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Hydrodynamic Limits

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Abstract. This article reviews recent progress on the derivation of the fundamental PDE models in fluid mechanics from the Boltzmann equation.

1. Introduction

The subject of hydrodynamic limits goes back to the work of the founders of the kinetic theory of gases, J. Clerk Maxwell and L. Boltzmann. At a time when the existence of atoms was controversial, kinetic theory could explain how to estimate the size of a gas molecule from macroscopic data such as the viscosity of the gas. Later, D. Hilbert formulated the question of hydrodynamic limits as a mathematical problem, giving an example in his 6th problem on the axiomatization of physics [25]. In Hilbert's own words "[...] Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes [...] which lead from the atomistic view to the laws of motion of continua". Hilbert himself attacked the problem in [26], as an application of his own work on integral equations.

We should mention that there are several interpretations of what is meant by "the atomistic view" in Hilbert's problem. One can either choose molecular dynamics (i.e., the N-body problem of classical mechanics with elastic collisions, assuming all bodies to be spherical and of equal mass); another possibility is to start from the Boltzmann equation of the kinetic theory of gases (which is what Hilbert himself did in [26]). However, one should be aware that the Boltzmann equation is not itself a "first principle" of physics, but is a low density limit of molecular dynamics – which can be considered as a first principle within the theory of classical, nonrelativistic mechanics.

The problem of hydrodynamic limits is to obtain rigorous derivations of macroscopic models such as the fundamental PDEs of fluid mechanics from a microscopic description of matter, either molecular dynamics or the kinetic theory of gases. The situation can be summarized by the following diagram:

$$
\begin{tikzcd}
\boxed{\text{MOLECULAR DYNAMICS}} \arrow[r] \arrow[dr] & \boxed{\text{KINETIC THEORY}} \arrow[d] \\
& \boxed{\text{HYDRODYNAMICS}}
\end{tikzcd}
$$

First, we recall that a rigorous derivation of the Boltzmann equation from molecular dynamics on short time intervals (i.e., the horizontal arrow in the diagram above) was obtained by O.E. Lanford in [30]. Hence, although not a first principle itself, the Boltzmann equation can be rigorously derived from first principles and therefore has more physical legitimacy than phenomenological models (such as lattice gases).

On the other hand, "formal" derivations of the Euler system for compressible fluids from molecular dynamics were discussed by C.B. Morrey in [37]. Later on, S.R.S. Varadhan and his collaborators considered stochastic variants of molecular gas dynamics and obtained rigorous derivations of macroscopic PDE models from these variants: see for instance [49] and the references therein, notably [39].

In the present work, we shall mostly restrict our attention to derivations of the fundamental PDEs of fluid mechanics from the Boltzmann equation. Perhaps the most complete result in this direction is the derivation of the Navier-Stokes equations for incompressible flows from the Boltzmann equation. Indeed, unlike in the case of other hydrodynamic models, this derivation is valid for all physically admissible data, without any restriction on the regularity or the size of the solutions considered. We conclude this presentation with a quick survey of other recent results and open problems on hydrodynamic limits of kinetic models.

2. The Navier-Stokes equations

The Navier-Stokes equations govern incompressible flows of a viscous fluid. In the sequel, we only consider the case of a fluid with constant density that can be set equal to 1 without loss of generality. The unknown is the velocity field $u \equiv u(t,x) \in \mathbf{R}^3$, where $t \in \mathbf{R}_+$ and $x \in \mathbf{R}^3$ are the time and space variables.

In the absence of external forces (such as electromagnetic forces, gravity…) the velocity field u satisfies

$$\begin{aligned} &\operatorname{div}_x u = 0\,, \\ &\partial_t u + (u\cdot\nabla_x)u + \nabla_x p = \nu\Delta_x u\,, \end{aligned} \tag{2.1}$$

where $\nu > 0$ is a constant called the "kinematic viscosity". Here, the notation $(u\cdot\nabla_x)u$ designates the parallel derivative of u along itself, whose coordinates are given by

$$((u\cdot\nabla_x)u)^i := \sum_{j=1}^{3} u^j \frac{\partial u^i}{\partial x^j}\,.$$

In physical terms, the first equality in (2.1) is the incompressibility condition, while the second equality is the motion equation – i.e., Newton's second law of motion applied to an infinitesimal volume of the fluid.

Observe that, for any C^1 divergence-free vector field v on $\mathbf{R}^3$

$$((v \cdot \nabla_x)v)^i = \sum_{j=1}^{3} v^j \frac{\partial v^i}{\partial x^j} = \sum_{j=1}^{3} \frac{\partial (v^i v^j)}{\partial x^j} =: (\operatorname{div}_x (v \otimes v))^i \ .$$

The expression $\operatorname{div}_x(v \otimes v)$ defines a (vector-valued) distribution on $\mathbf{R}^3$ if $v \in L^2(\mathbf{R}^3)$, and it coincides with $(v \cdot \nabla_x)v$ if v is of class C^1 on $\mathbf{R}^3$. This remark justifies the following notion of weak solution of the Navier-Stokes equations.

Definition 2.1. A weak solution of the Navier-Stokes equations is a vector-field[1] $u \in C(\mathbf{R}_+; w\text{–}L^2(\mathbf{R}^3;\mathbf{R}^3))$ which satisfies

$$\begin{aligned} \operatorname{div}_x u = 0\,, \\ \partial_t u + \operatorname{div}_x(u \otimes u) - \nu \Delta_x u = -\nabla_x p\,, \end{aligned} \tag{2.2}$$

in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3$, for some $p \in \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^3)$.

In fact, the term $-\nabla_x p$ is the Lagrange multiplier associated to the constraint $\operatorname{div}_x u = 0$. In other words, the motion equation in (2.2) should be viewed as

$$\partial_t u + \nabla_x u - \nu \Delta_x u = 0 \text{ modulo gradient fields.}$$

After these preliminaries, we can state Leray's existence result of a global weak solution for the incompressible Navier-Stokes equations.

Theorem 2.2 (J. Leray [31])**.** *For each $u^{in} \in L^2(\mathbf{R}^3;\mathbf{R}^3)$ such that $\operatorname{div}_x u^{in} = 0$, there exists a weak solution of the Navier-Stokes equations satisfying the initial data $u\big|_{t=0} = u^{in}$. Moreover, this solution verifies the "energy inequality"*

$$\tfrac{1}{2} \int_{\mathbf{R}^3} |u(t,x)|^2 dx + \nu \int_0^t \int_{\mathbf{R}^3} |\nabla_x u(s,x)|^2 dx ds \le \tfrac{1}{2} \int_{\mathbf{R}^3} |u^{in}(x)|^2 dx \tag{2.3}$$

for each $t > 0$.

Notice that the scalar function $p \equiv p(t,x)$ (the pressure) is not an unknown in the Navier-Stokes equations, since it is defined (modulo a constant) in terms of u by the relation

$$-\Delta_x p = \operatorname{div}_x((u \cdot \nabla_x)u)\,.$$

Whether Leray solutions of the Navier-Stokes equations are uniquely determined by their initial data is still unknown. Likewise, it is still unknown whether any Leray solution of the Navier-Stokes equations with smooth initial data remains smooth for all subsequent times. However, if the Cauchy problem (2.1) has a smooth solution u with $\nabla_x u \in L^\infty(\mathbf{R}_+ \times \mathbf{R}^3)$, any Leray solution of (2.1) must coincide with u.

Observe that, for smooth solutions of the Navier-Stokes equations decaying sufficiently fast as $|x| \to +\infty$, the energy inequality (2.3) is in fact an equality, as can be seen by taking the scalar product of both sides of the motion equation in (2.1) with u and integrating over $[0,t] \times \mathbf{R}^3$.

[1] The notation w–L^p designates the L^p space endowed with its weak topology.

3. The Boltzmann equation

In kinetic theory, the dynamics of a gas of (like) hard spheres is described by the Boltzmann equation. It governs the evolution of the number density $F \equiv F(t,x,v) \geq 0$, the 1-particle phase-space density of the gas molecules at time t. In other words, $F(t,x,v)$ is the density at time $t \geq 0$ (with respect to the Lebesgue measure $dxdv$ in $\mathbf{R}^3 \times \mathbf{R}^3$) of the gas molecules located at the position $x \in \mathbf{R}^3$ that have velocity $v \in \mathbf{R}^3$.

In the absence of external forces (such as electromagnetic forces, gravity...) the Boltzmann equation for F is

$$\partial_t F + v \cdot \nabla_x F = \mathcal{C}(F) \tag{3.1}$$

where $\mathcal{C}(F)$ is the Boltzmann collision integral.

Collisions other than binary are neglected in the Boltzmann equation, and these collisions are viewed as purely instantaneous and local. Indeed, in the kinetic theory of gases, the molecular radius is neglected everywhere in the description of the collision process except in the expression of the scattering cross-section. An important consequence of these physical assumptions is that $\mathcal{C}$ is a bilinear operator acting only on the v-variable in F.

For a gas of hard spheres, the collision integral is given by the expression[2]

$$\mathcal{C}(F)(v) = \iint_{\mathbf{R}^3 \times \mathbf{S}^2} (F(v')F(v'_*) - F(v)F(v_*))|v - v_*| dv_* d\sigma \,, \tag{3.2}$$

where the velocities v' and v'_* are defined in terms of $v, v_* \in \mathbf{R}^3$ and $\sigma \in \mathbf{S}^2$ by

$$\begin{aligned} v' &\equiv v'(v, v_*, \sigma) = \tfrac{1}{2}(v + v_*) + \tfrac{1}{2}|v - v_*|\sigma \,, \\ v'_* &\equiv v'_*(v, v_*, \sigma) = \tfrac{1}{2}(v + v_*) - \tfrac{1}{2}|v - v_*|\sigma \,. \end{aligned} \tag{3.3}$$

Perhaps the most important result on the structure of the Boltzmann collision integral is

Boltzmann's H Theorem. *Assume that* $F \equiv F(v) > 0$ *a.e. is rapidly decaying and such that* $\ln F$ *has polynomial growth as* $|v| \to +\infty$. *Then*

$$R(F) = -\int_{\mathbf{R}^3} \mathcal{C}(F) \ln F dv \geq 0 \,.$$

Moreover, the following conditions are equivalent:

$$R(F) = 0 \;\Leftrightarrow\; \mathcal{C}(F) = 0 \text{ a.e.} \;\Leftrightarrow\; F \text{ is a Maxwellian},$$

i.e., there exists $\rho, \theta > 0$ *and* $u \in \mathbf{R}^3$ *such that*

$$F(v) = \mathcal{M}_{(\rho,u,\theta)}(v) := \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v-u|^2}{2\theta}} \qquad \text{a.e. in } v \in \mathbf{R}^3 \,.$$

From the physical viewpoint, the nonnegative quantity $R(F)$ represents the entropy production rate.

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[2]In this formula, the molecular radius is chosen as the unit of length.

All hydrodynamic limits of the kinetic theory of gases considered in the present work bear on solutions of the Boltzmann equation that are fluctuations of some uniform Maxwellian state. We henceforth choose this uniform equilibrium state to be

$$M = \mathcal{M}_{(1,0,1)} \qquad \text{(the centered, reduced Gaussian distribution)}$$

without loss of generality. The size of the number density fluctuations around the equilibrium state M will be measured in terms of the relative entropy of the number density relatively to M, whose definition is recalled below.

Definition 3.1. Given two measurable functions $f \geq 0$ and $g > 0$ a.e. on $\mathbf{R}^3 \times \mathbf{R}^3$, the relative entropy of f relative to g is

$$H(f|g) = \iint_{\mathbf{R}^3 \times \mathbf{R}^3} \left[f \ln\left(\frac{f}{g}\right) - f + g \right] dx dv \geq 0\,.$$

(Notice that the integrand is a nonnegative measurable function, so that the integral is a well-defined element of $[0, +\infty]$.)

In [15], R. DiPerna and P.-L. Lions defined the following notion of a weak solution of the Boltzmann equation.

Definition 3.2. A renormalized solution of the Boltzmann equation (3.1) is a nonnegative function $F \in C(\mathbf{R}_+; L^1_{\mathrm{loc}}(\mathbf{R}^3 \times \mathbf{R}^3))$ such that

$$\frac{\mathcal{C}(F)}{1+F} \in L^1_{\mathrm{loc}}(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)\,,$$

and that satisfies the equality

$$(\partial_t + v \cdot \nabla_x) \ln(1+F) = \frac{\mathcal{C}(F)}{1+F}$$

in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3$.

The motivation for this definition is that the collision integral acts as the convolution of F with itself in the v variable, and as a pointwise product in the t and x variables. Since the natural estimates for solutions of the Boltzmann equation are bounds on

$$\int_{|x| \leq r} \int_{\mathbf{R}^3} (1+|v|^2) F(t,x,v) dx dv\,,$$

the collision integral $\mathcal{C}(F)$ may not be defined as a distribution on $\mathbf{R}_+^* \times \mathbf{R}^3 \times \mathbf{R}^3$ for such Fs. But the expression $\frac{\mathcal{C}(F)}{1+F}$ is homogeneous of degree one for F large, and happens to be well defined for any number density F that satisfies the natural bounds for solutions of the Boltzmann equation.

Theorem 3.3 (P.-L. Lions [33])**.** *For each $F^{in} \geq 0$ a.e. such that $H(F^{in}|M) < +\infty$, there exists a renormalized solution F of the Boltzmann equation* (3.1) *with initial data* $F\big|_{t=0} = F^{in}$. *This renormalized solution satisfies, for each* $t > 0$, *the "entropy inequality"*

$$H(F(t)|M) + \int_0^t \int_{\mathbf{R}^3} R(F)(s,x)dxds \le H(F^{in}|M)\,. \tag{3.4}$$

If F is a smooth solution of the Boltzmann equation that satisfies the assumptions of Boltzmann's H Theorem for all $t > 0$ and converges to M as $|x| \to +\infty$ rapidly enough, the entropy inequality (3.4) is in fact an equality. This fact alone suggests that there is a deep analogy between Leray solutions of the Navier-Stokes equations in 3 space dimensions and renormalized solutions of the Boltzmann equation. In fact, as we shall see below, Leray's theory can be seen as asymptotic to the DiPerna-Lions theory of renormalized solutions in some appropriate hydrodynamic limit.

4. From Boltzmann to Navier-Stokes

The incompressible Navier-Stokes equations can be formally derived from the Boltzmann equation as follows. According to Hilbert's prescription [26] for the hydrodynamic limit of the Boltzmann equation leading to the Euler system for compressible fluids, the solution of the Boltzmann equation is sought as a formal series

$$F(t,x,v) = \mathcal{M}_{(1,\epsilon u(\epsilon^2 t,\epsilon x),1)}(v) + \sum_{n\ge 2} \epsilon^n F_n(\epsilon^2 t, \epsilon x, v)$$

where u solves the incompressible Navier-Stokes equations (2.1) and F_n depends on t and x through $\nabla^k_{t,x} u$, $k = 0, \dots, n$.

In other words, the incompressible Navier-Stokes equations are derived from the Boltzmann equation in a regime of small, slowly varying fluctuations of number density about a uniform Maxwellian state, which, in the present case, is chosen to be the centered reduced Gaussian distribution $M = \mathcal{M}_{(1,0,1)}$.

This formal argument was discussed by Y. Sone in [47] for the steady problem, and by C. Bardos, F. Golse and C.D. Levermore [3] for the evolution problem (this latter reference also treated the case of an external conservative force leading to a coupling with a drift-diffusion equation for the temperature field).

Later, a rigorous derivation based on a truncated variant of Hilbert's formal solution above, following a method originally used by R. Caflisch for the compressible Euler limit of the Boltzmann equation (see [11]) was sketched by A. DeMasi, R. Esposito and J. Lebowitz in [13]. However, this derivation has the same shortcomings as the original Caflisch method: first, it gives solutions of the Boltzmann equation that fail to be everywhere nonnegative[3] and therefore lose physical meaning. Also, this derivation holds only on the time interval

[3]R. Esposito informed the author that this could probably be remedied by supplementing Hilbert's formal solution with initial layer terms, as done by Lachowicz [28] in the context of the compressible Euler limit; however, there is no written account of this so far.

on which the limiting solution of the Navier-Stokes equations is smooth. As mentioned above, based on current knowledge of the Navier-Stokes equations, we do not know whether this method leads to a derivation of the Navier-Stokes equations that is valid globally in time.

However, if one gives up the idea of working with Hilbert's formal solution and uses instead an energy method based on intrinsic quantities pertaining to the theory of Boltzmann's equation – essentially the relative entropy and the entropy production – one arrives at the following global result.

Theorem 4.1. *Let* $u^{in} \in L^2(\mathbf{R}^3;\mathbf{R}^3)$ *be such that* $\operatorname{div}_x u^{in} = 0$. *For each* $\epsilon > 0$, *let* $F_\epsilon \equiv F_\epsilon(t,x,v)$ *be a renormalized solution of the Boltzmann equation* (3.1) *with initial data*

$$F_\epsilon(0,x,v) = \mathcal{M}_{(1,\epsilon u^{in}(\epsilon x),1)}(v)\,.$$

Then the family of vector fields $u_\epsilon \equiv u_\epsilon(t,x) \in \mathbf{R}^3$ *defined by*

$$u_\epsilon(t,x) = \frac{1}{\epsilon}\int_{\mathbf{R}^3} vF_\epsilon\left(\frac{t}{\epsilon^2},\frac{x}{\epsilon},v\right)dv$$

is weakly relatively compact in $L^1_{\mathrm{loc}}(\mathbf{R}_+ \times \mathbf{R}^3;\mathbf{R}^3)$ *and each of its limit points as* $\epsilon \to 0$ *is a Leray solution of the incompressible Navier-Stokes equations* (2.1) *with initial data* u^{in} *and viscosity*

$$\nu = \tfrac{1}{5}\mathcal{D}^*(v\otimes v - \tfrac{1}{3}|v|^2 I)\,, \tag{4.1}$$

where $\mathcal{D}^*$ *is the Legendre dual of the Dirichlet form of the collision integral* $\mathcal{C}$ *linearized at* M.

The Dirichlet form of the collision integral linearized at M is easily found to be

$$\mathcal{D}(\Phi) = \tfrac{1}{8}\iiint_{\mathbf{R}^3\times\mathbf{R}^3\times\mathbf{S}^2} |\Phi + \Phi_* - \Phi' - \Phi'_*|^2 |v - v_*| MM_* dv dv_* d\sigma\,.$$

(Here Φ_*, Φ' and Φ'_* designate resp. $\Phi(v_*)$, $\Phi(v')$ and $\Phi(v'_*)$, where v' and v'_* are defined in (3.3).) The formula above holds for $\Phi \in C_c(\mathbf{R}^3_v; M_3(\mathbf{R}))$, with $|\cdot|$ denoting the Hilbert-Schmidt norm on matrices:

$$|A|^2 = \operatorname{trace}(A^T A)\,, \qquad A \in M_3(\mathbf{R})\,.$$

It can be extended to the form domain of the linearized collision integral, which is $L^2((1+|v|)Mdv)$.

Remark. The definition of u_ϵ consists in intertwining the evolution of the Boltzmann equation with the invariance group of the Navier-Stokes equations – we recall that, if $u \equiv u(t,x)$ is a solution of the Navier-Stokes equations, then $T_\lambda u :\equiv \lambda u(\lambda^2 t, \lambda x)$ is also a solution of the Navier-Stokes equations for each $\lambda > 0$.

The theorem above was proved by F. Golse and L. Saint-Raymond [22] in the case of Maxwell molecules; the extension to all hard potentials with Grad's cutoff assumption (including the hard sphere case described in the present paper) can be found in [23].

A general strategy for proving global hydrodynamic limits leading to incompressible models was proposed by C. Bardos, F. Golse and C.D. Levermore [5]. This method was based on a priori bounds deduced from the entropy inequality together with some appropriate compactness results. In [5], the incompressible Navier-Stokes limit was obtained under two additional assumptions which, at the time, were left unverified. In addition, only the stationary case was considered in [5]: indeed, high frequency oscillations in time due to the presence of acoustic waves may destroy the compactness of number density fluctuations as $\epsilon \to 0$.

Subsequently, several intermediate results were obtained on this limit. In [34], P.-L. Lions and N. Masmoudi succeeded in controlling the acoustic waves, and proved a result analogous to Theorem 4.1 under the same unverified assumptions as in [5]. In [18], F. Golse and C.D. Levermore went further in the direction of a complete proof by observing that the local conservation laws of momentum and energy could be recovered in the limit $\epsilon \to 0$ instead of being postulated on the renormalized solutions of the Boltzmann equation for each $\epsilon > 0$, as was done in [5]. At the same time, L. Saint-Raymond was able to prove the Navier-Stokes limit for the BGK model of the Boltzman equation [43],[44]. These contributions contained one important idea used in the proof of Theorem 4.1.

Finally, we should also mention that C. Bardos and S. Ukai [7] obtained a complete derivation of the Navier-Stokes equations for the Boltzmann equation in the case of small initial data for the Navier-Stokes equations – at variance with the strategy outlined in [5], the proof by Bardos and Ukai rests on the spectral analysis of the linearized equation, instead of energy bounds and compactness estimates. Unlike Theorem 4.1, this method cannot be applied to initial data of arbitrary size.

5. Sketch of the convergence proof

First, we recast the Boltzmann equation (3.1) in the hydrodynamic time and space variables. In other words, consider the relative number density fluctuation g_ϵ defined by

$$g_\epsilon(t,x,v) = \frac{F_\epsilon\left(\frac{t}{\epsilon^2}, \frac{x}{\epsilon}, v\right) - M(v)}{\epsilon M(v)}, \qquad \text{where } M(v) = \tfrac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}. \tag{5.1}$$

In terms of g_ϵ, the Boltzmann equation (3.1) becomes

$$\epsilon\partial_t g_\epsilon + v\cdot\nabla_x g_\epsilon + \frac{1}{\epsilon}\mathcal{L}g_\epsilon = \mathcal{Q}(g_\epsilon, g_\epsilon), \tag{5.2}$$

where the linearized collision operator $\mathcal{L}$ and the quadratic operator $\mathcal{Q}$ are defined in terms of the collision integral $\mathcal{C}$ by the formulas

$$\mathcal{L}g = -M^{-1}D\mathcal{C}[M](Mg), \qquad \mathcal{Q}(g,g) = \tfrac{1}{2}M^{-1}D^2\mathcal{C}[M](Mg, Mg). \tag{5.3}$$

Notice that, since F_ϵ is a renormalized solution of (3.1), its fluctuation g_ϵ does not satisfy (5.2), but a renormalized form thereof. However, for the sake of simplicity, we proceed as if g_ϵ did satisfy (5.2). In other words, this amounts to assuming that, for each $\epsilon > 0$, the number density F_ϵ is a classical solution of the Boltzmann equation, *without uniform regularity bounds in the vanishing ϵ limit.* In some sense, this lack of uniformity is the essential difficulty to overcome in this type of problem.

We recall the following important property of the linearized collision operator.

Lemma 5.1 (Hilbert [26])**.** *The operator $\mathcal{L}$ is a nonnegative, Fredholm, self-adjoint unbounded operator on $L^2(\mathbf{R}^3; Mdv)$ with*

$$\ker \mathcal{L} = \operatorname{span}\{1, v_1, v_2, v_3, |v|^2\}\,.$$

5.1. Step 1: Asymptotic fluctuations. First, we seek the asymptotic form of the number density fluctuations g_ϵ in the vanishing ϵ limit.

Multiplying the Boltzmann equation (5.2) by ϵ and letting $\epsilon \to 0$ suggests that

$$g_\epsilon \to g \text{ in the sense of distributions on } \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3 \text{ with } \mathcal{L}g = 0\,.$$

By Hilbert's lemma, g is an *infinitesimal Maxwellian*, i.e., is of the form

$$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \tfrac{1}{2}\theta(t,x)(|v|^2 - 3)\,. \tag{5.4}$$

Notice that g is parametrized by its own moments, since

$$\rho = \langle g\rangle\,, \quad u = \langle vg\rangle\,, \quad \text{and } \theta = \langle(\tfrac{1}{3}|v|^2 - 1)g\rangle\,,$$

where the bracket notation designates the Gaussian integral:

$$\langle\phi\rangle = \int_{\mathbf{R}^3} \phi(v)M(v)dv\,.$$

5.2. Step 2: Local conservation laws. Next, we use an extremely important feature of the Boltzmann collision integral.

Proposition 5.2. *For each measurable $f \equiv f(v)$ rapidly decaying at infinity (in the v-variable), the collision integral satisfies*

$$\int_{\mathbf{R}^3} \mathcal{C}(f)dv = \int_{\mathbf{R}^3} v_k\mathcal{C}(f)dv = \int_{\mathbf{R}^3} |v|^2\mathcal{C}(f)dv = 0\,, \quad k = 1,2,3\,. \tag{5.5}$$

Assuming that, for each $\epsilon > 0$, the solution F_ϵ satisfies the decay assumption in the above proposition, the first relation entails the continuity equation

$$\epsilon\partial_t\langle g_\epsilon\rangle + \operatorname{div}_x\langle vg_\epsilon\rangle = 0\,.$$

Passing to the limit in the sense of distributions in this continuity equation, we obtain

$$\operatorname{div}_x\langle vg\rangle = 0\,, \quad \text{or equivalently } \operatorname{div}_x u = 0\,, \tag{5.6}$$

which is the incompressibility condition in the Navier-Stokes equations.

The second relation in (5.5) together with entropy production controls implies that

$$\partial_t\langle vg_\epsilon\rangle + \mathrm{div}_x\left(\langle vg_\epsilon\rangle\otimes\langle vg_\epsilon\rangle\right) - \nu\Delta_x\langle vg_\epsilon\rangle \to 0 \text{ modulo gradients} \tag{5.7}$$

in the sense of distributions on $\mathbf{R}_+^*\times\mathbf{R}^3$. This leads to the Navier-Stokes motion equation in the limit as $\epsilon\to 0$.

Indeed, denoting $A(v) = v\otimes v - \frac{1}{3}|v|^2 I$ (the traceless part of $v\otimes v$), the second relation in (5.5) implies that

$$\partial_t\langle vg_\epsilon\rangle + \mathrm{div}_x\frac{1}{\epsilon}\langle A(v)g_\epsilon\rangle + \nabla_x\frac{1}{\epsilon}\langle\tfrac{1}{3}|v|^2 g_\epsilon\rangle = 0\,. \tag{5.8}$$

Observe that $A\perp\mathrm{span}\{1, v_1, v_2, v_3, |v|^2\}$; by Hilbert's lemma, there exists a unique symmetric matrix field $\hat{A}$ in the domain of $\mathcal{L}$ such that

$$\mathcal{L}\hat{A} = A\,, \qquad \text{with } \hat{A}\perp\ker\mathcal{L}\,.$$

Since $\mathcal{L}$ is self-adjoint on $L^2(Mdv)$,

$$\begin{aligned}\frac{1}{\epsilon}\langle A(v)g_\epsilon\rangle &= \frac{1}{\epsilon}\langle(\mathcal{L}\hat{A})(v)g_\epsilon\rangle\\ &= \left\langle\hat{A}(v)\frac{1}{\epsilon}\mathcal{L}g_\epsilon\right\rangle = \langle\hat{A}\mathcal{Q}(g_\epsilon,g_\epsilon)\rangle - \langle\hat{A}(\epsilon\partial_t + v\cdot\nabla_x)g_\epsilon\rangle\,.\end{aligned} \tag{5.9}$$

Let Π be the orthogonal projection on $\ker\mathcal{L}$ in $L^2(\mathbf{R}^3;Mdv)$: for each $\phi\in L^2(\mathbf{R}^3;Mdv)$, one has

$$\Pi\phi = \langle\phi\rangle + v\cdot\langle v\phi\rangle + \tfrac{1}{2}(|v|^2-3)\langle(\tfrac{1}{3}|v|^2-1)\phi\rangle\,.$$

Because of step 1, one expects that g_ϵ can be replaced by Πg_ϵ as $\epsilon\to 0$ in the right-hand side of (5.9). Hence

$$\begin{aligned}\frac{1}{\epsilon}\langle A(v)g_\epsilon\rangle &\simeq \langle\hat{A}\mathcal{Q}(\Pi g_\epsilon,\Pi g_\epsilon)\rangle - \langle\hat{A}v\cdot\nabla_x\Pi g_\epsilon\rangle\\ &= \langle\hat{A}\mathcal{Q}(\Pi g_\epsilon,\Pi g_\epsilon)\rangle - \langle\hat{A}\otimes A\rangle : \nabla_x\langle vg_\epsilon\rangle\end{aligned}$$

in some sense as $\epsilon\to 0$. The contraction in the last term of the right-hand side of the equality above bears on the indices of A and $\nabla_x\langle vg_\epsilon\rangle$; in other words, with the convention of repeated indices,

$$(\langle\hat{A}\otimes A\rangle : \nabla_x\langle vg_\epsilon\rangle)_{ij} = \langle\hat{A}_{ij}A_{kl}\rangle\partial_{x_k}\langle v_l g_\epsilon\rangle\,.$$

The nonlinear term is simplified as follows.

Lemma 5.3. *For each $\phi\in\ker\mathcal{L}$, one has*

$$\mathcal{Q}(\phi,\phi) = \tfrac{1}{2}\mathcal{L}(\phi^2)\,.$$

Proof. Differentiate twice the relation

$$\mathcal{C}(\mathcal{M}_{(\rho,u,\theta)}) = 0$$

with respect to the parameters ρ, u and θ. See [4] for a complete argument. $\square$

Eventually, we arrive at the formula

$$\begin{aligned}\frac{1}{\epsilon}\langle A(v)g_\epsilon\rangle &\simeq \tfrac{1}{2}\langle \hat{A}\mathcal{L}((\Pi g_\epsilon)^2)\rangle - \langle \hat{A}\otimes A\rangle : \nabla_x\langle vg_\epsilon\rangle \\ &= \tfrac{1}{2}\langle A|\Pi g_\epsilon|^2\rangle - \langle \hat{A}\otimes A\rangle : \nabla_x\langle vg_\epsilon\rangle \\ &= \langle vg_\epsilon\rangle\otimes\langle vg_\epsilon\rangle - \tfrac{1}{3}|\langle vg_\epsilon\rangle|^2 I - \nu D(\langle vg_\epsilon\rangle)\,,\end{aligned} \tag{5.10}$$

where $\nu = \frac{1}{10}\langle \hat{A} : A\rangle$ and, for each vector field $\xi \equiv \xi(x) \in \mathbf{R}^3$

$$D(\xi) = \nabla_x\xi + (\nabla_x\xi)^T - \tfrac{2}{3}(\mathrm{div}_x\,\xi)I\,.$$

Substituting the formula (5.10) for the momentum flux in (5.8), and taking into account the incompressibility condition (5.6), we arrive at the asymptotic momentum conservation law (5.7).

Actually, we do not know whether renormalized solutions of the Boltzmann equation (3.1) satisfy the local conservation laws of momentum and energy that Proposition 5.2 would entail in the case of classical solutions of (3.1) that are rapidly decaying as $|v| \to +\infty$. Instead of following exactly the argument described above, one must consider an approximate local conservation law of momentum modulo a defect term that vanishes as $\epsilon \to 0$. This leads to technical complications much too intricate to be described here.

5.3. Compactness arguments. The DiPerna-Lions entropy inequality gives *a priori* bounds on the number density fluctuations that are uniform in ϵ; it was proved in [5] that

$$(1+|v|^2)g_\epsilon \text{ is weakly relatively compact in } L^1_{\mathrm{loc}}(\mathbf{R}_+\times\mathbf{R}^3_x; L^1(\mathbf{R}^3_v))\,.$$

Hence, modulo extracting subsequences, for each $\phi \equiv \phi(v) = O(|v|^2)$ as $|v| \to +\infty$, one has

$$\phi g_\epsilon \to \phi g \text{ weakly in } L^1_{\mathrm{loc}}(\mathbf{R}_+\times\mathbf{R}^3_x; L^1(\mathbf{R}^3_v))\,,$$

and this justifies passing to the limit in expressions that are linear in g_ϵ.

It remains to pass to the limit in the nonlinear term, i.e., to justify that

$$\mathrm{div}(\langle vg_\epsilon\rangle\otimes\langle vg_\epsilon\rangle) \to \mathrm{div}(\langle vg\rangle\otimes\langle vg\rangle) \quad \text{modulo gradients as } \epsilon\to 0$$

and this requires a.e. pointwise, instead of weak convergence.

Perhaps the main compactness argument in the proof is a "velocity averaging" lemma, a typical example of which (in a time-independent situation) is as follows:

Lemma 5.4 (F. Golse, L. Saint-Raymond [21])**.** *Let $f_n \equiv f(x,v)$ be a bounded sequence in $L^1(\mathbf{R}^D_x; L^p(\mathbf{R}^D_v))$ for some $p > 1$ such that the sequence $v\cdot\nabla_x f_n$ is bounded in $L^1(\mathbf{R}^D\times\mathbf{R}^D)$. Then*

- *the sequence f_n is weakly relatively compact in $L^1_{\mathrm{loc}}(\mathbf{R}^D\times\mathbf{R}^D)$; and*
- *for each $\phi \in C_c(\mathbf{R}^D)$, the sequence of moments*

$$\int_{\mathbf{R}^D} f_n(x,v)\phi(v)dv \text{ is strongly relatively compact in } L^1_{\mathrm{loc}}(\mathbf{R}^D)\,.$$

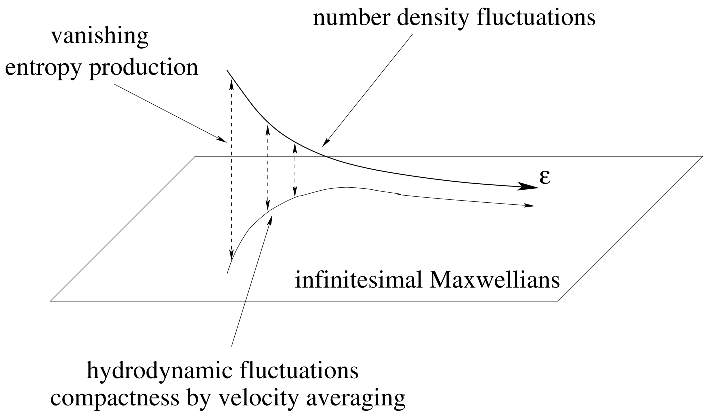

Figure 1. Convergence of the number density fluctuations

With the compactness lemma above, the a.e. pointwise convergence of the number density fluctuations g_ϵ (modulo extraction of a subsequence) is essentially obtained as follows: first, the entropy production bound inferred from (3.4) implies that g_ϵ approaches the manifold of infinitesimal Maxwellians, i.e., the class of functions of the form (5.4) a.e. pointwise. Since an infinitesimal Maxwellian f is parametrized by its velocity averages

$$\int_{\mathbf{R}^3} fMdv\,, \quad \int_{\mathbf{R}^3} vfMdv\,, \quad \int_{\mathbf{R}^3} (\tfrac{1}{3}|v|^2 - 1)fMdv\,,$$

one concludes by applying Lemma 5.4. The situation is summarized in Figure 1.

The idea of gaining compactness in the strong topology by velocity averaging in the context of transport equations is due to F. Golse, B. Perthame and R. Sentis, and appeared for the first time in [20]. This first result was an L^2-variant of the lemma above, and was proved with Fourier techniques, by controlling the small divisors involving the symbol of $v\cdot\nabla_x$. Independently, the regularity of the spherical harmonic coefficients of the solution of the radiative transfer equation was studied in [1]. Later, a systematic study of the regularity and compactness of velocity averages of solutions of transport equations in L^p for all $p \in [1, +\infty)$ appeared in [19]. The L^1-variant of velocity averaging contained in [19] was one of the key arguments in the proof by R.J. DiPerna and P.-L. Lions of global existence of a renormalized solution of the Boltzmann equation in [15].

More recently, velocity averaging results have been generalized to cases where f_n is bounded in $L^p(\mathbf{R}^D_x \times \mathbf{R}^D_v)$ and $v\cdot\nabla_x f_n = \operatorname{div}_x g_n$, with g_n relatively compact in $L^p(\mathbf{R}^D_x; W^{-m,p}(\mathbf{R}^D_v))$ for some $p \in (1, +\infty)$: see [16], [41], [14]. These results are proved with various techniques from harmonic analysis: see Chapter 1 in [8] for a survey as of 2000. This class of results is of considerable importance in the so-called "kinetic formulation" of hyperbolic conservation laws, a topic in some sense analogous to hydrodynamic limits: see [40] for a detailed introduction to this very active research field.

As for the $L^1_x(L^p_v)$ case considered in the lemma above, its proof is based on a representation of the solution in physical space (instead of Fourier space). One of the key ideas in the proof of this result is that the group generated by $v \cdot \nabla_x$, defined by the formula

$$e^{tv\cdot\nabla_x}\phi(x,v) = \phi(x+tv,v)$$

exchanges x- and v-regularity for $t \neq 0$. This implies dispersion estimates "à la Strichartz" (see [12], and also Chapter 1 in [8]); the proof of the velocity averaging lemma above is based on these dispersion estimates together with an interpolation argument somewhat reminiscent of [32]. A preliminary version of Lemma 5.4 was used in [43].

6. Other hydrodynamic limits

Hydrodynamic models other than the incompressible Navier-Stokes equations can also be derived from the Boltzmann equation. Here are some examples.

6.1. The incompressible Euler limit. Let $u^{in} \equiv u^{in}(x) \in \mathbf{R}^3$ satisfy $u^{in} \in H^3(\mathbf{R}^3, \mathbf{R}^3)$ and $\operatorname{div}_x u^{in} = 0$; let $u \in C([0,T); H^3(\mathbf{R}^3,\mathbf{R}^3))$ be the maximal solution of the incompressible Euler equations (see Kato [27])

$$\begin{gathered} \partial_t u + (u\cdot\nabla_x)u + \nabla_x p = 0\,, \qquad \operatorname{div}_x u = 0\,, \\ u\big|_{t=0} = u^{in}\,. \end{gathered} \tag{6.1}$$

These equations can be derived from the Boltzmann equation in the following manner.

Theorem 6.1 (L. Saint-Raymond [45])**.** *For each $\epsilon > 0$, let $\delta_\epsilon = \epsilon^a$ with $a \in (0,1)$ and let F^{in}_ϵ be defined as*

$$F^{in}_\epsilon(x,v) = \mathcal{M}_{(1,\delta_\epsilon u^{in}(\epsilon x),1)}(v)\,.$$

Let F_ϵ be a renormalized solution of the Boltzmann equation (3.1) *with initial data $F_\epsilon\big|_{t=0} = F^{in}_\epsilon$. Then, in the limit as $\epsilon \to 0$, one has*

$$\frac{1}{\delta_\epsilon}\int_{\mathbf{R}^3} vF_\epsilon\left(\frac{t}{\epsilon\delta_\epsilon},\frac{x}{\epsilon},v\right)dv \to u(t,x)$$

in $L^\infty([0,T'];L^1_{\mathrm{loc}}(\mathbf{R}^3))$ for each $T' \in (0,T)$ as $\epsilon \to 0$, where u is the maximal solution of (6.1) *on $[0,T)\times\mathbf{R}^3$.*

The proof of this result differs from that of the Navier-Stokes limit. In particular, under the scaling assumption leading to the incompressible Euler equations, the entropy production rate in the Boltzmann equation does not balance the action of the streaming operator on F_ϵ, which makes it impossible to apply the velocity averaging compactness lemma as in the Navier-Stokes limit. Here, the compactness of hydrodynamic fluctuations is obtained as a consequence of the stability (under perturbations of the initial data) of smooth solutions of the incompressible Euler equations. This theorem is proved by a variant of the relative entropy method (see H.-T. Yau [50] on the hydrodynamic limit of interacting diffusions on a lattice). Preliminary versions of the theorem above can be found in [8] and [34]; see also [42] for the BGK model of the Boltzmann equation.

However, the main feature of the relative entropy method is that the target equation (in this case the incompressible Euler equations) should have local smooth solutions.

6.2. The acoustic limit. Here is another example of a hydrodynamic limit of the Boltzmann equation, leading to a model for compressible fluids. Consider the acoustic system

$$\begin{aligned} \partial_t\rho + \operatorname{div}_x u &= 0\,, \\ \partial_t u + \nabla_x(\rho+\theta) &= 0\,, \qquad (\rho,u,\theta)\big|_{t=0} = (\rho^{in},u^{in},\theta^{in})\,. \\ \tfrac{3}{2}\partial_t\theta + \operatorname{div}_x u &= 0\,, \end{aligned} \tag{6.2}$$

The initial data satisfies

$$\rho^{in},\theta^{in}\in L^2(\mathbf{R}^3)\,,\quad u^{in}\in L^2(\mathbf{R}^3;\mathbf{R}^3)\,.$$

Clearly, the system above essentially reduces to a system of uncoupled wave equations for $\rho+\theta$ and the potential in the Helmholtz decomposition[4] of u, so that the Cauchy problem has a unique solution

$$(\rho,u,\theta)\in C(\mathbf{R};L^2(\mathbf{R}^3)\times L^2(\mathbf{R}^3;\mathbf{R}^3)\times L^2(\mathbf{R}^3))\,.$$

Moreover, the solution map $U(t)$ defined by

$$U(t)(\rho^{in},u^{in},\theta^{in}) = (\rho(t,\cdot),u(t,\cdot),\theta(t,\cdot))$$

is a unitary group on $L^2(\mathbf{R}^3)\times L^2(\mathbf{R}^3;\mathbf{R}^3)\times L^2(\mathbf{R}^3)$.

Theorem 6.2 (F. Golse – C.D. Levermore [18])**.** *Let $\delta_\epsilon>0$ satisfy $\delta_\epsilon|\ln\delta_\epsilon|^{1/2}=o(\sqrt{\epsilon})$, and consider, for each $\epsilon>0$,*

$$F_\epsilon^{in}(x,v) = \mathcal{M}_{(1+\delta_\epsilon\rho^{in}(\epsilon x),\delta_\epsilon u^{in}(\epsilon x),1+\delta_\epsilon\theta^{in}(\epsilon x))}(v)\,.$$

Let F_ϵ be a renormalized solution relative to M of the Boltzmann equation (3.1)*. Then, in the limit as $\epsilon\to 0$, one has*

$$\frac{1}{\delta_\epsilon}\int_{\mathbf{R}^3}\left(F_\epsilon\left(\frac{t}{\epsilon},\frac{x}{\epsilon},v\right)-M\right)\begin{pmatrix}1\\ v\\ (\frac13|v|^2-1)\end{pmatrix}dv\to\begin{pmatrix}\rho(t,x)\\ u(t,x)\\ \theta(t,x)\end{pmatrix}$$

in $L^1_{\mathrm{loc}}(\mathbf{R}_+\times\mathbf{R}^3)$, where (ρ,u,θ) is the solution of the acoustic system (6.2)*.*

[4]I.e., $u=u_0-\nabla_x\phi$ with $\operatorname{div}_x u_0=0$.

The proof of this result follows the same pattern as that of the incompressible Navier-Stokes limit. Unfortunately, the condition on the size of the number density fluctuations δ_ϵ is not optimal. A formal argument similar to steps 1–2 in the proof of the incompressible Navier-Stokes limit suggests that the same conclusion should hold under the assumption that only $\delta_\epsilon \to 0$ as $\epsilon \to 0$. Since we do not know whether renormalized solutions of the Boltzmann equation (3.1) satisfy the local conservation laws implied by Proposition 5.2 in the case of classical solutions of (3.1) that are rapidly decaying as $|v| \to +\infty$, the analogue of step 2 in the proof of the incompressible Navier-Stokes limit involves variants of these local conservation laws of momentum and energy modulo defect terms that vanish as $\epsilon \to 0$, provided that δ_ϵ satisfies the stronger assumption $\delta_\epsilon |\ln \delta_\epsilon|^{1/2} = o(\sqrt{\epsilon})$.

6.3. Models involving a heat equation. In fact, the result obtained in [22] or in [23] leads to the Navier-Stokes equations coupled with a drift-diffusion equation for (fluctuations of) the temperature field, i.e., the Navier-Stokes-Fourier system

$$\begin{aligned} \partial_t u + \operatorname{div}_x(u \otimes u) + \nabla_x p = \nu \Delta_x u \,, \qquad \operatorname{div}_x u = 0 \,, \\ \partial_t \theta + \operatorname{div}_x(u\theta) = \kappa \Delta_x \theta \,. \end{aligned} \tag{6.3}$$

The heat conductivity κ is given by a formula similar to (4.1), i.e.,

$$\kappa = \tfrac{4}{15} \mathcal{D}^* \left(\tfrac{1}{2}(|v|^2 - 5)v\right) .$$

A rigorous derivation of the linear variant of this system (i.e., the Stokes-Fourier system) from renormalized solutions of the Boltzmann equation can be found in [18]; previously, the evolution Stokes equations (for the velocity field only) had been similarly obtained by P.-L. Lions and N. Masmoudi in [34].

More elaborate asymptotic limits leading to a viscous heating term in the right-hand side of the drift-diffusion equation for the temperature field have been formally derived from the Boltzmann equation in [6], but obtaining a complete mathematical argument justifying this derivation remains a real challenge.

7. Open problems

An outstanding open problem in this field is the derivation of the Euler equations for compressible fluids from the Boltzmann equation. The compressible Euler system (for a perfect monatomic gas) is

$$\begin{aligned} \partial_t \rho + \operatorname{div}_x(\rho u) = 0 \,, \\ \partial_t(\rho u) + \operatorname{div}_x(\rho u \otimes u) + \nabla_x(\rho\theta) = 0 \,, \\ \partial_t\left(\rho\left(\tfrac{1}{2}|u|^2 + \tfrac{3}{2}\theta\right)\right) + \operatorname{div}_x\left(\rho u\left(\tfrac{1}{2}|u|^2 + \tfrac{3}{2}\theta\right)\right) = 0 \,, \end{aligned} \tag{7.1}$$

where $\rho \equiv \rho(t,x) \geq 0$ is the density of the fluid at time t and position x, while $\theta \equiv \theta(t,x) > 0$ is the temperature field and $u \equiv u(t,x) \in \mathbf{R}^3$ the velocity field.

This is a system of conservation laws with an entropy

$$\eta(\rho,u,\theta)=\rho\ln\left(\frac{\rho}{\theta^{3/2}}\right)$$

that is a convex function of ρ, ρu and $\rho(\frac{1}{2}|u|^2+\frac{3}{2}\theta)$ (the conserved densities). Hence (7.1) is a symmetrizable hyperbolic system, for which the Cauchy problem has local smooth solutions: see for instance the book by A. Majda [36]. It is known that, for a large class of initial data, the solution of (7.1) becomes singular in finite time (see [46]). Yet, the existence of global weak solutions of (7.1) is still unknown – and a major open problem of the theory of hyperbolic systems.

However, in the case where ρ, u and θ only depend upon one space variable (say, x_1), global existence of a weak solution to (7.1) for which η decreases across shock waves has been proved for initial data with small total variation. This result stems from Glimm's remarkable paper [17] and is due to T.-P. Liu [35].

So far, solutions of (7.1) have been derived from solutions of the Boltzmann equation (3.1) in the regularity phase: see [38], [11], [28]. The idea is to start from initial data of the form

$$F_\epsilon^{in}(x,v)=\mathcal{M}_{(\rho^{in}(\epsilon x),u^{in}(\epsilon x),\theta^{in}(\epsilon x))}$$

parametrized by $\epsilon>0$. For each $\epsilon>0$, let F_ϵ be a solution of (3.1) such that $F_\epsilon\big|_{t=0}=F_\epsilon^{in}$; then, one shows that the hydrodynamic moments of F_ϵ

$$\int_{\mathbf{R}^3}F_\epsilon\left(\frac{t}{\epsilon},\frac{x}{\epsilon},v\right)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix}dv\to\begin{pmatrix}\rho(t,x)\\ \rho u(t,x)\\ \rho(|u|^2+3\theta)(t,x)\end{pmatrix}$$

as $\epsilon\to 0$, where (ρ,u,θ) is the solution of (7.1) with initial data $(\rho^{in},u^{in},\theta^{in})$. The convergence above is of course local in time – at best over the lifespan of a smooth solution of (7.1).

It would be of considerable interest to derive the global BV solutions constructed by T.-P. Liu from the Boltzmann equation. As in the case of the incompressible Euler limit of the Boltzmann equation, the entropy production bound entailed by Boltzmann's H Theorem does not balance the action of the streaming operator on the number density: the compactness of hydrodynamic moments of the number density is probably to be sought in some stability property of BV solutions of the compressible Euler system. Most likely, such a theory should use Bressan's remarkable results in that direction (see [9], [10]).

Another open problem would be to improve Theorem 6.2, by relaxing the unphysical assumption made on the size of the number density fluctuations δ_ϵ to reach the physically natural condition that $\delta_\epsilon\to 0$ as $\epsilon\to 0$. This will probably require more information on the local conservation laws of momentum and energy for renormalized solutions of the Boltzmann equation. Such information would most likely be an important prerequisite for progress on the compressible Euler limit.

Finally, we have only treated evolution problems in this paper. In fact, steady problems are perhaps even more important for applications (as in aerodynamics). For instance, it is well known that, for any force field $f \equiv f(x) \in L^2(\Omega; \mathbf{R}^3)$ such that $\operatorname{div}_x f = 0$, the steady incompressible Navier-Stokes equations in a smooth, bounded open domain $\Omega \subset \mathbf{R}^3$

$$\begin{aligned} &-\nu \Delta_x u = f - \nabla_x p - (u \cdot \nabla_x) u\,, \quad \operatorname{div}_x u = 0\,, \qquad x \in \Omega\,, \\ &\quad u\big|_{\partial\Omega} = 0 \end{aligned} \tag{7.2}$$

has at least one classical solution $u \equiv u(x) \in H^2(\Omega, \mathbf{R}^3)$, obtained by a Leray-Schauder fixed point argument (see for instance [29]). Unfortunately, the parallel theory for the Boltzmann equation is not as advanced: see however the classical papers by Guiraud [24], and more recent work by L. Arkeryd and A. Nouri (see for instance [2]). Yet, the fact that the solutions of (7.2) are more regular than in the case of the evolution problem could be of considerable help in the context of the hydrodynamic limit. A rather exhaustive description of these kinds of problems (at the formal level) may be found in the recent monograph by Y. Sone [48]

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