$ of Boltzmann's collision integral is defined as follows

$$\mathcal{L}_{\mathcal{M}_{(\rho,u,\theta)}} f = -2\mathcal{M}_{(\rho,u,\theta)}^{-1} \mathcal{B}(\mathcal{M}_{(\rho,u,\theta)}, \mathcal{M}_{(\rho,u,\theta)} f), \tag{3.42}$$

where $\mathcal{B}$ is the bilinear operator obtained by polarization from the Boltzmann collision integral. In other words,

$$\mathcal{L}_{\mathcal{M}_{(\rho,u,\theta)}} f = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \big(f + f_* - f' - f_*'\big) b(v - v_*, \omega) \mathcal{M}_{(\rho,u,\theta)}(v_*) \, dv_* \, d\omega, \tag{3.43}$$

where f_*, f' and f_*' are the values of f at v_*, v' and v_*', respectively, and where v' and v_*' are determined in terms of v, v_* and ω by the usual collision relations (3.3).

The dependence on the parameters ρ, u and θ of the linearized collision integral is handled most easily by using the translation and scaling invariance of $\mathcal{L}$.

3.6.1. *Translation and scaling invariance of* $\mathcal{L}_{\mathcal{M}_{(\rho,u,\theta)}}$.   We introduce the following notation for the actions of translation and scaling transformations on functions defined on $\mathbb{R}^3$:

$$\tau_u f(v) = f(v - u), \qquad m_\lambda f(v) = \lambda^{-3} f\left(\frac{v}{\lambda}\right). \tag{3.44}$$

For instance, with these notations

$$\mathcal{M}_{(\rho,u,\theta)} = \rho m_{\sqrt{\theta}} \tau_u \mathcal{M}_{(1,0,1)}. \tag{3.45}$$

Given a collision kernel $b \equiv b(z,\omega)$, we denote by $\mathcal{B}^b$ Boltzmann's collision integral defined by this collision kernel. With the notation so defined, a straightforward change of variables in the collision integral leads to the following relation

$$\tau_u \mathcal{B}^b(\Phi,\Phi) = \mathcal{B}^b(\tau_u \Phi, \tau_u \Phi),$$
$$m_\lambda \mathcal{B}^b(\Phi,\Phi) = \lambda^3 \mathcal{B}^{m_\lambda b}(m_\lambda \Phi, m_\lambda \Phi) \tag{3.46}$$

for each continuous, rapidly decaying $\Phi \equiv \Phi(v)$, where, in the expression $m_\lambda b$, it is understood that the scaling transformation acts on the first argument of b, i.e., on the relative velocity. The analogous formula for the linearized collision operator is

$$m_{\sqrt{\theta}} \tau_u \mathcal{L}^b_{\mathcal{M}_{(1,0,1)}} \phi = \rho \theta^{3/2} \mathcal{L}^{m_{\sqrt{\theta}} b}_{\mathcal{M}_{(\rho,u,\theta)}} (m_{\sqrt{\theta}} \tau_u \phi). \tag{3.47}$$

This relation shows that it is enough to study the linearization of the collision integral at the centered reduced Gaussian

$$M = \mathcal{M}_{(1,0,1)}$$

with an arbitrary collision kernel b.

3.6.2. *Rotational invariance of $\mathcal{L}_{\mathcal{M}_{(1,0,1)}}$.* The orthogonal group $O_3(\mathbb{R})$ acts on functions on $\mathbb{R}^3$ by the formula

$$f_R(v) = f(R^T v), \quad R \in O_3(\mathbb{R}), v \in \mathbb{R}^3; \tag{3.48}$$

likewise its action on vector fields is defined by

$$U_R(v) = RU(R^T v), \quad R \in O_3(\mathbb{R}), v \in \mathbb{R}^3, \tag{3.49}$$

while its action on symmetric matrix fields is given by

$$S_R(v) = RS(R^T v)R^T, \quad R \in O_3(\mathbb{R}), v \in \mathbb{R}^3. \tag{3.50}$$

The Boltzmann collision integral is obviously invariant under the action of $O_3(\mathbb{R})$ – indeed, the microscopic collision process is isotropic. In fact, an elementary change of variables in the collision integral shows that

$$\mathcal{B}(\Phi_R, \Phi_R) = \mathcal{B}(\Phi,\Phi)_R \tag{3.51}$$

for each continuous, rapidly decaying Φ. Since the centered unit Gaussian $M = \mathcal{M}_{(1,0,1)}$ is a radial function, this rotation invariance property goes over to $\mathcal{L}_M$

$$\mathcal{L}_M(\phi_R) = (\mathcal{L}_M\phi)_R. \tag{3.52}$$

Extending $\mathcal{L}_M$ to act componentwise on vector or matrix fields on $\mathbb{R}^3$, one finds that

$$\mathcal{L}_M(U_R) = (\mathcal{L}_M U)_R \tag{3.53}$$

for continuous, rapidly decaying vector fields U and

$$\mathcal{L}_M(S_R) = (\mathcal{L}_M S)_R \tag{3.54}$$

for continuous, rapidly decaying symmetric matrix fields S, where the notations U_R and S_R are as in (3.48)–(3.50).

   As we shall see below, this $O_3(\mathbb{R})$-invariance of $\mathcal{L}_M$ has important consequences: it implies in particular that the viscosity and heat conductivity are scalar quantities (and not matrices).

3.6.3. *The Fredholm property.*   Henceforth, we assume that the collision kernel b satisfies a hard cut-off assumption in the sense of Grad [64], i.e., there exists $\alpha \in [0, 1]$ and $C_{\mathrm{b}} > 0$ such that, for a.e. $z \in \mathbb{R}^3$ and $\omega \in \mathbb{S}^2$, one has

$$0 < b(z, \omega) \leqslant C_{\mathrm{b}}(1 + |z|)^\alpha \quad \text{and}$$
$$\int_{\mathbb{S}^2} b(z, \omega)\, \mathrm{d}\omega \geqslant \frac{1}{C_{\mathrm{b}}}(1 + |z|)^\alpha. \tag{3.55}$$

Consider $\mathcal{L}$, the linearization of the Boltzmann collision integral at the centered, reduced Gaussian state M above. (Notice that we have discarded the dependence of $\mathcal{L}$ on b and M for notational simplicity.) From (3.43), we infer that $\mathcal{L}$ can be split as the sum of a local (multiplication) operator and of an integral operator, as follows

$$\mathcal{L}\phi(v) = a(|v|)\phi(v) - \mathcal{K}\phi(v), \tag{3.56}$$

where a is the collision frequency

$$a(|v|) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} b(v - v_*, \omega) M_*\, \mathrm{d}v_*\, \mathrm{d}\omega. \tag{3.57}$$

The nonlocal operator $\mathcal{K}$ is further split into two parts

$$\mathcal{K}\phi = \mathcal{K}_1\phi - \mathcal{K}_2\phi, \tag{3.58}$$

where

$$\mathcal{K}_1\phi = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \phi_* b(v - v_*, \omega) M_* \, dv_* \, d\omega$$

$$\mathcal{K}_2\phi = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} (\phi' + \phi_*') b(v - v_*, \omega) M_* \, dv_* \, d\omega \tag{3.59}$$

$$= 2 \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \phi' b(v - v_*, \omega) M_* \, dv_* \, d\omega.$$

(This last formula is not entirely obvious; it rests on a change of variables that will be explained later.) It is clear that $\mathcal{K}_1$ is a compact operator on $L^2(M \, dv)$; that $\mathcal{K}_2$ shares the same property is much less obvious, and was proved by Hilbert in the hard sphere case. Fifty years later, Grad introduced the cut-off assumption which now bears his name and used it in particular to extend Hilbert's result to all cut-off potentials.

LEMMA 3.9 (Hilbert [69], Grad [64]). *Assume that b is a collision kernel that satisfies the hard cut-off assumption* (3.55). *Then the operator $\mathcal{K}_2$ is compact on $L^2(M \, dv)$.*

We shall not give the proof of the Hilbert–Grad lemma here, since it is rather long and technical; the interested reader is referred to the lucid exposition by Glassey [50], or to the treatise by Cercignani, Illner and Pulvirenti [28] for the hard sphere case.

Instead, we shall digress from our discussion of the linearized collision integral and mention a new result by Lions which can be viewed as a nonlinear analogue of the Hilbert–Grad result.

LEMMA 3.10 (Lions [84]). *Assume that b is the collision kernel of a hard sphere gas*: $b(z, \omega) = |z \cdot \omega|$, *and consider the gain term in the Boltzmann collision integral*

$$\mathcal{B}_+(F, F) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} F' F_*' |(v - v_*) \cdot \omega| \, dv_* \, d\omega.$$

Then $\mathcal{B}_+$ maps $L^2_{\mathrm{comp}}(\mathbb{R}^3)$ continuously into $H^1_{\mathrm{loc}}(\mathbb{R}^3)$.

Actually, this is not exactly Lions' result; in the reference [84], only compactly supported collision kernels are considered. The proof in [84] is based upon the L^2-continuity of Fourier integral operators of order 0. The statement above was later proved by Bouchut and Desvillettes [17] by a very simple and elegant argument.

Deriving the Hilbert–Grad lemma from the Lions lemma would require additional estimates in order to handle contributions from large $|v|$. Since obtaining these estimates requires essentially as much work as does the proof of the Hilbert–Grad lemma, we do not claim that the Lions lemma leads to a simpler proof of the compactness of $\mathcal{K}$; it is however of great independent interest.

The main properties of $\mathcal{L}$ are now summarized in the following theorem.

THEOREM 3.11. *Assume that the collision kernel b satisfies the hard cut-off assumption (3.55). Then $\mathcal{L}$ is an unbounded self-adjoint nonnegative Fredholm operator, with domain $D(\mathcal{L}) = L^2(a^2 M\,dv)$ (a being the collision frequency defined in (3.57)). Its nullspace is the space of collision invariants*

$$\operatorname{Ker}\mathcal{L} = \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}.$$

Finally, $\mathcal{L}$ satisfies the following relative coercivity property on $(\operatorname{Ker}\mathcal{L})^{\perp}$: there exists $C_0 > 0$ such that, for each $\phi \in L^2(aM\,dv)$ – the form domain of $\mathcal{L}$ – one has

$$\int_{\mathbb{R}^3} \phi\mathcal{L}\phi M\,dv \geqslant C_0 \int_{\mathbb{R}^3} (\phi - \Pi\phi)^2 aM\,dv, \tag{3.60}$$

where Π is the $L^2(M\,dv)$-orthogonal projection on $\operatorname{Ker}\mathcal{L}$.

SKETCH OF THE PROOF. Let us briefly explain how these various facts are established. It follows from the Hilbert–Grad lemma that $\mathcal{K}$ is a compact operator on $L^2(M\,dv)$. Hence $\mathcal{L} = a - \mathcal{K}$ is an unbounded Fredholm operator with domain $D(\mathcal{L}) = L^2(a^2 M\,dv)$. That $\mathcal{L}$ is self-adjoint comes from the symmetries of Boltzmann's collision integral, especially from the computation of

$$\int_{\mathbb{R}^3} \mathcal{B}(F, F)\phi\,dv$$

as explained in Proposition 3.1. By the same argument, one sees that

$$\int_{\mathbb{R}^3} \phi\mathcal{L}\phi M\,dv = \frac{1}{4} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(\phi + \phi_* - \phi' - \phi'_*\right)^2 bM\,dv M_*\,dv_*\,d\omega \geqslant 0 \tag{3.61}$$

so that $\mathcal{L}$ is a nonnegative operator; moreover, if $\mathcal{L}\phi = 0$, the integral on the right-hand side of the above equality vanishes, so that

$$\phi + \phi' - \phi' - \phi'_* = 0 \quad \text{a.e. in } v, v_*, \omega.$$

Hence ϕ is a collision invariant (see Definition 3.5) and this characterizes $\operatorname{Ker}\mathcal{L}$, see Proposition 3.6. As for the last inequality, the spectral theory of compact operators implies that $\mathcal{L}$ has a spectral gap in $L^2(M\,dv)$

$$\int_{\mathbb{R}^3} \phi\mathcal{L}\phi M\,dv \geqslant C^* \int_{\mathbb{R}^3} (\phi - \Pi\phi)^2 M\,dv. \tag{3.62}$$

On the other hand, by continuity of $\mathcal{K}$ on $L^2(M\,dv)$, one has

$$\int_{\mathbb{R}^3} \phi \mathcal{L}\phi M\,dv = \int_{\mathbb{R}^3} (\phi - \Pi\phi)\mathcal{L}(\phi - \Pi\phi)M\,dv$$

$$\geqslant \int_{\mathbb{R}^3} (\phi - \Pi\phi)^2 aM\,dv - \|\mathcal{K}\|^2 \int_{\mathbb{R}^3} (\phi - \Pi\phi)^2 M\,dv.$$

Combining both inequalities leads to the announced estimate. $\square$

The improved spectral gap estimate above is due to Bardos, Caflisch and Nicolaenko [6]. Also, an elegant explicit estimate for the $L^2(M\,dv)$ spectral gap can be found in [5].

In fact, it is interesting to notice that the weighted spectral gap estimate (3.60) is in some sense more intrinsic than the unweighted one. Indeed, in the case of soft cut-off potentials, (3.62) is false, but (3.60) (which is of course a weaker statement since the collision frequency vanishes for large velocities in the soft potential case) holds true: this was proved in [58], based on decay estimates on the gain term in the linearized collision integral due to Grad and Caflisch [23]. In other words, while the spectral gap estimate (**??**) holds for hard potentials only, the weighted spectral gap estimate holds for hard as well as soft potentials, in the cut-off case.

An important consequence of Theorem 3.11 is that the integral equation

$$\mathcal{L}\phi = \psi, \quad \psi \in L^2(M\,dv), \tag{3.63}$$

satisfies the *Fredholm alternative*:

- either $\psi \perp \operatorname{Ker}\mathcal{L}$, in which case (3.63) has a unique solution

$$\phi_0 \in L^2(a^2 M\,dv) \cap (\operatorname{Ker}\mathcal{L})^\perp;$$

  then any solution of (3.63) is of the form

$$\phi = \phi_0 + \phi_1, \quad \text{where } \phi_1 \text{ is an arbitrary element of } \operatorname{Ker}\mathcal{L};$$

- or $\psi \notin (\operatorname{Ker}\mathcal{L})$, in which case (3.63) has no solution.

EXAMPLE. Consider the vector field

$$B(v) = \frac{1}{2}\big(|v|^2 - 5\big)v$$

and the matrix field

$$A(v) = v \otimes v - \frac{1}{3}|v|^2 I.$$

Clearly

$$A_{jk} \perp \operatorname{Ker}\mathcal{L}, \qquad B_l \perp \operatorname{Ker}\mathcal{L}, \qquad A_{jk} \perp B_l, \quad j,k,l = 1,2,3. \tag{3.64}$$

In fact, more is true

$$\int_{\mathbb{R}^3} A(v) f(|v|^2) M \, \mathrm{d}v = 0,$$

$$\int_{\mathbb{R}^3} A(v) v f(|v|^2) M \, \mathrm{d}v = 0,$$

$$\int_{\mathbb{R}^3} B(v) f(|v|^2) M \, \mathrm{d}v = 0,$$

$$\int_{\mathbb{R}^3} B(v) \cdot v M \, \mathrm{d}v = 0.$$

The second and third formulas are obvious since A is even and B odd. As for the first formula, observe that A is an isotropic matrix, in the sense that $A(Rv) = RA(v)R^T$ for each $R \in O_3(\mathbb{R})$ – with the notation (3.50) for the action of $O_3(\mathbb{R})$ on symmetric matrices, $A_R = A$ for each $R \in O_3(\mathbb{R})$. Hence the matrix

$$\int_{\mathbb{R}^3} A(v) f(|v|^2) M \, \mathrm{d}v$$

commutes with any $R \in O_3(\mathbb{R})$ – as can be seen by changing v into Rv in the above integral – and is therefore a scalar multiple of the identity matrix. But

$$\mathrm{trace} \int_{\mathbb{R}^3} A(v) f(|v|^2) M \, \mathrm{d}v = \int_{\mathbb{R}^3} \mathrm{trace}\, A(v) f(|v|^2) M \, \mathrm{d}v = 0$$

and hence this scalar multiple of the identity matrix is null. The fourth and last formula is based on the following elementary recursion formula for Gaussian integrals

$$\int_{\mathbb{R}^3} |v|^n M \, \mathrm{d}v = (n+1) \int_{\mathbb{R}^3} |v|^{n-2} M \, \mathrm{d}v, \quad n \geqslant 2 \tag{3.65}$$

(to see this, use spherical coordinates and integrate by parts).

In particular, the Fredholm alternative implies the existence of a matrix field $\widehat{A}$ and of a vector field $\widehat{B}$ such that

$$\begin{aligned} \mathcal{L}\widehat{A} = A \quad \text{and} \quad \widehat{A} \perp \mathrm{Ker}\,\mathcal{L}, \\ \mathcal{L}\widehat{B} = B \quad \text{and} \quad \widehat{B} \perp \mathrm{Ker}\,\mathcal{L}. \end{aligned} \tag{3.66}$$

Observe that

$$\mathcal{L}(\widehat{A}_R) = A_R = A, \qquad \widehat{A}_R \perp \mathrm{Ker}\,\mathcal{L} \quad \text{for all } R \in O_3(\mathbb{R}),$$

$$\mathcal{L}(\widehat{B}_R) = B_R = B, \qquad \widehat{B}_R \perp \mathrm{Ker}\,\mathcal{L} \quad \text{for all } R \in O_3(\mathbb{R}),$$

so that, by the uniqueness part in the Fredholm alternative

$$\widehat{A}_R = \widehat{A} \quad \text{and} \quad \widehat{B}_R = \widehat{B} \quad \text{for all } R \in O_3(\mathbb{R}).$$

An elementary geometric argument (see [34]) shows the existence of two scalar functions

$$\mathbf{a}: \mathbb{R}_+ \to \mathbb{R}, \qquad \mathbf{b}: \mathbb{R}_+ \to \mathbb{R}$$

such that

$$\widehat{A}(v) = \mathbf{a}\big(|v|^2\big) A(v) \quad \text{and} \quad \widehat{B}(v) = \mathbf{b}\big(|v|^2\big) B(v). \tag{3.67}$$

As we shall see further, the viscosity and heat conductivity of a gas are expressed as Gaussian integrals of the scalar functions $\mathbf{a}$ and $\mathbf{b}$, and therefore are scalar quantities themselves.

4. Hydrodynamic scalings for the Boltzmann equation

This short section introduces in particular the Knudsen number, *Kn*, a dimensionless parameter of considerable importance for the derivation of hydrodynamic models from the kinetic theory of gases.

4.1. *Notion of a rarefied gas*

Think of a monatomic gas as a cloud of $\mathcal{N}$ hard spheres of radius r confined in a container of volume V; we shall call *excluded volume* the volume V_e that the $\mathcal{N}$ gas molecules would occupy if tightly packed somewhere in the container (as oranges in a grocery store). Clearly

$$\frac{4}{3}\mathcal{N}\pi r^3 \leqslant V_e \leqslant \mathcal{N}(2r)^3,$$

and we shall call a *perfect gas* one for which $V_e \ll V$. It is well known that the equation of state for a perfect gas is given by the Boyle–Mariotte law

$$p = k\rho\theta,$$

where p, ρ and θ are the pressure, the density and the temperature in the gas, while k designates the Boltzmann constant ($k = 1.38 \cdot 10^{-23}$ J K^{-1}).

EXAMPLE. For a monatomic gas at room temperature and atmospheric pressure, about $\mathcal{N} = 10^{20}$ molecules with radius 10^{-8} cm are to be found in a volume V of 1 cm^3. Hence, the excluded volume satisfies $V_e \leqslant 10^{20} \cdot (2 \cdot 10^{-8})^3 = 8 \cdot 10^{-4}$ cm^3, so that $V_e \ll V$.

Another important notion is that of *mean free path*: it is the average distance between two successive collisions for one gas molecule picked at random. There is more than one precise definition of that notion, in particular, because there are several choices of probability measures for computing the mean. One choice could be to use the empirical notion of mean: Pick a typical particle, wait until after this particle has collided $n \gg 1$ times with other particles; the ratio of the distance traveled by the particle between the initial time and the last collision time to the number n of collisions should converge, in the large n limit, to one notion of mean free path. This definition, however, does not provide us with an easy way of estimating the mean free path, because it is fairly difficult to compute the trajectory of one particle in the cloud of the $\mathcal{N} - 1$ other particles.

Intuitively, one expects that the larger the number of particles in the container, the smaller the mean free path; likewise, the bigger the particles, the smaller the mean free path. This suggests using the following formula to estimate the order of magnitude of the mean free path,

$$\text{mean free path} \simeq \frac{1}{(\mathcal{N}/(V - V_{\mathrm{e}})) \times \mathcal{A}},$$

where $\mathcal{A}$ is the surface area of the section of the particles. A mathematical justification of the above formula for the mean free path can be found in [111]; see also [39].

EXAMPLE. For the same monatomic gas as above, $\mathcal{N} = 10^{20}$ molecules, $V - V_{\mathrm{e}} \simeq V = 1$ cm^3, while $\mathcal{A} = \pi \cdot (10^{-8})^2$ cm^2. This gives a mean free path of the order of $3 \cdot 10^{-5}$ cm $\ll 1$ cm (the size of the container). Hence, a gas molecule will bump into roughly 10^4 other particles when traveling a distance comparable to the size of the container.

The example above shows that even in the case of a perfect gas (i.e., for an excluded volume negligible when compared to the size of the container) any given particle can collide with a large number of other particles. See, for instance, [2] for more information on the importance of the excluded volume in this context.

EXAMPLE. For the same monatomic gas as above, lower the pressure from $p = 1$ atm to $p = 10^{-4}$ atm. Then, only $\mathcal{N} = 10^{16}$ molecules are to be found in the container, which gives a mean free path of the order of 0.3 cm, comparable to the size of the container.

These examples suggest that the natural way of measuring the degree of rarefaction in the gas is by using the Knudsen number, *Kn*, defined as the ratio of the mean free path to the size of the container, or more generally, as

$$Kn = \frac{\text{mean free path}}{\text{macroscopic length scale}}.$$

In other words,

- a rarefied gas is a gas for which $Kn \gtrsim 1$, while
- a gas in hydrodynamic regime satisfies $Kn \ll 1$.

4.2. *The dimensionless Boltzmann equation*

In the discussion above, the introduction of the Knudsen number was based on physical arguments. We shall see below that it also appears naturally when writing the Boltzmann equation in dimensionless variables.

Choose a macroscopic length scale L and time scale T, and a reference temperature Θ. This defines *two* velocity scales:

- one is the speed at which some macroscopic portion of the gas is transported over a distance L in time T, i.e.,

$$U = \frac{L}{T};$$

- the other one is the *thermal speed* of the molecules with energy $\frac{3}{2}k\Theta$; in fact, it is more natural to define this velocity scale as

$$c = \sqrt{\frac{5}{3}\frac{k\Theta}{m}},$$

  m being the molecular mass, which is the *speed of sound* in a monatomic gas at the temperature Θ.

Define next the dimensionless variables involved in the Boltzmann equation, i.e., the dimensionless time, space and velocity variables as

$$\hat{t} = \frac{t}{T}, \qquad \hat{x} = \frac{x}{L} \quad \text{and} \quad \hat{v} = \frac{v}{c}.$$

Define also the dimensionless number density

$$\widehat{F}(\hat{t}, \hat{x}, \hat{v}) = \frac{L^3 c^3}{\mathcal{N}^3} F(t, x, v),$$

where $\mathcal{N}$ is the total number of gas molecules in a volume L^3. Finally, we must rescale the collision kernel b. As mentioned earlier, $b(z, \omega)$ is the relative velocity multiplied by the scattering cross-section of the gas molecules; define

$$\hat{b}(\hat{z}, \omega) = \frac{1}{c \times \pi r^2} b(z, \omega) \quad \text{with } \hat{z} = \frac{z}{c},$$

where r is the molecular radius.

If f satisfies the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \big(F' F'_* - F F_*\big) b(v - v_*, \omega) \, \mathrm{d}v_* \, \mathrm{d}\omega,$$

then

$$\frac{L}{cT}\partial_{\hat{t}}\widehat{F} + \hat{v}\cdot\nabla_{\hat{x}}\widehat{F} = \frac{\mathcal{N}\pi r^2}{L^2}\iint_{\mathbb{R}^3\times\mathbb{S}^2}\left(\widehat{F}'\widehat{F}'_* - \widehat{F}\widehat{F}_*\right)\hat{b}(\hat{v} - \hat{v}_*, \omega)\,\mathrm{d}\hat{v}_*\,\mathrm{d}\omega.$$

The factor multiplying the collision integral is

$$L\times\frac{\mathcal{N}\times\pi r^2}{L^3} = \frac{L}{\text{mean free path}} = \frac{1}{Kn},$$

where Kn is the Knudsen number defined above. The factor multiplying the time derivative

$$\frac{(1/T)\times L}{c} =: St$$

is called the *kinetic Strouhal number* (by analogy with the notion of Strouhal number used in the dynamics of vortices). Hence the dimensionless form of the Boltzmann equation (see Section 2.9 in [115]) is

$$St\,\partial_{\hat{t}}\widehat{F} + \hat{v}\cdot\nabla_{\hat{x}}\widehat{F} = \frac{1}{Kn}\iint_{\mathbb{R}^3\times\mathbb{S}^2}\left(\widehat{F}'\widehat{F}'_* - \widehat{F}\widehat{F}_*\right)\hat{b}(\hat{v} - \hat{v}_*, \omega)\,\mathrm{d}\hat{v}_*\,\mathrm{d}\omega. \tag{4.1}$$

There is some arbitrariness in the way the length, time and temperature scales L, T, Θ are chosen. For instance, if $F^{\mathrm{in}} \equiv F^{\mathrm{in}}(x, v)$ is the initial number density (at time $t = 0$), one can choose

$$\frac{1}{L} = \frac{\iint_{\mathbb{R}^3\times\mathbb{R}^3}|\nabla_x F^{\mathrm{in}}|\,\mathrm{d}x\,\mathrm{d}v}{\iint_{\mathbb{R}^3\times\mathbb{R}^3}F^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v},$$

$$U = \frac{\int_{\mathbb{R}^3}|\int_{\mathbb{R}^3}vF^{\mathrm{in}}\,\mathrm{d}v|\,\mathrm{d}x}{\iint_{\mathbb{R}^3\times\mathbb{R}^3}F^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v},$$

$$\Theta = \frac{1/(3k)\iint_{\mathbb{R}^3\times\mathbb{R}^3}m|v|^2 F^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v}{\iint_{\mathbb{R}^3\times\mathbb{R}^3}F^{\mathrm{in}}\,\mathrm{d}x\,\mathrm{d}v},$$

and define $T = L/U$. In addition to Sone's book [115], we also refer to the Introduction of [10] for a presentation of the Boltzmann equation in dimensionless variables.

All hydrodynamic limits of the Boltzmann equation correspond to situations where the Knudsen number, Kn, satisfies

$$Kn \ll 1.$$

In other words, the Knudsen number governs the transition from the kinetic theory of gases to hydrodynamic models, just as the Reynolds number in fluid mechanics governs the transition from laminar to turbulent flows, except that the hydrodynamic limit is much better understood than the latter situation.

But there is no universal prescription for the Strouhal number in the context of the hydrodynamic limit; as we shall see below, various hydrodynamic regimes can be derived from the Boltzmann equation by appropriately tuning the Strouhal number.

5. Compressible limits of the Boltzmann equation: Formal results

In this section, we study the dimensionless Boltzmann equation (4.1) in the case where

$$Kn = \varepsilon \ll 1, \qquad St = 1.$$

The dimensionless Boltzmann equation takes the form

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} \mathcal{B}(F_\varepsilon, F_\varepsilon), \tag{5.1}$$

where

$$\mathcal{B}(F, F) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \left(F' F'_* - F F_* \right) b(v - v_*, \omega) \, dv_* \, d\omega.$$

The collision kernel b is assumed to satisfy Grad's hard cut-off assumption

$$0 < b(z, \omega) \leqslant C_b \left(1 + |z| \right)^\alpha \quad \text{and} \quad \int_{\mathbb{S}^2} b(z, \omega) \, d\omega \geqslant \frac{1}{C_b} \left(1 + |z| \right)^\alpha \tag{5.2}$$

for a.e. $z \in \mathbb{R}^3$ and $\omega \in \mathbb{S}^2$, where $\alpha \in [0, 1]$ and $C_b > 0$.

5.1. *The compressible Euler limit: The Hilbert expansion*

In [69], Hilbert proposed to seek the solution of (5.1) as a formal power series in ε,

$$F_\varepsilon(t, x, v) = \sum_{k \geqslant 0} \varepsilon^k F_k(t, x, v) \tag{5.3}$$

with $F_k \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3; \mathcal{S}(\mathbb{R}^3))$ for each $k \geqslant 0$.

The equations governing the coefficients F_k are obtained by inserting the right-hand side of (5.3) in (5.1) and equating the coefficients multiplying the successive powers of ε.

Order ε^{-1}.   One finds

$$0 = \mathcal{B}(F_0, F_0),$$

hence F_0 is a local Maxwellian, meaning that there exists $\rho \equiv \rho(t, x) > 0, \theta \equiv \theta(t, x) > 0$ and $u \equiv u(t, x) \in \mathbb{R}^3$ such that

$$F_0(t, x, v) = \mathcal{M}_{(\rho(t,x), u(t,x), \theta(t,x))}(v) \quad \text{a.e.}$$

Order ε^0.   One finds

$$(\partial_t + v \cdot \nabla_x) F_0 = \mathcal{B}(F_1, F_0) + \mathcal{B}(F_0, F_1)$$

which can be recast in terms of the linearization at the local Maxwellian F_0 of the collision integral (see (3.42)) as

$$\mathcal{L}_{F_0}\left(\frac{F_1}{F_0}\right) = -(\partial_t + v \cdot \nabla_x) \ln F_0. \tag{5.4}$$

This is precisely an integral equation for F_1/F_0 of the form (3.63) studied above.

Let us compute the right-hand side of the above equality, i.e., the expression

$$(\partial_t + v \cdot \nabla_x) \ln F_0.$$

For convenience, we shall denote by V the vector

$$V = \frac{1}{\sqrt{\theta}}(v - u).$$

Then

$$(\partial_t + v \cdot \nabla_x) \ln \mathcal{M}_{(\rho, u, \theta)}$$

$$= \frac{1}{\rho}(\partial_t + v \cdot \nabla_x)\rho - \frac{3}{2}\frac{1}{\theta}(\partial_t + v \cdot \nabla_x)\theta$$

$$+ \frac{v - u}{\theta}(\partial_t + v \cdot \nabla_x)u + \frac{|v - u|^2}{2\theta^2}(\partial_t + v \cdot \nabla_x)\theta.$$

We shall rearrange the right-hand side of the above equality and express it as a linear combination of the functions

$$1, \qquad V_j, \qquad \frac{1}{2}(|V|^2 - 3), \qquad A(V)_{kl} \quad \text{and} \quad B(V)_j, \quad j, k, l = 1, 2, 3.$$

One finds that

$$(\partial_t + v \cdot \nabla_x) \ln \mathcal{M}_{(\rho, u, \theta)}$$

$$= \frac{1}{\rho}(\partial_t \rho + u \cdot \nabla_x \rho + \rho \operatorname{div}_x u)$$

$$+ \frac{1}{\sqrt{\theta}} V \cdot \left(\partial_t u + u \cdot \nabla_x u + \nabla_x \theta + \frac{\theta}{\rho} \nabla_x \rho\right)$$

$$+ \frac{1}{2}(|V|^2 - 3)\frac{1}{\theta}\left(\partial_t \theta + u \cdot \nabla_x \theta + \frac{2}{3}\theta \operatorname{div}_x u\right)$$

$$+ A(V) : \nabla_x u + 2B(V) \cdot \nabla_x \sqrt{\theta}. \tag{5.5}$$

Because of the orthogonality relations (3.64), the last two terms on the right-hand side of (5.5) belong to $(\mathrm{Ker}\,\mathcal{L}_{F_0})^{\perp}$, while the first three terms there are in $\mathrm{Ker}\,\mathcal{L}_{F_0}$. Therefore, the solvability condition for the Fredholm integral equation (5.4) consists in setting to 0 the coefficients of the functions 1, V_j and $\frac{1}{2}(|V|^2 - 3)$, i.e.,

$$\partial_t \rho + u \cdot \nabla_x \rho + \rho \,\mathrm{div}_x\, u = 0,$$

$$\partial_t u + u \cdot \nabla_x u + \nabla_x \theta + \frac{\theta}{\rho} \nabla_x \rho = 0,$$

$$\partial_t \theta + u \cdot \nabla_x \theta + \frac{2}{3} \theta \,\mathrm{div}_x\, u = 0,$$

or in other words,

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0,$$

$$\partial_t u + u \cdot \nabla_x u + \frac{1}{\rho} \nabla_x(\rho \theta) = 0, \qquad (5.6)$$

$$\partial_t \theta + u \cdot \nabla_x \theta + \frac{2}{3} \theta \,\mathrm{div}_x\, u = 0.$$

We recognize the system of Euler equations for a compressible fluid (2.7), in the case of a perfect monatomic gas; i.e., for $\gamma = 0$, so that the pressure law and the internal energy are given respectively by

$$\text{pressure} = \rho \theta \quad \text{and} \quad \text{internal energy} = \frac{3}{2} \theta.$$

Assuming that ρ, u and θ satisfy these Euler equations, we solve for F_1 the Fredholm integral equation (5.4) to find

$$F_1 = -\frac{1}{\rho} F_0 \big(\mathbf{a}(\theta, |V|) A(V) \cdot \nabla_x u + 2\mathbf{b}(\theta, |V|) B(V) \cdot \nabla_x \sqrt{\theta} \,\big)$$

$$+ F_0 \left(\frac{\rho_1}{\rho} + \frac{1}{\sqrt{\theta}} V \cdot u_1 + \frac{1}{2}(|V|^2 - 3) \frac{\theta_1}{\theta} \right).$$

The second term on the right-hand side of the formula giving F_1 represents the arbitrary element of $\mathrm{Ker}\,\mathcal{L}_{F_0}$ that appears in the general solution of the integral equation (3.63), while the scalar quantities $\mathbf{a}(\theta, V)$ and $\mathbf{b}(\theta, V)$ satisfy (see (3.67))

$$\mathcal{L}_{F_0}\big(\mathbf{a}(\theta, |V|) A(V)\big) = A(V) \quad \text{and} \quad \mathcal{L}_{F_0}\big(\mathbf{b}(\theta, |V|) B(V)\big) = B(V).$$

More precisely, let b be the collision kernel satisfying (5.2) considered in this chapter. Applying the notation in Section 3.6.1, and especially (3.44), we define scalar functions $\mathbf{a}(\theta, \cdot)$ and $\mathbf{b}(\theta, \cdot)$ as in (3.67) with a collision kernel $m_{1/\sqrt{\theta}}b$.

Order ε^1.   One finds that

$$\partial_t F_1 + v \cdot \nabla_x F_1 - \mathcal{B}(F_1, F_1) = 2\mathcal{B}(F_0, F_2)$$

which can be put in the form

$$\mathcal{L}_{F_0}\left(\frac{F_2}{F_0}\right) = \mathcal{Q}_{F_0}\left(\frac{F_1}{F_0}, \frac{F_1}{F_0}\right) - \frac{1}{F_0}(\partial_t + v \cdot \nabla_x)F_1$$

and this is therefore a Fredholm integral equation of the type (3.63). Here $\mathcal{Q}_{F_0}$ is the Boltzmann collision integral intertwined with the multiplication by the local Maxwellian F_0, i.e.,

$$\mathcal{Q}_{F_0}(\phi, \phi) = F_0^{-1}\mathcal{B}(F_0\phi, F_0\phi).$$

For this equation to have a solution, one must verify the compatibility conditions

$$\partial_t \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F_1 \, dv + \text{div}_x \int_{\mathbb{R}^3} v \otimes \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F_1 \, dv = 0.$$

These five compatibility conditions are five PDEs for the five unknown functions ρ_1, u_1 and θ_1.

Order ε^n.   One finds

$$\partial_t F_n + v \cdot \nabla_x F_n - \sum_{k+l=n, 1 \leqslant k, l, \leqslant n} \mathcal{B}(F_k, F_l) = 2\mathcal{B}(F_0, F_{n+1})$$

which is a Fredholm equation of the same type as above.

Here again, the compatibility condition reduces to

$$\partial_t \int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F_n \, dv + \text{div}_x \int_{\mathbb{R}^3} v \otimes \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F_n \, dv = 0.$$

More generally, the compatibility condition at order $n + 1$ (to guarantee the existence of F_{n+1}) provides the system of five equations satisfied by that part of F_n which belongs to the nullspace of $\mathcal{L}_{F_0}$. See [115] or Chapter V, Section 2 in [27] for more on Hilbert's expansion.

Conclusion.   The Hilbert expansion method shows that the leading order in ε of Hilbert's formal solution (5.3) of the scaled Boltzmann equation (5.1) with Strouhal number $St = 1$ and Knudsen number $Kn = \varepsilon \ll 1$ is a local Maxwellian state whose parameters are governed by the Euler equations of gas dynamics (for a perfect monatomic gas).

In general, one cannot hope that Hilbert's formal power series (5.3) has a positive radius of convergence. Yet, Hilbert's expansion method can be used to obtain a rigorous derivation of the Euler equations of gas dynamics from the Boltzmann equation, as we shall see in Section 9.

5.2. *The compressible Navier–Stokes limit: The Chapman–Enskog expansion*

In this subsection, we shall seek higher-order (in ε) corrections to the compressible Euler system. The Hilbert expansion presented above is not well suited for this purpose, because linear combinations of collision invariants (i.e., hydrodynamic modes) appear at each order in ε instead of being all concentrated in the leading order term.

For that reason, we shall use a slightly different expansion method, the Chapman–Enskog expansion. Thus, we seek a solution of the scaled Boltzmann equation (5.1) as a Chapman–Enskog formal power series,

$$F_\varepsilon(t, x, v) = \sum_{n \geqslant 0} \varepsilon^n F^{(n)}\big[\vec{P}(t, x)\big](v), \tag{5.7}$$

parametrized by the vector $\vec{P}$ of conserved densities of F_ε.

NOTATION. $F^n[\vec{P}(t, x)](v)$ designates a quantity that depends smoothly on $\vec{P}$ and any finite number of its derivatives with respect to the x-variable at the same point (t, x), and on the v-variable.

In particular, $F^n[\vec{P}(t, x)](v)$ does not contain time-derivatives of $\vec{P}$: the Chapman–Enskog method is based on eliminating $\partial_t \vec{P}$ in favor of x-derivatives via conservation laws satisfied by $\vec{P}$.

That $\vec{P}$ is the vector of conserved densities of F_ε means that

$$\int_{\mathbb{R}^3} F^{(0)}\big[\vec{P}\big](v) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} \mathrm{d}v = \vec{P},$$

$$\int_{\mathbb{R}^3} F^{(n)}\big[\vec{P}\big](v) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} \mathrm{d}v = \vec{0}, \quad n \geqslant 1. \tag{5.8}$$

These conserved densities satisfy a formal system of conservation laws of the form

$$\partial_t \vec{P} = \sum_{n \geqslant 0} \varepsilon^n \operatorname{div}_x \Phi^{(n)}\big[\vec{P}\big], \tag{5.9}$$

where the formal fluxes $\Phi^{(n)}$ are obtained from the local conservation laws associated to the Boltzmann equation by the formulas

$$\Phi^{(n)}\big[\vec{P}\big] = -\int_{\mathbb{R}^3} v \otimes \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} F^{(n)}\big[\vec{P}\big](v)\, dv \qquad (5.10)$$

for $n \geqslant 0$.

Let us analyze the first orders in ε of the Chapman–Enskog expansion.

Order 0.   One has

$$\mathcal{B}\big(F^{(0)}\big[\vec{P}\big], F^{(0)}\big[\vec{P}\big]\big) = 0 \quad \text{and thus} \quad F^{(0)}\big[\vec{P}\big] = \mathcal{M}_{(\rho,u,\theta)},$$

here

$$\vec{P} = \begin{pmatrix} \rho \\ \rho u \\ \rho(\frac{1}{2}|u|^2 + \frac{3}{2}\theta) \end{pmatrix}, \qquad \Phi^{(0)}\big[\vec{P}\big] = -\begin{pmatrix} \rho u \\ \rho u^{\otimes 2} + \rho\theta I \\ \rho u(\frac{1}{2}|u|^2 + \frac{5}{2}\theta) \end{pmatrix}.$$

Hence the formal conservation law at order 0 is

$$\partial_t \vec{P} = \operatorname{div}_x \Phi^{(0)}\big[\vec{P}\big] \quad \operatorname{mod} \mathrm{O}(\varepsilon)$$

whose explicit form is

$$\partial_t \rho + u \cdot \nabla_x \rho + \rho \operatorname{div}_x u = 0,$$

$$\partial_t u + (u \cdot \nabla_x)u + \frac{1}{\rho}\nabla_x(\rho\theta) = 0 \quad \operatorname{mod} \mathrm{O}(\varepsilon), \qquad (5.11)$$

$$\partial_t \theta + u \cdot \nabla_x \theta + \frac{2}{3}\theta \operatorname{div}_x u = 0.$$

In other words, the 0th order of the Chapman–Enskog expansion gives the compressible Euler system, as does the Hilbert expansion.

The Hilbert and Chapman–Enskog methods differ at order 1 in ε, as we shall see below.

Order 1.   One has

$$(\partial_t + v \cdot \nabla_x)F^{(0)}\big[\vec{P}\big] = 2\mathcal{B}\big(F^{(0)}\big[\vec{P}\big], F^{(1)}\big[\vec{P}\big]\big). \qquad (5.12)$$

Using the formal conservation law at order 0, we eliminate $\partial_t F^{(0)}[\vec{P}]$ and replace it with x-derivatives of $F^{(0)}[\vec{P}]$, by using (5.5) and the conservation laws at order 0 (5.11) (i.e., Euler's system) as follows

$$(\partial_t + v \cdot \nabla_x)\mathcal{M}_{(\rho,u,\theta)}$$

$$= \mathcal{M}_{(\rho,u,\theta)}\big(A(V) : D(u) + 2B(V) \cdot \nabla_x\sqrt{\theta}\,\big) + \mathrm{O}(\varepsilon) \qquad (5.13)$$

with the notation

$$V = \frac{v - u}{\sqrt{\theta}}, \qquad A(V) = V \otimes V - \frac{1}{3}|v|^2 I, \qquad B(V) = \frac{1}{2}V(|V|^2 - 5),$$

and where $D(u)$ is the traceless part of the deformation tensor of u

$$D(u) = \frac{1}{2}\left(\nabla_x u + (\nabla_x u)^T - \frac{2}{3}\operatorname{div}_x u I\right).$$

In view of (5.12) and (5.13), $F^{(1)}[\vec{P}]$ is determined by the conditions

$$A(V) : D(u) + 2B(V) \cdot \nabla_x \sqrt{\theta} = -\mathcal{L}_{\mathcal{M}_{(\rho,u,\theta)}}\left(\frac{F^{(1)}[\vec{P}]}{\mathcal{M}_{(\rho,u,\theta)}}\right),$$

$$\int F^{(1)}[\vec{P}](v) \begin{pmatrix} 1 \\ v \\ \frac{1}{2}|v|^2 \end{pmatrix} dv = 0.$$

By Hilbert's lemma, $\mathcal{L}_{\mathcal{M}_{(1,u,\theta)}}$ is a Fredholm operator on $L^2(M\,dv)$; thus

$$F^{(1)}[\vec{P}](v) = -\mathcal{M}_{(1,u,\theta)}\left(\mathbf{a}(\theta, |V|)A(V) : D(u) + 2\mathbf{b}(\theta, |V|)B(V) \cdot \nabla_x \sqrt{\theta}\right),$$

where we recall that the scalar functions $\mathbf{a}$ and $\mathbf{b}$ are defined by

$$\mathcal{L}_{\mathcal{M}_{(1,u,\theta)}}\left(\mathbf{a}(\theta, |V|)A(V)\right) = A(V) \quad \text{and} \quad \mathbf{a}(\theta, |V|)A(V) \perp \operatorname{Ker} \mathcal{L}_{\mathcal{M}_{(1,u,\theta)}}$$

while

$$\mathcal{L}_{\mathcal{M}_{(1,u,\theta)}}\left(\mathbf{b}(\theta, |V|)B(V)\right) = B(V) \quad \text{and} \quad \mathbf{b}(\theta, |V|)B(V) \perp \operatorname{Ker} \mathcal{L}_{\mathcal{M}_{(1,u,\theta)}}.$$

Hence the first-order correction to the fluxes in the formal conservation law is

$$\Phi^{(1)}[\vec{P}] = \begin{pmatrix} 0 \\ \mu(\theta)D(u) \\ \mu(\theta)D(u) \cdot u + \kappa(\theta)\nabla_x \theta \end{pmatrix}.$$

Therefore, the formal conservation law at first order is

$$\partial_t \vec{P} = \operatorname{div}_x \Phi^{(0)}[\vec{P}] + \varepsilon \operatorname{div}_x \Phi^{(1)}[\vec{P}] \quad \mod \mathrm{O}(\varepsilon^2),$$

i.e., the compressible Navier–Stokes system with $O(\varepsilon)$ dissipation terms

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0,$$

$$\partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x(\rho \theta) = \varepsilon \, \mathrm{div}_x\big(\mu D(u)\big) \quad \mathrm{mod}\, O\big(\varepsilon^2\big),$$

$$\partial_t\left(\rho\left(\frac{1}{2}|u|^2 + \frac{3}{2}\theta\right)\right) + \mathrm{div}_x\left(\rho u\left(\frac{1}{2}|u|^2 + \frac{5}{2}\theta\right)\right) \tag{5.14}$$

$$= \varepsilon \, \mathrm{div}_x(\kappa \nabla_x \theta) + \varepsilon \, \mathrm{div}_x\big(\mu D(u) \cdot u\big).$$

The viscosity μ and heat conduction κ are computed as follows

$$\theta \int_{\mathbb{R}^3} \mathbf{a}\big(\theta, |V|\big) A_{ij}(V) A_{kl}(V) \mathcal{M}_{(1,u,\theta)} \, \mathrm{d}v = \mu(\theta)\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{kl}\right),$$

$$\theta \int_{\mathbb{R}^3} \mathbf{b}\big(\theta, |V|\big) B_i(V) B_j(V) \mathcal{M}_{(1,u,\theta)} \, \mathrm{d}v = \kappa(\theta)\delta_{ij},$$

or in other words,

$$\mu(\theta) = \frac{2}{15}\theta \int_0^{+\infty} \mathbf{a}(\theta, r) r^6 \mathrm{e}^{-r^2/2} \frac{\mathrm{d}r}{\sqrt{2\pi}},$$

$$\kappa(\theta) = \frac{1}{6}\theta \int_0^{+\infty} \mathbf{b}(\theta, r) r^4 \big(r^2 - 5\big)^2 \mathrm{e}^{-r^2/2} \frac{\mathrm{d}r}{\sqrt{2\pi}}. \tag{5.15}$$

In the hard sphere case, one finds that the viscosity and heat conduction are of the form

$$\mu(\theta) = \mu_0 \sqrt{\theta}, \qquad \kappa(\theta) = \kappa_0 \sqrt{\theta},$$

where μ_0 and ν_0 are positive constants.

Conclusion. The Chapman–Enskog expansion method shows that the first-order (in ε) correction to the compressible Euler system in the limit of (5.1) – with $St = 1$ and $Kn = \varepsilon \to 0$ – is the compressible Navier–Stokes system (5.14).

Notice that the class of Navier–Stokes systems obtained in this way is by no means the most general: only the equation of state of a perfect monatomic gas can be obtained in this way, as in the case of the compressible Euler limit of the Boltzmann equation. In addition, the viscous dissipation tensor obtained in this way involves *only one* viscosity coefficient instead of two.

Finally, the Chapman–Enskog expansion can be pushed further to obtain higher-order corrections to the compressible Euler equations. For instance, the second-order correction to the compressible Euler equations is a system known as *the Burnett equations*; further corrections have also been computed and are known as *the super-Burnett equations*. See Chapter 5, Section 3 in [27] and Section 25 in [63] for more material on the Chapman–Enskog expansion, as well as for a comparison with Hilbert's expansion.

It should be noted however that these further corrections to the Euler system, beyond the Navier–Stokes system, i.e., the Burnett and super-Burnett equations, are in general not well posed, so that their practical interest in fluid mechanics is unclear. Therefore, we shall not pursue this line of investigation. However, Levermore recently proposed a subtle modification of the Chapman–Enskog expansion which leads to well-posed variants of the Burnett systems [82].

5.3. *The compressible Euler limit: The moment method*

We shall now present a method for deriving hydrodynamic equations from the Boltzmann equation that differs from either the Hilbert or Chapman–Enskog expansions. It consists of passing to the limit as the Knudsen number vanishes in the local conservation laws of mass, momentum and energy that are satisfied by "well-behaved" solutions of the Boltzmann equation. We describe this method on the derivation of the compressible Euler equations from the scaled Boltzmann equation (5.1).

Start from the Cauchy problem for the Boltzmann equation in the periodic box

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon}\mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3,$$

$$F_\varepsilon|_{t=0} = \mathcal{M}_{(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}})}.$$

(5.16)

As before, the collision kernel b in the Boltzmann collision integral satisfies the cut-off assumption (5.2).

THEOREM 5.1. *Let $\rho^{\text{in}} \geqslant 0$ a.e., u^{in} and $\theta^{\text{in}} > 0$ a.e. be such that*

$$\int_{\mathbb{T}^3} \rho^{\text{in}}\big(1 + |u^{\text{in}}|\big)\big(|u^{\text{in}}|^2 + \theta^{\text{in}} + \big|\ln \rho^{\text{in}}\big| + \big|\ln \theta^{\text{in}}\big|\big)\, dx < +\infty.$$

For each $\varepsilon > 0$, let F_ε be a solution of (5.16) *that satisfies the local conservation laws of mass, momentum, and energy, as well as the local entropy relation. Assume that*

$$F_\varepsilon \to F \quad a.e.$$

as well as

$$\int_{\mathbb{R}^3} F_\varepsilon\, dv \to \int_{\mathbb{R}^3} F\, dv \qquad in\ C\big(\mathbb{R}_+; \mathcal{D}'(\mathbb{T}^3)\big),$$

$$\int_{\mathbb{R}^3} v F_\varepsilon\, dv \to \int_{\mathbb{R}^3} v F\, dv \qquad in\ C\big(\mathbb{R}_+; \mathcal{D}'(\mathbb{T}^3)\big),$$

$$\int_{\mathbb{R}^3} |v|^2 F_\varepsilon\, dv \to \int_{\mathbb{R}^3} |v|^2 F\, dv \quad in\ C\big(\mathbb{R}_+; \mathcal{D}'(\mathbb{T}^3)\big),$$

while

$$\int_{\mathbb{R}^3} v \otimes v F_\varepsilon \, dv \to \int_{\mathbb{R}^3} v \otimes v F \, dv \quad \text{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{T}^3\big),$$

$$\int_{\mathbb{R}^3} v|v|^2 F_\varepsilon \, dv \to \int_{\mathbb{R}^3} v|v|^2 F \, dv \quad \text{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{T}^3\big),$$

$$\int_{\mathbb{R}^3} F_\varepsilon \ln F_\varepsilon \, dv \to \int_{\mathbb{R}^3} F \ln F \, dv \quad \text{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{T}^3\big),$$

$$\int_{\mathbb{R}^3} v F_\varepsilon \ln F_\varepsilon \, dv \to \int_{\mathbb{R}^3} v F \ln F \, dv \quad \text{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{T}^3\big),$$

as $\varepsilon \to 0$. Then

$$F = \mathcal{M}_{(\rho, u, \theta)},$$

where (ρ, u, θ) is an entropic solution of the compressible Euler system

$$\partial_t \rho + \operatorname{div}_x(\rho u) = 0,$$

$$\partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x(\rho \theta) = 0, \tag{5.17}$$

$$\partial_t\left(\rho\left(\frac{1}{2}|u|^2 + \frac{3}{2}\theta\right)\right) + \operatorname{div}_x\left(\rho u\left(\frac{1}{2}|u|^2 + \frac{5}{2}\theta\right)\right) = 0,$$

that satisfies the initial condition

$$(\rho, u, \theta)|_{t=0} = \big(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}\big). \tag{5.18}$$

PROOF. The moment method involves three steps.

Step 1 (Entropy production bound implies convergence to local equilibrium).   The entropy relation in the 3-torus implies the entropy production bound

$$\frac{1}{4} \int_0^t \int_{\mathbb{T}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \big(F_\varepsilon' F_{\varepsilon*}' - F_\varepsilon F_{\varepsilon*}\big) \ln\left(\frac{F_\varepsilon' F_{\varepsilon*}'}{F_\varepsilon F_{\varepsilon*}}\right) b \, dv \, dv_* \, d\omega$$

$$\leqslant \varepsilon H\big(\mathcal{M}_{(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}})} \,\big|\, \overline{\mathcal{M}}\big), \tag{5.19}$$

where $\overline{\mathcal{M}}$ is any global Maxwellian state, for instance, one could choose

$$\overline{\mathcal{M}} = \mathcal{M}_{(\overline{\rho^{\text{in}}}, \overline{u^{\text{in}}}, \overline{\theta^{\text{in}}})},$$

where

$$\overline{\rho^{\text{in}}} = \int_{\mathbb{T}^3} \rho^{\text{in}}(x) \, dx,$$

$$\overline{u^{\mathrm{in}}} = \frac{1}{\overline{\rho^{\mathrm{in}}}} \int_{\mathbb{T}^3} \rho^{\mathrm{in}} u^{\mathrm{in}}(x)\,\mathrm{d}x,$$

$$\overline{\theta^{\mathrm{in}}} = \frac{1}{\overline{\rho^{\mathrm{in}}}} \int_{\mathbb{T}^3} \rho^{\mathrm{in}} \left(\frac{1}{3} \big| u^{\mathrm{in}} - \overline{u^{\mathrm{in}}} \big|^2 + \theta^{\mathrm{in}} \right)(x)\,\mathrm{d}x.$$

Next we apply Fatou's lemma: assuming that $F_\varepsilon \to F$ a.e. on $\mathbb{R}_+ \times \mathbb{T}^3 \times \mathbb{R}^3$, one has

$$0 \leqslant \int_0^t \int_{\mathbb{T}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \big(F' F'_* - F F_* \big) \ln \left(\frac{F' F'_*}{F F_*} \right) b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s$$

$$\leqslant \varliminf_{\varepsilon \to 0} \int_0^t \int_{\mathbb{T}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \big(F'_\varepsilon F'_{\varepsilon*} - F_\varepsilon F_{\varepsilon*} \big) \ln \left(\frac{F'_\varepsilon F'_{\varepsilon*}}{F_\varepsilon F_{\varepsilon*}} \right) b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s$$

$$= 0.$$

Hence F is a local Maxwellian, i.e.,

$$F(t, x, v) = \mathcal{M}_{(\rho(t,x), u(t,x), \theta(t,x))}(v)$$

for some $\rho(t, x) \geqslant 0$, $\theta(t, x) > 0$ and $u(t, x) \in \mathbb{R}^3$.

Step 2 (Passing to the limit in the local conservation laws).   For each positive ε, the number density F_ε satisfies the local conservation laws recalled below

$$\partial_t \int_{\mathbb{R}^3} F_\varepsilon\,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v F_\varepsilon\,\mathrm{d}v = 0,$$

$$\partial_t \int_{\mathbb{R}^3} v F_\varepsilon\,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v \otimes v F_\varepsilon\,\mathrm{d}v = 0, \tag{5.20}$$

$$\partial_t \int_{\mathbb{R}^3} \frac{1}{2} |v|^2 F_\varepsilon\,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v \frac{1}{2} |v|^2 F_\varepsilon\,\mathrm{d}v = 0.$$

It follows from our assumptions and Step 1 that

$$\int_{\mathbb{R}^3} F_\varepsilon\,\mathrm{d}v \to \int_{\mathbb{R}^3} \mathcal{M}_{(\rho, u, \theta)}\,\mathrm{d}v = \rho,$$

$$\int_{\mathbb{R}^3} v F_\varepsilon\,\mathrm{d}v \to \int_{\mathbb{R}^3} v \mathcal{M}_{(\rho, u, \theta)}\,\mathrm{d}v = \rho u,$$

$$\int_{\mathbb{R}^3} v \otimes v F_\varepsilon\,\mathrm{d}v \to \int_{\mathbb{R}^3} v \otimes v \mathcal{M}_{(\rho, u, \theta)}\,\mathrm{d}v = \rho(u \otimes u + \theta I),$$

$$\int_{\mathbb{R}^3} v |v|^2 F_\varepsilon\,\mathrm{d}v \to \int_{\mathbb{R}^3} v |v|^2 \mathcal{M}_{(\rho, u, \theta)}\,\mathrm{d}v = \rho u \left(\frac{1}{2} |u|^2 + \frac{5}{2} \theta \right),$$

in the limit as $\varepsilon \to 0$. Hence the functions ρ and θ, and the vector field u satisfy the system of PDEs (5.17). It also satisfies the initial condition (5.18) because the convergence of the conserved densities is locally uniform in t.

Step 3 (Passing to the limit in the local entropy relation).   Finally, we recall the local entropy relation

$$\partial_t \int_{\mathbb{R}^3} F_\varepsilon \ln F_\varepsilon \, \mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v F_\varepsilon \ln F_\varepsilon \, \mathrm{d}v = -\text{local entropy production rate} \leqslant 0.$$

It follows from our assumptions that

$$\int_{\mathbb{R}^3} F_\varepsilon \ln F_\varepsilon \, \mathrm{d}v \to \int_{\mathbb{R}^3} \mathcal{M}_{(\rho,u,\theta)} \ln \mathcal{M}_{(\rho,u,\theta)} \, \mathrm{d}v$$

$$= \rho \ln\left(\frac{\rho}{\theta^{3/2}}\right) - \frac{3}{2}\big(1 + \ln(2\pi)\big)\rho,$$

$$\int_{\mathbb{R}^3} v F_\varepsilon \ln F_\varepsilon \, \mathrm{d}v \to \int_{\mathbb{R}^3} v \mathcal{M}_{(\rho,u,\theta)} \ln \mathcal{M}_{(\rho,u,\theta)} \, \mathrm{d}v$$

$$= \rho u \ln\left(\frac{\rho}{\theta^{3/2}}\right) - \frac{3}{2}\big(1 + \ln(2\pi)\big)\rho u,$$

so that, by passing to the limit in the local entropy relation, on account of the continuity equation in (5.17), one arrives at the differential inequality

$$\partial_t\left(\rho \ln\left(\frac{\rho}{\theta^{3/2}}\right)\right) + \mathrm{div}_x\left(\rho u \ln\left(\frac{\rho}{\theta^{3/2}}\right)\right) \leqslant 0. \tag{5.21}$$

In other words, (ρ, u, θ) is a solution of the compressible Euler equations that satisfies the Lax–Friedrichs entropy condition. □

Theorem 5.1 and its proof are taken from [7].

5.4. *The acoustic limit*

Start from the Boltzmann equation with the same scaling as before, but with initial data that are small perturbations of a uniform Maxwellian equilibrium. By Galilean invariance, one can assume without loss of generality that this uniform equilibrium is the centered reduced Gaussian

$$M(v) = \mathcal{M}_{(1,0,1)}(v) = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{-|v|^2/2}.$$

Thus, the problem (5.16) reduces to

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} \mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3,$$

$$F_\varepsilon|_{t=0} = \mathcal{M}_{(1+\eta_\varepsilon \rho^{\mathrm{in}}, \eta_\varepsilon u^{\mathrm{in}}, 1+\eta_\varepsilon \theta^{\mathrm{in}})},$$

(5.22)

where $0 < \varepsilon \ll 1$ and $0 < \eta_\varepsilon \ll 1$. The same moment method as above shows that the limiting behavior of the solution to (5.22) under these assumptions is governed by the acoustic system.

THEOREM 5.2. *Assume that ρ^{in}, u^{in} and θ^{in} belong to $L^2(\mathbb{T}^3)$ and that $\eta_\varepsilon \to 0$ as $\varepsilon \to 0$. For each $\varepsilon > 0$, let F_ε be a solution of (5.22) that satisfies the local conservation laws of mass momentum and energy, as well as the local entropy relation.*

Assume that

$$g_\varepsilon = \frac{F_\varepsilon - M}{\eta_\varepsilon M} \to g \quad \text{in the sense of distributions,}$$

while

$$\eta_\varepsilon \mathcal{B}(F_\varepsilon - M, F_\varepsilon - M) \to 0 \quad \text{in the sense of distributions,}$$

as well as

$$\int_{\mathbb{R}^3} g_\varepsilon M \, \mathrm{d}v \to \int_{\mathbb{R}^3} g M \, \mathrm{d}v \qquad \text{in } C(\mathbb{R}_+; \mathcal{D}'(\mathbb{T}^3)),$$

$$\int_{\mathbb{R}^3} v g_\varepsilon M \, \mathrm{d}v \to \int_{\mathbb{R}^3} v g M \, \mathrm{d}v \qquad \text{in } C(\mathbb{R}_+; \mathcal{D}'(\mathbb{T}^3)),$$

$$\int_{\mathbb{R}^3} |v|^2 g_\varepsilon M \, \mathrm{d}v \to \int_{\mathbb{R}^3} |v|^2 g M \, \mathrm{d}v \quad \text{in } C(\mathbb{R}_+; \mathcal{D}'(\mathbb{T}^3)),$$

and

$$\int_{\mathbb{R}^3} v \otimes v g_\varepsilon M \, \mathrm{d}v \to \int_{\mathbb{R}^3} v \otimes v g M \, \mathrm{d}v \quad \text{in } \mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3),$$

$$\int_{\mathbb{R}^3} v |v|^2 g_\varepsilon M \, \mathrm{d}v \to \int_{\mathbb{R}^3} v |v|^2 g M \, \mathrm{d}v \quad \text{in } \mathcal{D}'(\mathbb{R}_+^* \times \mathbb{T}^3)$$

as $\varepsilon \to 0$. Then

$$g(t, x, v) = \rho(t, x) + u(t, x) \cdot v + \theta(t, x) \frac{1}{2}(|v|^2 - 3),$$

where (ρ, u, θ) is the solution of the acoustic system

$$\partial_t \rho + \mathrm{div}_x u = 0,$$
$$\partial_t u + \nabla_x (\rho + \theta) = 0, \tag{5.23}$$
$$\frac{3}{2} \partial_t \theta + \mathrm{div}_x u = 0,$$

that satisfies the initial condition

$$(\rho, u, \theta)|_{t=0} = \left(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}\right). \tag{5.24}$$

The proof of this theorem is an easy variant of the formal compressible Euler limit, and is left to the reader. See [53] for the missing details of this formal proof.

6. Incompressible limits of the Boltzmann equation: Formal results

So far, we have discussed various limits of the kinetic theory of gases leading to hydrodynamic models for compressible fluids that satisfy the equation of state of perfect gases. As we shall see in this section, incompressible hydrodynamic models describing *incompressible flows* of perfect gases can also be derived from the Boltzmann equation.

6.1. *The incompressible Navier–Stokes limit*

The scaling on the Boltzmann equation that leads to the incompressible Navier–Stokes equations in the hydrodynamic limit is defined by

$$Kn = St = \varepsilon \ll 1.$$

However, this scaling is not sufficient by itself: as in all long time scalings, one should assume that the length and time scales L and T that enter the definition of St capture the speed of the fluid motion. In other words, situations where

$$\left|u(t, x)\right| \gg \frac{L}{T} \quad \text{must be excluded,}$$

where u is the bulk velocity of the gas, i.e.,

$$u(t, x) = \frac{\int_{\mathbb{R}^3} v F \, \mathrm{d}v}{\int_{\mathbb{R}^3} F \, \mathrm{d}v}.$$

In other words, since $St = (L/T)/\text{speed of sound}$, one must take

$$Ma = \mathrm{O}(St).$$

The case $Ma \sim St$ corresponds to the largest possible velocity field compatible with the above condition, and therefore leads to the Navier–Stokes equation, while the case $Ma = \mathrm{o}(\varepsilon)$ leads to the linearized version of the Navier–Stokes equations, i.e., the Stokes equations.

Hence the complete Navier–Stokes scaling is

$$Kn = St = Ma = \varepsilon \ll 1. \tag{6.1}$$

We consider therefore the scaled Boltzmann equation posed on the spatial domain $\mathbb{R}^3$ with uniform Maxwellian equilibrium at infinity – without loss of generality, this Maxwellian equilibrium is assumed to be the centered reduced Gaussian distribution

$$M(v) = \mathcal{M}_{(1,0,1)}(v) = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{-|v|^2/2}.$$

The problem to be studied is therefore

$$\varepsilon \, \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} \mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3,$$

$$F_\varepsilon(t, x, v) \to M \quad \text{as } |x| \to +\infty. \tag{6.2}$$

That $Ma = \mathrm{O}(\varepsilon)$ is seen on the number density F_ε, and not on the Boltzmann equation itself. Here is an example of number density with $Ma = \mathrm{O}(\varepsilon)$.

EXAMPLE 1. Take F_ε of the form

$$F_\varepsilon(t, x, v) = \mathcal{M}_{(1,\varepsilon u(t,x),1)}(v).$$

Indeed,

$$\frac{\int_{\mathbb{R}^3} v F_\varepsilon \, \mathrm{d}v}{\int_{\mathbb{R}^3} F_\varepsilon \, \mathrm{d}v} := \varepsilon u.$$

The speed of sound for the state of the gas described by F_ε is $\sqrt{\frac{5}{3}\theta}$ where

$$\theta(t, x) :\equiv \frac{\int_{\mathbb{R}^3} (1/3)|v - \varepsilon u|^2 F_\varepsilon \, \mathrm{d}v}{\int_{\mathbb{R}^3} F_\varepsilon \, \mathrm{d}v} = 1.$$

Hence the Mach number for the state of the gas associated to F_ε is

$$\frac{\varepsilon |u(t,x)|}{\sqrt{(5/3)\theta(t,x)}} = \mathrm{O}(\varepsilon).$$

Here is another example that also involves fluctuations of temperature.

EXAMPLE 2. Take F_ε of the form

$$F_\varepsilon(t, x, v) = \mathcal{M}_{\left(1 - \varepsilon \theta(t,x), \frac{\varepsilon u(t,x)}{1 - \varepsilon \theta(t,x)}, \frac{1}{1 - \varepsilon \theta(t,x)}\right)}(v)$$

with $\theta \in L^\infty(\mathbb{R}_+ \times \mathbb{R}^3)$ and $\varepsilon \|\theta\|_{L^\infty} < 1$. One easily checks that

$$\frac{\int_{\mathbb{R}^3} v F_\varepsilon \, dv}{\int_{\mathbb{R}^3} F_\varepsilon \, dv} = \varepsilon u + O(\varepsilon^2),$$

while

$$\frac{\int_{\mathbb{R}^3} (1/3)|v - \varepsilon u(t,x)/(1 - \varepsilon \theta(t,x))|^2 F_\varepsilon \, dv}{\int_{\mathbb{R}^3} F_\varepsilon \, dv} = 1 + \varepsilon \theta + O(\varepsilon^2).$$

Hence the Mach number is

$$\frac{\varepsilon u + O(\varepsilon^2)}{\sqrt{(5/3)(1 + \varepsilon \theta + O(\varepsilon^2))}} = O(\varepsilon).$$

More generally, if F_ε is a number density of the form

$$F_\varepsilon = M(1 + \varepsilon g_\varepsilon) \quad \text{such that} \quad g_\varepsilon \geq -\frac{1}{\varepsilon} \quad \text{a.e.,} \tag{6.3}$$

one can check that, provided that $\|g_\varepsilon\|_{L^\infty} = O(1)$, the Mach number for the state of the gas defined by F_ε is $O(\varepsilon)$.

Hence, we shall supplement the scaled Boltzmann equation (6.2) with the initial condition

$$F_\varepsilon|_{t=0} \equiv \mathcal{M}_{\left(1 - \varepsilon \theta^{\text{in}}(x), \frac{\varepsilon u^{\text{in}}(x)}{1 - \varepsilon \theta^{\text{in}}(x)}, \frac{1}{1 - \varepsilon \theta^{\text{in}}(x)}\right)}(v), \tag{6.4}$$

where

$$\text{div}_x u^{\text{in}} = 0.$$

For each $\varepsilon > 0$, define the number density fluctuation

$$g_\varepsilon = \frac{F_\varepsilon - M}{\varepsilon M}.$$

In terms of the number density fluctuation g_ε, the Boltzmann equation (6.2) takes the form

$$\varepsilon \, \partial_t g_\varepsilon + v \cdot \nabla_x g_\varepsilon + \frac{1}{\varepsilon} \mathcal{L}_M g_\varepsilon = \mathcal{Q}_M(g_\varepsilon, g_\varepsilon), \quad t > 0, x, v \in \mathbb{R}^3, \tag{6.5}$$

where $\mathcal{Q}_M$ is the collision integral intertwined with the multiplication by M

$$\mathcal{Q}_M(\phi, \phi) = M^{-1}\mathcal{B}(M\phi, M\phi). \tag{6.6}$$

We shall also need notation for moments

$$\langle \phi \rangle = \int_{\mathbb{R}^3} \phi(v)M(v)\,dv.$$

Hence the local conservation laws of mass momentum and energy satisfied by F_ε take the form

$$\varepsilon\,\partial_t\langle g_\varepsilon \rangle + \mathrm{div}_x\langle vg_\varepsilon \rangle = 0 \qquad \text{(mass)},$$

$$\varepsilon\,\partial_t\langle vg_\varepsilon \rangle + \mathrm{div}_x\langle v \otimes vg_\varepsilon \rangle = 0 \qquad \text{(momentum)}, \tag{6.7}$$

$$\varepsilon\,\partial_t\left\langle \frac{1}{2}|v|^2 g_\varepsilon \right\rangle + \mathrm{div}_x\left\langle v\frac{1}{2}|v|^2 g_\varepsilon \right\rangle = 0 \quad \text{(energy)}.$$

THEOREM 6.1 (Bardos, Golse and Levermore [8,9]). *For each $\varepsilon > 0$, let F_ε be a solution of* (6.2)–(6.4). *Assume that*

$$\frac{F_\varepsilon - M}{\varepsilon M} \to g \quad \text{in the sense of distributions on } \mathbb{R}_+^* \times \mathbb{R}^3,$$

and that F_ε satisfies the local conservation laws of mass, momentum and energy, and that

$$\langle vg_\varepsilon \rangle \to \langle vg \rangle \quad \text{and} \quad \langle (|v|^2 - 5)g_\varepsilon \rangle \to \langle (|v|^2 - 5)g \rangle \quad \text{in } C(\mathbb{R}_+, \mathcal{D}'(\mathbb{R}^3))$$

while

$$\mathcal{L}_M g_\varepsilon \to \mathcal{L}_M g \quad \text{in } \mathcal{D}'(\mathbb{R}_+^* \times \mathbb{R}^3)$$

and all formally small terms vanish in the sense of distributions as $\varepsilon \to 0$. Assume further that

$$\langle v \otimes vg_\varepsilon \rangle \to \langle v \otimes vg \rangle, \qquad \langle Bg_\varepsilon \rangle \to \langle Bg \rangle,$$

$$\langle \widehat{A}\mathcal{Q}_M(g_\varepsilon, g_\varepsilon) \rangle \to \langle \widehat{A}\mathcal{Q}_M(g_\varepsilon, g_\varepsilon) \rangle \quad \text{and} \quad \langle \widehat{A} \otimes vg_\varepsilon \rangle \to \langle \widehat{A} \otimes vg \rangle,$$

$$\langle \widehat{B}\mathcal{Q}_M(g_\varepsilon, g_\varepsilon) \rangle \to \langle \widehat{B}\mathcal{Q}_M(g_\varepsilon, g_\varepsilon) \rangle \quad \text{and} \quad \langle \widehat{B} \otimes vg_\varepsilon \rangle \to \langle \widehat{B} \otimes vg \rangle$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^3$.

   Then g is of the form

$$g(t, x, v) = u(t, x) \cdot v + \theta(t, x)\frac{1}{2}(|v|^2 - 5),$$

where (u, θ) satisfy the incompressible Navier–Stokes–Fourier system

$$\partial_t u + \mathrm{div}_x(u \otimes u) + \nabla_x p = \nu \Delta_x u, \qquad \mathrm{div}_x u = 0,$$
$$\big(\partial_t \theta + \mathrm{div}_x(u\theta)\big) = \kappa \Delta_x \theta, \tag{6.8}$$

where

$$\nu = \frac{1}{10}\langle \widehat{A} : A \rangle = \frac{2}{15}\int_0^{+\infty} \mathbf{a}(1, r) r^6 \mathrm{e}^{-r^2/2}\,\frac{\mathrm{d}r}{\sqrt{2\pi}},$$
$$\kappa = \frac{2}{15}\langle \widehat{B} \cdot B \rangle = \frac{1}{15}\int_0^{+\infty} \mathbf{b}(1, r) r^2\big(r^2 - 5\big)^2 \mathrm{e}^{-r^2/2}\,\frac{\mathrm{d}r}{\sqrt{2\pi}}. \tag{6.9}$$

We recall that

$$A(v) = v \otimes v - \frac{1}{3}|v|^2 I, \qquad B(v) = \frac{1}{2}\big(|v|^2 - 5\big)v, \tag{6.10}$$

and that there exists $\widehat{A}$ and $\widehat{B} \in L^2(\mathbb{R}^3, M\,\mathrm{d}v)$ uniquely determined by

$$\mathcal{L}_M \widehat{A} = A, \qquad \widehat{A} \perp \mathrm{Ker}\,\mathcal{L}_M,$$
$$\mathcal{L}_M \widehat{B} = B, \qquad \widehat{B} \perp \mathrm{Ker}\,\mathcal{L}_M. \tag{6.11}$$

Furthermore, there exists two scalar functions $\mathbf{a}$ and $\mathbf{b}$ such that

$$\widehat{A}(v) = \mathbf{a}\big(1, |v|\big)A(v) \quad \text{and} \quad \widehat{B}(v) = \mathbf{b}\big(1, |v|\big)B(v). \tag{6.12}$$

PROOF OF THEOREM 6.1. We shall use the moment method, although either the Hilbert or the Chapman–Enskog expansions would also lead to the incompressible Navier–Stokes limit. However, the moment method is the closest to a complete (instead of formal) convergence proof for global solutions without restriction on the size of the initial data.

Step 1 (Asymptotic form of the fluctuations).   Multiply the Boltzmann equation (6.5) by ε, so that

$$\varepsilon^2\,\partial_t g_\varepsilon + \varepsilon v \cdot \nabla_x g_\varepsilon + \mathcal{L}_M g_\varepsilon = \varepsilon \mathcal{Q}_M(g_\varepsilon, g_\varepsilon).$$

Passing to the limit as $\varepsilon \to 0$ in the above equation, we arrive at

$$\mathcal{L}_M g = 0.$$

Hence, for a.e. $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$, $g(t, x, \cdot) \in \mathrm{Ker}\,\mathcal{L}_M$, which means that g is of the form

$$g(t, x, v) = \rho(t, x) + u(t, x) \cdot v + \theta(t, x)\frac{1}{2}\big(|v|^2 - 3\big). \tag{6.13}$$

Step 2 (Passing to the limit in the local conservation laws).   Passing to the limit in (6.7) we arrive at

$$\operatorname{div}_x \langle vg \rangle = 0,$$

$$\operatorname{div}_x \langle v \otimes vg \rangle = 0,$$

$$\operatorname{div}_x \langle v|v|^2 g \rangle = 0.$$

Together with the relation (6.13), the first and the last relation reduce to the incompressibility condition for u

$$\operatorname{div}_x u = 0. \tag{6.14}$$

Together with the relation (6.13), the second relation gives

$$\nabla_x (\rho + \theta) = 0.$$

Since $g \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3, M\, dv))$, this implies the Boussinesq relation

$$\rho + \theta = 0. \tag{6.15}$$

With this last relation, the asymptotic form of g becomes

$$g(t, x, v) = u(t, x) \cdot v + \theta(t, x) \frac{1}{2}(|v|^2 - 5). \tag{6.16}$$

It remains to derive the motion and heat equations from the local conservation laws. To do this, we recast the local conservation law of momentum as

$$\partial_t \langle vg_\varepsilon \rangle + \operatorname{div}_x \frac{1}{\varepsilon} \langle Ag_\varepsilon \rangle + \nabla_x \frac{1}{\varepsilon} \left\langle \frac{1}{3} |v|^2 g_\varepsilon \right\rangle = 0,$$

and we combine the local conservation laws of mass and energy into

$$\partial_t \left\langle \frac{1}{2}(|v|^2 - 5) g_\varepsilon \right\rangle + \operatorname{div}_x \frac{1}{\varepsilon} \langle Bg_\varepsilon \rangle = 0.$$

One easily checks with (6.16) that

$$\langle vg_\varepsilon \rangle \to \langle vg \rangle = u,$$

$$\left\langle \frac{1}{2}(|v|^2 - 5) g_\varepsilon \right\rangle \to \left\langle \frac{1}{2}(|v|^2 - 5) g \right\rangle = \frac{5}{2}\theta \tag{6.17}$$

in $C(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}^3))$ as $\varepsilon \to 0$.

Next we pass to the limit in the flux terms. Since $\mathcal{L}_M$ is self-adjoint, one has

$$\frac{1}{\varepsilon}\langle Ag_\varepsilon\rangle = \frac{1}{\varepsilon}\langle(\mathcal{L}_M\widehat{A})g_\varepsilon\rangle = \left\langle\widehat{A}\frac{1}{\varepsilon}\mathcal{L}_M g_\varepsilon\right\rangle.$$

Then we eliminate the term $\frac{1}{\varepsilon}\mathcal{L}_M g_\varepsilon$ with (6.5)

$$\left\langle\widehat{A}\frac{1}{\varepsilon}\mathcal{L}_M g_\varepsilon\right\rangle = \langle\widehat{A}\mathcal{Q}_M(g_\varepsilon, g_\varepsilon)\rangle - \langle\widehat{A}(\varepsilon\partial_t + v\cdot\nabla_x)g_\varepsilon\rangle$$

so that, passing to the limit as $\varepsilon\to 0$ leads to

$$\frac{1}{\varepsilon}\langle Ag_\varepsilon\rangle \to \langle\widehat{A}\mathcal{Q}_M(g,g)\rangle - \langle\widehat{A}v\cdot\nabla_x g\rangle \quad \text{in } \mathcal{D}'\bigl(\mathbb{R}_+^*\times\mathbb{R}^3\bigr). \tag{6.18}$$

Likewise

$$\frac{1}{\varepsilon}\langle Bg_\varepsilon\rangle \to \langle\widehat{B}\mathcal{Q}_M(g,g)\rangle - \langle\widehat{B}v\cdot\nabla_x g\rangle \quad \text{in } \mathcal{D}'\bigl(\mathbb{R}_+^*\times\mathbb{R}^3\bigr), \tag{6.19}$$

as $\varepsilon\to 0$.

With (6.16), one easily finds that

$$\langle\widehat{A}v\cdot\nabla_x g\rangle = \nu\left(\nabla_x u + (\nabla_x u)^T - \frac{2}{3}(\mathrm{div}_x u)I\right),$$

$$\langle\widehat{B}v\cdot\nabla_x g\rangle = \frac{5}{2}\kappa\nabla_x\theta, \tag{6.20}$$

where ν and κ are given by (6.9).

The nonlinear term is slightly more difficult. Its computation involves in particular the following classical lemma.

LEMMA 6.2. *Let* $\phi\in\mathrm{Ker}\,\mathcal{L}_M$, *then*

$$\mathcal{Q}_M(\phi,\phi) = \frac{1}{2}\mathcal{L}_M\bigl(\phi^2\bigr).$$

PROOF. Differentiate twice the relation

$$\mathcal{B}(\mathcal{M}_{(\rho,u,\theta)}, \mathcal{M}_{(\rho,u,\theta)}) = 0$$

with respect to ρ, u and θ. See, for instance, [25] or [9] for the missing details. $\qquad\square$

Then

$$\langle\widehat{A}\mathcal{Q}_M(g,g)\rangle = \frac{1}{2}\langle\widehat{A}\mathcal{L}_M\bigl(g^2\bigr)\rangle = \frac{1}{2}\langle(\mathcal{L}_M\widehat{A})g^2\rangle = \frac{1}{2}\langle Ag^2\rangle = u\otimes u - \frac{1}{3}|u|^2 I, \tag{6.21}$$

and likewise

$$\langle \widehat{B} \mathcal{Q}_M(g,g) \rangle = \frac{1}{2} \langle \widehat{B} \mathcal{L}_M(g^2) \rangle = \frac{1}{2} \langle (\mathcal{L}_M \widehat{B}) g^2 \rangle = \frac{1}{2} \langle B g^2 \rangle = \frac{5}{2} u\theta. \tag{6.22}$$

Observe that

$$\operatorname{div}_x \left(u \otimes u - \frac{1}{3} |u|^2 I \right) = \operatorname{div}_x(u \otimes u) - \frac{1}{3} \nabla_x |u|^2,$$

while

$$\operatorname{div}_x \left(\nabla_x u + (\nabla_x u)^{\mathrm{T}} - \frac{2}{3} (\operatorname{div}_x u) I \right) = \Delta_x u + \nabla_x(\operatorname{div}_x u) - \frac{2}{3} \nabla_x(\operatorname{div}_x u)$$
$$= \Delta_x u,$$

because of the divergence-free condition on u.

Gathering (6.17)–(6.22), we arrive at

$$\partial_t u + \operatorname{div}_x(u \otimes u) - \nu \Delta_x u = 0 \quad \text{modulo gradients},$$
$$\partial_t \theta + \operatorname{div}_x(u\theta) - \kappa \Delta_x \theta = 0. \qquad \square$$

6.2. *The incompressible Stokes and Euler limits*

By the same moment method, one can derive other incompressible models from the Boltzmann equation. We just state the results below without giving the proofs (which are anyway simpler than that of the Navier–Stokes limit).

6.2.1. *The Stokes limit.*   The Stokes–Fourier system is the linearization about $u = 0$ and $\theta = 0$ of the Navier–Stokes–Fourier system. Thus, in order to derive the Stokes–Fourier system from the Boltzmann equation, one keeps the same scaling as for the incompressible Navier–Stokes limit on the Boltzmann equation, i.e.,

$$Kn = St = \varepsilon \ll 1$$

and one scales the Mach number as

$$Ma = \eta_\varepsilon = \mathrm{o}(\varepsilon).$$

In other words, we start from the following Cauchy problem

$$\varepsilon \, \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} \mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3,$$
$$F_\varepsilon(t, x, v) \to M \quad \text{as } |x| \to +\infty, \tag{6.23}$$

with initial condition

$$F_\varepsilon|_{t=0} = \mathcal{M}_{\left(1-\eta_\varepsilon\theta^{\mathrm{in}},\frac{\eta_\varepsilon u^{\mathrm{in}}}{1-\eta_\varepsilon\theta^{\mathrm{in}}},\frac{1}{1-\eta_\varepsilon\theta^{\mathrm{in}}}\right)}. \tag{6.24}$$

In this subsection, the number density fluctuation g_ε is defined to be

$$g_\varepsilon = \frac{F_\varepsilon - M}{\eta_\varepsilon M}.$$

THEOREM 6.3 (Bardos, Golse and Levermore [9]). *For each $\varepsilon > 0$, let F_ε be a solution of (6.23)–(6.24). Assume that*

$$\frac{F_\varepsilon - M}{\eta_\varepsilon M} \to g \quad \text{in the sense of distributions on } \mathbb{R}_+^* \times \mathbb{R}^3,$$

that F_ε satisfies the local conservation laws of mass, momentum and energy, and that

$$\langle vg_\varepsilon\rangle \to \langle vg\rangle \quad \text{and} \quad \langle(|v|^2 - 5)g_\varepsilon\rangle \to \langle(|v|^2 - 5)g\rangle \quad \text{in } C\big(\mathbb{R}_+, \mathcal{D}'(\mathbb{R}^3)\big)$$

while

$$\mathcal{L}_M g_\varepsilon \to \mathcal{L}_M g \quad \text{in } \mathcal{D}'\big(\mathbb{R}_+^* \times \mathbb{R}^3\big)$$

and all formally small terms vanish in the sense of distributions as $\varepsilon \to 0$. Assume further that

$$\langle v \otimes vg_\varepsilon\rangle \to \langle v \otimes vg\rangle, \qquad \langle Bg_\varepsilon\rangle \to \langle Bg\rangle,$$
$$\big\langle\widehat{A} \otimes vg_\varepsilon\big\rangle \to \big\langle\widehat{A} \otimes vg\big\rangle \quad \text{and} \quad \big\langle\widehat{B} \otimes vg_\varepsilon\big\rangle \to \big\langle\widehat{B} \otimes vg\big\rangle$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^3$.

  Then g is of the form

$$g(t, x, v) = u(t, x) \cdot v + \theta(t, x)\frac{1}{2}\big(|v|^2 - 5\big),$$

where (u, θ) satisfy the incompressible Stokes–Fourier system

$$\partial_t u + \nabla_x p = \nu\Delta_x u, \qquad \mathrm{div}_x u = 0,$$
$$\partial_t\theta = \kappa\Delta_x\theta, \tag{6.25}$$

with ν and κ given by formula (6.9).

6.2.2. *The incompressible Euler limit.*   The incompressible Euler equations are formally the inviscid limit of the Navier–Stokes equations. Thus, in order to derive the incompressible Euler equations from the Boltzmann equation, one chooses a scaling that increases the strength of the nonlinear term. In other words, one takes

$$St = Ma = \varepsilon \quad \text{while } Kn = \eta_\varepsilon = \text{o}(\varepsilon).$$

In fact, this is consistent with the von Karman relation, which relates the Mach, Knudsen and Reynolds numbers as follows

$$Kn = \frac{Ma}{Re}. \tag{6.26}$$

Indeed, one gets $Re = \varepsilon/\eta_\varepsilon \to +\infty$ as $\varepsilon \to 0$: therefore, the limiting equation obtained in this way is incompressible (since $Ma \to 0$) and inviscid (since $Re \to +\infty$).

   In other words, we start from the following Cauchy problem

$$\varepsilon \, \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\eta_\varepsilon} \mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3,$$

$$F_\varepsilon(t, x, v) \to M \quad \text{as } |x| \to +\infty, \tag{6.27}$$

with initial condition

$$F_\varepsilon|_{t=0} = \mathcal{M}_{(1-\varepsilon\theta^{\text{in}}, \frac{\varepsilon u^{\text{in}}}{1-\varepsilon\theta^{\text{in}}}, \frac{1}{1-\varepsilon\theta^{\text{in}}})}. \tag{6.28}$$

In this subsection, the number density fluctuation g_ε is defined to be

$$g_\varepsilon = \frac{F_\varepsilon - M}{\varepsilon M}.$$

THEOREM 6.4 (Bardos, Golse and Levermore [9]). *For each $\varepsilon > 0$, let F_ε be a solution of* (6.27)–(6.28). *Assume that*

$$\frac{F_\varepsilon - M}{\varepsilon M} \to g \quad \text{in the sense of distributions on } \mathbb{R}_+^* \times \mathbb{R}^3,$$

that F_ε satisfies the local conservation laws of mass, momentum and energy, and that

$$\langle v g_\varepsilon \rangle \to \langle v g \rangle \quad \text{and} \quad \langle (|v|^2 - 5) g_\varepsilon \rangle \to \langle (|v|^2 - 5) g \rangle \quad \text{in } C(\mathbb{R}_+, \mathcal{D}'(\mathbb{R}^3))$$

while

$$\mathcal{L}_M g_\varepsilon \to \mathcal{L}_M g \quad \text{in } \mathcal{D}'(\mathbb{R}_+^* \times \mathbb{R}^3)$$

and all formally small terms vanish in the sense of distributions as $\varepsilon \to 0$. Assume further that

$$\langle v \otimes v g_\varepsilon \rangle \to \langle v \otimes v g \rangle, \qquad \langle B g_\varepsilon \rangle \to \langle B g \rangle,$$

$$\left\langle \widehat{A} \mathcal{Q}_M(g_\varepsilon, g_\varepsilon) \right\rangle \to \left\langle \widehat{A} \mathcal{Q}_M(g_\varepsilon, g_\varepsilon) \right\rangle \quad and \quad \left\langle \widehat{B} \mathcal{Q}_M(g_\varepsilon, g_\varepsilon) \right\rangle \to \left\langle \widehat{B} \mathcal{Q}_M(g_\varepsilon, g_\varepsilon) \right\rangle$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^3$.

Then g is of the form

$$g(t, x, v) = u(t, x) \cdot v + \theta(t, x) \frac{1}{2} (|v|^2 - 5),$$

where (u, θ) satisfy the system

$$\partial_t u + \mathrm{div}_x (u \otimes u) + \nabla_x p = 0, \qquad \mathrm{div}_x u = 0,$$

$$\partial_t \theta + \mathrm{div}_x (u\theta) = 0. \tag{6.29}$$

6.3. *Other incompressible models*

There are many possible variants of the incompressible Navier–Stokes–Fourier limit described above. To begin with, it is possible to include a conservative force in the Boltzmann equation. The scaling is as follows. Start from equation

$$\varepsilon \, \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon - \varepsilon \nabla_x \phi(x) \cdot \nabla_v F_\varepsilon = \frac{1}{\varepsilon} \mathcal{B}(F_\varepsilon, F_\varepsilon),$$

where $\phi \equiv \phi(x)$ is a given (smooth) potential. Writing the (x, v)-derivative in the Boltzmann equation above as a Poisson bracket, i.e.,

$$v \cdot \nabla_x F_\varepsilon - \varepsilon \nabla_x \phi(x) \cdot \nabla_v F_\varepsilon = \left\{ \frac{1}{2} |v|^2 + \varepsilon \phi(x); F_\varepsilon \right\}$$

suggests to seek the solution F_ε in the form

$$F_\varepsilon(t, x, v) = \mathrm{e}^{-\varepsilon \phi(x)} M(v) \bigl(1 + \varepsilon g_\varepsilon(t, x, v) \bigr).$$

Indeed,

$$\mathrm{e}^{-\varepsilon \phi(x)} M(v) = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{-|v|^2/2 - \varepsilon \phi(x)}$$

is both a Maxwellian and an element of the nullspace of the Poisson bracket $\{ \frac{1}{2} |v|^2 + \varepsilon \phi(x); \cdot \}$.

The same formal argument as above shows that

$$g_\varepsilon(t, x, v) \to u(t, x) \cdot v + \theta(t, x)\frac{1}{2}\big(|v|^2 - 5\big),$$

where u and θ satisfy

$$\partial_t u + \mathrm{div}_x(u \otimes u) + \nabla_x p - \theta \nabla_x \phi = \nu \Delta_x u,$$

$$\partial_t \theta + \mathrm{div}_x(u\theta) + \frac{2}{5} u \cdot \nabla_x \phi = \kappa \Delta_x \theta,$$

with

$$\nu = \frac{1}{10}\langle \widehat{A} : A \rangle \quad \text{and} \quad \kappa = \frac{2}{15}\langle \widehat{B} \cdot B \rangle$$

as in the incompressible Navier–Stokes–Fourier limit theorem above. Of course, this is in agreement with the discussion in Section 2.7. We refer the interested reader to [8] for more details on the formal derivation.

In still another variant of the Navier–Stokes–Fourier limit presented above, it is possible to recover viscous heating terms as in Section 2.6. As explained in that subsection, viscous heating terms should appear when the fluctuations of velocity field are of the order of the square root of temperature fluctuations. In [13], Levermore and Bardos used the following very elegant approach: start from the Boltzmann equation in the Navier–Stokes scaling (6.2) and seek the number density F_ε as

$$F_\varepsilon(t, x, v) = M(v)\big(1 + \varepsilon g_\varepsilon^-(t, x, v) + \varepsilon^2 g_\varepsilon^+(t, x, v)\big),$$

where g_ε^- is odd in v while g_ε^+ is even in v. Because the Boltzmann collision integral is rotation invariant (see Section 3.6.2 and especially (3.51)),

$$\mathcal{B}(\Phi^+, \Phi^+) \text{ and } \mathcal{B}(\Phi^-, \Phi^-) \text{ are even in } v, \text{ while}$$
$$\mathcal{B}(\Phi^-, \Phi^+) \text{ and } \mathcal{B}(\Phi^+, \Phi^-) \text{ are odd in } v.$$

Levermore and Bardos gave a formal argument showing that

$$g_\varepsilon^-(t, x, v) \to u(t, x) \cdot v,$$

while

$$g_\varepsilon^+(t, x, v) \to \rho(t, x) + \frac{1}{2}\big(|u(t, x)|^2 + 3\theta(t, x)\big)\left(\frac{1}{3}|v|^2 - 1\right)$$

$$+ \frac{1}{2} A : u(t, x) \otimes u(t, x) - \widehat{A} : \nabla_x u(t, x),$$

where ρ, u and θ satisfy the relations

$$\operatorname{div}_x u = 0, \qquad p = \rho + \theta,$$

$$\partial_t u + \operatorname{div}_x (u \otimes u) + \nabla_x p = \nu \Delta_x u,$$

$$\partial_t \left(\frac{5}{2}\theta + \frac{1}{2}|u|^2 - p \right) + \operatorname{div}_x \left(u \left(\frac{5}{2}\theta + \frac{1}{2}|u|^2 \right) \right)$$

$$= \frac{5}{2}\kappa \Delta_x \theta + \mu \operatorname{div}_x \big(\big(\nabla_x u + (\nabla_x u)^{\mathrm{T}} \big) \cdot u \big).$$

In the system above, ν and κ are given by the same formulas as in the Navier–Stokes–Fourier limit theorem, i.e., (6.9).

7. Mathematical theory of the Cauchy problem for hydrodynamic models

In this section we have gathered a few mathematical results bearing on the various hydrodynamic models that appear as limits of the Boltzmann equation. We shall leave aside the compressible Navier–Stokes system, since its derivation from the Boltzmann equation leads to dissipation terms that are of the order of the Knudsen number, and therefore vanish in the hydrodynamic limit. Put in other words, the compressible Navier–Stokes system is an asymptotic expansion of the Boltzmann equation in the Knudsen number, and not a limit thereof. Hence, a mathematical treatment of this limit would involve existence results on the compressible Navier–Stokes system that are uniform in the Reynolds and Péclet numbers, which is beyond current knowledge on this model at the time of this writing.

7.1. *The Stokes and acoustic systems*

We begin with the simplest hydrodynamic models
- the Stokes–Fourier system, and
- the acoustic system,

which are variants of the heat and the wave equations.

7.1.1. *The Stokes–Fourier system.*   Consider the Stokes–Fourier system

$$\partial_t u + \nabla_x p = \nu \Delta_x u, \qquad \operatorname{div}_x u = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^3,$$

$$\partial_t \theta = \kappa \Delta_x \theta,$$

$$(u, \theta)|_{t=0} = (u^{\mathrm{in}}, \theta^{\mathrm{in}}),$$

where ν and $\kappa > 0$.

THEOREM 7.1. *For each* $(u^{\mathrm{in}}, \theta^{\mathrm{in}}) \in L^2(\mathbb{R}^3)$ *such that* $\operatorname{div}_x u^{\mathrm{in}} = 0$, *there exists a unique solution* (u, θ) *of the Stokes–Fourier system such that*

$$(u, \theta) \in C\big(\mathbb{R}_+; L^2(\mathbb{R}^3)\big) \quad and \quad p \in C\big(\mathbb{R}_+^*; L^2(\mathbb{R}^3)/\mathbb{R}\big).$$

In addition, the pressure p is constant and

$$(u, \theta) \in C^\infty\big(\mathbb{R}_+^* \times \mathbb{R}^3\big).$$

PROOF. Applying the divergence to both sides of the motion equation gives

$$\Delta_x p = 0;$$

since for each $t > 0$ the function $p(t, \cdot) \in L^2(\mathbb{R}^3)/\mathbb{R}$ is harmonic, it is a constant in variable x. Since the divergence operator commutes with the heat operator on $\mathbb{R}^3$, the Stokes–Fourier system above reduces to a system of uncoupled heat equations, whence the announced result.   □

7.1.2. *The acoustic system.*   Consider the acoustic system

$$\partial_t \rho + \text{div}_x u = 0,$$

$$\partial_t u + \nabla_x (\rho + \theta) = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^3,$$

$$\frac{3}{2} \partial_t \theta + \text{div}_x u = 0,$$

$$(\rho, u, \theta)|_{t=0} = \big(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}\big).$$

THEOREM 7.2. *For each $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \in L^2(\mathbb{R}^3)$, there exists a unique solution (ρ, u, θ) of the acoustic system such that*

$$(\rho, u, \theta) \in C\big(\mathbb{R}_+; L^2(\mathbb{R}^3)\big).$$

PROOF. Applying the Helmholtz decomposition to $u(t, \cdot)$ gives

$$u(t, \cdot) = u^s(t, \cdot) - \nabla_x \phi(t, \cdot), \qquad \text{div}_x u_s = 0.$$

Hence the acoustic system becomes

$$\partial_t \rho - \Delta_x \phi = 0, \qquad \frac{3}{2} \partial_t \theta - \Delta_x \phi = 0,$$

$$\partial_t \phi - \rho - \theta = 0, \qquad \partial_t u_s = 0.$$

Hence

$$\partial_t \phi - \rho - \theta = 0, \qquad \partial_t (\rho + \theta) - \frac{5}{3} \Delta_x \phi = 0,$$

$$\partial_t \left(\rho - \frac{3}{2} \theta \right) = 0.$$

Therefore, letting $\psi = \rho + \theta$, we arrive at

$$\partial_{tt}\phi - \frac{5}{3}\Delta_x \phi = 0,$$

$$\partial_{tt}\psi - \frac{5}{3}\Delta_x \psi = 0.$$

This is a system of two uncoupled wave equations; then, (ρ, u, θ) is reconstructed from ψ and $\mu \equiv \mu(x) = \rho(t, x) - \frac{3}{2}\theta(t, x)$ by the formulas

$$\rho(t, x) = \frac{3}{5}\psi(t, x) + \frac{2}{5}\mu(x),$$

$$\theta(t, x) = \frac{2}{5}\big(\psi(t, x) - \mu(x)\big),$$

while

$$u(t, x) = u^{\mathrm{s}}(0, x) - \nabla_x \phi(t, x).$$

Applying the classical theory of the Cauchy problem for the wave equations satisfied by ϕ and ψ leads to the announced result.   □

7.2. *The incompressible Navier–Stokes equations*

Next, we consider the incompressible Navier–Stokes equations. The mathematical theory of the Cauchy problem for these equations was developed by J. Leray in the early 1930s. Here is a quick summary of his results, see for instance [30,45,86] for more information on this subject.

Consider the Navier–Stokes equations

$$\partial_t u + \mathrm{div}_x(u \otimes u) + \nabla_x p = \nu\Delta_x u, \qquad \mathrm{div}_x u = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{R}^D,$$

$$u|_{t=0} = u^{\mathrm{in}},$$

where $\nu > 0$.

We begin with the three-dimensional case.

THEOREM 7.3 (Leray [80]). *For each $u^{\mathrm{in}} \in L^2(\mathbb{R}^3)$ such that $\mathrm{div}_x u^{\mathrm{in}} = 0$, there exists a solution in the sense of distributions to the Cauchy problem for the Navier–Stokes equations such that*

$$u \in C\big(\mathbb{R}_+; w - L^2(\mathbb{R}^3)\big) \cap L^2\big(\mathbb{R}_+; H^1(\mathbb{R}^3)\big).$$

Moreover, the function $t \mapsto \|u(t, \cdot)\|_{L^2}$ *is nonincreasing on* $\mathbb{R}_+$ *and satisfies, for each* $t > 0$, *the Leray energy inequality*

$$\frac{1}{2}\|u(t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^D} \nu |\nabla_x u(s, x)|^2 \, \mathrm{d}x \, \mathrm{d}s \leqslant \frac{1}{2}\|u^{\mathrm{in}}\|_{L^2}^2.$$

A solution of the Navier–Stokes equations in the sense of distributions that belongs to $C(\mathbb{R}_+; w - L^2(\mathbb{R}^3)) \cap L^2(\mathbb{R}_+; H^1(\mathbb{R}^3))$ and satisfies the Leray energy inequality is called a "Leray solution".

In fact, a modification of Leray's original argument allows constructing weak solutions that satisfy the *local* energy inequality

$$\partial_t \frac{1}{2}|u|^2 + \mathrm{div}_x\left(u\left(\frac{1}{2}|u|^2 + p\right)\right) + \nu|\nabla_x u|^2 \leqslant \nu \Delta_x\left(\frac{1}{2}|u|^2\right) \tag{7.1}$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^3$.

It is not known whether Leray solutions are uniquely determined by their initial data; however, Leray was able to prove that regular solutions of the Navier–Stokes equations are unique within the class of Leray solutions.

THEOREM 7.4 (Leray [80]). *Let* $u^{\mathrm{in}} \in L^2(\mathbb{R}^3)$ *such that* $\mathrm{div}_x u^{\mathrm{in}} = 0$. *Assume that there exists a classical solution,*

$$v \in C^1\big(\mathbb{R}_+; H^1(\mathbb{R}^3)\big) \cap C\big(\mathbb{R}_+; H^2(\mathbb{R}^3)\big) \quad \text{such that} \quad \nabla_x v \in L^\infty\big(\mathbb{R}_+ \times \mathbb{R}^3\big),$$

of the Navier–Stokes equations with initial condition

$$v|_{t=0} = u^{\mathrm{in}}.$$

Then, any Leray solution u *of the Navier–Stokes equations with initial data*

$$u|_{t=0} = u^{\mathrm{in}}$$

coincides with v *a.e.*

Whether the space dimension is $D = 2$ or $D = 3$ leads to fundamental differences in the regularity theory for the Navier–Stokes equations.

THEOREM 7.5 (Leray [79]). *Let* $u^{\mathrm{in}} \in L^2(\mathbb{R}^2)$ *such that* $\mathrm{div}_x u^{\mathrm{in}} = 0$. *Then there exists a unique weak solution* u *to the Navier–Stokes equations with initial data*

$$u|_{t=0} = u^{\mathrm{in}}$$

such that

$$u \in C\big(\mathbb{R}_+; L^2(\mathbb{R}^2)\big) \cap L^2\big(\mathbb{R}_+; H^1(\mathbb{R}^2)\big).$$

Furthermore, this solution is smooth for t > 0

$$u \in C^\infty\big(\mathbb{R}_+^* \times \mathbb{R}^2\big)$$

and satisfies the energy equality

$$\frac{1}{2}\big\|u(t)\big\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^D} \nu\big|\nabla_x u(s,x)\big|^2 \,dx\,ds = \frac{1}{2}\big\|u^{\mathrm{in}}\big\|_{L^2}^2$$

for each t ⩾ 0.

In space dimension 3, it is an outstanding open problem to determine whether Leray solutions with C^∞ initial data remain C^∞ for all times.

What is known to this date is the following partial regularity theorem which improves on an earlier result by Scheffer [112].

THEOREM 7.6 (Caffarelli, Kohn and Nirenberg [21]). *In space dimension* 3, *let u be a Leray solution of the Navier–Stokes equations that satisfies the local variant of Leray's energy inequality* (7.1). *Let the singular set of u be*

$$S(u) = \big\{(t,x) \in \mathbb{R}_+ \times \mathbb{R}^3 \,\big|\, u \text{ is not bounded in a neighborhood of } (t,x)\big\}.$$

Then, $S(u)$ *has parabolic Hausdorff dimension less than* 1.

This definition of the singular set $S(u)$ comes from a bootstrap argument due to Serrin [113] showing that, if a Leray solution u of the Navier–Stokes equations is bounded in $B((t,x), R)$, then u is C^∞ in $B((t,x), R/2)$.

The parabolic Hausdorff dimension is defined through coverings with translates of $(-r^2, r^2) \times B(0,r)$ in $\mathbb{R} \times \mathbb{R}^3$ (the usual Hausdorff dimension being defined with balls for the Euclidean metric of $\mathbb{R}^4$).

This result implies that the singular set $S(u)$ must be smaller than a curve in space–time: in other words singularities of solutions to the Navier–Stokes equations in space dimension 3 are rare (if they exist at all).

7.3. *The incompressible Navier–Stokes–Fourier system*

By mimicking the compactness method in the proof of the Leray existence theorem, we can also treat the case of the Navier–Stokes–Fourier system

$$\partial_t u + \mathrm{div}_x(u \otimes u) + \nabla_x p = \nu \Delta_x u, \qquad \mathrm{div}_x u = 0, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^3,$$

$$\partial_t \theta + \mathrm{div}_x(u\theta) = \kappa \Delta_x \theta, \quad (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^3,$$

$$(u, \theta)|_{t=0} = \big(u^{\mathrm{in}}, \theta^{\mathrm{in}}\big),$$

where $\kappa > 0$ and $\nu > 0$.

The analogue of Leray's theorem for the Navier–Stokes–Fourier system is the following one.

THEOREM 7.7. *For each $(u^{\mathrm{in}}, \theta^{\mathrm{in}}) \in L^2(\mathbb{R}^3)$ such that $\mathrm{div}_x\, u^{\mathrm{in}} = 0$, there exists a solution in the sense of distributions to the Cauchy problem for the Navier–Stokes–Fourier system such that*

$$(u, \theta) \in C\big(\mathbb{R}_+; w - L^2(\mathbb{R}^3)\big) \cap L^2\big(\mathbb{R}_+; H^1(\mathbb{R}^3)\big).$$

This solution satisfies, for each $t > 0$

$$\frac{1}{2}\big\|u(t)\big\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^D} \nu \big|\nabla_x u(s, x)\big|^2 \, \mathrm{d}x \, \mathrm{d}s \leqslant \frac{1}{2}\big\|u^{\mathrm{in}}\big\|_{L^2}^2,$$

$$\frac{1}{2}\big\|\theta(t)\big\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^D} \kappa \big|\nabla_x \theta(s, x)\big|^2 \, \mathrm{d}x \, \mathrm{d}s \leqslant \frac{1}{2}\big\|\theta^{\mathrm{in}}\big\|_{L^2}^2.$$

7.4. *The compressible Euler system*

The compressible Euler system is a quasilinear hyperbolic system. The existence and uniqueness theory for this system is not entirely satisfying in its present state, especially in space dimension greater than or equal to 2. More information on the theory of hyperbolic system of conservation laws can be found for instance in [32,78] and [19].

Consider the Cauchy problem for the compressible Euler system (with perfect monatomic gas equation of state)

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0,$$

$$\partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x(\rho\theta) = 0, \tag{7.2}$$

$$\partial_t\left(\rho\left(\frac{1}{2}|u|^2 + \frac{3}{2}\theta\right)\right) + \mathrm{div}_x\left(\rho\left(\frac{1}{2}|u|^2 + \frac{5}{2}\theta\right)\right) = 0.$$

We begin with a local existence and uniqueness result which is a particular case of a general theorem on quasilinear *symmetrizable systems*. The theory of symmetric hyperbolic system was developed very early by Friedrichs; the importance of the notion of symmetrizable systems was recognized by Godunov [52], and then by Friedrichs and Lax [43] – see also [46] and [81] for more information on the theory of hyperbolic systems. The result below comes from [91]; the case of general systems is studied in [78] and [72].

THEOREM 7.8. *Let $D \geqslant 1$ and $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}}) \in H^m(\mathbb{R}^D)$ with $m > D/2 + 1$. There exists $T > 0$ and a unique solution $(\rho, u, \theta) \in C([0, T]; H^m(\mathbb{R}^D)) \cap C^1([0, T]; H^{m-1}(\mathbb{R}^D))$ of the compressible Euler system in the sense of distributions on $(0, T) \times \mathbb{R}^D$.*

Such solutions are regular (since $H^m(\mathbb{R}^D) \subset C^{[m-D/2]}(\mathbb{R}^D)$ by Sobolev embedding). In general, one does not expect that regular solutions to (7.2) may exist for all times. Singularities – such as shock waves – are expected to appear in finite time for a large class of smooth initial data. Here is an interesting result in this direction, due to Sideris [114].

THEOREM 7.9. *Let $D = 3$, and let $R > 0$. Pick the initial data $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \in C^\infty(\mathbb{R}^3)$ to be such that $\rho^{\text{in}}, \theta^{\text{in}} > 0$ on $\mathbb{R}^3$ with*

$$\text{supp}(\rho^{\text{in}} - 1) \subset B(0, R), \qquad \text{supp}(u^{\text{in}}) \subset B(0, R),$$

$$\text{supp}(\theta^{\text{in}} - 1) \subset B(0, R) \quad \text{and} \quad \rho^{\text{in}} \geqslant (\theta^{\text{in}})^{3/2} \quad \text{on } \mathbb{R}^3.$$

Assume further the existence of $R_0 \in (0, R)$ such that

$$\int_{|x|>r} \frac{(|x| - r)^2}{|x|} (\rho^{\text{in}}(x) - 1) \, dx > 0$$

and

$$\int_{|x|>r} \frac{(|x|^2 - r^2)}{|x|^3} \rho^{\text{in}}(x) x \cdot u^{\text{in}}(x) \, dx \geqslant 0$$

for each $r \in (R_0, R)$. Then the life-span T of the C^1 solution to (7.2) with such initial data is finite.

In dimension greater than or equal to 2, there is no satisfying theory of weak solutions that could extend classical solutions after blow-up time.

At variance, in the one-dimensional case, there is a rather complete theory of weak solutions, that are obtained as superpositions of interacting Riemann problems (i.e., a Cauchy problem for (5.6) whose initial data is a step function with only one jump). Liu studied the compressible Euler system in space dimension 1 written in Lagrangian coordinates. Denoting by a the Lagrangian particle label, this system reads

$$\partial_t V - \partial_a U = 0,$$

$$\partial_t U + \partial_a \left(\frac{\Theta}{V} \right) = 0,$$

$$\partial_t \left(\frac{1}{2} U^2 + \frac{3}{2} \Theta \right) + \partial_a \left(\frac{\Theta U}{V} \right) = 0, \tag{7.3}$$

$$(V, U, \Theta)|_{t=0} = (V^{\text{in}}, U^{\text{in}}, \Theta^{\text{in}}),$$

with the notation

$$V(t, a) = \frac{1}{\rho(t, X(t, 0, a))},$$

$$U(t, a) = u\big(t, X(t, 0, a)\big),$$

$$\Theta(t, a) = \theta\big(t, X(t, 0, a)\big),$$

where $t \mapsto X(t, 0, a)$ is the path of the particle which is at the position a at time $t = 0$.

The following existence result is based on Glimm's algorithm [51] for constructing *BV* solutions to hyperbolic systems of conservation laws in the one-dimensional case that are global in time for initial data of small total variation.

THEOREM 7.10 (Liu [89]). *Assume that* $(V^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \in BV(\mathbb{R})$ *with* $\theta^{\text{in}} \geqslant \theta_* > 0$ *while* $V^{\text{in}} \leqslant V^*$ *on* $\mathbb{R}$. *There exists* $\eta_0 > 0$ *such that the Cauchy problem (7.3) has a global weak solution provided that* $TV(V^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \leqslant \eta_0$. *Moreover, this solution is "entropic", i.e.,*

$$\partial_t \ln\big(V^{2/3}\Theta\big) \geqslant 0.$$

(*In other words, the entropy density cannot decrease along particle paths.*)

7.5. *The incompressible Euler equations*

There are essentially two main directions in the mathematical theory of the incompressible Euler equations:

- the PDE viewpoint, and
- the geometric viewpoint.

In the geometric viewpoint, Euler's equations for incompressible fluids in the periodic box $\mathbb{T}^D$ are the equations of geodesics on the group of volume preserving diffeomorphisms of $\mathbb{T}^D$, endowed with the metric defined by the kinetic energy. We shall say nothing of this part of the theory, for which we refer the reader to the beautiful book by Arnold and Khesin [4].

Instead, we shall just recall a few results on the Euler equations as nonlinear PDEs on $\mathbb{T}^D$. The reader is advised to read [92] and [86] for more information on that topic.

The incompressible Euler equations are

$$\partial_t u + \text{div}_x(u \otimes u) + \nabla_x p = 0, \qquad \text{div}_x u = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{T}^D,$$

$$u|_{t=0} = u^{\text{in}}.$$

We begin with a local existence result for classical solutions in the three-dimensional case, due to Kato.

THEOREM 7.11 (Kato [71]). *Let* $u^{\text{in}} \in H^m(\mathbb{T}^3)$, $m \in \mathbb{N}$, $m \geqslant 3$, *such that* $\text{div}_x u^{\text{in}} = 0$. *Then, there exists* $T > 0$ *and a unique local solution* u *of the incompressible Euler equations with initial data* u^{in}, *such that*

$$u \in C\big([0, T); H^m(\mathbb{T}^3)\big) \cap AC_{\text{loc}}\big([0, T); H^{m-1}(\mathbb{T}^3)\big).$$

Whether the life-span of such solutions is finite is an outstanding open problem in the theory of nonlinear PDEs. What is known is that the blow-up time, if finite, does not depend on the regularity index m, as shown by the following beautiful result.

THEOREM 7.12 (Beale, Kato and Majda [15]). *Under the same assumptions as in the previous theorem, if T is finite, then*

- *either*

$$\int_0^T \big\| \operatorname{curl}_x u(t,\cdot) \big\|_{L^\infty} \, \mathrm{d}t = +\infty,$$

- *or $u \in C([0,T]; H^m(\mathbb{T}^3))$.*

(*In the second case, the solution u can be extended to an interval of time $[0,T')$ with $T' > T$.*)

In the two-dimensional case, there is global existence and uniqueness of a classical solution to the Cauchy problem for the incompressible Euler equations.

THEOREM 7.13 (Yudovich [123]). *Let $u^{\mathrm{in}} \in H^m(\mathbb{T}^2)$, $m \in \mathbb{N}$, $m \geqslant 3$, such that $\operatorname{div}_x u^{\mathrm{in}} = 0$. Then there exists a unique global solution u of the incompressible Euler equations with initial data u^{in} such that*

$$u \in C\big(\mathbb{R}_+; H^m(\mathbb{T}^2)\big) \cap C^1\big(\mathbb{R}_+; H^{m-1}(\mathbb{T}^2)\big).$$

A good reference on the two-dimensional case of the incompressible Euler equations is [29].

In view of the importance of the vorticity field $\operatorname{curl}_x u$ for the regularity of the solution to the incompressible Euler equations, the main difference between the two- and the three-dimensional cases comes from the following observation.

If one considers the two-dimensional flow as given by a three-dimensional velocity field u of the form

$$u(t,x) = \begin{pmatrix} u_1(t,x_1,x_2) \\ u_2(t,x_1,x_2) \\ 0 \end{pmatrix},$$

the vorticity field is

$$\operatorname{curl}_x u(t,x) = \begin{pmatrix} 0 \\ 0 \\ \omega(t,x_1,x_2) \end{pmatrix},$$

where

$$\omega(t,x_1,x_2) = \partial_{x_1} u_2(t,x_1,x_2) - \partial_{x_2} u_1(t,x_1,x_2).$$

The scalar quantity ω satisfies the transport equation

$$\partial_t \omega + u \cdot \nabla_x \omega = 0.$$

The maximum principle holds for the transport equation above, so that

$$\|\omega\|_{L^\infty(\mathbb{R}_+ \times \mathbb{T}^2)} = \big\|\omega(0, \cdot)\big\|_{L^\infty(\mathbb{T}^2)}.$$

Hence, in the two-dimensional case, if u^{in} is sufficiently regular, the vorticity is globally bounded and therefore, by the Beale–Kato–Majda criterion, the regularity of the initial data is propagated for all times.

In the three-dimensional case, the vorticity $\text{curl}_x u$ is a *vector* field that satisfies the analogue of the scalar transport equation above for vectors

$$\partial_t \text{curl}_x u + (u \cdot \nabla_x) \text{curl}_x u = \nabla_x u \cdot \text{curl}_x u.$$

The length of the vector $\text{curl}_x u$ can be amplified, or damped, by the matrix $\nabla_x u$: this mechanism is called "vortex stretching" and so far, there is no satisfying method for controlling it. Therefore, there is no a priori bound on the vorticity as in the two-dimensional case, and this is why the question of global existence or finite-time blow-up for classical solutions to the incompressible Euler equations in the three-dimensional case remains very much open.

Finally, we conclude this section with an important class of global solutions on the periodic box to the incompressible Euler equations. Choose

$$U(t, x) = \begin{pmatrix} u_1(t, x_1, x_2) \\ u_2(t, x_1, x_2) \\ w(t, x_1, x_2) \end{pmatrix}, \quad (t, x) \in \mathbb{R}_+ \times \mathbb{T}^3,$$

where

$$u(t, x_1, x_2) = \begin{pmatrix} u_1(t, x_1, x_2) \\ u_2(t, x_1, x_2) \end{pmatrix}$$

is a C^1-solution of the two-dimensional incompressible Euler equations on $\mathbb{R}_+ \times \mathbb{T}^2$. If w satisfies the transport equation

$$\partial_t w + \text{div}_x(wu) = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{T}^2,$$

the vector field U solves the three-dimensional incompressible Euler equation on $\mathbb{R}_+ \times \mathbb{T}^3$. Such solutions are referred to as $2D$–$3C$ *solutions of the incompressible Euler equations* – see, in particular, Section S4.3, pp. 150–153 of [86] for an interesting application of 2D–3C solutions to the problem of a priori estimates on the Euler equations.

8. Mathematical theory of the Cauchy problem for the Boltzmann equation

8.1. *Global classical solutions for "small" data*

In this subsection, we briefly review two early existence theories for the Boltzmann equation:

- Ukai's theory for perturbations of uniform Maxwellian states; and
- the Illner–Shinbrot theory for small perturbations of the vacuum state.

Consider the Cauchy problem for the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3,$$

$$F|_{t=0} = F^{\text{in}}.$$

8.1.1. *Small perturbations of the vacuum.*   The collision kernel b in the collision integral is supposed to satisfy

$$0 \leqslant b(z, \omega) \leqslant C\left(1 + |z|\right) \quad \text{for a.e. } (z, \omega) \in \mathbb{R}^3 \times \mathbb{S}^2.$$

THEOREM 8.1 (Illner and Shinbrot [70]). *Pick $c > 0$; there exists $\eta > 0$ such that for each $C \in (0, \eta)$ and each initial data F^{in} satisfying*

$$0 \leqslant F^{\text{in}}(x, v) \leqslant C e^{-c(|x|^2 + |v|^2)}, \quad x, v \in \mathbb{R}^3,$$

then the Cauchy problem for the Boltzmann equation with initial data F^{in} has a unique global (mild) solution.

The key to this result is to dominate the solution F of the Boltzmann equation with a Maxwellian traveling wave $e^{-c|x-tv|^2}$.

Obviously this result is not useful in the context of incompressible hydrodynamic limits: it bears on rarefied clouds of gas that expand in the vacuum, and therefore never approach any global Maxwellian equilibrium.

8.1.2. *Small perturbations of a global Maxwellian state.*   The following result, due to Ukai, is the first global existence and uniqueness result proved on the (space inhomogeneous) Boltzmann equation. It bears on the dynamics of a gas whose state is a perturbation of a global Maxwellian equilibrium. It uses a detailed spectral analysis of the linearization of the collision integral at the background Maxwellian state, see [40].

THEOREM 8.2 (Ukai [116]). *Assume that $b(z, \omega) = |z \cdot \omega|$ (hard sphere case). Let $s > 3/2$, and $\beta > 3$; define*

$$\|f\|_{s,\beta} = \sup_{v \in \mathbb{R}^3} \left(1 + |v|\right)^\beta \left\|f(\cdot, v)\right\|_{H^s(\mathbb{R}^3)}$$

and let

$$H_\beta^s = \left\{ f \in L_{\mathrm{loc}}^\infty\big(\mathbb{R}_v^3; H_x^s\big) \,\big|\, \|f\|_{s,\beta} < +\infty \right\}.$$

There exists $\eta > 0$ such that, for any $f^{\mathrm{in}} \equiv f^{\mathrm{in}}(x, v)$ satisfying

$$\big\| f^{\mathrm{in}} \big\|_{s,\beta} < \eta \quad and \quad f^{\mathrm{in}} \geqslant -\mathcal{M}_{(1,0,1)}^{1/2},$$

the Cauchy problem for the Boltzmann equation with initial data

$$F^{\mathrm{in}} = \mathcal{M}_{(1,0,1)} + \mathcal{M}_{(1,0,1)}^{1/2} f^{\mathrm{in}}$$

has a unique solution F such that

$$F \in L^\infty\big(\mathbb{R}_+; H_\beta^s\big) \cap C\big(\mathbb{R}_+; H_{\beta-\varepsilon}^{s-\varepsilon}\big) \cap C^1\big(\mathbb{R}_+; H_{\beta-1-\varepsilon}^{s-1-\varepsilon}\big)$$

for each $\varepsilon > 0$.

Ukai's theory describes the evolution of number densities that are close to a uniform Maxwellian state, and therefore, one could think of using it in the context of incompressible hydrodynamic limits. However, one should bear in mind that the parameter η that monitors the size of the initial number density fluctuation is not uniform in the Knudsen number, so that applying Ukai's ideas to derive, say, the incompressible Navier–Stokes equations from the Boltzmann equation requires nontrivial modifications, due to Bardos and Ukai [14].

Hence, for the purpose of deriving hydrodynamic models from the Boltzmann equation, it is desirable to have at one's disposal a global existence theory based on a priori estimates that are uniform in the Knudsen number. The only such existence theory so far is the DiPerna–Lions theory of weak solutions of the Boltzmann equation that is described below.

8.2. *The DiPerna–Lions theory*

As already mentioned in our presentation of the Boltzmann equation, the collision integral is local in t and x and an integral operator in v. In other words, the collision integral acts as a multiplication operator in the variables (t, x), and as kind of convolution in the variable v.

On the other hand, the natural a priori bound for the number density is

$$F \in L_t^\infty(L \ln L_x).$$

For such an F, expressions like

$$F^2 \quad or \quad F \int_{\mathbb{R}^3} F \, \mathrm{d}v$$

are defined only as measurable functions, and not as distributions. Hence the collision integral does not define a distribution for all number densities that satisfy the natural a priori bounds on solutions of the Boltzmann equation.

To get around this, DiPerna and Lions proposed to use the following notion of solution.

DEFINITION 8.3. A nonnegative function $F \in C(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ is a renormalized solution of the Boltzmann equation if

$$\frac{\mathcal{B}(F, F)}{\sqrt{1 + F}} \in L^1_{\text{loc}}(\,\mathrm{d}t \,\mathrm{d}x \,\mathrm{d}v),$$

and if, for each $\Gamma \in C^1(\mathbb{R}_+)$ such that

$$\Gamma'(Z) \leqslant \frac{C}{\sqrt{1 + Z}} \quad \text{for all } Z \geqslant 0,$$

one has

$$(\partial_t + v \cdot \nabla_x)\Gamma(F) = \Gamma'(F)\mathcal{B}(F, F)$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3$.

In this subsection, we shall consider collision kernels that satisfy the following weak cut-off assumption

$$\lim_{|v| \to +\infty} \frac{1}{1 + |v|^2} \int_{|v_*| < R} \int_{\mathbb{S}^2} b(v - v_*, \omega) \,\mathrm{d}\omega \,\mathrm{d}v_* = 0 \quad \text{for each } R > 0. \tag{8.1}$$

The following result was proved by DiPerna and Lions in [37].

THEOREM 8.4. *Let* $F^{\text{in}} \equiv F(x, v) \geqslant 0$ *a.e. on* $\mathbb{R}^3 \times \mathbb{R}^3$ *satisfy*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(1 + |x|^2 + |v|^2 + \left|\ln F^{\text{in}}\right|\right) F^{\text{in}} \,\mathrm{d}x \,\mathrm{d}v < +\infty.$$

Then, there exists a renormalized solution of the Boltzmann equation such that $F|_{t=0} = F^{\text{in}}$. *This renormalized solution satisfies*
- *the continuity equation* (*local conservation of mass*)

$$\partial_t \int_{\mathbb{R}^3} F \,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v F \,\mathrm{d}v = 0;$$

- *the global conservation of momentum: for each* $t \geqslant 0$,

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} v F(t) \,\mathrm{d}v \,\mathrm{d}x = \iint_{\mathbb{R}^3 \times \mathbb{R}^3} v F^{\text{in}} \,\mathrm{d}v \,\mathrm{d}x;$$

- *the energy inequality*: *for each $t \geqslant 0$,*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{1}{2} |v|^2 F(t) \, dv \, dx \leqslant \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{1}{2} |v|^2 F^{\text{in}} \, dv \, dx;$$

- *and the H inequality*: *for each $t > 0$, one has*

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} F \ln F(t) \, dv \, dx$$

$$+ \int_0^t \int_{\mathbb{R}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \left(F' F'_* - F F_* \right) \ln\left(\frac{F' F'_*}{F F_*} \right) b \, dv \, dv_* \, d\omega \, dx \, ds$$

$$\leqslant \iint_{\mathbb{R}^3 \times \mathbb{R}^3} F^{\text{in}} \ln F^{\text{in}} \, dv \, dx.$$

A complete description of the proof of the DiPerna–Lions theorem is beyond the scope of the present work. We shall just explain the main ideas in it.

8.2.1. *The role of the normalizing nonlinearity.*   Assume that $F \equiv F(t, x, v) \geqslant 0$ a.e. on $\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ satisfies

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \left(1 + |v|^2 \right) F(t, x, v) \, dv \, dx \leqslant C \quad \text{for a.e. } t \geqslant 0, \tag{8.2}$$

and

$$\int_0^T \int_{\mathbb{R}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \left(F' F'_* - F F_* \right) \ln\left(\frac{F' F'_*}{F F_*} \right) b \, dv \, dv_* \, d\omega \, dx \, ds \leqslant C_T \tag{8.3}$$

for each $T > 0$. A solution to the Boltzmann equation satisfies the first inequality (by the global conservation law of energy) and the second estimate by the entropy production bound deduced from Boltzmann's H-theorem.

Then

$$\frac{|\mathcal{B}(F, F)|}{\sqrt{1 + F}} \in L^1_{\text{loc}} \left(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3 \right).$$

Indeed, we first recall the elementary inequality

$$\left(\sqrt{X} - \sqrt{Y} \right)^2 \leqslant \frac{1}{4} (X - Y)(\ln X - \ln Y) \quad \text{for each } X, Y > 0.$$

Then

$$\left| F' F'_* - F F_* \right| = \left(\sqrt{F' F'_*} + \sqrt{F F_*} \right) \left| \sqrt{F' F'_*} - \sqrt{F F_*} \right|$$

$$\leqslant \left(\sqrt{F' F'_*} - \sqrt{F F_*} \right)^2 + 2 \sqrt{F F_*} \left| \sqrt{F' F'_*} - \sqrt{F F_*} \right|.$$

Hence

$$\int_0^T \int_{\mathbb{R}^3} \int_{|v|<R} \frac{|\mathcal{B}(F,F)|}{\sqrt{1+F}} \, dv \, dx \, dt$$

$$\leqslant \int_0^T \int_{\mathbb{R}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \left(\sqrt{F'F'_*} - \sqrt{FF_*}\right)^2 b \, dv \, dv_* \, d\omega \, dx \, ds$$

$$+ 2\left(\int_0^T \int_{\mathbb{R}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3} \left(\sqrt{F'F'_*} - \sqrt{FF_*}\right)^2 b \, dv \, dv_* \, d\omega \, dx \, ds\right)^{1/2}$$

$$\times \left(\int_0^T \iint_{\mathbb{R}^3 \times \mathbb{R}^3} F_* \left(\iint_{|v|<R, |\omega|=1} b(v-v_*, \omega) \, d\omega \, dv\right) dv_* \, dx \, ds\right)^{1/2}.$$

By using the entropy production bound (8.3) and the weak cut-off assumption above on the collision kernel b, this inequality becomes

$$\int_0^T \int_{\mathbb{R}^3} \int_{|v|<R} \frac{|\mathcal{B}(F,F)|}{\sqrt{1+F}} \, dv \, dx \, dt$$

$$\leqslant C_T + 2\sqrt{C_T}$$

$$\times \left(\int_0^T \iint_{\mathbb{R}^3 \times \mathbb{R}^3} F_* \left(\iint_{|v|<R, |\omega|=1} b(v-v_*, \omega) \, d\omega \, dv\right) dv_* \, dx \, ds\right)^{1/2}$$

$$\leqslant C_T + 2\sqrt{C_T} \left(\int_0^T \iint_{\mathbb{R}^3 \times \mathbb{R}^3} F_* O\left(1 + |v_*|^2\right) dv_* \, dx \, ds\right)^{1/2}$$

$$\leqslant C_T + 2\sqrt{C_T O(C)}.$$

In the sequel, we restrict our attention for simplicity to the case of bounded collision kernels

$$0 < b(z, \omega) \leqslant C_b, \quad (z, \omega) \in \mathbb{R}^3 \times \mathbb{S}^2. \tag{8.4}$$

8.2.2. *Compactness by velocity averaging.* A renormalized solution of the Boltzmann equation is obtained as a limit point of the sequence F_n of solutions to the regularized Boltzmann equation

$$\partial_t F_n + v \cdot \nabla_x F_n = \frac{\mathcal{B}(F_n, F_n)}{1 + (1/n) \int_{\mathbb{R}^3} F_n \, dv},$$

$$F_n|_{t=0} = F^{\text{in}} \tag{8.5}$$

in the weak topology of $L^1_{\text{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$. As in the case of all nonlinear PDEs, the main difficulty is to pass to the limit in the nonlinear term – here, in the collision integral. This requires a class of compactness results on the Boltzmann equation that are somehow adapted to the collision integral. In particular, since the collision integral acts as a

convolution operator in the variable v and a multiplication operator in the variables t and x, one should seek compactness "with respect to the t and x variables only" – a notion which, of course, remains to be defined.

The appropriate class of compactness theorems was discovered a few years before being applied to the Boltzmann equation. They are known as "velocity averaging results", and were first introduced by Golse, Perthame and Sentis in [56] within the context of the diffusion approximation of the neutron transport equation. (Independently, analogous regularity results for the coefficients of the spherical harmonic expansion of the solution to the free transport equation were announced in [1].)

Obviously, whatever compactness in the strong topology of L^1_{loc} is to be found on a sequence of solutions to the Boltzmann equation (regularized or not) has to come from the streaming (free transport) operator $\partial_t + v \cdot \nabla_x$.

Being hyperbolic, the transport operator $v \cdot \nabla_x$ propagates singularities along characteristics. Therefore, at first sight it seems hopeless that one might obtain any regularizing effect from the free streaming part of the Boltzmann equation, or of any other similar kinetic model. One can think of the following elementary example.

EXAMPLE. Let $f \in L^2(\mathbb{R})$; for a.e. $x, v \in \mathbb{R}^2$, define $F(x, v) = f(v_2 x_1 - v_1 x_2)$. Clearly, $F \in L^2_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R}^2)$ and $v \cdot \nabla_x F = 0$. However, since f can be any function in $L^2(\mathbb{R})$, $F \notin H^s_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R}^2)$ for any $s > 0$, although $v \cdot \nabla_x F \in L^2_{\mathrm{loc}}(\mathbb{R}^2 \times \mathbb{R}^2)$.

The key to obtaining regularizing effects from the transport operator $v \cdot \nabla_x$ is to seek those effects not on the number density itself, but on velocity averages thereof, in other words, on the macroscopic densities. Here is the prototype of all velocity averaging results.

THEOREM 8.5 (Golse, Perthame and Sentis [56]). *Let F_ε be a bounded family in $L^2(\mathbb{R}^D \times \mathbb{R}^D)$. Assume that the family $v \cdot \nabla_x F_\varepsilon$ is also bounded in $L^2(\mathbb{R}^D \times \mathbb{R}^D)$. Then, for each $\phi \in L^2(\mathbb{R}^D)$, the family of moments ρ_ε defined by*

$$\rho_\varepsilon(x) = \int_{\mathbb{R}^D} F_\varepsilon(x, v)\phi(v)\, dv$$

is relatively compact in $L^2_{\mathrm{loc}}(\mathbb{R}^D)$.

PROOF. Set $G_\varepsilon = F_\varepsilon + v \cdot \nabla_x F_\varepsilon$, and let $\widehat{F}_\varepsilon$ and $\widehat{G}_\varepsilon$ denote respectively the Fourier transforms of F_ε and G_ε in the x variable only. The assumptions on F_ε imply that the family G_ε is bounded in $L^2(\mathbb{R}^D \times \mathbb{R}^D)$, and that

$$\hat{\rho}_\varepsilon(\xi) = \int_{\mathbb{R}^D} \frac{\widehat{G}_\varepsilon(\xi, v)\phi(v)\, dv}{1 + iv \cdot \xi}.$$

We need to study how $\hat{\rho}_\varepsilon(\xi)$ decays for $|\xi|$ large. By the Cauchy–Schwarz inequality

$$\left|\hat{\rho}_\varepsilon(\xi)\right|^2 \leqslant \frac{1}{m(\xi)} \int_{\mathbb{R}^D} \left|\widehat{G}_\varepsilon(\xi, v)\right|^2 dv,$$

with

$$\frac{1}{m(\xi)} := \int_{\mathbb{R}^D} \frac{|\phi(v)|^2 \, \mathrm{d}v}{1 + |v \cdot \xi|^2} = \int_{\mathbb{R}^D} \frac{|\phi(v)|^2 \, \mathrm{d}v}{1 + |v \cdot \omega|^2 |\xi|^2},$$

where $\omega = \xi/|\xi| \in \mathbb{S}^{D-1}$ for all $\xi \in \mathbb{R}^D \setminus \{0\}$. The latter integral is a decreasing family indexed by $|\xi|$ of continuous functions of ω; this family vanishes pointwise in ω as $|\xi| \to +\infty$ by dominated convergence. By Dini's theorem, it vanishes uniformly in $\omega \in \mathbb{S}^{D-1}$, and therefore $m(\xi) \to +\infty$ as $|\xi| \to +\infty$. Since the family

$$\int_{\mathbb{R}^D} \left|\hat{\rho}_\varepsilon(\xi)\right|^2 m(\xi) \, \mathrm{d}\xi \leqslant \iint_{\mathbb{R}^D \times \mathbb{R}^D} \left|\widehat{G}_\varepsilon(\xi, v)\right|^2 \mathrm{d}\xi \, \mathrm{d}v$$

is bounded by Plancherel's theorem, ρ_ε is relatively compact in $L^2_{\mathrm{loc}}(\mathbb{R}^D)$ (by a variant of Rellich's compactness theorem). $\qquad\square$

Since the operator $(I + v \cdot \nabla_x)^{-1}$ (which maps G on the solution F of $F + v \cdot \nabla_x F = G$) is a contraction mapping on both $L^1(\mathbb{R}^D \times \mathbb{R}^D)$ and $L^\infty(\mathbb{R}^D \times \mathbb{R}^D)$, the velocity averaging result above also holds in L^p for all $p \in (1, +\infty)$ by interpolation. However, it fails in L^1, as the following example shows. (It also fails in L^∞, see [55], p. 124.)

EXAMPLE ([55], pp. 123–124). Consider G_ε, a bounded family of $L^1(\mathbb{R}^D \times \mathbb{R}^D)$, and for each ε, let F_ε be the solution to $F_\varepsilon + v \cdot \nabla_x F_\varepsilon = G_\varepsilon$. Assume that $G_\varepsilon \to \delta_0 \otimes \delta_{v^*}$ weakly, where $|v^*| = 1$. Then both F_ε and $v \cdot \nabla_x F_\varepsilon$ are bounded in $L^1(\mathbb{R}^D \times \mathbb{R}^D)$ and

$$\rho_\varepsilon(x) = \int_{\mathbb{R}^D} F_\varepsilon(x, v) \, \mathrm{d}v = \int_0^{+\infty} \int_{\mathbb{R}^D} \mathrm{e}^{-t} G_\varepsilon(x - tv, v) \, \mathrm{d}v \, \mathrm{d}t$$

so that, for any test function $\phi \in C_c(\mathbb{R}^D)$,

$$\int_{\mathbb{R}^D} \rho_\varepsilon(x) \phi(x) \, \mathrm{d}x \to \int_0^{+\infty} \mathrm{e}^{-t} \phi\left(tv^*\right) \mathrm{d}t$$

as $\varepsilon \to 0$. Hence ρ_ε converges weakly to a density carried by the half-line $\mathbb{R}_+ v^*$, so that in particular the family ρ_ε is not relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^D)$.

This example rests on the possible build-up of concentrations in F_ε and $v \cdot \nabla_x F_\varepsilon$. If such concentrations are ruled out, the same interpolation argument as above entails the following L^1 variant of velocity averaging.

PROPOSITION 8.6 (Golse, Lions, Perthame and Sentis [55]). *Let F_ε be a family of measurable functions on $\mathbb{R}^D \times \mathbb{R}^D$ such that, for each compact subset K of $\mathbb{R}^D$, both families F_ε and $v \cdot \nabla_x F_\varepsilon$ are uniformly integrable on $K \times \mathbb{R}^D$. Then the family ρ_ε defined by*

$$\rho_\varepsilon(x) = \int_{\mathbb{R}^D} F_\varepsilon(x, v) \, \mathrm{d}v$$

is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^D)$.

PROOF. Let $\chi \equiv \chi(x)$ belongs to $C^\infty_c(\mathbb{R}^D)$. Set

$$G_\varepsilon(x, v) = \chi(x) F_\varepsilon(x, v) + v \cdot \nabla_x\big(\chi(x) F_\varepsilon(x, v)\big)$$

and, for each $\lambda > 0$, decompose $\chi \rho_\varepsilon$ as follows

$$\chi \rho_\varepsilon = \chi \rho^>_{\lambda,\varepsilon} + \chi \rho^<_{\lambda,\varepsilon}$$

with

$$\chi \rho^>_{\lambda,\varepsilon} = \int_{|v| \leqslant \lambda} (I + v \cdot \nabla_x)^{-1}(G_\varepsilon \mathbb{1}_{|G_\varepsilon| > \lambda}) \, \mathrm{d}v + \int_{|v| > \lambda} \chi F_\varepsilon \, \mathrm{d}v,$$

$$\chi \rho^<_{R,\lambda,\varepsilon} = \int_{|v| \leqslant \lambda} (I + v \cdot \nabla_x)^{-1}(G_\varepsilon \mathbb{1}_{|G_\varepsilon| \leqslant \lambda}) \, \mathrm{d}v.$$

The assumptions on F_ε imply that G_ε is uniformly integrable, so that $G_\varepsilon \mathbb{1}_{|G_\varepsilon| > \lambda} \to 0$ and $\chi F_\varepsilon \mathbb{1}_{|v| > \lambda} \to 0$ in $L^1(\mathbb{R}^D \times \mathbb{R}^D)$ uniformly in ε as $\lambda \to +\infty$; thus $\chi \rho^>_{l,\varepsilon} \to 0$ in $L^1(\mathbb{R}^D)$ uniformly in ε as $\lambda \to +\infty$.

On the other hand, for each λ, the family $G_\varepsilon \mathbb{1}_{|G_\varepsilon| > \lambda}$ indexed by ε is bounded in $L^2(\mathbb{R}^D)$; thus, by velocity averaging in L^2, $\chi \rho^<_{\lambda,\varepsilon}$ is relatively compact in $L^2(\mathbb{R}^D)$ – and therefore in $L^1(\mathbb{R}^D)$, since it has support in $\mathrm{supp}\,\chi$ which is compact.

Hence $\chi \rho_\varepsilon$ is relatively compact in $L^1(\mathbb{R}^D)$. Since χ is arbitrary, this eventually implies that ρ_ε is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^D)$.   □

In fact, by a further interpolation argument, one can get rid of the assumption of uniform integrability on derivatives.

THEOREM 8.7 (Golse and Saint-Raymond [60]). *Let F_ε be a family of measurable functions on $\mathbb{R}^D \times \mathbb{R}^D$ such that, for each compact subset K of $\mathbb{R}^D$, the family F_ε is uniformly integrable on $K \times \mathbb{R}^D$, while $v \cdot \nabla_x F_\varepsilon$ is bounded in $L^1(K \times \mathbb{R}^D)$. Then the family ρ_ε defined by*

$$\rho_\varepsilon(x) = \int_{\mathbb{R}^D} F_\varepsilon(x, v) \, \mathrm{d}v$$

is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}^D)$.

PROOF. Without loss of generality, assume that all the F_ε are supported in $K \times \mathbb{R}^D$. Write

$$\rho_\varepsilon(x) = \lambda \int_{\mathbb{R}^D} (\lambda I + v \cdot \nabla_x)^{-1} F_\varepsilon(x, v) \, \mathrm{d}v$$

$$+ \int_{\mathbb{R}^D} (\lambda I + v \cdot \nabla_x)^{-1}(v \cdot \nabla_x F_\varepsilon)(x, v) \, \mathrm{d}v.$$

Since

$$\left\| (\lambda I + v \cdot \nabla_x)^{-1} \right\|_{\mathcal{L}(L^1_{x,v})} \leqslant \frac{1}{\lambda},$$

the second term on the right-hand side of the equality above is $O(1/\lambda)$ in L^1_x uniformly in ε, while the first term is strongly relatively compact in L^1_x for each $\lambda > 0$ by the previous proposition. Hence the family ρ_ε is strongly relatively compact in L^1_x. $\square$

There are several extensions and variants of the velocity averaging results recalled here, see for instance [36,38,47–49,106]. Except for the extension of the above results to the evolution problem, which is needed in the construction of renormalized solutions to the Boltzmann equation, we shall not discuss these extensions in the present notes, but refer the interested reader to Chapter 1 of [18] for a survey of that theory as of 2000.

Here is the analogue of the L^1-variant of velocity averaging for evolution problems.

THEOREM 8.8. *Consider $F_\varepsilon \equiv F_\varepsilon(t, x, v)$, a family of measurable functions on $\mathbb{R}_+ \times \mathbb{R}^D \times \mathbb{R}^D$ such that, for each $T > 0$ and each compact $K \subset \mathbb{R}^D$, F_ε is uniformly integrable on $[0, T] \times K \times \mathbb{R}^D$ while $(\partial_t + v \cdot \nabla_x)F_\varepsilon$ is bounded in $L^1([0, T] \times K \times \mathbb{R}^D)$. Then the family ρ_ε defined by*

$$\rho_\varepsilon(t, x) = \int_{\mathbb{R}^D} F_\varepsilon(t, x, v) \, dv$$

is relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^D)$.

The proof is a straightforward variant of the arguments for the steady transport operator $v \cdot \nabla_x$ given above.

8.2.3. *Conclusion.* Let us briefly explain how the renormalization procedure is combined with compactness by velocity averaging in the proof of the DiPerna–Lions theorem.

Choose as normalizing nonlinearity the function

$$\Gamma_\delta(Z) = \frac{1}{\delta} \ln(1 + \delta Z), \quad \delta > 0,$$

and consider the truncated Boltzmann equation (8.5). We leave it to the reader to verify that, under the condition (8.4), the map

$$F \mapsto \frac{\mathcal{B}(F, F)}{1 + (1/n) \int_{\mathbb{R}^3} F \, dv}$$

is Lipschitz continuous on $L^1(\mathbb{R}^3_x \times \mathbb{R}^3_v)$, so that the truncated Boltzmann equation (8.5) has a global solution. Because the truncation factor

$$\frac{1}{1 + (1/n) \int_{\mathbb{R}^3} F \, dv}$$

is independent of v, the symmetries of the collision integral that imply the local conservation laws of mass, momentum and energy, and the H-theorem hold also for the truncated collision integral

$$\frac{\mathcal{B}(F, F)}{1 + (1/n) \int_{\mathbb{R}^3} F \, dv}$$

so that, in particular,

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \big(1 + |x - tv|^2 + |v|^2\big) F_n(t, x, v) \, dx \, dv$$

$$= \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \big(1 + |x|^2 + |v|^2\big) F^{\text{in}}(x, v) \, dx \, dv$$

for all $t \geqslant 0$, while

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} F_n \ln F_n(t, x, v) \, dx \, dv \leqslant \iint_{\mathbb{R}^3 \times \mathbb{R}^3} F^{\text{in}} \ln F^{\text{in}}(x, v) \, dx \, dv.$$

As explained in Case 4 of Section 3.3, this implies the existence of a positive constant C such that

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \big(1 + |x|^2 + |v|^2\big) F_n(t, x, v) \, dx \, dv \leqslant C\big(1 + t^2\big) \quad \text{and}$$

$$\iint_{\mathbb{R}^3 \times \mathbb{R}^3} F_n |\ln F_n|(t, x, v) \, dx \, dv \leqslant C\big(1 + t^2\big) \quad \text{for all } t \geqslant 0. \tag{8.6}$$

Hence F_n is weakly relatively compact in $L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$ by the Dunford–Pettis theorem. Since, for each $\delta > 0$, one has

$$\Gamma_\delta(Z) \leqslant Z \quad \text{for each } Z \geqslant 0,$$

the sequence of functions $\Gamma_\delta(F_n)$ is also weakly relatively compact in $L^1_{\text{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$. On the other hand, the discussion in Section 8.2.1 shows that

$$(\partial_t + v \cdot \nabla_x) \Gamma_\delta(F_n) = \frac{\mathcal{B}(F_n, F_n)}{(1 + \delta F_n)(1 + (1/n) \int_{\mathbb{R}^3} F_n \, dv)}$$

is bounded in $L^1_{\text{loc}}(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3)$. By velocity averaging (Theorem 8.8), this implies that the sequence

$$\int_{\mathbb{R}^3} \Gamma_\delta(F_n) \, dv$$

is strongly relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3))$. On the other hand, because of the bound (8.6),

$$\int_{\mathbb{R}^3} F_n \, dv - \int_{\mathbb{R}^3} \Gamma_\delta(F_n) \, dv \to 0$$

in $L^1_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3))$ as $\delta \to 0$, uniformly in $n \geqslant 1$. Hence we conclude that the sequence

$$\int_{\mathbb{R}^3} F_n \, dv$$

is strongly relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}_+; L^1(\mathbb{R}^3))$.

Hence, modulo extraction of a subsequence

$$F_n \to F \quad \text{weakly in } L^1_{\mathrm{loc}}\big(\mathbb{R}_+ \times \mathbb{R}^3 \times \mathbb{R}^3\big)$$

while

$$\int_{\mathbb{R}^3} F_n \, dv \to \int_{\mathbb{R}^3} F \, dv \quad \text{strongly in } L^1_{\mathrm{loc}}\big(\mathbb{R}_+ \times \mathbb{R}^3\big).$$

Since the collision integral acts as a convolution in the v-variable and a multiplication operator in the t and x variables, this compactness theorem implies that, for each $\phi \in C_{\mathrm{c}}(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} \mathcal{B}(F_n, F_n)\phi \, dv \to \int_{\mathbb{R}^3} \mathcal{B}(F, F)\phi \, dv \quad \text{in measure on } [0, T] \times K$$

for each $T > 0$ and each compact $K \subset \mathbb{R}^3$. At first sight, this is not enough to pass to the limit in the sense of distributions in both sides of the truncated Boltzmann equation (8.5).

Instead, one integrates the truncated Boltzmann equation along characteristics by treating the gain term in the truncated collision integral as a source term. One easily sees that the limit F is a supersolution of the limiting Boltzmann equation integrated along characteristics; notice that this does not make use of the renormalization procedure. That F is a subsolution is more involved, we refer to [37] for a complete proof.

8.3. *Variants of the DiPerna–Lions theory*

The original DiPerna–Lions theorem considers a cloud of gas that expands in the vacuum, without any restriction on its degree of rarefaction, i.e., for an initial number density of arbitrary size that has finite mass, energy, second moment in x and entropy. As explained in Section 8.1.1, this kind of situation is not compatible with incompressible hydrodynamic limits.

For that purpose, we describe below two variants of the DiPerna–Lions theory that are particularly relevant in the context of incompressible hydrodynamic limits.

8.3.1. *The periodic box.*   The first variant of the DiPerna–Lions theory that we discuss here is the case of the spatial domain $\mathbb{T}^3$ (the periodic box). The collision kernel b satisfies the weak cut-off condition (8.1). Let M be the centered reduced Gaussian

$$M(v) = \frac{1}{(2\pi)^{3/2}} \mathrm{e}^{-|v|^2/2}.$$

We shall say that $F \in C(\mathbb{R}_+; L^1(\mathbb{R}^3 \times \mathbb{R}^3))$ is a *renormalized solution* of the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3,$$

$$F|_{t=0} = F^{\mathrm{in}},$$

relative to M if and only if, for each normalizing nonlinearity $\Gamma \in C^1(\mathbb{R}_+)$ such that

$$\Gamma'(Z) \leqslant \frac{C}{\sqrt{1+Z}}, \quad Z \geqslant 0,$$

one has

$$M(\partial_t + v \cdot \nabla_x)\Gamma\left(\frac{F}{M}\right) = \Gamma'\left(\frac{F}{M}\right)\mathcal{B}(F, F)$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3$.

THEOREM 8.9. *Let $F^{\mathrm{in}} \geqslant 0$ a.e. be a measurable function such that $H(F^{\mathrm{in}}|M) < +\infty$. There exists a renormalized solution F relative to M of the Cauchy problem for the Boltzmann equation with initial data F^{in}. This solution satisfies*

- *the continuity equation*

$$\partial_t \int_{\mathbb{R}^3} F \,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v F \,\mathrm{d}v = 0;$$

- *the following variant of the local conservation of momentum*

$$\partial_t \int_{\mathbb{R}^3} v F \,\mathrm{d}v + \mathrm{div}_x \int_{\mathbb{R}^3} v \otimes v F \,\mathrm{d}v + \mathrm{div}_x \, m = 0,$$

  where $m \in L^\infty(\mathbb{R}_+; \mathcal{M}(\mathbb{T}^3, M_3(\mathbb{R})))$ with values in nonnegative symmetric matrices;
- *the following energy relation*

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{1}{2}|v|^2 F(t, x, v) \,\mathrm{d}x \,\mathrm{d}v + \frac{1}{2} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \mathrm{trace}\big(m(t)\big)$$

$$= \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{1}{2}|v|^2 F^{\mathrm{in}}(x, v) \,\mathrm{d}x \,\mathrm{d}v$$

for each $t > 0$;

- *and the H inequality: for each $t > 0$, one has*

$$\frac{1}{4}\int_0^t \int_{\mathbb{T}^3}\!\!\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \big(F'F'_* - FF_*\big)\ln\!\left(\frac{F'F'_*}{FF_*}\right)b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s$$

$$\leqslant H\big(F^{\mathrm{in}}\big|M\big) - H\big(F(t)\big|M\big) - \frac{1}{2}\int \operatorname{trace}\big(m(t)\big).$$

The above result – especially the existence of the defect measure m – is due to Lions and Masmoudi [88].

8.3.2. *The Euclidean space with uniform Maxwellian state at infinity.* The next variant of the DiPerna–Lions theory that we consider is the case of a spatial domain that is the Euclidean space with uniform Maxwellian equilibrium at infinity. Consider the Cauchy problem

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3,$$

$$F(t, x, v) \to M \quad \text{as } |x| \to +\infty,$$

$$F|_{t=0} = F^{\mathrm{in}},$$

where M is the centered reduced Gaussian

$$M(v) = \frac{1}{(2\pi)^{3/2}}\mathrm{e}^{-|v|^2/2}.$$

Here again, the collision kernel b satisfies the weak cut-off assumption (8.1). The notion of "renormalized solution relative to M" of the Cauchy problem above is the same as in the case of the periodic box.

THEOREM 8.10 (Lions [83]). *Let $F^{\mathrm{in}} \geqslant 0$ a.e. be a measurable function such that $H(F^{\mathrm{in}}|M) < +\infty$. There exists a renormalized solution F relative to M of the Cauchy problem for the Boltzmann equation with initial data F^{in}. This solution satisfies*

- *the continuity equation*

$$\partial_t \int_{\mathbb{R}^3} F\,\mathrm{d}v + \operatorname{div}_x \int_{\mathbb{R}^3} vF\,\mathrm{d}v = 0;$$

- *and the H-inequality: for each $t > 0$, one has*

$$\frac{1}{4}\int_0^t \int_{\mathbb{R}^3}\!\!\iiint_{\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3} \big(F'F'_* - FF_*\big)\ln\!\left(\frac{F'F'_*}{FF_*}\right)b\,\mathrm{d}v\,\mathrm{d}v_*\,\mathrm{d}\omega\,\mathrm{d}x\,\mathrm{d}s$$

$$\leqslant H\big(F^{\mathrm{in}}\big|M\big) - H\big(F(t)\big|M\big).$$

9. The Hilbert expansion method: Application to the compressible Euler limit

In this section we describe a first method for deriving hydrodynamic models from the Boltzmann equation. In spite of its numerous shortcomings (which we shall discuss at the end of the present section), this method is extremely robust, and can be applied to various kinetic models other than the Boltzmann equation.

Start from the Boltzmann equation in the compressible Euler scaling

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} \mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3. \tag{9.1}$$

For simplicity, we only consider in this section the case of a hard sphere gas, so that the collision kernel b in Boltzmann's collision integral $\mathcal{B}$ is given by the expression

$$b(z, \omega) = |z \cdot \omega|, \quad (z, \omega) \in \mathbb{R}^3 \times \mathbb{S}^2.$$

Consider next the compressible Euler system for a perfect monatomic gas

$$\partial_t \rho + \mathrm{div}_x(\rho u) = 0,$$

$$\partial_t(\rho u) + \mathrm{div}_x(\rho u \otimes u) + \nabla_x(\rho \theta) = 0,$$

$$\partial_t \left(\rho \left(\frac{1}{2}|u|^2 + \frac{3}{2}\theta \right) \right) + \mathrm{div}_x \left(\rho u \left(\frac{1}{2}|u|^2 + \frac{5}{2}\theta \right) \right) = 0, \tag{9.2}$$

$$(\rho, u, \theta)|_{t=0} = \left(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}} \right),$$

where

$$\rho^{\mathrm{in}}, u^{\mathrm{in}} \text{ and } \theta^{\mathrm{in}} \in H^5(\mathbb{T}^3), \qquad \rho^{\mathrm{in}} \geqslant 0 \quad \text{and} \quad \theta^{\mathrm{in}} > 0 \quad \text{on } \mathbb{T}^3. \tag{9.3}$$

Let (ρ, u, θ) be the solution to (9.2) predicted by Theorem 7.8 under the assumption (9.3), and call $T > 0$ its lifespan. Finally, define the local Maxwellian

$$E(t, x, v) = \mathcal{M}_{(\rho(t,x), u(t,x), \theta(t,x))}.$$

THEOREM 9.1 (Caflisch [22]). *There exists ε_0 such that, for each $\varepsilon \in (0, \varepsilon_0)$, there is a unique solution F_ε of the Boltzmann equation* (9.1) *on $[0, T) \times \mathbb{T}^3 \times \mathbb{R}^3$ satisfying the estimate*

$$\sup_{0 \leqslant t \leqslant T'} \| F(t, \cdot, \cdot) - E(t, \cdot, \cdot) \|_{L^2(\mathbb{T}^3 \times \mathbb{R}^3)} = \mathrm{O}(\varepsilon) \quad \text{as } \varepsilon \to 0^+$$

for each $T' < T$.

We shall not give a complete proof of this result, which is fairly technical. Instead, we simply discuss below the main ideas in it.

First, the solution F_ε is sought as a truncated Hilbert expansion, plus a remainder term

$$F_\varepsilon(t, x, v) = \sum_{k=0}^{6} \varepsilon^k F_k(t, x, v) + \varepsilon^3 R_\varepsilon(t, x, v), \tag{9.4}$$

such that the last term in the truncated expansion satisfies

$$\int_{\mathbb{R}^3} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} F_6 \, dv = 0. \tag{9.5}$$

We recall from our discussion in Section 5.1 that the projection of a term F_k in Hilbert's expansion on the space of collision invariants is determined by postulating the existence of the next term in that expansion, i.e., F_{k+1}. In the truncated expansion above, we obviously do not postulate the existence of a F_7, so that there is a certain amount of arbitrariness in the choice of F_6, which is resolved by condition (9.5). The other terms F_k, $k = 0, \ldots, 5$, are computed as in Section 5.1.

Next we write an equation for the remainder R_ε: inserting the right-hand side of (9.4) in the scaled Boltzmann equation (9.1) we arrive at

$$(\partial_t + v \cdot \nabla_x) R_\varepsilon = \frac{2}{\varepsilon} \mathcal{B}(F_0, R_\varepsilon) + 2\mathcal{B}\left(\sum_{k=1}^{6} \varepsilon^{k-1} F_k, R_\varepsilon\right) + \varepsilon^2 \mathcal{B}(R_\varepsilon, R_\varepsilon)$$
$$+ \sum_{k+l \geqslant 7} \varepsilon^{k+l-4} \mathcal{B}(F_k, F_l) - \varepsilon^3 (\partial_t + v \cdot \nabla_x) F_6. \tag{9.6}$$

Indeed, for $k = 0, \ldots, 5$, the terms F_k are chosen as explained in Section 5.1, so that

$$(\partial_t + v \cdot \nabla_x) \sum_{k=0}^{5} \varepsilon^k F_k = \sum_{l+m \leqslant 6} \varepsilon^{l+m-1} \mathcal{B}(F_l, F_m).$$

Notice that, in (9.6), the nonlinear term is multiplied by an ε^2 factor: hence (9.6) is a weakly nonlinear equation. However, the linear term in (9.6) is

$$\frac{2}{\varepsilon} \mathcal{B}(F_0, R_\varepsilon)$$

at leading order, which defines a nonpositive operator in a weighted L^2 space in the v-variable. Specifically, the operator $R \mapsto \mathcal{B}(F_0, R)$ is self-adjoint in $L^2(\mathbb{R}^3; F_0^{-1} \, dv)$. A slight difficulty in this setting is that the weight F_0^{-1} depends on (t, x).

To avoid this, we use a slightly different definition of the linearized collision integral than in Section 3.6. Define

$$\mathbf{L}_M \phi = -2M^{-1/2} \mathcal{B}(M, M^{1/2} \phi),$$

where M is a Maxwellian density. As in Section 3.6, the operator $\mathbf{L}_M$ is split into

$$\mathcal{L}_M \phi(v) = a(|v|)\phi(v) - \mathbf{K}_M \phi(v),$$

where

$$a_M(|v|) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} |(v - v_*) \cdot \omega| M_* \, dv_* \, d\omega$$

and

$$\mathbf{K}_M \phi(v) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \left(\sqrt{M}\phi_* - \sqrt{M'_*}\phi' - \sqrt{M'}\phi'_*\right) |(v - v_*) \cdot \omega| \sqrt{M_*} \, dv_* \, d\omega.$$

The properties of $\mathbf{L}_M$ are summarized in the following theorem.

THEOREM 9.2. *The operator $\mathbf{K}_M$ is compact on $L^2(\mathbb{R}^3)$, while $a_M(|v|) \sim a^*|v|$ as $|v| \to +\infty$. Hence the operator $\mathbf{L}_M$ is an unbounded self-adjoint Fredholm operator on $L^2(\mathbb{R}^3)$ with domain $L^2(\mathbb{R}^3; (1 + |v|)\, dv)$ and nullspace*

$$\operatorname{Ker} \mathbf{L}_M = \operatorname{span}\left\{\sqrt{M}, \sqrt{M}v_1, \sqrt{M}v_2, \sqrt{M}v_3, \sqrt{M}|v|^2\right\}.$$

The advantage in using $\mathbf{L}_M$ instead of the operator $R \mapsto \mathcal{B}(F_0, R)$ is that the former operator is self-adjoint on the unweighted space $L^2(\mathbb{R}^3)$.

Formulating (9.6) in terms of the operator $\mathbf{L}_M$ leads to an equation of the form

$$(\partial_t + v \cdot \nabla_x)\left(E^{-1/2}R_\varepsilon\right) + \frac{1}{\varepsilon}\mathbf{L}_E\left(E^{-1/2}R_\varepsilon\right)$$

$$= \varepsilon^2 \mathbf{Q}_E\left(E^{-1/2}R_\varepsilon, E^{-1/2}R_\varepsilon\right) + 2E^{-1/2}\mathcal{B}\left(\sum_{k=1}^{6} \varepsilon^{k-1}F_k, R_\varepsilon\right)$$

$$+ \sum_{k+l \geqslant 7} \varepsilon^{k+l-4} E^{-1/2}\mathcal{B}(F_k, F_l) - \varepsilon^3 E^{-1/2}(\partial_t + v \cdot \nabla_x)F_6$$

$$+ \frac{1}{2}R_\varepsilon(\partial_t + v \cdot \nabla_x)\ln E, \tag{9.7}$$

where E is the local Maxwellian whose parameters solve (9.2) and the quadratic operator $\mathbf{Q}_M$ is defined by

$$\mathbf{Q}_M(\phi, \phi) = M^{-1/2}\mathcal{B}\left(\sqrt{M}\phi, \sqrt{M}\phi\right).$$

The last term on the right-hand side of (9.7) is the most annoying one, since it grows like

$$\frac{1}{2}|v|^3 \frac{|\nabla_x \theta|}{\theta^2} R_\varepsilon;$$

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in particular, it cannot be controlled by the damping part of the linearized collision integral, since $a_M(|v|) = O(|v|)$ as $|v| \to +\infty$.

To overcome this difficulty, R. Caflisch introduced a new Maxwellian state $\mathbf{M}$ defined by

$$\mathbf{M}(v) = \mathcal{M}_{(1,0,\hat{\theta})}(v) = \frac{1}{(2\pi\hat{\theta})^{3/2}} e^{-|v|^2/2\hat{\theta}},$$

where

$$\hat{\theta} = 2\|\theta\|_{L^\infty}.$$

Hence, there exists a constant C, that depends on $\|\rho\|_{L^\infty}$, $\|u\|_{L^\infty}$ and $\|1/\theta\|_{L^\infty}$ such that

$$E \leqslant C\mathbf{M}.$$

Next, one decomposes R_ε as

$$R_\varepsilon = \sqrt{E} r_\varepsilon + \sqrt{\mathbf{M}} q_\varepsilon,$$

where the new unknowns r_ε and q_ε are governed by the coupled system

$$(\partial_t + v \cdot \nabla_x) r_\varepsilon = -\frac{1}{\varepsilon} \mathbf{L}_E r_\varepsilon + \mathbb{1}_{|v| \leqslant c} \frac{1}{\varepsilon} \sigma \mathbf{K_M} q_\varepsilon,$$

$$(\partial_t + v \cdot \nabla_x) q_\varepsilon = -\mathbf{M}^{-1/2} r_\varepsilon (\partial_t + v \cdot \nabla_x) E^{1/2} - \frac{1}{\varepsilon}(a_\mathbf{M} - \mathbb{1}_{|v|>c} \mathbf{K_M}) q_\varepsilon$$

$$+ 2\mathbf{M}^{-1/2} \mathcal{B}\left(\sum_{k=1}^{6} \varepsilon^{k-1} F_k, \mathbf{M}^{1/2}(\sigma r_\varepsilon + q_\varepsilon) \right) \tag{9.8}$$

$$+ 2\mathbf{Q_M}(\sigma r_\varepsilon + q_\varepsilon, \sigma r_\varepsilon + q_\varepsilon) + \varepsilon^2 s,$$

where

$$\sigma = \sqrt{\frac{E}{\mathbf{M}}},$$

where c is a truncation parameter to be chosen below, and the source term s is

$$s = \mathbf{M}^{-1/2} \sum_{k+l \geqslant 7} \varepsilon^{k+l-6} \mathcal{B}(F_k, F_l) - \varepsilon \mathbf{M}^{-1/2}(\partial_t + v \cdot \nabla_x) F_6.$$

This system is solved for the initial condition

$$r_\varepsilon|_{t=0} = q_\varepsilon|_{t=0} = 0$$

by a fixed point argument in the norm

$$\|\phi\|_{r,s} = \sup_{\xi \in \mathbb{R}^3} \left(1 + |\xi|\right)^r \left\| f(\cdot, \xi) \right\|_{H^s(\mathbb{R}^3)}$$

for $s > 3/2$. We refer to [22] for the complete proof.

REMARKS. Several remarks are in order.

1. The construction in Caflisch's theorem leads to a solution of the Boltzmann equation that exists and approximates the Maxwellian built on the solution of the compressible Euler system for as long as the solution of that system exists and remains smooth. This is very satisfying: should there be a blow-up in finite time in the solution of the Boltzmann equation, it cannot happen before the onset of singularities in the Euler system.

2. There is however a rather unpleasant feature in Caflisch's construction: the solution of the Boltzmann equation so constructed is, in general, not everywhere nonnegative, and therefore loses physical meaning. This is most easily seen on the initial data for F_ε in the form (9.4): in Caflisch's paper, $R_\varepsilon|_{t=0} = 0$, so that, in the particular case of Maxwell's molecules, $F_\varepsilon|_{t=0}$ is a polynomial in v that is not (at least in general) everywhere nonnegative. It could be that an improvement of Caflisch's ansatz with initial layers as in [73] helps avoiding this; however, there is no mention of this difficulty in either [22] or [73].

3. The interested reader is invited to compare Caflisch's theorem with an earlier result by Nishida [103], who obtained the compressible Euler limit of the Boltzmann equation by an abstract Cauchy–Kowalewski argument (in the style of Nirenberg and Ovsyannikov, see, for instance, [102]). Nishida's result is as follows: consider the scaled Boltzmann equation (9.1) with an initial data F^{in} analytic in x with enough decay in v that is a perturbation of some absolute Maxwellian. Then, there exists a family of solutions of the scaled Boltzmann parametrized by $\varepsilon > 0$ that lives on some interval of time independent of $\varepsilon > 0$. This family of solutions converges to the Maxwellian built on an analytic solution of the compressible Euler system in the vanishing ε limit. However, the lifespan of Nishida's family of solutions of the scaled Boltzmann equation is not known to coincide with the blow-up time of the limiting smooth solution of the compressible Euler system. See also [118] for an improved variant of Nishida's result.

10. The relative entropy method: Application to the incompressible Euler limit

As explained in Section 9, all methods based on asymptotic expansions in the Knudsen number apply only to situations where the solutions of both the Boltzmann equation and the limiting hydrodynamic equation are smooth. The present section introduces a new method for cases where only the solution of the target equation (i.e., the hydrodynamic model) is smooth.

We shall explain how this method applies to the incompressible Euler limit of the Boltzmann equation. Consider the Boltzmann equation in the incompressible Euler scaling

$$\varepsilon \, \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon^q} \mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3,$$

$$F_\varepsilon|_{t=0} = \mathcal{M}(1, \varepsilon u^{\text{in}}, 1), \tag{10.1}$$

where $q > 1$ and $u^{\text{in}} \equiv u^{\text{in}}(x)$ is a divergence-free vector field on $\mathbb{T}^3$. Here, the collision kernel b is supposed to satisfy assumption (3.55) (i.e., to come from a hard cut-off potential) as well as the additional condition

$$\inf_{(z, \omega) \in \mathbb{R}^3 \times \mathbb{S}^2} \frac{b(z, \omega)}{|(z/|z|) \cdot \omega|} > 0. \tag{10.2}$$

Throughout this section, we denote by M the centered reduced Gaussian distribution in v, i.e.,

$$M(v) = \mathcal{M}_{(1,0,1)}(v) = \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}.$$

Next, we introduce a new concept of limit that is especially well adapted to all incompressible hydrodynamic limits of the Boltzmann equation.

DEFINITION 10.1 (Bardos, Golse and Levermore [10]). A family $g_\varepsilon \equiv g_\varepsilon(x, v)$ of $L^1_{\text{loc}}(\mathbb{T}^3 \times \mathbb{R}^3; M \, dx \, dv)$ is said to converge entropically to g at rate ε if

- $1 + \varepsilon g_\varepsilon \geqslant 0$ a.e. on $\mathbb{T}^3 \times \mathbb{R}^3$ for each $\varepsilon > 0$,
- $g_\varepsilon \to g$ weakly in $L^1_{\text{loc}}(\mathbb{T}^3 \times \mathbb{R}^3; M \, dx \, dv)$,
- and

$$\frac{1}{\varepsilon^2} H\big(M(1 + \varepsilon g_\varepsilon)\big|M\big) \to \frac{1}{2} \iint_{\mathbb{T}^3 \times \mathbb{R}^3} g(x, v)^2 \, dx \, dv$$

as $\varepsilon \to 0$.

After these preliminaries, we can state the incompressible Euler limit theorem.

THEOREM 10.2 (Saint-Raymond [110]). *Assume that* $u^{\text{in}} \in H^3(\mathbb{T}^3)$ *is a divergence-free vector field, and let* u *be the maximal solution of the incompressible Euler equations*

$$\partial_t u + u \cdot \nabla_x u + \nabla_x p = 0, \qquad \text{div}_x u = 0,$$

$$u|_{t=0} = u^{\text{in}}$$

on $[0, T) \times \mathbb{T}^3$. *For* $\varepsilon > 0$, *let* F_ε *be a renormalized solution relative to* M *of the scaled Boltzmann equation* (10.1). *Then, for each* $t \in [0, T)$,

$$\frac{F_\varepsilon(t, x, v) - M(v)}{\varepsilon M(v)} \to u(t, x) \cdot v$$

entropically at rate ε as $\varepsilon \to 0$.

The relative entropy method was used for the first time to derive incompressible hydrodynamic models from the Boltzmann equation in Chapter 2 of [18], [54] and in [88] – however, the results obtained in these references were incomplete since the proofs used additional controls not known to be satisfied by renormalized solutions of the Boltzmann equation.

Of course, if u is a 2D–3C solution of the incompressible Euler equations (see Section 7.5), then $T = +\infty$ and the incompressible Euler limit is global.

Let us now describe the main ideas in the relative entropy method.

First, assuming that F_ε is a classical solution to the scaled Boltzmann equation (10.1), we compute

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{1}{\varepsilon^2} H(F_\varepsilon | \mathcal{M}_{(1,\varepsilon u,1)}) = -\frac{1}{\varepsilon^2} \int_{\mathbb{T}^3} D(u) : \int_{\mathbb{R}^3} (v - \varepsilon u)^{\otimes 2} F_\varepsilon \,\mathrm{d}v\,\mathrm{d}x$$

$$+ \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \nabla_x p \cdot \int_{\mathbb{R}^3} (v - \varepsilon u) F_\varepsilon \,\mathrm{d}v\,\mathrm{d}x.$$

Now, this identity is not known to be true if F_ε is a renormalized solution. What is known instead is the following variant of it (see Theorem 8.10); for each $t \in [0, T)$, one has

$$\frac{1}{\varepsilon^2} H(F_\varepsilon | \mathcal{M}_{(1,\varepsilon u,1)})(t) + \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \mathrm{trace}\big(m_\varepsilon(t)\big)$$

$$\leqslant -\frac{1}{\varepsilon^2} \int_0^t \int_{\mathbb{T}^3} D(u) : \int_{\mathbb{R}^3} (v - \varepsilon u)^{\otimes 2} F_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \qquad (10.3)$$

$$+ \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^3} \nabla_x p \cdot \int_{\mathbb{R}^3} (v - \varepsilon u) F_\varepsilon \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s - \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^3} D(u) : m_\varepsilon(s) \,\mathrm{d}s.$$

The key argument is the following lemma.

LEMMA 10.3. *For each $T' \in [0, T)$ there exists $C_{T'} \geqslant 1$ such that*

$$\frac{1}{\varepsilon^2} \int_0^t \iint_{\mathbb{T}^3 \times \mathbb{R}^3} \big| D(u) : (v - \varepsilon u)^{\otimes 2} F_\varepsilon \big| \,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s$$

$$\leqslant \frac{C_{T'}}{\varepsilon^2} \int_0^t \big\| D(u) \big\|_{L^\infty} H(F_\varepsilon | \mathcal{M}_{(1,\varepsilon u,1)})(s) \,\mathrm{d}s + \mathrm{o}(1)$$

uniformly in $t \in [0, T']$ as $\varepsilon \to 0$.

Define

$$X_\varepsilon(t) = \frac{1}{\varepsilon^2} H(F_\varepsilon | \mathcal{M}_{(1,\varepsilon u,1)})(t) + \frac{1}{\varepsilon} \int_{\mathbb{T}^3} \mathrm{trace}\big(m_\varepsilon(t)\big);$$

it follows from (10.3) and Lemma 10.3 that, for each $T' \in [0, T)$ and each $t \in [0, T']$, one has

$$X_\varepsilon(t) \leqslant C_{T'} \big\| D(u) \big\|_{L^\infty([0,T']\times\mathbb{T}^3)} \int_0^t X_\varepsilon(s)\,\mathrm{d}s$$

$$+ \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^3} \nabla_x p \cdot \int_{\mathbb{R}^3} (v - \varepsilon u) F_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s + \mathrm{o}(1)_{L^\infty([0,T'])}.$$

Hence

$$X_\varepsilon(t) \leqslant \mathrm{e}^{C_{T'}\|D(u)\|_{L^\infty([0,T']\times\mathbb{T}^3)}} \left| \frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^3} \nabla_x p \cdot \int_{\mathbb{R}^3} (v - \varepsilon u) F_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \right|$$

$$+ \mathrm{o}(1)_{L^\infty([0,T'])}. \tag{10.4}$$

With this inequality, one concludes as follows. The sequence of fluctuations

$$\frac{1}{\varepsilon}(F_\varepsilon - M) \text{ is relatively compact in } w^* - L^\infty\big(\mathbb{R}_+; w - L^1\big((1 + |v|^2)\,\mathrm{d}x\,\mathrm{d}v\big)\big)$$

so that, modulo extraction of a subsequence

$$\int_{\mathbb{R}^3} F_\varepsilon\,\mathrm{d}v \to 1 \qquad \text{in } L^\infty\big(\mathbb{R}_+; L^1\big(\mathbb{T}^3\big)\big)$$

and

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^3} v F_\varepsilon\,\mathrm{d}v \to U \quad \text{in } w^* - L^\infty\big(\mathbb{R}_+; w - L^1\big(\mathbb{T}^3\big)\big).$$

Because of the local conservation of mass that is satisfied by the renormalized solution F_ε, one has

$$\mathrm{div}_x\, U = 0.$$

Hence

$$\frac{1}{\varepsilon} \int_0^t \int_{\mathbb{T}^3} \nabla_x p \cdot \int_{\mathbb{R}^3} (v - \varepsilon u) F_\varepsilon\,\mathrm{d}v\,\mathrm{d}x\,\mathrm{d}s \to \int_0^t \int_{\mathbb{T}^3} \nabla_x p \cdot (U - u)\,\mathrm{d}x\,\mathrm{d}s = 0$$

so that

$$X_\varepsilon(t) \to 0 \quad \text{as } \varepsilon \to 0 \text{ for each } t \in \big[0, T'\big].$$

By convexity and weak limit, one has

$$\frac{1}{2} \big\| (U - u)(t) \big\|_{L^2(\mathbb{T}^3)}^2 \leqslant \varliminf_{\varepsilon \to 0} \frac{1}{\varepsilon^2} H(F_\varepsilon | \mathcal{M}_{(1,\varepsilon u,1)})(t)$$

for each $t \in [0, T']$. Therefore, passing to the limit in (10.4) shows that $U = u$, as announced.

We shall not describe the proof of Lemma 10.3, which is really technical. In this proof, one has to master the difficulties created by the high velocity tails of the number density; this is done by using decay estimates due to Grad and Caflisch [23] on the gain term of the linearized collision operator, we refer to [110] for a complete proof.

But the difficulties in the proof of Lemma 10.3 are special to the Boltzmann equation; the main line of the relative entropy method is as described above. This method is due to Yau [122] who used it for the hydrodynamic limit of Ginzburg–Landau models.

11. Applications of the moment method

The moment method is based on compactness results which are used to pass to the limit in the local conservation laws of mass momentum and energy as the Knudsen number vanishes. This method does not require estimates other than the natural bounds on mass, energy, entropy and entropy production. On principle, it could therefore be used when both the solutions of the scaled Boltzmann equation and of the limiting hydrodynamic equations are not known to be regular.

First, we state the various theorems on hydrodynamic limits that can be proved in this way.

11.1. *The acoustic limit*

We start from the Boltzmann equation in the acoustic scaling posed in the periodic box

$$\partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} \mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3,$$

$$F_\varepsilon|_{t=0} = F_\varepsilon^{\text{in}}. \tag{11.1}$$

Assume that b satisfies the weak cut-off assumption (8.1) as well as the bound

$$0 < \int_{\mathbb{S}^2} b(z, \omega) \, d\omega \leqslant C_b \big(1 + |z|^2\big)^\beta \quad \text{a.e., } z \in \mathbb{R}^3, \tag{11.2}$$

for some $\beta \in [0, 1]$.

We assume that

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} F_\varepsilon^{\text{in}} \, dx \, dv = 1,$$

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} v F_\varepsilon^{\text{in}} \, dx \, dv = 0, \tag{11.3}$$

$$\iint_{\mathbb{T}^3 \times \mathbb{R}^3} \frac{1}{2} |v|^2 F_\varepsilon^{\text{in}} \, dx \, dv = \frac{3}{2}.$$

Set M to be the centered, reduced Gaussian distribution

$$M(v) = \frac{1}{(2\pi)^{3/2}} e^{-|v|^2/2}, \tag{11.4}$$

so that

$$\int_{\mathbb{R}^3} M \, dx \, dv = 1,$$

$$\int_{\mathbb{R}^3} v M \, dx \, dv = 0,$$

$$\int_{\mathbb{R}^3} \frac{1}{2} |v|^2 M \, dx \, dv = \frac{3}{2}.$$

The acoustic limit of the Boltzmann equation (11.1) is given by the following theorem.

THEOREM 11.1 (Golse and Levermore [53]). *Let $\delta_\varepsilon > 0$ be such that*

$$\delta_\varepsilon \to 0 \quad and \quad \delta_\varepsilon |\ln \delta_\varepsilon|^{\beta/2} = o(\varepsilon^{1/2})$$

as $\varepsilon \to 0$. Assume that

$$\frac{F_\varepsilon^{\text{in}}(x, v) - M}{\delta_\varepsilon M} \to \rho^{\text{in}}(x) + u^{\text{in}}(x) \cdot v + \theta^{\text{in}}(x) \frac{1}{2}(|v|^2 - 3)$$

entropically at rate δ_ε. For each $\varepsilon > 0$, let F_ε be a renormalized solution relative to M of the scaled Boltzmann equation (11.1) with initial data $F_\varepsilon^{\text{in}}$.
  Then, for each $t \geqslant 0$, the family

$$\frac{F_\varepsilon(t, x, v) - M}{\delta_\varepsilon M} \to \rho(t, x) + u(t, x) \cdot v + \theta(t, x) \frac{1}{2}(|v|^2 - 3)$$

entropically at rate δ_ε, where (ρ, u, θ) is a solution of the acoustic system

$$\partial_t \rho + \text{div}_x u = 0,$$

$$\partial_t u + \nabla_x (\rho + \theta) = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{T}^3,$$

$$\frac{3}{2} \partial_t \theta + \text{div}_x u = 0,$$

with initial data

$$(\rho, u, \theta)|_{t=0} = (\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}).$$

Notice that our assumptions on the family of initial data imply that

$$\left(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}\right) \in L^2\left(\mathbb{T}^3\right),$$

and that

$$\int_{\mathbb{T}^3} \rho^{\text{in}}(x)\, dx = 0, \qquad \int_{\mathbb{T}^3} u^{\text{in}}(x)\, dx = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} \theta^{\text{in}}(x)\, dx = 0.$$

In particular, the existence and uniqueness theory for the acoustic system described in Section 7.1.2 applies here, and the solution (ρ, u, θ) satisfies

$$(\rho, u, \theta) \in C\left(\mathbb{R}_+; L^2\left(\mathbb{T}^3\right)\right)$$

and

$$\int_{\mathbb{T}^3} \rho^{\text{in}}(x)\, dx = 0, \qquad \int_{\mathbb{T}^3} u^{\text{in}}(x)\, dx = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} \theta^{\text{in}}(x)\, dx = 0.$$

11.2. *The Stokes–Fourier limit*

We start from the Boltzmann equation in the Stokes scaling posed in the periodic box

$$\varepsilon\, \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} \mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{T}^3 \times \mathbb{R}^3,$$

$$F_\varepsilon|_{t=0} = F_\varepsilon^{\text{in}}. \tag{11.5}$$

Assume that the collision kernel b comes from a hard cut-off potential, i.e., that it satisfies (5.2) for some $\alpha \in [0, 1]$. Assume further that, for each $\varepsilon > 0$, the initial data $F_\varepsilon^{\text{in}}$ satisfies the relations (11.3).

THEOREM 11.2 (Golse and Levermore [53]). *Let $\delta_\varepsilon > 0$ be such that*

$$\delta_\varepsilon \to 0 \quad \text{and} \quad \delta_\varepsilon |\ln \delta_\varepsilon|^\alpha = \text{o}(\varepsilon)$$

as $\varepsilon \to 0$. Assume that

$$\frac{F_\varepsilon^{\text{in}}(x, v) - M}{\delta_\varepsilon M} \to u^{\text{in}}(x) \cdot v + \theta^{\text{in}}(x) \frac{1}{2}\left(|v|^2 - 5\right)$$

entropically at rate δ_ε, where u^{in} satisfies

$$\text{div}_x\, u^{\text{in}} = 0$$

and where M is the centered, reduced Gaussian distribution (11.4). *For each $\varepsilon > 0$, let F_ε be a renormalized solution relative to M of the scaled Boltzmann equation* (11.5) *with initial data $F_\varepsilon^{\text{in}}$.*

Then, for each $t \geqslant 0$, the family

$$\frac{F_\varepsilon(t, x, v) - M}{\delta_\varepsilon M} \to u(t, x) \cdot v + \theta(t, x)\frac{1}{2}\bigl(|v|^2 - 5\bigr)$$

entropically at rate δ_ε, where (u, θ) is a solution of the Stokes–Fourier system

$$\partial_t u + \nabla_x p = \nu \Delta_x u, \qquad \text{div}_x u = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{T}^3,$$

$$\partial_t \theta = \kappa \Delta_x \theta,$$

with initial data

$$(u, \theta)|_{t=0} = \bigl(u^{\text{in}}, \theta^{\text{in}}\bigr).$$

The viscosity and heat conductivity are given by

$$\nu = \frac{1}{10} \int_{\mathbb{R}^3} \widehat{A} : AM \, dv, \qquad \kappa = \frac{2}{15} \int_{\mathbb{R}^3} \widehat{B} \cdot BM \, dv,$$

see also formula (6.9) *for expressions of these quantities in terms of the functions* **a** *and* **b** *defined in* (6.12).

We recall that $\widehat{A}$ and $\widehat{B}$ are defined in terms of

$$A(v) = v \otimes v - \frac{1}{3}|v|^2 I, \qquad B = \frac{1}{2}\bigl(|v|^2 - 5\bigr)v$$

by

$$\mathcal{L}_M \widehat{A} = A, \qquad \widehat{A} \perp \text{Ker}\,\mathcal{L}_M,$$

$$\mathcal{L}_M \widehat{B} = B, \qquad \widehat{B} \perp \text{Ker}\,\mathcal{L}_M.$$

Our assumptions on the family of initial data imply that

$$\bigl(u^{\text{in}}, \theta^{\text{in}}\bigr) \in L^2\bigl(\mathbb{T}^3\bigr),$$

while

$$\int_{\mathbb{T}^3} u^{\text{in}}(x) \, dx = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} \theta^{\text{in}}(x) \, dx = 0.$$

Hence, the existence and uniqueness theory for the Stokes–Fourier system described in Section 7.1.1 applies here, and the solution (u, θ) satisfies

$$(u, \theta) \in C\big(\mathbb{R}_+; L^2\big(\mathbb{T}^3\big)\big) \cap C^\infty\big(\mathbb{R}_+^* \times \mathbb{R}^3\big)$$

and

$$\int_{\mathbb{T}^3} u^{\mathrm{in}}(x) \, \mathrm{d}x = 0 \quad \text{and} \quad \int_{\mathbb{T}^3} \theta^{\mathrm{in}}(x) \, \mathrm{d}x = 0.$$

11.3. *The Navier–Stokes–Fourier limit*

We start from the Boltzmann equation in the Navier–Stokes scaling, posed in the Euclidean space, with Maxwellian equilibrium at infinity

$$\varepsilon \, \partial_t F_\varepsilon + v \cdot \nabla_x F_\varepsilon = \frac{1}{\varepsilon} \mathcal{B}(F_\varepsilon, F_\varepsilon), \quad (t, x, v) \in \mathbb{R}_+^* \times \mathbb{R}^3 \times \mathbb{R}^3,$$

$$F_\varepsilon(t, x, v) \to M \quad |x| \to +\infty, \tag{11.6}$$

$$F_\varepsilon|_{t=0} = F_\varepsilon^{\mathrm{in}},$$

where M is the centered, reduced Gaussian distribution (11.4).

Assume that the collision kernel b comes from a hard cut-off potential, i.e., that it satisfies (5.2) for some $\alpha \in [0, 1]$.

THEOREM 11.3 (Golse and Saint-Raymond [61,62]). *Assume that*

$$\frac{F_\varepsilon^{\mathrm{in}}(x, v) - M}{\varepsilon M} \to u^{\mathrm{in}}(x) \cdot v + \theta^{\mathrm{in}}(x) \frac{1}{2}\big(|v|^2 - 5\big)$$

entropically at rate ε, where u^{in} satisfies

$$\operatorname{div}_x u^{\mathrm{in}} = 0.$$

For each $\varepsilon > 0$, let F_ε be a renormalized solution relative to M of the scaled Boltzmann equation (11.5) *with initial data $F_\varepsilon^{\mathrm{in}}$.*

Then the family

$$\left(\frac{1}{\varepsilon} \int_{\mathbb{R}^3} v F_\varepsilon(t, x, v) \, \mathrm{d}v, \frac{1}{\varepsilon} \int_{\mathbb{R}^3} \left(\frac{1}{3}|v|^2 - 1\right)\big(F_\varepsilon(t, x, v) - M\big) \, \mathrm{d}v\right)$$

is weakly relatively compact in $L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^3)$, and each of its limit points as $\varepsilon \to 0$ is

$$(u, \theta) \in C\big(\mathbb{R}_+, w - L^2\big(\mathbb{R}^3\big)\big) \cap L^2\big(\mathbb{R}_+; H^1\big(\mathbb{R}^3\big)\big),$$

a solution of the Navier–Stokes–Fourier system

$$\partial_t u + \mathrm{div}_x(u \otimes u) + \nabla_x p = \nu \Delta_x u, \qquad \mathrm{div}_x u = 0, \quad (t, x) \in \mathbb{R}_+^* \times \mathbb{T}^3,$$

$$\partial_t \theta + \mathrm{div}_x(u\theta) = \kappa \Delta_x \theta,$$

with initial data

$$(u, \theta)|_{t=0} = (u^{\mathrm{in}}, \theta^{\mathrm{in}}).$$

The viscosity and heat conductivity are given by

$$\nu = \frac{1}{10} \int_{\mathbb{R}^3} \widehat{A} : AM \, \mathrm{d}v, \qquad \kappa = \frac{2}{15} \int_{\mathbb{R}^3} \widehat{B} \cdot BM \, \mathrm{d}v$$

again, *see* (6.9) *for expressions of these quantities in terms of the functions* **a** *and* **b** *defined in* (6.12). *Moreover, this solution* (u, θ) *satisfies, for each* $t > 0$, *the inequality*

$$\frac{1}{2} \|u(t)\|_{L^2}^2 + \frac{5}{4} \|\theta(t)\|_{L^2}^2 + \int_0^t \int_{\mathbb{R}^3} (\nu |\nabla_x u|^2 + \kappa |\nabla_x \theta|^2) \, \mathrm{d}x \, \mathrm{d}s$$

$$\leqslant \frac{1}{2} \|u^{\mathrm{in}}\|_{L^2}^2 + \frac{5}{4} \|\theta^{\mathrm{in}}\|_{L^2}^2.$$

In particular, if $\theta^{\mathrm{in}} = 0$, this theorem shows that any weak limit point of

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^3} v F_\varepsilon(t, x, v) \, \mathrm{d}v$$

in $L^1_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^3)$ as $\varepsilon \to 0$ is a Leray solution of the Navier–Stokes equations with initial data u^{in}.

This theorem explains the following observation by Lions: "[...] the global existence of [renormalized] solutions [...] can be seen as the analogue for Boltzmann's equation to the pioneering work on the Navier–Stokes equations by J. Leray" (see [84], p. 432).

11.4. *Sketch of the proof of the Navier–Stokes–Fourier limit by the moment method*

The proof of the Navier–Stokes–Fourier limit theorem above involves many ideas developed in a sequence of papers over the past 15 years:

- the BGL program was defined in [10]; this reference provided the general entropy and entropy production estimates used to control the number density fluctuation and its distance to local equilibrium; as a result, the evolution Stokes and the steady Navier–Stokes motion equations were derived under the assumption that the renormalized solutions to the Boltzmann equation considered satisfy the local conservation of momentum as well as a nonlinear compactness estimate (for the Navier–Stokes limit) that will be described below in more details;

- under the same assumptions as in [10], Lions and Masmoudi [87] were able to derive the evolution Navier–Stokes motion equations, by a kind of "compensated compactness" argument bearing on fast oscillating acoustic waves; they also introduced a slightly modified notion of renormalized solution which led them to a complete derivation of the evolution Stokes motion equation;
- in [11], it was observed for the first time that the local conservation law of momentum could be *proved* in the hydrodynamic limit, thereby relieving the need for assuming that local conservation law at the level of the renormalized solutions of the Boltzmann equation; this led to a complete proof of the acoustic limit for bounded collision kernels; a more complete understanding of how the local conservation laws of both momentum and energy could be proved in the hydrodynamic limit was eventually reached in [53]; the latter reference provided an essentially optimal derivation of the Stokes motion and energy equations as well as a derivation of the acoustic system that allowed for the most general hard cut-off potentials; however, the acoustic system was established only under some unphysical restriction on the scaling of the number density fluctuation;
- in [107,109], Saint-Raymond gave a complete derivation of both the Navier–Stokes motion and energy equations for the BGK model with constant relaxation time; her proof was based on obtaining for the first time some weaker analogue of the nonlinear compactness assumption used in [10];
- finally, a complete derivation of the incompressible Navier–Stokes motion and heat equations from the Boltzmann equation was proposed for the first time in [61] for bounded collision kernels (such as occurring in the case of cut-off Maxwell molecules); this reference used all the methods constructed in the previous works mentioned above, together with a new velocity averaging method specific to the L^1 case and that amplified Saint-Raymond's observation in [107]; this result was later extended to all hard cut-off potentials (including hard spheres) in [62].

It is this last reference that we describe below; although its scope is more general than that of [61], it involves a new idea for handling unbounded collision kernels that actually simplifies the discussion in [61].

11.4.1. *A priori estimates.* In this subsection, we quickly list the a priori estimates on the family F_ε of solutions of the Boltzmann equation that are uniform in $\varepsilon > 0$. As can be seen from Theorem 8.10, the only such estimate comes from the DiPerna–Lions variant of Boltzmann's H-theorem,

$$
H\big(F_\varepsilon(t)\big|M\big) + \frac{1}{4\varepsilon^2} \int_0^t \int_{\mathbb{R}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \big(F_\varepsilon' F_{\varepsilon*}' - F_\varepsilon F_{\varepsilon*}\big)
$$

$$
\times \ln\bigg(\frac{F_\varepsilon' F_{\varepsilon*}'}{F_\varepsilon F_{\varepsilon*}}\bigg) b \, \mathrm{d}v \, \mathrm{d}v_* \, \mathrm{d}\omega \, \mathrm{d}x \, \mathrm{d}s
$$

$$
\leqslant H\big(F_\varepsilon^{\mathrm{in}}\big|M\big). \tag{11.7}
$$

We shall further transform (11.7) with the two following inequalities.

Pointwise inequalities.   For each ξ and $\eta \in \mathbb{R}_+^*$,

$$\left(\sqrt{\xi} - 1\right)^2 \leqslant \frac{1}{4}(\xi \ln \xi - \xi + 1) \tag{11.8}$$

while

$$\left(\sqrt{\xi} - \sqrt{\eta}\right)^2 \leqslant \frac{1}{4}(\xi - \eta)(\ln \xi - \ln \eta). \tag{11.9}$$

Since $(F_\varepsilon - M)/(\varepsilon M)$ converges entropically at rate ε, one has

$$H\left(F_\varepsilon^{\mathrm{in}} \middle| M\right) \leqslant C^{\mathrm{in}} \varepsilon^2. \tag{11.10}$$

This bound and the relative entropy inequality (11.7) entail the following entropy bound for each $t > 0$

$$H\left(F_\varepsilon(t) \middle| M\right) \leqslant C^{\mathrm{in}} \varepsilon^2 \tag{11.11}$$

and the following entropy production bounds

$$\int_0^{+\infty} \int_{\mathbb{R}^3} \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \left(F'_\varepsilon F'_{\varepsilon*} - F_\varepsilon F_{\varepsilon*}\right) \ln\left(\frac{F'_\varepsilon F'_{\varepsilon*}}{F_\varepsilon F_{\varepsilon*}}\right) b \, dv \, dv_* \, d\omega \, dx \, dt$$
$$\leqslant 4 C^{\mathrm{in}} \varepsilon^4. \tag{11.12}$$

Introducing the relative number density and relative number density fluctuations,

$$G_\varepsilon = \frac{F_\varepsilon}{M} \quad \text{and} \quad g_\varepsilon = \frac{F_\varepsilon - M}{\varepsilon M}, \tag{11.13}$$

the two pointwise inequalities (11.8) and (11.9) convert the entropy and entropy production bounds into the two uniform a priori estimates

$$\int_{\mathbb{R}^3} \left\langle \left(\sqrt{G_\varepsilon(t)} - 1\right)^2 \right\rangle dx \leqslant \frac{1}{4} C^{\mathrm{in}} \varepsilon^2 \tag{11.14}$$

and

$$\int_0^{+\infty} \int_{\mathbb{R}^3} \left\langle\!\left\langle \left(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\right)^2 \right\rangle\!\right\rangle dx \, dt \leqslant C^{\mathrm{in}} \varepsilon^4. \tag{11.15}$$

The importance of the two a priori bounds above (11.14) and (11.15) in the derivation of the Navier–Stokes limit cannot be overestimated. In fact, various analogues of these bounds were used earlier in the context of nonlinear diffusion limits, see [12] and especially [59].

The importance of the entropy and entropy dissipation bounds for hydrodynamic limits of diffusive type was noticed. We recall that

$$\langle \phi \rangle = \int_{\mathbb{R}^3} \phi(v) M(v) \, \mathrm{d}v,$$

and we further introduce the notation

$$\langle\langle \Phi \rangle\rangle = \iiint_{\mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{S}^2} \Phi(v, v_*, \omega) \, \mathrm{d}\mu(v, v_*, \omega),$$

where

$$\mathrm{d}\mu(v, v_*, \omega) = b(v - v_*, \omega) M(v) \, \mathrm{d}v \, M(v_*) \, \mathrm{d}v_* \, \mathrm{d}\omega.$$

In the sequel, we outline the main ideas in the derivation of the Navier–Stokes motion equation in Theorem 11.3, since the derivation of the heat equation is essentially analogous.

Of course, this proof more or less follows the formal argument presented in Section 6.1. However, several key properties of the solutions to the scaled Boltzmann equation used in this formal argument – such as, for instance, the local conservation laws of momentum and energy – are not known to be satisfied by renormalized solutions. Hence the proof sketched below differs noticeably from the formal argument in several places, yet the general idea remains essentially the same.

11.4.2. *Normalizing functions.* As explained in Section 8.2, the Boltzmann equation can be equivalently renormalized with any admissible nonlinearity whose derivative saturates the quadratic growth of the collision integral.

Throughout the proof of the Navier–Stokes limit theorem, we shall essentially use two kinds of normalizing nonlinearities

- compactly supported nonlinearities that coincide with the identity near the reference Maxwellian state, and
- variants of the maximal, i.e., square-root renormalization.

Nonlinearities of the first kind are used to define the renormalized form of the Boltzmann equation in which one passes to the vanishing ε limit, while the square-root normalization is used to establish compactness properties of the family of solutions to the scaled Boltzmann equation.

The first kind of normalizing nonlinearities is defined through the class of bump functions $\gamma \in C^\infty(\mathbb{R}_+)$ such that

$$\gamma|_{[0,3/2]} \equiv 1, \qquad \gamma|_{[2,+\infty)} \equiv 0, \qquad \gamma \text{ is nonincreasing on } \mathbb{R}_+. \tag{11.16}$$

The Boltzmann equation is then renormalized with the nonlinearity

$$\Gamma(Z) = (Z - 1)\gamma(Z);$$

later on, we denote

$$\hat{\gamma}(Z) = \frac{d}{dZ}\big((Z-1)\gamma(Z)\big) = \Gamma'(Z). \tag{11.17}$$

The scaled Boltzmann equation renormalized with Γ is put in the form

$$\partial_t(g_\varepsilon \gamma_\varepsilon) + \frac{1}{\varepsilon}v\cdot\nabla_x(g_\varepsilon \gamma_\varepsilon) = \frac{1}{\varepsilon^3}\hat{\gamma}_\varepsilon \mathcal{Q}_M(G_\varepsilon, G_\varepsilon), \tag{11.18}$$

where we have denoted

$$\gamma_\varepsilon = \gamma(G_\varepsilon), \qquad \hat{\gamma}_\varepsilon = \hat{\gamma}(G_\varepsilon),$$

and where $\mathcal{Q}$ designates the Boltzmann collision integral intertwined with the multiplication by M (see (6.6)),

$$\mathcal{Q}(G,G) = M^{-1}\mathcal{B}(MG, MG). \tag{11.17}$$

Later on, we shall pass to the limit in the momentum equation deduced from (11.18).

The second class of normalizing nonlinearities that we shall use to establish compactness properties of the number density fluctuations G_ε is defined as

$$\Gamma_\zeta(Z) = \sqrt{\zeta + Z}, \quad \zeta > 0,$$

where the parameter ζ will later be adapted to ε.

11.4.3. *Governing equations for moments of g_ε.* As explained in Theorem 8.10, renormalized solutions to the Boltzmann equation satisfy the local conservation of mass (i.e., the continuity equation); in terms of the number density fluctuation g_ε, this local conservation law is expressed as

$$\varepsilon\,\partial_t\langle g_\varepsilon\rangle + \mathrm{div}_x\langle vg_\varepsilon\rangle = 0. \tag{11.19}$$

Now, the entropy bound (11.11) implies that

$$\big(1 + |v|^2\big)g_\varepsilon \text{ is relatively compact in } w - L^1_{\mathrm{loc}}\big(dt\,dx; L^1(M\,dv)\big). \tag{11.20}$$

Before saying a few words on (11.20), let us explain how we use it. Modulo extraction of a subsequence, one has

$$g_\varepsilon \to g \quad \text{in } w - L^1_{\mathrm{loc}}\big(dt\,dx; L^1\big((1 + |v|^2)M\,dv\big)\big)$$

and hence

$$\langle g_\varepsilon\rangle \to \langle g\rangle \quad \text{and} \quad \langle vg_\varepsilon\rangle \to \langle vg\rangle \quad \text{in } w - L^1_{\mathrm{loc}}(dt\,dx).$$

Passing to the limit as $\varepsilon \to 0$ in (11.19) leads to

$$\operatorname{div}_x \langle vg \rangle = 0,$$

so that, denoting

$$u = \langle vg \rangle$$

the relation above is the incompressibility condition in the Navier–Stokes equations, i.e.,

$$\operatorname{div}_x u = 0.$$

Let us go back to (11.20) and explain how it follows from the entropy bound (11.11), see [10] for more details on this. Define

$$h(z) = (1 + z)\ln(1 + z) - z, \quad z > -1.$$

In terms of h, the entropy bound (11.11) is expressed as

$$\frac{1}{\varepsilon^2} \int_{\mathbb{R}^3} \langle h(\varepsilon g_\varepsilon(t)) \rangle \, \mathrm{d}x \leqslant C^{\mathrm{in}}.$$

If the entropy bound (11.11) was equivalent to an $L^p(M \, \mathrm{d}v \, \mathrm{d}x)$ bound for some $p > 1$, Hölder's inequality would imply that $(1 + |v|)^2 g_\varepsilon$ is bounded in $L^\infty(\mathrm{d}t; L^r(M \, \mathrm{d}v \, \mathrm{d}x))$ for some $r > 1$, since $(1 + |v|^2) \in L^q(M \, \mathrm{d}v)$ for each $q \in (1, +\infty)$. However, the entropy control (11.11) on g_ε is weaker than an $L^p(M \, \mathrm{d}v \, \mathrm{d}x)$ bound. But (11.20) follows from a careful use of Young's inequality

$$p|z| \leqslant \frac{\alpha}{\varepsilon^2} h(\varepsilon z) + \frac{1}{\alpha} h^*(p), \quad p > 0, z > -1, 0 < \varepsilon < \alpha,$$

where

$$h^*(p) = \mathrm{e}^p - p - 1$$

designates the Legendre dual of h. The compactness (11.20) follows from replacing z with g_ε and p with $\frac{1}{4}(1 + |v|^2)$ in the inequality above, letting then $\alpha \to 0$ in the inequality so obtained.

Let us now explain how the motion equation in the Navier–Stokes system is derived from the Boltzmann equation. This is of course the main part in the proof, and it involves several technicalities.

In particular, we shall need truncations in the velocity variable at a level that is tied to ε. For each function $\xi \equiv \xi(v)$ and each $K > 6$, we define

$$\xi_{K_\varepsilon}(v) = \xi(v)\mathbb{1}_{|v|^2 \leqslant K|\ln \varepsilon|}. \tag{11.21}$$

Multiplying each side of the scaled, renormalized Boltzmann equation (11.18) by each component of v_{K_ε} and averaging in v leads to

$$\partial_t \langle v g_\varepsilon \gamma_\varepsilon \rangle + \operatorname{div}_x \mathbf{F}_\varepsilon(A) + \nabla_x \frac{1}{\varepsilon} \left\langle \frac{1}{3} |v|^2_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \right\rangle = \mathbf{D}_\varepsilon(v), \tag{11.22}$$

where $\mathbf{F}_\varepsilon(A)$ is the truncated, renormalized traceless part of the momentum flux

$$\mathbf{F}_\varepsilon(A) = \frac{1}{\varepsilon} \langle A_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \tag{11.23}$$

while $\mathbf{D}_\varepsilon(v)$ is the momentum conservation defect

$$\mathbf{D}_\varepsilon(v) = \frac{1}{\varepsilon^3} \langle\!\langle v_{K_\varepsilon} \hat{\gamma}_\varepsilon \left(G'_\varepsilon G'_{\varepsilon*} - G_\varepsilon G_{\varepsilon*} \right) \rangle\!\rangle. \tag{11.24}$$

Notice that truncating large velocities in the number density, or large values thereof (which is what the renormalization procedure does) break the symmetries in the collision integral leading to the local conservation of momentum (see Proposition 3.1): this accounts for the defect $\mathbf{D}_\varepsilon(v)$ on the right-hand side of (11.22). As $\varepsilon \to 0$, $v_{K_\varepsilon} \to v$ while $G_\varepsilon \to 1$ so that $\hat{\gamma}_\varepsilon \to 1$; hence, the missing symmetries are restored in the integrand defining $\mathbf{D}_\varepsilon(v)$. Hence, one can hope that $\mathbf{D}_\varepsilon(v) \to 0$ as $\varepsilon \to 0$.

In fact, the strategy for establishing the Navier–Stokes limit theorem consists of the following three steps.

Step 1.   Prove that, modulo extraction of a subsequence

$$\langle v g_\varepsilon \gamma_\varepsilon \rangle \to \langle v g \rangle = u \quad \text{in } w - L^1_{\text{loc}}(\mathrm{d}t \, \mathrm{d}x),$$

while

$$P \langle v g_\varepsilon \gamma_\varepsilon \rangle \to u \quad \text{in } C\big(\mathbb{R}_+; \mathcal{D}'(\mathbb{R}^3)\big),$$

where P denotes the Leray projection, i.e., the orthogonal projection on divergence-free vector fields in $L^2(\mathbb{R}^3)$.

Step 2.   Likewise, prove that

$$\mathbf{D}_\varepsilon(v) \to 0 \quad \text{in } L^1_{\text{loc}}(\mathrm{d}t \, \mathrm{d}x).$$

Step 3.   Finally prove that

$$P\big(\operatorname{div}_x \mathbf{F}_\varepsilon(A)\big) \to P \operatorname{div}_x (u \otimes u) - \nu \Delta_x u \quad \text{in } \mathcal{D}'\big(\mathbb{R}^*_+ \times \mathbb{R}^3\big).$$

Once these three steps are completed, one applies P to both sides of (11.22), which gives

$$\partial_t P \langle v g_\varepsilon \gamma_\varepsilon \rangle + P\big(\operatorname{div}_x \mathbf{F}_\varepsilon(A)\big) = P \mathbf{D}_\varepsilon(v). \tag{11.25}$$

Taking limits in each term as $\varepsilon \to 0$ shows that u satisfies the Navier–Stokes motion equation. As for the initial condition, observe that it is guaranteed by the uniform convergence in t, i.e., by the second statement in Step 1.

11.4.4. *Vanishing of the momentum conservation defect.*   We start with Step 2, i.e., we explain how to prove the following proposition.

PROPOSITION 11.4. *Under the same assumptions as in Theorem* 11.3,

$$\mathbf{D}_\varepsilon(v) \to 0 \quad in \ L^1_{\mathrm{loc}}(\mathrm{d}t \, \mathrm{d}x).$$

First, we start from the elementary formula

$$G'_\varepsilon G'_{\varepsilon*} - G_\varepsilon G_{\varepsilon*}$$
$$= \left(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\right)\left(\sqrt{G'_\varepsilon G'_{\varepsilon*}} + \sqrt{G_\varepsilon G_{\varepsilon*}}\right)$$
$$= \left(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\right)^2 + 2\sqrt{G_\varepsilon G_{\varepsilon*}}\left(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\right)$$

and split the momentum conservation defect as

$$\mathbf{D}_\varepsilon(v) = \mathbf{D}^1_\varepsilon(v) + \mathbf{D}^2_\varepsilon(v)$$

with

$$\mathbf{D}^1_\varepsilon(v) = \frac{1}{\varepsilon^3}\left\langle\!\!\left\langle v_{K_\varepsilon} \hat{\gamma}_\varepsilon \left(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\right)^2\right\rangle\!\!\right\rangle$$

and

$$\mathbf{D}^2_\varepsilon(v) = \frac{2}{\varepsilon^3}\left\langle\!\!\left\langle v_{K_\varepsilon} \hat{\gamma}_\varepsilon \sqrt{G_\varepsilon G_{\varepsilon*}}\left(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\right)\right\rangle\!\!\right\rangle.$$

That $\mathbf{D}^1_\varepsilon(v) \to 0$ in $L^1_{\mathrm{loc}}(\mathrm{d}t \, \mathrm{d}x)$ follows from the entropy production estimate (11.15). Setting

$$\varXi_\varepsilon = \frac{1}{\varepsilon^2}\sqrt{G_\varepsilon G_{\varepsilon*}}\left(\sqrt{G'_\varepsilon G'_{\varepsilon*}} - \sqrt{G_\varepsilon G_{\varepsilon*}}\right)$$

we further split $\mathbf{D}^2_\varepsilon(v)$ as

$$\mathbf{D}^2_\varepsilon(v) = -\frac{2}{\varepsilon}\left\langle\!\!\left\langle v \mathbb{1}_{|v|^2 > K_\varepsilon} \hat{\gamma}_\varepsilon \varXi_\varepsilon\right\rangle\!\!\right\rangle + \frac{2}{\varepsilon}\left\langle\!\!\left\langle v \hat{\gamma}_\varepsilon\left(1 - \hat{\gamma}_{\varepsilon*}\hat{\gamma}'_\varepsilon\hat{\gamma}'_{\varepsilon*}\right)\varXi_\varepsilon\right\rangle\!\!\right\rangle$$
$$+ \frac{1}{\varepsilon}\left\langle\!\!\left\langle (v + v_*)\hat{\gamma}_\varepsilon\hat{\gamma}_{\varepsilon*}\hat{\gamma}'_\varepsilon\hat{\gamma}'_{\varepsilon*}\varXi_\varepsilon\right\rangle\!\!\right\rangle.$$

The first term is easily mastered by the entropy production estimate (11.15) and the following classical estimate on the tail of Gaussian integrals

$$\int_{\mathbb{R}^N} e^{-|v|^2/2}|v|^a \mathbb{1}_{|v|^2 > R}\, dv = O\big(R^{(a+N)/2-1}e^{-R/2}\big) \quad \text{as } R \to +\infty.$$

Observe that the integrand in the third term has the same symmetries as the original collision integrand (before truncation in $|v|$ and renormalization). It is also mastered by a combination of the entropy production estimate (11.15) with the Gaussian tail estimate above.

The most difficult part in the analysis of the momentum conservation defect is by far the second term in the decomposition of $\mathbf{D}_\varepsilon^2(v)$ above. That it vanishes in $L^1_{\text{loc}}(dt\, dx)$ as $\varepsilon \to 0$ ultimately relies upon the following estimate.

Nonlinear compactness estimate.

$$\big(1 + |v|\big)^\alpha \left(\frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}\right)^2 \text{ is uniformly integrable on } [0, T] \times K \times \mathbb{R}^3 \qquad (11.26)$$

for the measure $dt\, dx\, M\, dv$, for each $T > 0$ and each compact $K \subset \mathbb{R}^3$, where α is the relative velocity exponent that appears in the hard cut-off assumption (3.55) on the collision kernel b.

We shall not give further details on the proof that $\mathbf{D}_\varepsilon^2(v) \to 0$ in $L^1_{\text{loc}}(dt\, dx)$, which is based on the above nonlinear compactness estimate together with the entropy production bound (11.15).

Let us however say a few words on the nonlinear compactness estimate itself. The relative entropy bound (11.11) is essentially as good as an $L^\infty(dt; L^2(M\, dv\, dx))$ bound on g_ε *on the set of* (t, x, v)*'s such that* $G_\varepsilon(t, x, v) = O(1)$. Elsewhere, it essentially reduces to an $O(\varepsilon)$ bound in $L^\infty(dt; L^1(M\, dv\, dx))$, which is quite not enough for the Navier–Stokes limit. This is why the first works on this limit *assumed* some variant of this nonlinear compactness estimate. For instance, in either [10] or [87], it was assumed that

$$\big(1 + |v|^2\big)\frac{g_\varepsilon^2}{1 + G_\varepsilon} \text{ is uniformly integrable on } [0, T] \times K \times \mathbb{R}^3, \qquad (11.27)$$

whereas all that was known on this quantity was the estimate

$$\big(1 + |v|^2\big)\frac{g_\varepsilon^2}{1 + G_\varepsilon} = O\big(|\ln \varepsilon|\big) \quad \text{in } L^1_{\text{loc}}\big(dt\, dx; L^1(M\, dv)\big)$$

(see [10]). This led to a decomposition of the number density fluctuation as

$$g_\varepsilon = g_\varepsilon^\flat + \varepsilon g_\varepsilon^\sharp,$$

where the "good" part of the fluctuation is

$$g_\varepsilon^\flat = \frac{g_\varepsilon}{0.5 + 0.5 G_\varepsilon} = \mathrm{O}(1) \quad \text{in } L^\infty\big(\mathrm{d}t; L^2(M \,\mathrm{d}v \,\mathrm{d}x)\big),$$

while the "bad" part is

$$g_\varepsilon^\sharp = \frac{g_\varepsilon^2}{1 + G_\varepsilon} = \mathrm{O}(1) \quad \text{in } L^\infty\big(\mathrm{d}t; L^1(M \,\mathrm{d}v \,\mathrm{d}x)\big).$$

In later works – for instance in [107,109] and [61] – this decomposition was slightly modified as follows. Pick a bump function $\gamma \in C_c^\infty(\mathbb{R}_+^*)$ such that

$$\gamma|_{[\frac{3}{4}, \frac{5}{4}]} \equiv 1, \quad \operatorname{supp}(\gamma) \subset \left[\frac{1}{2}, \frac{3}{2}\right] \text{ and } 0 \leqslant \gamma \leqslant 1,$$

and define

$$g_\varepsilon^\flat = g_\varepsilon \gamma(G_\varepsilon), \qquad g_\varepsilon^\sharp = \frac{1 - \gamma(G_\varepsilon)}{\varepsilon} g_\varepsilon.$$

It was proved in [61] that

$$\big|g_\varepsilon^\flat\big|^2 \text{ is uniformly integrable on } [0, T] \times K \times \mathbb{R}^3 \tag{11.28}$$

for the measure $\mathrm{d}t \,\mathrm{d}x \, M \,\mathrm{d}v$, while

$$g_\varepsilon^\sharp = \mathrm{O}\!\left(\frac{1}{\ln|\ln\varepsilon|}\right) \quad \text{in } L_{\mathrm{loc}}^1\big(\mathrm{d}t \,\mathrm{d}x; L^1(M \,\mathrm{d}v)\big). \tag{11.29}$$

Observe the difference between these last two controls and (11.27): with the new definition of $g_\varepsilon^\flat$ and $g_\varepsilon^\sharp$, it is no longer true that $|g_\varepsilon^\flat|^2 \leqslant C g_\varepsilon^\sharp$, while $(\frac{g_\varepsilon}{1+G_\varepsilon})^2 \leqslant \frac{g_\varepsilon^2}{1+G_\varepsilon}$, so that (11.27) actually entailed that the square of the good part in the old flat-sharp decomposition is uniformly integrable, even with a quadratic weight in v. In fact, the techniques in [61] did not allow adding a quadratic weight in v as in (11.27), so that this compactness assumption remained unproved; fortunately, it was possible to complete the proof of the Navier–Stokes limit for cut-off Maxwell molecules with only the bounds (11.28)–(11.29), and the weighted estimate

$$\big(1 + |v|\big)^s\big(1 - \gamma(G_\varepsilon)\big) = \mathrm{O}\!\left(\frac{\varepsilon^2}{\sqrt{\ln|\ln\varepsilon|}}\right) \quad \text{in } L_{\mathrm{loc}}^1\big(\mathrm{d}t \,\mathrm{d}x; L^1(M \,\mathrm{d}v)\big). \tag{11.30}$$

This control shows that the set where the bad part of the number density fluctuation dominates is small in weighted v-space. There is a definite lack of symmetry between the controls (11.29), bearing on large values of g_ε, and (11.30), bearing on large $|v|$'s. This lack of symmetry is remedied in the most recent variant (11.26) of these nonlinear compactness estimates (see [62]), we shall return to this when sketching the proof of (11.26).

11.4.5. *The asymptotic momentum flux.* With the vanishing of conservation defects (Step 2 in the proof of the Navier–Stokes limit) settled in the previous section, we turn our attention to Step 3, i.e., passing to the limit in the divergence of the momentum flux modulo gradients. This is by far the most difficult part of our analysis, and does require several preparations. In the present subsection, we reduce the momentum flux to some asymptotic normal form, to which we eventually apply compactness results to be described later.

LEMMA 11.5. *Let Π be the $L^2(M\,dv)$-orthogonal projection on $\mathrm{Ker}\,\mathcal{L}$; then, under the same assumptions as in Theorem* 11.3,

$$\mathbf{F}_\varepsilon(A) = \left\langle A\left(\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle - 2\left\langle \widehat{A}\frac{1}{\varepsilon^2}\mathcal{Q}_M\big(\sqrt{G_\varepsilon},\sqrt{G_\varepsilon}\big)\right\rangle + \mathrm{o}(1)_{L^1_{\mathrm{loc}}(dt\,dx)},$$

where we recall that the tensor field $\widehat{A}$ is defined by

$$\widehat{A} \perp \mathrm{Ker}\,\mathcal{L}_M \quad and \quad \mathcal{L}_M\big(\widehat{A}\big) = A = v\otimes v - \frac{1}{3}|v|^2 I.$$

The proof of this lemma is based upon splitting the momentum flux as

$$\mathbf{F}_\varepsilon(A) = \frac{1}{\varepsilon}\left\langle A_{K_\varepsilon}\gamma_\varepsilon\frac{G_\varepsilon-1}{\varepsilon}\right\rangle$$

$$= \left\langle A_{K_\varepsilon}\gamma_\varepsilon\left(\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle + \frac{2}{\varepsilon}\left\langle A_{K_\varepsilon}\gamma_\varepsilon\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right\rangle$$

$$= \mathbf{F}^1_\varepsilon(A) + \mathbf{F}^2_\varepsilon(A),$$

as a consequence of the elementary identity

$$\frac{1}{\varepsilon}(G_\varepsilon-1) = \frac{1}{\varepsilon}\big(\sqrt{G_\varepsilon}-1\big)\big(\sqrt{G_\varepsilon}+1\big)$$

$$= \frac{1}{\varepsilon}\big(\sqrt{G_\varepsilon}-1\big)^2 + \frac{2}{\varepsilon}\big(\sqrt{G_\varepsilon}-1\big).$$

Then, one applies the following corollary of the nonlinear compactness estimate (11.26).

COROLLARY 11.6. *Under the same assumptions as in Theorem* 11.3,

$$\frac{\sqrt{G_\varepsilon}-1}{\varepsilon} - \Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon} \to 0 \quad in\ L^2\big(dt\,dx; L^2\big((1+|v|^\alpha)M\,dv\big)\big)$$

as $\varepsilon \to 0$.

With the corollary above, one can show that the term $\mathbf{F}_\varepsilon^1(A)$ in the decomposition of the momentum flux is asymptotically close to

$$\left\langle A\left(\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle$$

(notice that the high velocity truncation is disposed of since $\Pi(\sqrt{G_\varepsilon}-1)/\varepsilon$ has at most polynomial growth in v as $|v|\to+\infty$). That the second term $\mathbf{F}_\varepsilon^2(A)$ is close to

$$\left\langle\widehat{A}\frac{1}{\varepsilon^2}\mathcal{Q}_M\big(\sqrt{G_\varepsilon},\sqrt{G_\varepsilon}\big)\right\rangle$$

uses Lemma 6.2.

Next, we explain how Lemma 11.5 is used in the proof of the Navier–Stokes limit. To begin with, since

$$\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\simeq\frac{1}{2}g_\varepsilon\gamma_\varepsilon,$$

one has

$$\left\langle A\left(\Pi\frac{\sqrt{G_\varepsilon}-1}{\varepsilon}\right)^2\right\rangle\simeq\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\otimes\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle-\frac{1}{3}\big|\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\big|^2I.$$

On the other hand, the entropy production estimate (11.15) implies that, modulo extraction of a subsequence, one has

$$\frac{1}{\varepsilon^2}\big(\sqrt{G_\varepsilon'G_{\varepsilon*}'}-\sqrt{G_\varepsilon G_{\varepsilon*}}\big)\to q$$

in $w-L^2(\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}\mu)$. Passing to the limit in the scaled, renormalized Boltzmann equation (11.18) entails the relation

$$\iint_{\mathbb{R}^3\times\mathbb{S}^2}qb(v-v_*,\omega)M\,\mathrm{d}v_*\,\mathrm{d}\omega$$

$$=v\cdot\nabla_x g=\frac{1}{2}A:\nabla_x u+\text{terms that are odd in }v.$$

Eventually we arrive at the following asymptotic form of the momentum flux.

PROPOSITION 11.7. *Under the same assumptions as in Theorem* 11.3, *one has*

$$\mathbf{F}_\varepsilon(A)=\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\otimes\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle-\frac{1}{3}\big|\langle v_{K_\varepsilon}g_\varepsilon\gamma_\varepsilon\rangle\big|^2I$$

$$-\nu\big(\nabla_x u+(\nabla_x u)^T\big)+\mathrm{o}(1)_{L^1_{\mathrm{loc}}(\mathrm{d}t\,\mathrm{d}x)},$$

where

$$u = \langle vg \rangle \quad \text{and} \quad g = \lim_{\varepsilon \to 0} g_\varepsilon \quad \text{in } w - L^1_{\text{loc}}\big(\mathrm{d}t\,\mathrm{d}x; L^1(M\,\mathrm{d}v)\big).$$

11.4.6. *Strong compactness of* $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$. In order to pass to the limit in the quadratic term $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$ and to conclude that

$$P \operatorname{div}_x \left(\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle - \frac{1}{3} \big| \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \big|^2 I \right)$$

$$\to P \operatorname{div}_x \left(u \otimes u - \frac{1}{3} |u|^2 I \right)$$

in the sense of distributions on $\mathbb{R}^*_+ \times \mathbb{R}^3$ as $\varepsilon \to 0$, the weak convergence properties of g_ε established so far are clearly insufficient. One needs instead some *strong* compactness properties on the family $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$.

(a) *Strong compactness in the x-variable.* Velocity averaging is the natural way to obtain compactness in the space variable x for kinetic equations in the parabolic scaling (11.6).

For the purpose of studying the compactness of $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$ in the x-variable, we use the following variant of the L^2-based velocity averaging theorem.

LEMMA 11.8. *Let ϕ_ε be a bounded family in $L^2_{\text{loc}}(\mathrm{d}t\,\mathrm{d}x; L^2(M\,\mathrm{d}v))$ such that $|\phi_\varepsilon|^2$ is locally uniformly integrable on $\mathbb{R}^*_+ \times \mathbb{R}^3 \times \mathbb{R}^3$ for the Lebesgue measure. Assume that*

$$(\varepsilon\,\partial_t + v \cdot \nabla_x)\phi_\varepsilon \text{ is bounded in } L^1_{\text{loc}}(\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v).$$

Then, for each $\psi \in L^2(M\,\mathrm{d}v)$, the family $\langle \phi_\varepsilon \psi \rangle$ is relatively compact in $L^2_{\text{loc}}(\mathrm{d}t\,\mathrm{d}x)$ with respect to the x-variable, meaning that, for each $T > 0$ and each compact $K \subset \mathbb{R}^3$, one has

$$\iint_{[0,T] \times K} \big| \langle \phi_\varepsilon \psi \rangle(t, x+y) - \langle \phi_\varepsilon \psi \rangle(t, x) \big|^2 \, \mathrm{d}t\,\mathrm{d}x \to 0$$

as $y \to 0$ uniformly in ε.

See [61] for the proof, which is somewhat similar to the L^1 case of velocity averaging recalled in Section 8.2.2 (especially Proposition **??**).

Now, we apply the lemma above to

$$\phi_\varepsilon = \frac{\sqrt{\varepsilon^c + G_\varepsilon} - 1}{\varepsilon}$$

since

$$(\varepsilon\,\partial_t + v \cdot \nabla_x)\phi_\varepsilon = \frac{1}{\varepsilon^2} \frac{\mathcal{Q}(G_\varepsilon, G_\varepsilon)}{2\sqrt{\varepsilon^c + G_\varepsilon}} = \mathrm{O}(1)_{L^1_{\text{loc}}(\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}v)}$$

for $c \in (1, 2)$, by the entropy production estimate (11.15). Since

$$\frac{\sqrt{\varepsilon^c + G_\varepsilon} - 1}{\varepsilon} \simeq \frac{1}{2} g_\varepsilon \gamma_\varepsilon,$$

applying the velocity averaging lemma above leads to the following compactness ("in the x-variable") result.

PROPOSITION 11.9. *Under the same assumptions as in Theorem* 11.3, *for each $T > 0$ and $K \subset \mathbb{R}^3$ compact, one has*

$$\iint_{[0,T] \times K} \left| \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle (t, x + y) - \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle (t, x) \right|^2 dt\, dx \to 0$$

uniformly in ε as $y \to 0$.

(b) *Strong compactness in the t-variable.*   It remains to obtain compactness in the time variable. As we shall see, the solenoidal part of $\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$ is strongly compact in the t-variable, but its orthogonal complement, which is a gradient field, is not because of high frequency oscillations in t.

PROPOSITION 11.10. *Under the assumptions of Theorem* 11.3, *modulo extraction of a subsequence, one has*

$$P \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \to u$$

in $C(\mathbb{R}_+; w - L_x^2)$ and in $L^2_{\text{loc}}(dt\, dx)$ as $\varepsilon \to 0$.

PROOF. Indeed, Proposition 11.9 and the translation invariance of the Leray projection P imply that

$$\iint_{[0,T] \times K} \left| P \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle (t, x + y) - P \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle (t, x) \right|^2 dt\, dx \to 0 \qquad (11.31)$$

uniformly in ε as $y \to 0$. On the other hand, the conservation law (11.25) implies that

$$\partial_t \int_{\mathbb{R}^3} P \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \cdot \xi\, dx = O(1) \quad \text{in } L^1_{\text{loc}}(dt), \qquad (11.32)$$

for each compactly supported, solenoidal vector field $\xi \in H^3(\mathbb{R}^3)$, since we know from Lemma 11.5 and the bounds (11.14) and (11.15), that $\mathbf{F}_\varepsilon(A)$ is bounded in $L^1_{\text{loc}}(dt\, dx)$. Also,

$$g_\varepsilon \gamma_\varepsilon \leqslant (1 + \sqrt{2}) \frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}$$

so that (11.14) implies that

$$\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle = \mathrm{O}(1) \quad \text{in } B\big(\mathbb{R}_+; L^2(\mathbb{R}^3)\big), \tag{11.33}$$

where $B(X, Y)$ denotes the class of bounded maps from X to Y.

Since the class of H^3, compactly supported solenoidal vector fields is dense in that of all H^3 solenoidal vector fields (see Appendix A of [86]), (11.33) and (11.32) imply that

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \text{ is relatively compact in } C\big(\mathbb{R}_+; w - L^2(\mathbb{R}^3)\big), \tag{11.34}$$

by a variant of Ascoli's theorem that can be found in Appendix C of [86].

As for the $L^2_{\text{loc}}(\mathrm{d}t \, \mathrm{d}x)$ compactness, observe that (11.34) implies that

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \star \chi_\delta \text{ is relatively compact in } L^2_{\text{loc}}(\mathrm{d}t \, \mathrm{d}x),$$

where χ_δ designates any mollifying sequence and $\star$ is the convolution in the x-variable only. Hence

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \cdot P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \star \chi_\delta \to Pu \cdot Pu \star \chi_\delta$$

in $w - L^1_{\text{loc}}(\mathrm{d}t \, \mathrm{d}x)$ as $\varepsilon \to 0$. By (11.31),

$$P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \star \chi_\delta \to P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle$$

in $L^2_{\text{loc}}(\mathrm{d}t \, \mathrm{d}x)$ uniformly in ε as $\delta \to 0$. With this, we conclude that

$$\big| P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \big|^2 \to |Pu|^2 \quad \text{in } w - L^1_{\text{loc}}(\mathrm{d}t \, \mathrm{d}x)$$

which implies that $P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \to Pu$ strongly in $L^2_{\text{loc}}(\mathrm{d}t \, \mathrm{d}x)$. □

Next, consider

$$\nabla_x \pi_\varepsilon = \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle - P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle.$$

Since

$$\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \to u \quad \text{in } w - L^2_{\text{loc}}(\mathrm{d}t \, \mathrm{d}x) \quad \text{and} \quad \text{div}_x \, u = 0$$

one has

$$\nabla_x \pi_\varepsilon \to 0 \quad \text{in } w - L^2_{\text{loc}}(\mathrm{d}t \, \mathrm{d}x) \tag{11.35}$$

as $\varepsilon \to 0$. Decompose then

$$P \, \text{div}_x \big(\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \big)$$

$$= P \, \text{div}_x \big(P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \big) + P \, \text{div}_x \big(P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes \nabla_x \pi_\varepsilon \big)$$

$$+ P \, \text{div}_x \big(\nabla_x \pi_\varepsilon \otimes P\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \big) + P \, \text{div}_x \big(\nabla_x \pi_\varepsilon \otimes \nabla_x \pi_\varepsilon \big).$$

By Proposition 11.10, the first term converges to $P \operatorname{div}_x(u \otimes u)$ in the sense of distributions, while the second and third terms converge to 0 in the sense of distributions because of (11.35).

As for the last term, let $\zeta_\delta = \xi_\delta \star \xi_\delta \star \xi_\delta$, where ξ_δ is an approximate identity. One can prove that

$$\varepsilon \, \partial_t \zeta_\delta \star_x \nabla_x \pi_\varepsilon + \nabla_x \zeta_\delta \star_x \left\langle \frac{1}{3} |v|^2_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \right\rangle \to 0 \quad \text{in } L^1_{\text{loc}}\big(\mathbb{R}_+; H^1_{\text{loc}}(\mathbb{R}^3)\big),$$

$$\varepsilon \, \partial_t \zeta_\delta \star_x \left\langle \frac{1}{3} |v|^2_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \right\rangle + \frac{5}{3} \Delta_x \zeta_\delta \star_x \pi_\varepsilon \to 0 \quad \text{in } L^1_{\text{loc}}\big(\mathbb{R}_+; H^1_{\text{loc}}(\mathbb{R}^3)\big)$$

as a result of (11.22), the vanishing of momentum and energy conservation defects (see Proposition 11.4 for the momentum, and proceed analogously for the energy) and the fact that $\mathbf{F}_\varepsilon(A)$ is bounded in $L^1_{\text{loc}}(dt \, dx)$ (see Lemma 11.5, and the bounds (11.14) and (11.15)). From the above system, Lions and Masmoudi observed in [87] that

$$\operatorname{div}_x(\nabla_x \zeta_\delta \star_x \pi_\varepsilon \otimes \nabla_x \zeta_\delta \star_x \pi_\varepsilon)$$

$$= \frac{1}{2} \nabla_x \left(|\nabla_x \zeta_\delta \star_x \pi_\varepsilon|^2 - \frac{5}{3} \zeta_\delta \star_x \left\langle \frac{1}{3} |v|^2_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \right\rangle^2 \right) + \text{o}(1)_{L^1_{\text{loc}}(dt \, dx)}.$$

Together with the uniform compactness "in the x-variable" proved in Proposition 11.9, this implies that

$$P \operatorname{div}_x(\nabla_x \pi_\varepsilon \otimes \nabla_x \pi_\varepsilon) \to 0$$

in the sense of distributions. Collecting the observations above, we have just proved the following proposition.

PROPOSITION 11.11. *Under the assumptions of Theorem 11.3, modulo extraction of a subsequence, one has*

$$P \operatorname{div}_x \big(\langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \otimes \langle v_{K_\varepsilon} g_\varepsilon \gamma_\varepsilon \rangle \big) \to P \operatorname{div}_x(u \otimes u)$$

in the sense of distributions on $\mathbb{R}_+^* \times \mathbb{R}^3$ as $\varepsilon \to 0$.

With this proposition, we have completed all three steps in the proof of the Navier–Stokes limit (Theorem 11.3).

11.5. *The nonlinear compactness estimate*

It only remains to prove the nonlinear compactness estimate (11.26), on which the two most important steps in the proof of the Navier–Stokes limit – i.e., the vanishing of conservation defects and the limiting form of the momentum flux – are based.

This nonlinear compactness estimate results from new ideas on velocity averaging in L^1; consistently with our presentation of this subject in Section 8.2.2, we shall describe these ideas on the steady transport equation.

We start with a definition of the notion of partial uniform integrability in a product space.

DEFINITION 11.12. Let μ and ν be positive Borel measures on $\mathbb{R}^N$; we say that a family ϕ_ε of elements of $L^1(\mathbb{R}^N_x \times \mathbb{R}^N_y; d\mu(x)\,d\nu(y))$ is uniformly integrable in the variable y if

$$\int_{\mathbb{R}^N} \left(\sup_{\nu(A) < \eta} \int_A |\phi_\varepsilon(x,y)|\,d\nu(y) \right) d\mu(x) \to 0$$

as $\eta \to 0$, uniformly in ε.

EXAMPLE. Assume that ν is a finite measure; then, for each $p > 1$, any bounded family in $L^1(d\mu(x); L^p(d\nu(y)))$ is uniformly integrable in y. Indeed, if ϕ_ε is any such family, one has

$$\int_{\mathbb{R}^N} \left(\sup_{\nu(A) < \eta} \int_A |\phi_\varepsilon(x,y)|\,d\nu(y) \right) d\mu(x) \leqslant \eta^{1/p'} \int_{\mathbb{R}^N} \|\phi_\varepsilon(x,\cdot)\|_{L^q(d\nu)}\,dx,$$

where $p' = p/(p-1)$, by applying Hölder's inequality to the inner integral.

As usual, we shall say that a family $\phi_\varepsilon \in L^1(\mathbb{R}^N_x \times \mathbb{R}^N_y; d\mu(x)\,d\nu(y))$ is locally uniformly integrable in y if, for each compact $K \subset \mathbb{R}^N \times \mathbb{R}^N$, the family $\mathbb{1}_K \phi_\varepsilon$ is uniformly integrable in y.

With this notion of partial uniform integrability, we can formulate an important improvement of the L^1-variant of velocity averaging stated in Proposition 8.6 and Theorem 8.7.

THEOREM 11.13 (Golse and Saint-Raymond [60]). *Let f_ε be a bounded family in $L^1_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^N; dx\,dy)$ such that*
- *the family $v \cdot \nabla_x f_\varepsilon$ is bounded in $L^1_{\text{loc}}(\mathbb{R}^N \times \mathbb{R}^N; dx\,dy)$, and*
- *the family f_ε is locally uniformly integrable in the variable v.*

Then
(1) *the family f_ε is locally uniformly integrable (in both variables (x, v)), and*
(2) *for each compactly supported $\phi \in L^\infty(\mathbb{R}^N)$, the family of averages*

$$\int_{\mathbb{R}^N} f_\varepsilon(x,v)\phi(v)\,dv$$

is relatively compact in $L^1_{\text{loc}}(\mathbb{R}^N; dx)$.

This result amplifies an earlier remark by Saint-Raymond who observed in [107] that, under the extra assumption that f_ε is bounded in $L^1_x(L^\infty_y)$, the family of averages

$$\int_{\mathbb{R}^N} f_\varepsilon(x,v)\phi(v)\,dv$$

is locally uniformly integrable.

SKETCH OF THE PROOF OF THEOREM 11.13. We shall explain how to prove Saint-Raymond's result under the assumptions of the theorem above.

Step 1.   Let $\chi \equiv \chi(t, x, v)$ be the solution of the free transport equation

$$\partial_t \chi + v \cdot \nabla_x \chi = 0, \quad (x, v) \in \mathbb{R}^N \times \mathbb{R}^N, t > 0,$$

$$\chi(0, x, v) = \mathbb{1}_A(x), \quad (x, v) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Clearly, $\chi(t, x, v) = \mathbb{1}_A(x - tv)$ for each $t > 0$; it can therefore be put in the form

$$\chi(t, x, v) = \mathbb{1}_{A_{t,x}}(v), \quad t > 0, (x, v) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where, for each $t > 0$,

$$A_{t,x} = \left\{ v \in \mathbb{R}^N \,\middle|\, x - tv \in A \right\}.$$

Further, $A_{t,x}$ is measurable and, for each $t > 0$ and $x \in \mathbb{R}^N$, one has

$$|A_{t,x}| = \int_{\mathbb{R}^N} \chi(t, x, v)\, dv = \int_{\mathbb{R}^N} \mathbb{1}_A(x - tv)\, dv = \frac{1}{t^N} \int_{\mathbb{R}^N} \mathbb{1}_A(z)\, dz = \frac{|A|}{t^N}.$$

Step 2.   Without loss of generality, assume that f_ε and ϕ_ε in the statement of the theorem are nonnegative, and that all the f_ε's are supported in the same compact K of $\mathbb{R}^N \times \mathbb{R}^N$. Then

$$\int_A \int_{\mathbb{R}^N} f_\varepsilon(x, v)\phi(v)\, dv\, dx$$

$$= \int_{\mathbb{R}^N} \int_{A_{t,x}} f_\varepsilon(x, v)\phi(v)\, dv\, dx$$

$$- \int_0^t \iint_{\mathbb{R}^N \times \mathbb{R}^N} \chi(s, x, v) v \cdot \nabla_x f_\varepsilon(x, v)\phi(v)\, dx\, dv\, ds \quad (11.36)$$

as can be seen by integrating by parts the second integral on the right-hand side of the equality above.

   Pick $\eta > 0$ arbitrarily small; the second integral on the right-hand side of (11.36) satisfies

$$\left| \int_0^t \iint_{\mathbb{R}^N \times \mathbb{R}^N} \chi(s, x, v) v \cdot \nabla_x f_\varepsilon(x, v)\phi(v)\, dx\, dv\, ds \right| \leqslant t \|v \cdot \nabla_x f_\varepsilon\|_{L^1} \|\phi\|_{L^\infty}$$

and therefore can be made less than η by choosing

$$0 < t < \frac{\eta}{1 + \|v \cdot \nabla_x f_\varepsilon\|_{L^1} \|\phi\|_{L^\infty}}.$$

For this $t > 0$, the first integral on the right-hand side of (11.36) satisfies

$$\int_{\mathbb{R}^N}\!\!\int_{A_{t,x}} f_\varepsilon(x,v)\phi(v)\,\mathrm{d}v\,\mathrm{d}x \to 0$$

as $|A| \to 0$ uniformly in ε, since f_ε is uniformly integrable in v and $|A_{t,x}| = |A|/t^N$, as established in Step 1.

Therefore, for each $\eta > 0$, there exists $\alpha > 0$ such that $|A| < \alpha$ implies that

$$\int_A\!\!\int_{\mathbb{R}^N} f_\varepsilon(x,v)\phi(v)\,\mathrm{d}v\,\mathrm{d}x \leqslant 2\eta$$

uniformly in $\varepsilon > 0$, which entails that the family of averages

$$\int_{\mathbb{R}^N} f_\varepsilon(x,v)\phi(v)\,\mathrm{d}v$$

is uniformly integrable. $\qquad\square$

The nonlinear compactness estimate (11.26) will of course be obtained from statement (1) in Theorem 11.13. In fact, we first observe that, for each $c \in (1,2)$,

$$\phi_\varepsilon^\delta = \left(\frac{\sqrt{\varepsilon^c + G_\varepsilon} - 1}{\varepsilon}\right)^2 \gamma\left(\varepsilon\delta\left(\frac{\sqrt{\varepsilon^c + G_\varepsilon} - 1}{\varepsilon}\right)\right)$$

satisfies

$$\phi_\varepsilon^\delta = \mathrm{O}(1) \quad \text{in } L_t^\infty\big(L^1(M\,\mathrm{d}v\,\mathrm{d}x)\big),$$

while

$$(\varepsilon\,\partial_t + v\cdot\nabla_x)\phi_\varepsilon^\delta = \mathrm{O}(1) \quad \text{in } L_{\mathrm{loc}}^1(\mathrm{d}t\,\mathrm{d}x\,M\,\mathrm{d}v).$$

We next let $\delta \to 0$ and remove the ε^c from under the square root (in that order) so that

$$\phi_\varepsilon^\delta \simeq \left(\frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}\right)^2.$$

In order to apply Theorem 11.13 to ϕ_ε^δ, it remains to prove that this family is uniformly integrable in the v-variable. In fact, we prove the following results.

PROPOSITION 11.14. *Under the assumptions of Theorem* 11.3, *for each $T > 0$ and each compact $K \subset \mathbb{R}^3$, the family*

$$\left(1 + |v|\right)^\alpha \left(\frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}\right)^2$$

is uniformly integrable on $[0, T] \times K \times \mathbb{R}^3$ *for the measure* $dt \, dx \, M \, dv$. (*We recall that* α *is the relative velocity exponent in the hard cut-off assumption on the collision kernel* b, *see* (3.55).)

This proposition improves upon the result in [61], that applied to cut-off Maxwell molecules only (i.e., to the case where $\alpha = 0$). Its proof is fairly technical, so that we shall just sketch the main idea in it.

Start from the identity

$$\mathcal{L}_M\left(\frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}\right)$$

$$= \varepsilon \mathcal{Q}_M\left(\frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}, \frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}\right) - \frac{1}{\varepsilon} \mathcal{Q}_M\left(\sqrt{G_\varepsilon}, \sqrt{G_\varepsilon}\right). \tag{11.37}$$

Next, we recall the Bardos–Caflisch–Nicolaenko spectral gap for $\mathcal{L}_M$ in weighted space (see Theorem 3.11)

$$\langle \phi \mathcal{L}_M \phi \rangle \geqslant C_0 \big\langle (1 + |v|)^\alpha \phi^2 \big\rangle, \quad \phi \in (\text{Ker } \mathcal{L}_M)^\perp$$

together with the following continuity estimate for $\mathcal{Q}$ (see [57])

$$\big\| \mathcal{Q}_M(\phi, \phi) \big\|_{L^2((1+|v|)^{-\alpha} M \, dv)} \leqslant C \|\phi\|_{L^2(M \, dv)} \|\phi\|_{L^2((1+|v|)^\alpha M \, dv)}.$$

Using both estimates in the identity above leads to the following control

$$\left(1 - \mathrm{O}(\varepsilon) \left\| \frac{\sqrt{G_\varepsilon} - 1}{\varepsilon} \right\|_{L^2(M \, dv)}\right) \left\| \frac{\sqrt{G_\varepsilon} - 1}{\varepsilon} - \Pi \frac{\sqrt{G_\varepsilon} - 1}{\varepsilon} \right\|_{L^2((1+|v|)^\alpha M \, dv)}$$

$$\leqslant \mathrm{O}(\varepsilon)_{L^2_{t,x}} + \mathrm{O}(\varepsilon) \left\| \frac{\sqrt{G_\varepsilon} - 1}{\varepsilon} \right\|^2_{L^2(M \, dv)}.$$

This control suggests that

$$\frac{\sqrt{G_\varepsilon} - 1}{\varepsilon} \text{ is close to its hydrodynamic projection } \Pi \frac{\sqrt{G_\varepsilon} - 1}{\varepsilon}$$

precisely in that weighted L^2 space that appears in the statement of Proposition 11.14. Since the hydrodynamic projection $\Pi \frac{\sqrt{G_\varepsilon}-1}{\varepsilon}$ is as regular in v as one can hope for (being a quadratic polynomial in v), this eventually entails uniform integrability in v once the difficulties related to the (t, x) dependence in the estimate above have been handled. As we already indicated above, the remaining part of the proof is too technical to be described here, and we refer the interested reader to [62] for a complete argument.

Instead of giving more details on this proof, we comment on the differences between the nonlinear compactness estimate (11.28)–(11.30) obtained in [61] in the case of cut-off Maxwell molecules, and the most recent variant of such controls obtained in [62],

namely (11.26). In fact, the main difference between these controls lies in the method for obtaining uniform integrability in v.

In [61], the decomposition (11.37) was replaced with

$$G_\varepsilon = \left(G_\varepsilon - \frac{\mathcal{A}^+(MG_\varepsilon, MG_\varepsilon)}{M\langle G_\varepsilon \rangle} \right) + \frac{\mathcal{A}^+(MG_\varepsilon, MG_\varepsilon)}{M\langle G_\varepsilon \rangle} \tag{11.38}$$

(in fact, with a more technical variant of (11.38) involving several truncations), where $\mathcal{A}^+$ was the gain part of a fictitious collision operator

$$\mathcal{A}^+(\phi, \phi) = \iint_{\mathbb{R}^3 \times \mathbb{S}^2} \phi' \phi'_* \big| \cos(v - v_*, \omega) \big| \, dv_* \, d\omega.$$

Roughly speaking the term

$$\left(G_\varepsilon - \frac{\mathcal{A}^+(MG_\varepsilon, MG_\varepsilon)}{M\langle G_\varepsilon \rangle} \right)$$

was easily controlled by the entropy production bound, while the term

$$\frac{\mathcal{A}^+(MG_\varepsilon, MG_\varepsilon)}{M\langle G_\varepsilon \rangle}$$

improved the regularity and decay in v due to estimates by Grad [63] and Caflisch [23] – that are linear analogues of the result by Lions on the smoothing properties of the gain part in the Boltzmann collision integral (Lemma 3.10).

However, the main drawback of the decomposition (11.38) was that it naturally involved the quantity

$$g_\varepsilon \gamma_\varepsilon + \frac{1 - \gamma_\varepsilon}{\varepsilon}$$

(see (2.17) in [61]), thereby leading to a dissymmetry between the roles of large v's and large values of g_ε in the estimates (11.28)–(11.30). This, as we already mentioned, was an obstruction in upgrading (11.28)–(11.29) with a $(1 + |v|)^\alpha$ weight as in (11.26).

The method used in [62] essentially differs from that of [61]; to begin with, we do not attempt to establish that the number density F_ε is close in some sense to its associated local equilibrium M_{F_ε} (i.e., the Maxwellian with same local density, energy and momentum as F_ε), an idea that is very natural in the context of the BGK model of the Boltzmann equation, see [107,109].

Instead, we seek to prove that some variant of the number density fluctuation, namely the quantity $\frac{\sqrt{G_\varepsilon} - 1}{\varepsilon} \simeq \frac{1}{2} g_\varepsilon$, becomes close to its associated infinitesimal Maxwellian, i.e., its $L^2(M \, dv)$-orthogonal projection on $\text{Ker} \, \mathcal{L}_M$. This new approach naturally leads to using the Dirichlet form of the linearized collision operator, whose relative coercivity transversally to $\text{Ker} \, \mathcal{L}_M$ provides exactly the $(1 + |v|)^\alpha$ weight needed in the proof.

12. Conclusions and open problems

This survey of the Boltzmann equation and its hydrodynamic limits remains incomplete in several respects. As already mentioned in the Introduction, we have chosen to emphasize global theories (such as the DiPerna–Lions theory for the Boltzmann equation, or the Leray theory for the Navier–Stokes equations) for the evolution problem posed on a spatial domain without boundaries. More realistic formulations of the hydrodynamic limits of the Boltzmann equation would certainly involve boundary value problems. At the formal level, the influence of boundaries on hydrodynamic limits is discussed at length in Sone's authoritative monograph [115].

More specifically, hydrodynamic limits of boundary value problems may, in some particular cases, involve boundary layer equations whose mathematical study is a subject of its own and has inspired a considerable amount of literature. These boundary layers are meant to forget the specific dependence of the boundary data upon the velocity variable so as to match it with the inner form of the number density, which, as explained in this survey, is a local Maxwellian or infinitesimal Maxwellian in the hydrodynamic limit. This is obviously not the place for a description of that theory, to begin with because these boundary layer are essentially steady (instead of evolution) problems. The interested reader could start with Chapter 5, Section 5 of [27] and then get acquainted with the mathematical theory of half-space problems as exposed in [6,26,31,57,58,93] and [119]. These works deal with linearized, or at best, weakly nonlinear problems. Nonlinear boundary layer equations with nonzero net mass flux at the boundary – typically in the case of a gas condensing on, or evaporating from a solid boundary – have been studied in depth numerically by Sone, Aoki and their collaborators, see Chapter 7 of [115] and the references therein. At the formal level, their theory is expected to agree with a general theory of boundary conditions for the Euler system of compressible fluids, yet to be formulated at the time of this writing.

In some cases however, boundary layer are not leading order effects. This is the case of the incompressible Stokes or Navier–Stokes limit of the Boltzmann equation for a gas in a container with specular or diffuse reflection, or Maxwell's accommodation condition at the boundary; see [96], which extends to such a boundary value problems the results in [53]. The analogue of the DiPerna–Lions theory for the boundary value problem is due to Mischler [97], this is by no means a straightforward extension of [37]. It requires rather subtle analytical tools because natural boundary conditions for the Boltzmann equation – except in the case of a specular reflection of gas molecule at the boundary – do not interact well with the renormalization procedure which is essential in controlling the collision integral.

Let us conclude with a few open problems, in addition to those already mentioned in the body of this article.

It would be of considerable interest to derive the global BV solutions constructed by T.-P. Liu from the Boltzmann equation. As in the case of the incompressible Euler limit of the Boltzmann equation, the entropy production bound entailed by Boltzmann's H-theorem does not balance the action of the streaming operator on the number density: the compactness of hydrodynamic moments of the number density is probably to be sought in some stability property of BV solutions of the compressible Euler system. Most likely, such a theory should use Bressan's remarkable results in that direction (see [19,20]).

As an indication of the level of difficulty of this problem, let us simply mention that the hydrodynamic limit of the Boltzmann equation leading to a single Riemann problem has not been established so far, except in the case of a single, weak shock wave. Shock profiles for the Boltzmann equation were formally discussed in [25]; for the hard sphere gas, a complete construction of these profiles in the weakly nonlinear regime can be found in a series of important papers by Nicolaenko [99–101]. All these constructions were based on a deep understanding of the algebra of the linearized Boltzmann equation in connection with the underlying saddle point structure of subsonic states in gas dynamics. They suggested some topological structure that remains to be unraveled and could ultimately lead to a construction of shock profiles of arbitrary strength. The extension to other cut-off molecular interactions is due to Caflisch and Nicolaenko [24]. However, all these constructions led to solutions of the Boltzmann equation whose positivity was not established. This latter problem was solved only very recently by Liu and Yu [90].

Another open problem would be to improve Theorem **??**, by relaxing the unphysical assumption made on the size of the number density fluctuations δ_ε to reach the physically natural condition that $\delta_\varepsilon \to 0$ as $\varepsilon \to 0$. This would probably require more information on the local conservation laws of momentum and energy for renormalized solutions of the Boltzmann equation. Such information would most likely be an important prerequisite for making progress on the compressible Euler limit.

Finally, we have only considered evolution problems in this survey. Although steady problems are beyond the scope of the present book, they are perhaps even more important in some applications (such as aerodynamics). For instance, it is well known that, for any divergence free force field $f \equiv f(x) \in L^2(\Omega; \mathbb{R}^3)$, the steady incompressible Navier–Stokes equations in a smooth, bounded open domain $\Omega \subset \mathbb{R}^3$

$$-\nu\Delta_x u = f - \nabla_x p - (u \cdot \nabla_x)u, \qquad \mathrm{div}_x\, u = 0, \quad x \in \Omega,$$

$$u|_{\partial\Omega} = 0$$

$$(12.1)$$

has at least one classical solution $u \equiv u(x) \in H^2(\Omega, \mathbb{R}^3)$, obtained by a Leray–Schauder fixed point argument (see, for instance, [74]). Unfortunately, the parallel theory for the Boltzmann equation is not as advanced: for one thing, at the time of this writing, there is no analogue of the DiPerna–Lions result for the steady Boltzmann equation with a prescribed external force field. The reader interested in those matters is advised to read the survey article by Maslova [94]; see also her book [95]; among classical references on the subject, there are some papers by Guiraud (see [65–67]), and more recent work by Arkeryd and Nouri (see, for instance, [3]). See also [117] for the case of a gas flow modeled by the Boltzmann equation around a convex body. Finally, a rather exhaustive description of such steady problems and their hydrodynamic limits (at the formal level) may be found in Sone's book [115].

In spite of the difficulties inherent to the steady Boltzmann equation, the fact that the solutions of (12.1) are more regular than in the case of the evolution problem could be of considerable help, at least in the context of the hydrodynamic limit.

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