The Lorentz gas

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$\boxed{\textbf{Lecture 4}}$

- Computation of the transition probability $P(s,h|h')$

- A kinetic equation in an extended phase space for the BG limit of the periodic Lorentz gas

Computation of the probability density $P(s,h|h')$

•Using (a consequence of) Birkhoff's ergodic theorem for the Gauss map, we have proved the existence of a transition probability density $P(s,h|h')$ independent of v such that

$$\boxed{\frac{1}{|\ln \eta|}\int_{\eta}^{1/4} f(T_r(h',v))\frac{dr}{r} \to \int_0^{\infty}\int_{-1}^{1} \Phi(s,h)P(s,h|h')dsdh}$$

a.e. in $v \in \mathbf{S}^1$ as $\eta \to 0^+$, for each $f \in C_c(\mathbf{R}_+^* \times [-1,1])$ and each $h' \in [-1,1]$.

•Applying Birkhoff's ergodic theorem to a function of $\epsilon q_{N(\alpha,\epsilon)}(\alpha)$ requires replacing $q_{N(\alpha,\epsilon)}(\alpha)$ by a truncated series involving only the quantities $d_n(\alpha)$s. This shows that the limit exists, but without computing it explicitly.

Theorem. (E. Caglioti, F.G. 2007) *The transition probability density $P(s,h|h')$ is given in terms of $a = \frac{1}{2}|h-h'|$ and $b = \frac{1}{2}|h+h'|$ by the explicit formula*

$$\boxed{\begin{aligned} P(s,h|h') = \frac{3}{\pi^2 sa}\Big[&\Big((s-\tfrac{1}{2}sa)\wedge(1+\tfrac{1}{2}sa)-(1\vee(\tfrac{1}{2}s+\tfrac{1}{2}sb)\Big)_+ \\ &+\Big((s-\tfrac{1}{2}sa)\wedge 1-((\tfrac{1}{2}s+\tfrac{1}{2}sb)\vee\Big(1-\tfrac{1}{2}sa\Big)\Big)_+ \\ &+sa\wedge|1-s|\mathbf{1}_{s<1}+(sa-|1-s|)_+\Big] \end{aligned}}$$

with the notations $x\wedge y = \min(x,y)$ *and* $x\vee y = \max(x,y)$.

Moreover, the function

$$(s,h,h') \mapsto (1+s)P(s,h|h') \textit{ belongs to } L^2(\mathbf{R}_+\times[-1,1]^2)$$

•In fact, the key result bears on the asymptotic distribution of 3-obstacle collision patterns:

Theorem. (E. Caglioti, F.G. 2007) *Define* $\mathbf{K} = [0,1]^3 \times \{\pm 1\}$*; then, for each* $F \in C(\mathbf{K})$

$$\frac{1}{|\ln \eta|}\int_\eta^{1/4} F((A,B,Q,\Sigma)(v,r))\frac{dr}{r} \to \mathcal{L}(F)$$
$$= \int_{\mathbf{K}} F(A,B,Q,\Sigma)d\mu(A,B,Q,\Sigma) \text{ a.e. in } v \in \mathbf{S}^1$$

as $\eta \to 0^+$*, where*

$$d\mu(A,B,Q,\Sigma) = d\nu(A,B,Q) \otimes \tfrac{1}{2}(\delta_{\Sigma=1} + \delta_{\Sigma=-1})$$
$$d\nu(A,B,Q) = \tfrac{12}{\pi^2}\mathbf{1}_{0<A<1}\mathbf{1}_{0<B<1-A}\mathbf{1}_{0<Q<\frac{1}{2-A-B}}\frac{dAdBdQ}{1-A}$$

•Maybe it is worth explaining why this measure is natural(!)

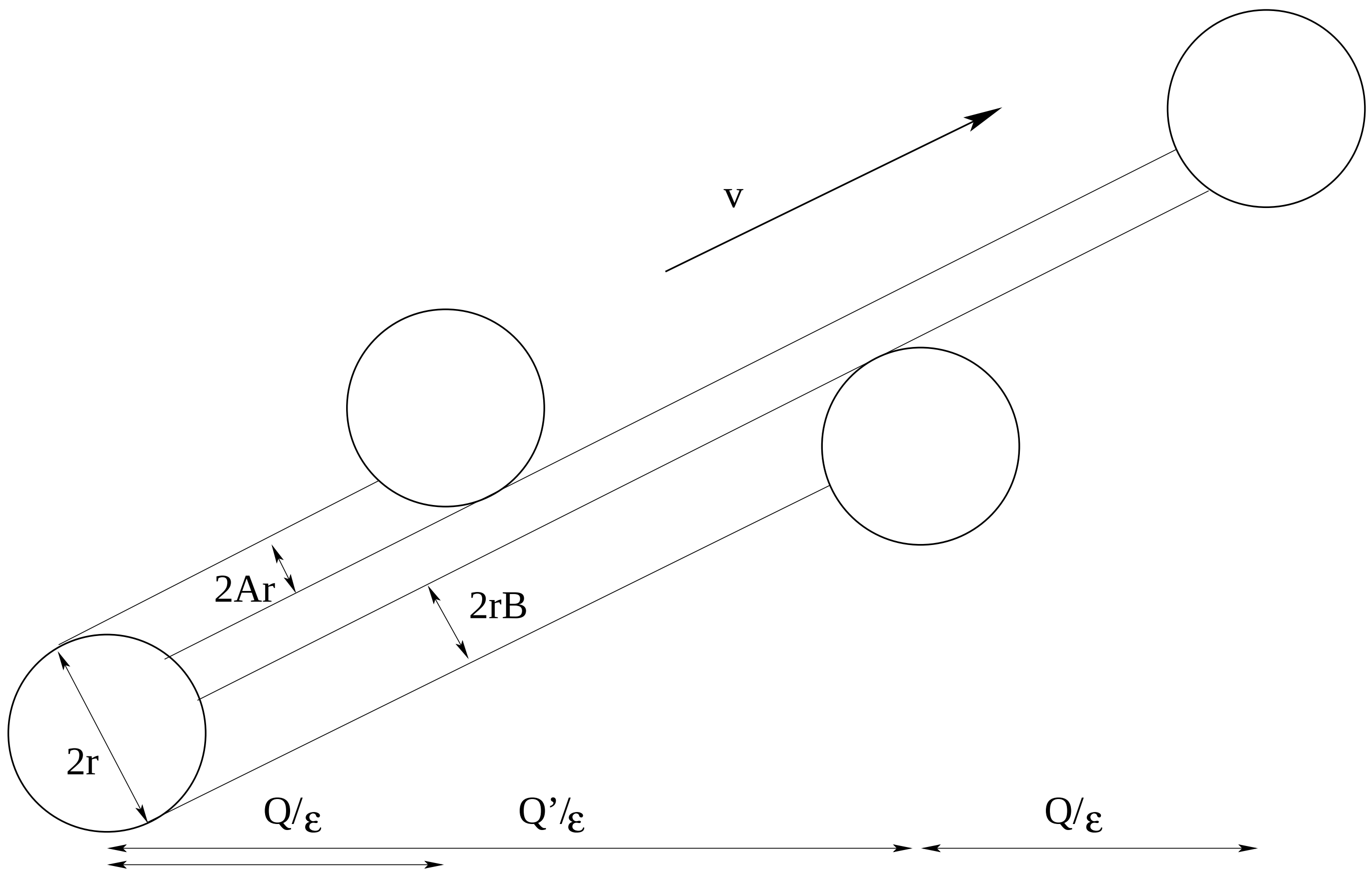
a) the constraints $0 < A < 1$ and $0 < B < 1 - A$ have obvious geometric meaning (see figure);

b) likewise, the total area of the 2-torus is the sum of the areas of the strips consisting of all orbits with the 3 possible lengths:

$$\begin{aligned} 1 = QA + Q'B + (Q + Q')(1 - A - B) &= Q(1 - B) + Q'(1 - A) \\ &\geq Q(2 - A - B) \end{aligned}$$

as $Q' \geq Q$ (see figure again);

c) the volume element $\frac{dAdBdQ}{1-A}$ means that the parameters A, $\frac{B}{1-A}$ (or equivalently B mod. $1 - A$) and Q are INDEPENDENT AND UNIFORMLY DISTRIBUTED in the largest subdomain of $[0, 1]^3$ that is compatible with the geometric constraints



The generic 3-obstacle pattern

$\bullet$Thm2 $\Rightarrow$ the explicit formula for the transition probability $P(s,h|h')$ in Thm1

Indeed, $P(s,h|h')dsdh$ is the image measure of $d\mu(A,B,Q,\Sigma)$ under the map

$$\mathbf{K} \ni (A,B,Q,\Sigma) \mapsto T_{(A,B,Q,\Sigma)}(h',v)$$

That $(1+s)P(s,h|h')$ is square integrable is proved by inspection — by using the explicit formula for $P(s,h|h')$.

METHOD OF PROOF FOR THM 1:

Since we know a priori that the transition probability $P(s,h|,h')$ is independent of v, we only have to compute

$$\lim_{r\to 0^+}\frac{1}{2\pi}\int_{\mathbf{S}^1} f(T_r(h',v))dv \left(=\int_0^\infty\int_{-1}^{1}\Phi(s,h)P(s,h|h')dsdh\right)$$

The method for computing this type of expression is based on

- Farey fractions (a.k.a. slow continued fractions)

- estimates for Kloosterman's sums, due to Boca-Zaharescu (2007)

Farey fractions

- Put a filtration on the set of rationals in $[0,1]$ as follows

$$\mathcal{F}_{\mathcal{Q}} = \{\tfrac{p}{q} \,|\, 0 \le p \le q \le \mathcal{Q}\,,\ \text{q.c.d.}(p,q) = 1\}$$

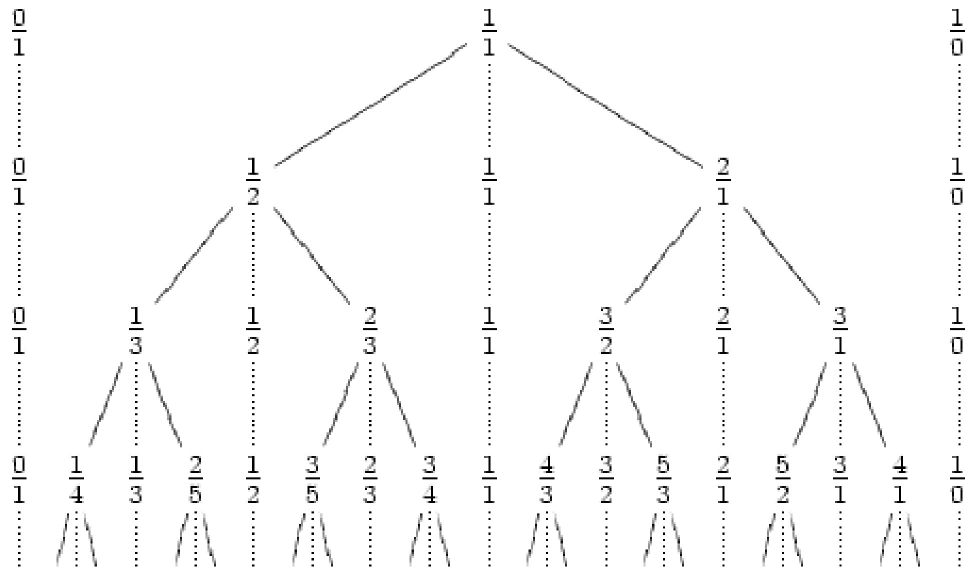
indexed in increasing order:

$$0 = \frac{0}{1} < \gamma_1 < \ldots < \gamma_j = \frac{p_j}{q_j} < \ldots < \gamma_{\varphi(\mathcal{Q})} = \frac{1}{1} = 1$$

(φ being Euler's totient function)

- <u>MEDIANT</u>: given $\gamma = \frac{p}{q}$ and $\widehat{\gamma} = \frac{\widehat{p}}{\widehat{q}}$ with $0 \le p \le q$, $0 \le \widehat{p} \le \widehat{q}$, and g.c.d.$(p,q) =$ g.c.d.$(\widehat{p},\widehat{q}) = 1$

$$\boxed{\text{mediant} = \gamma \oplus \widehat{\gamma} = \frac{p + \widehat{p}}{q + \widehat{q}} \in (\gamma, \widehat{\gamma})}$$


0/1 1/1 1/0
0/1 1/2 1/1 2/1 1/0
0/1 1/3 1/2 2/3 1/1 3/2 2/1 3/1 1/0
0/1 1/4 1/3 2/5 1/2 3/5 2/3 3/4 1/1 4/3 3/2 5/3 2/1 5/2 3/1 4/1 1/0

•Hence, if $\gamma = \frac{p}{q} < \widehat{\gamma} = \frac{\widehat{p}}{\widehat{q}}$ adjacent in $\mathcal{F}_{\mathcal{Q}}$, then

$$\widehat{a}q - a\widehat{q} = 1 \text{ and } q + \widehat{q} > \mathcal{Q}$$

Conversely, $q, \widehat{q}$ are denominators of adjacent fractions in $\mathcal{F}_{\mathcal{Q}}$ iff

$$\boxed{0 \leq q, \widehat{q} \leq \mathcal{Q}\,, \quad q + \widehat{q} > \mathcal{Q}\,, \quad \text{g.c.d.}(q, q') = 1}$$

•Given $\alpha \in (0,1) \setminus \mathbf{Q}$ and $\mathcal{Q} \geq 1$, there exists a unique pair of adjacent Farey fractions in $\mathcal{F}_{\mathcal{Q}}$, say, γ and γ' such that

$$\boxed{\gamma(\alpha, \mathcal{Q}) = \frac{p(\alpha, \mathcal{Q})}{q(\alpha, \mathcal{Q})} < \alpha < \widehat{\gamma}(\alpha, \mathcal{Q}) = \frac{\widehat{p}(\alpha, \mathcal{Q})}{\widehat{q}(\alpha, \mathcal{Q})}}$$

RELATION WITH CONTINUED FRACTIONS:

Pick $0 < \epsilon < 1$; we recall that, for each $\alpha \in (0,1) \setminus \mathbf{Q}$

$$N(\alpha, \epsilon) = \min\{n \in \mathbf{N} \,|\, d_n(\alpha) \leq \epsilon\}\,, \quad d_n(\alpha) = \mathrm{dist}(q_n(\alpha)\alpha, \mathbf{Z})$$

Set $\mathcal{Q} = [1/\epsilon]$, and let $\gamma(\alpha, \mathcal{Q}) < \hat{\gamma}(\alpha, \mathcal{Q})$ be the two adjacent Farey fractions in $\mathcal{F}_{\mathcal{Q}}$ surrounding α. Then

•one of the two integers $q(\alpha, \mathcal{Q})$ and $\hat{q}(\alpha, \mathcal{Q})$ is $q_{N(\alpha,\epsilon)}(\alpha)$

•the other is of the form

$$m q_{N(\alpha,\epsilon)} + q_{N(\alpha,\epsilon)-1} \text{ with } 0 \leq m \leq a_{N(\alpha,\epsilon)}(\alpha)$$

where we recall that

$$\alpha = [0; a_1, a_2, \ldots] = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \ldots}}$$

Setting $\alpha = v_2/v_1$ and $\epsilon = 2r/v_1$, we recall that definition

$$Q(v,r) = \epsilon q_{N(\alpha,\epsilon)}(\alpha) \in \{\epsilon q(\alpha, \mathcal{Q}), \epsilon \widehat{q}(\alpha, \mathcal{Q})\} \text{ with } \mathcal{Q} = [1/\epsilon]$$

and further define

$$D(v,r) = d_{N(\alpha,\epsilon)}/\epsilon = \text{dist}(\tfrac{1}{\epsilon}Q(v,r)\alpha, \mathbf{Z})/\epsilon$$

and

$$\tilde{Q}(v,r) = \epsilon \widehat{q}(\alpha, \mathcal{Q}) \text{ if } q_{N(\alpha,\epsilon)}(\alpha) = q(\alpha, \mathcal{Q})$$
$$\tilde{Q}(v,r) = \epsilon q(\alpha, \mathcal{Q}) \text{ if } q_{N(\alpha,\epsilon)}(\alpha) = \widehat{q}(\alpha, \mathcal{Q})$$

Now, we recall that $A(v,r) = 1 - D(v,r)$; moreover, we see that

$$\boxed{\begin{aligned} B(v,r) &= 1 - \frac{d_{N(\alpha,\epsilon)-1}(\alpha)}{\epsilon} - \left[\frac{1 - d_{N(\alpha,\epsilon)-1}(\alpha)/\epsilon}{D(v,r)}\right] D(v,r) \\ &= 1 - d_{N(\alpha,\epsilon)-1}(\alpha)/\epsilon \text{ mod. } D(v,r) \\ \\ &= 1 - \text{dist}(\tfrac{1}{\epsilon}\tilde{Q}(v,r)\alpha, \mathbf{Z})/\epsilon \text{ mod. } D(v,r) \end{aligned}}$$

To cut a long story short:

$$\boxed{F(A(v,r), B(v,r), Q(v,r)) = G(Q(v,r), \widehat{Q}(v,r), D(v,r))}$$

and we are left with the task of computing

$$\lim_{r \to 0+} \int_{\mathbf{S}^1_+} G(Q(v,r), \widehat{Q}(v,r), D(v,r))dv$$

where $\mathbf{S}^1_+$ is the first octant in the unit circle

The other octants in the unit circle give the same contribution by obvious symmetry arguments.

More specifically:

Lemma. *Let* $\alpha \in (0,1) \setminus \mathbf{Q}$*, and let* $\frac{p}{q} < \alpha < \frac{\widehat{p}}{\widehat{q}}$ *be the two adjacent Farey fractions in* $\mathcal{F}_{\mathcal{Q}}$ *surrounding* α*, with* $\mathcal{Q} = [1/\epsilon]$*. Then*

if $\frac{p}{q} < \alpha \leq \frac{\widehat{p}-\epsilon}{\widehat{q}}$ *then*

$$Q(v,r) = \epsilon q\,, \quad \tilde{Q}(v,r) = \epsilon \widehat{q}\,, \quad D(v,r) = \frac{1}{\epsilon}(\alpha q - p)$$

if $\frac{p+\epsilon}{q} < \alpha < \frac{\widehat{p}}{\widehat{q}}$ *then*

$$Q(v,r) = \epsilon \widehat{q}\,, \quad \tilde{Q}(v,r) = \epsilon q\,, \quad D(v,r) = \frac{1}{\epsilon}(\widehat{p} - \alpha \widehat{q})$$

if $\frac{p+\epsilon}{q} < \alpha \leq \frac{\widehat{p}-\epsilon}{\widehat{q}}$ *then*

$$Q(v,r) = \epsilon q \wedge \widehat{q}\,, \quad \tilde{Q}(v,r) = \epsilon q \vee \widehat{q}\,, \quad D(v,r) = \textit{dist}(\tfrac{1}{\epsilon}Q(v,r)\alpha, \mathbf{Z})$$

Therefore, assuming for simplicity

$$G(x,y,z) = g(x,y)H'(z) \text{ and } \epsilon = 1/\mathcal{Q}$$

one has

$$\begin{aligned}
&\int_{\mathbf{S}^1_+} G(Q(v,r), \widehat{Q}(v,r), D(v,r))dv \\
&= \sum_{\substack{0<q,\widehat{q}\leq\mathcal{Q}<q+\widehat{q} \\ g.c.d.(q,\widehat{q})=1}} \int_{p/q}^{(\widehat{p}-\epsilon)/\widehat{q}} g\left(\frac{q}{\mathcal{Q}}, \frac{\widehat{q}}{\mathcal{Q}}\right) H'(\mathcal{Q}(q\alpha - p))d\alpha \\
&\qquad + \text{three other similar terms} \\
&= \sum_{\substack{0<q,\widehat{q}\leq\mathcal{Q}<q+\widehat{q} \\ g.c.d.(q,\widehat{q})=1}} g\left(\frac{q}{\mathcal{Q}}, \frac{\widehat{q}}{\mathcal{Q}}\right) \frac{1}{q\mathcal{Q}} \left(H\left(\frac{1-q/\mathcal{Q}}{\widehat{q}/\mathcal{Q}}\right) - H(0)\right) \\
&\qquad + \text{three other similar terms}
\end{aligned}$$

Therefore, everything reduces to computing

$$\frac{1}{\mathcal{Q}^2} \sum_{\substack{0<q,\widehat{q}\leq\mathcal{Q}<q+\widehat{q} \\ g.c.d.(q,\widehat{q})=1}} \psi\left(\frac{q}{\mathcal{Q}}, \frac{\widehat{q}}{\mathcal{Q}}\right)$$

Lemma. (Boca-Zaharescu) *For $\psi \in C_c(\mathbf{R}^2)$, one has*

$$\boxed{\frac{1}{\mathcal{Q}^2} \sum_{\substack{0<q,\widehat{q}\leq\mathcal{Q}<q+\widehat{q} \\ g.c.d.(q,\widehat{q})=1}} \psi\left(\frac{q}{\mathcal{Q}}, \frac{\widehat{q}}{\mathcal{Q}}\right) \to \frac{6}{\pi^2} \iint_{0<x,y<1<x+y} \psi(x,y)\,dx\,dy}$$

in the limit as $\mathcal{Q} \to \infty$.

With the method outlined above, Boca and Zaharescu were able to compute the limiting distribution of free path length: remember that, in space dimension 2, we proved that

$$\frac{1}{|\ln \eta|}\int_{\eta}^{1/4} \text{Prob}(\{x \,|\, r\tau_r(x,v) > t\})\frac{dr}{r} \to \Phi(t)$$

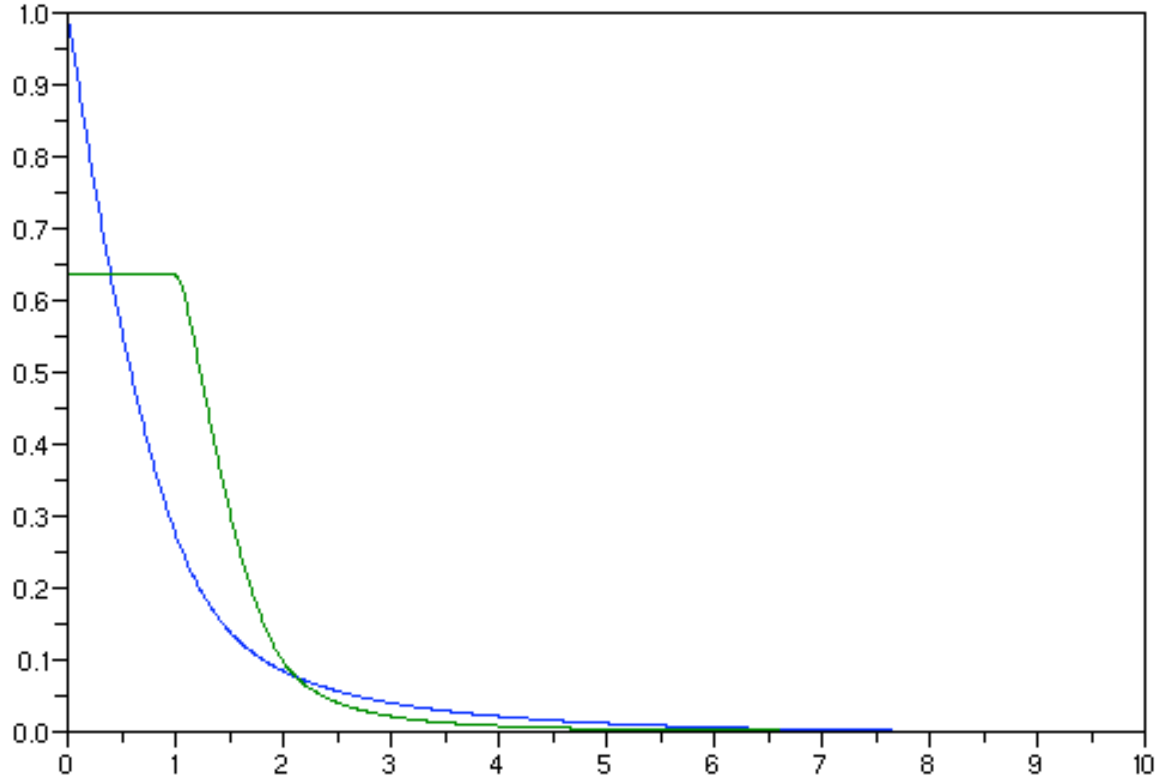
a.e. in $v \in \mathbf{S}^1$ as $\eta \to 0^+$.

Theorem. (Boca-Zaharescu, 2007) *For each* $t > 0$

$$\textit{Prob}(\{(x,v) \,|\, r\tau_r(x,v) > t) \to \Phi(t) = \frac{6}{\pi^2}\int_t^{\infty}(s-t)g(s)ds$$

where

$$\boxed{g(s) = \begin{cases} 1 & s \in [0,1] \\ \frac{1}{s} + 2\left(1-\frac{1}{s}\right)^2 \ln(1-\frac{1}{s}) - \frac{1}{2}\left|1-\frac{2}{s}\right|^2 \ln\left|1-\frac{2}{s}\right| & s \in (1,\infty) \end{cases}}$$



Graph of $\Phi(t)$ (blue curve) and $g(t) = \Phi''(t)$ (green curve)

A (plausible?) conjecture for the dynamics in the BG limit

For each $r \in]0, \frac{1}{2}[$, denote

$$\Gamma_r^+ = \{(x,v) \in \partial\Omega \times \mathbf{S}^1 \,|\, v \cdot n_x \geq 0\}\,, \quad d\gamma_r^+(x,v) = \frac{v \cdot n_x dx dv}{\int_{\Gamma_r^+} v \cdot n_x dx dv}$$

Consider the billiard map:

$$\mathbf{B}_r : \Gamma_r^+ \ni (x,v) \mapsto \mathbf{B}_r(x,v) = (x+\tau_r(x,v)v, \mathcal{R}[x+\tau_r(x,v)v]v) \in \Gamma_r^+$$

For $(x_0, v_0) \in \Gamma_r^+$, set

$$(x_n, v_n) = \mathbf{B}_r^n(x_0, v_0) \text{ and } \alpha_n = \min(|v_2/v_1|, |v_1/v_2|)$$

and define

$$b_r^n = (A, B, Q, N \text{ mod. } 2)(\alpha^n, r)\,, \quad n \in \mathbf{N}^*$$

Notation: $Q_n := \mathbf{R}^2 \times \mathbf{S}^1 \times \mathbf{R}_+ \times [-1,1] \times K^n$.

•We make the following asymptotic independence hypothesis: there exists a probability measure Π on $\mathbf{R}_+ \times [-1,1]$ such that , for each $n \geq 1$ and each $\Psi \in C(Q_n)$ with compact support

$$(H) \quad \begin{aligned} &\lim_{r \to 0^+} \int_{Z_r \times \mathbf{S}^1} \Psi(x, v, r\tau_r(\tfrac{x}{r}, v), h_r(\tfrac{x_1}{r}, v_1), b_r^1, \ldots, b_r^n) dx dv \\ &= \int_{Q_n} \Psi(x, v, \tau, h, \beta_1, \ldots, \beta_n) dx dv d\Pi(\tau, h) d\mu(\beta_1) \ldots d\mu(\beta_n) \end{aligned}$$

where

$$(x_0, v_0) = (x - \tau_r(x, -v)v, v)\,, \text{ and } h_r(\tfrac{x_1}{r}, v_1) = \sin(n_{x_1}, v_1)$$

If this holds, the iterates of the transfer map T_r are described by the Markov chain with transition probability $P(s, h|h')$. This leads to a kinetic equation on an extended phase space for the Boltzmann-Grad limit of the periodic 2D Lorentz gas:

$$F(t, x, v, s, h) =$$

density of particles with velocity v and position x at time t

that will hit an obstacle after time s, with impact parameter h

Theorem. (E. Caglioti, F.G. 2007) *Assume (H), and let* $f^{in} \geq 0$ *belong to* $C_c(\mathbf{R}^2 \times \mathbf{S}^1)$*. Then one has*

$$f_r \to \int_0^\infty \int_{-1}^1 F(\cdot,\cdot,\cdot,s,h)dsdh \text{ in } L^\infty(\mathbf{R}_+ \times \mathbf{R}^2 \times \mathbf{S}^1) \text{ weak-}*,$$

in the limit as $r \to 0^+$*, where* $F \equiv F(t,x,v,s,h)$ *is the solution of*

$$(\partial_t + v\cdot\nabla_x - \partial_s)F(t,x,v,s,h) = \int_{-1}^1 P(s,h|h')F(t,x,R[\pi - 2\arcsin(h')]v, 0, h')dh'$$

$$F(0,x,v,s,h) = f^{in}(x,v)\int_s^\infty \int_{-1}^1 P(\tau,h|h')dh'd\tau$$

with (x,v,s,h) *running through* $\mathbf{R}^2 \times \mathbf{S}^1 \times \mathbf{R}_+^* \times]-1,1[$*. The notation* $R[\theta]$ *designates the rotation of an angle* θ*.*

CONCLUSION:

Classical kinetic theory (Boltzmann theory for elastic, hard sphere collisions) is based on two fundamental principles

a) deflections in velocity at each collision are mutually independent and identically distributed

b) time intervals between collisions are mutually independent, independent of velocities, and exponentially distributed.

The BG limit of the periodic Lorentz gas provides an example of a non classical kinetic theory where

a') velocity deflections at each collision jointly form a Markov chain;

b') the time intervals between collisions are not independent of the velocity deflections

In both cases, collisions are purely local and instantaneous events (BG limit $\Rightarrow$ point particles)

In a recent preprint (arxiv0801.0612), J. Marklof and A. Strombergsson have proved the Markov property of the limiting process in extended phase space — in other words, assumption (H) — and extended it in higher dimensions