Invent. math. 155, 81–161 (2004)
DOI: 10.1007/s00222-003-0316-5

Inventiones mathematicae

The Navier–Stokes limit of the Boltzmann equation for bounded collision kernels

François Golse[1], **Laure Saint-Raymond**[2]

[1] Institut Universitaire de France, Département de Mathématiques et Applications, Ecole Normale Supérieure Paris, 45, rue d'Ulm, 75230 Paris cedex 05, France (e-mail: `golse@dma.ens.fr`)
[2] Laboratoire d'Analyse Numérique, Université Paris VI, 175, rue du Chevaleret, 75013 Paris, France (e-mail: `saintray@ann.jussieu.fr`)

Oblatum 10-IV-2002 & 12-V-2003
Published online: 9 September 2003 –

Abstract. The present work establishes a Navier–Stokes limit for the Boltzmann equation considered over the infinite spatial domain $\mathbf{R}^3$. Appropriately scaled families of DiPerna-Lions renormalized solutions are shown to have fluctuations whose limit points (in the w-L^1 topology) are governed by Leray solutions of the limiting Navier–Stokes equations. This completes the arguments in Bardos-Golse-Levermore [Commun. Pure Appl. Math. **46**(5), 667–753 (1993)] for the steady case, and in Lions-Masmoudi [Arch. Ration. Mech. Anal. **158**(3), 173–193 (2001)] for the time-dependent case.

Introduction

Hydrodynamic models such as the Euler or Navier–Stokes equations were first established by applying Newton's second law of motion to infinitesimal volume elements of the fluid under consideration. All equations from fluid dynamics can be obtained in this way – see Chap. I of [38]. However, this method fails to relate equations of state (expressing for example the pressure in terms of the density and temperature) or transport coefficients (like the heat conduction or the viscosity) to microscopic data (such as the laws governing molecular interactions). In the particular case of gas dynamics, kinetic theory allows one to express thermodynamic functions and transport coefficients for perfect gases in terms of purely mechanical data concerning collisions between the gas molecules.

In his 6th problem, Hilbert asked for a full mathematical justification of this procedure; in his own words [35]: "[...] Boltzmann's work on the

Mathematics Subject Classification (2000): 35Q35, 35Q30, 82C40

principles of mechanics suggests the problem of developing mathematically the limiting processes [...] which lead from the atomistic view to the laws of motion of continua".

The present work answers one part of this question, namely the limiting process leading from the Boltzmann equation of the classical kinetic theory of gases to the Navier–Stokes equations of incompressible fluids. This limit was first discussed by Y. Sone in [59] in the steady case on the basis of formal asymptotic expansions, and later by C. Bardos, F. Golse and C.D. Levermore in [5], [6] in the time dependent case by a systematic moment-closure method. The first complete mathematical proof of this limit is due to C. Bardos and S. Ukai [11] in the case of small initial data leading to smooth solutions; K. Asano [3] studied independently the same limit for short times. A complete justification of the Hilbert expansion for the incompressible Navier–Stokes limit method as in [59] has been given by A. DeMasi, R. Esposito and J. Lebowitz in [21]. However these methods fail to encompass the generality of all physically admissible initial data for either the Boltzmann or the Navier–Stokes equations, at least as long as it remains unknown whether initially smooth solutions to these equations may develop singularities in finite time. What is worse, the proof based on Hilbert's expansion [21] – at least in its present formulation – leads to solutions of the Boltzmann equation that are not a.e. nonnegative, which is incompatible with the original physical meaning of solutions of the Boltzmann equation as phase space densities. However, it seems[1] that one can obtain nonnegative solutions by adding initial layers to the truncated expansion used in [21], following the method of M. Lachowicz [37] for the compressible Euler limit.

That the Navier–Stokes equation of *incompressible* fluids can be derived from the Boltzmann equation may seem somewhat surprising, since the latter models *compressible* fluids – more precisely, perfect gases. However, it is well known (see for instance [36]) that, for compressible fluids in the low Mach number limit, fluctuations about an equilibrium state are governed by the equations of incompressible fluids. In other words, the limit considered in the present paper concerns incompressible *flows* of a compressible fluid.

At present, the only known theorems giving the global existence of solutions to either equations in the spatial domain $\mathbf{R}^3$ for all physically admissible initial data are those of J. Leray [40] in the case of the Navier–Stokes equations, and of R. DiPerna and P.-L. Lions [23] in the case of the Boltzmann equation. Both results lead to weak solutions to which the methods of either [11] or [21] cannot be applied.

For that reason, a program concerning the derivation of Leray (weak) solutions of the Navier–Stokes equations from DiPerna-Lions renormalized solutions of the Boltzmann equation was discussed in detail by C. Bardos, F. Golse and C.D. Levermore in [7]. There, this derivation is established rigorously in the time-discretized case under two conditions bearing on the

[1] R. Esposito, personal communication.

sequence of renormalized solutions considered. This method was extended by P.-L. Lions and N. Masmoudi to the time dependent case in [50], under the same two conditions. That these conditions hold is not guaranteed by the theory of the Boltzmann equation in its present state, so that the derivations in [7] and [50] remained incomplete. We refer to [62] for a recent survey of these issues.

In the present work, we show how to circumvent the need for both assumptions left unverified in either [7] or [50], thereby proving the Navier–Stokes limit of the Boltzmann equation (including a convection-diffusion equation for the temperature field) for all physically admissible initial data. Our discussion is restricted so far to bounded collision kernels, as in the case of cutoff Maxwellian molecules. Yet our methods could apply to more general cases which we hope to analyze in subsequent publications.

An alternate approach, proposed by J. Quastel and H.-T. Yau in [54] consists in deriving the Navier–Stokes equations from some stochastic lattice gas. Some of the methods in [54] might eventually prove useful in the context of hydrodynamic limits. However, this result in itself is somewhat remote from Hilbert's original question: indeed the microscopic model in [54] is neither a fundamental principle of physics nor a consequence thereof. On the contrary, the Boltzmann equation has been widely accepted and used as a legitimate microscopic model. In fact, it has been rigorously derived by O. Lanford from the Newtonian dynamics of a large number of spheres interacting by elastic collisions [39] – see also Chap. 4 of [19].

Notation for spaces. The notations L^p_x, $L^p(\mathbf{R}^D)$ and $L^p(dx)$ all designate $L^p(\mathbf{R}^D, dx)$. In general, whenever a positive Borel measure m is defined on a topological space X, the notation $L^p(dm)$ designates $L^p(X, dm)$. For any Banach space E, the notation w-E designates E endowed with its weak topology, while the notation w^*-E' designates E' (the topological dual of E) endowed with its weak-* topology.

Contents

1. Statement of the problem and main results

1.1. The Boltzmann equation. In kinetic theory, a gas is described by a function $F \equiv F(t, x, v) \geq 0$ measuring the density of gas molecules which at time $t \in \mathbf{R}_+$ are located at $x \in \mathbf{R}^3$ and have instantaneous velocity $v \in \mathbf{R}^3$. This function F, usually called the "distribution function" or the "number density", is governed by the Boltzmann equation

$$\partial_t F + v \cdot \nabla_x F = \mathcal{B}(F, F) \tag{1.1}$$

where $\mathcal{B}(F, F)$ is the Boltzmann collision integral. This collision integral acts only on the v-argument of the number density F and is given by the expression

$$\begin{aligned}&\mathcal{B}(F, F)(t, x, v) = \\ &\iint_{\mathbf{R}^3\times\mathbf{S}^2} (F'F'_1 - FF_1)b(v - v_1, \omega)|\cos(v - v_1, \omega)|d\omega dv_1 \,,\end{aligned} \tag{1.2}$$

where the notations F_1, F' and F'_1 designate respectively the values $F(t, x, v_1)$, $F(t, x, v')$ and $F(t, x, v'_1)$, with v' and v'_1 given in terms of $v_1 \in \mathbf{R}^3$ and $\omega \in \mathbf{S}^2$ by the formulas

$$v' = v - (v - v_1) \cdot \omega\,\omega\,, \quad v'_1 = v_1 + (v - v_1) \cdot \omega\,\omega\,. \tag{1.3}$$

These formulas give all possible solutions to the system with unknowns v' and v'_1

$$v' + v'_1 = v + v_1\,, \quad |v'|^2 + |v'_1|^2 = |v|^2 + |v_1|^2\,, \tag{1.4}$$

in terms of the data v and v_1 and of an arbitrary unit vector ω. The relations (1.4) are the conservation of momentum and kinetic energy for each binary collision between gas molecules (of like mass). The notation $d\omega$ designates the uniform measure on the sphere $\mathbf{S}^2$ normalized so that

(1.5)
$$\int_{\mathbf{S}^2} d\omega = 2\,, \text{ which implies that } \int_{\mathbf{S}^2} |\cos(z, \omega)|d\omega = 1 \text{ for all } z \in \mathbf{R}^3\,;$$

below, we use the notation

$$d\sigma_{v,v_1}(\omega) = |\cos(v - v_1, \omega)|d\omega\,. \tag{1.6}$$

The geometrical interpretation of the measure $d\sigma_{v,v_1}(\omega)$ is as follows. Let (v, v_1, v', v'_1) be any quadruple satisfying (1.4); clearly

$$|v'_1 - v'| = |v_1 - v|\,.$$

Whenever $v \neq v_1$, define $\sigma = \frac{v'_1 - v'}{|v_1 - v|}$. For $\omega \in \mathbf{S}^2$ as in the collision formulas (1.3), one has

$$|v'_1 - v'|\sigma = (v_1 - v) - 2(v_1 - v) \cdot \omega\,\omega\,.$$

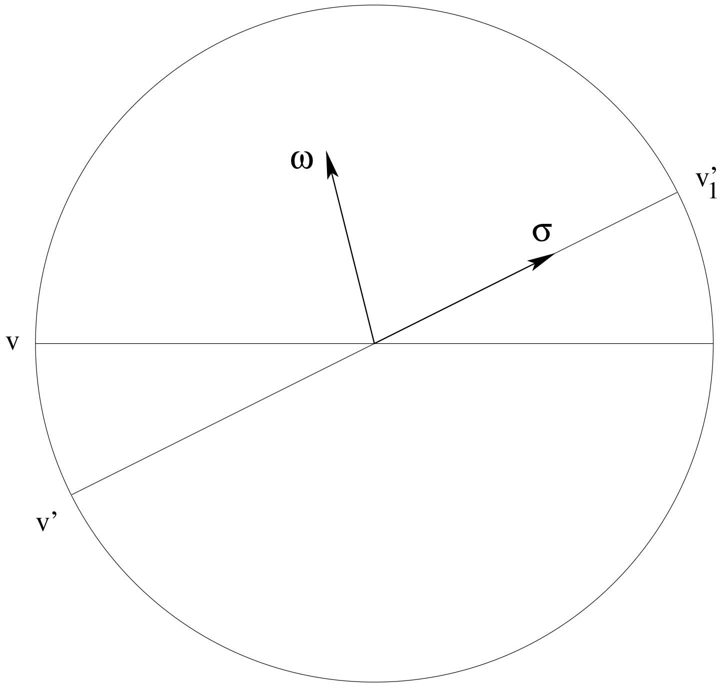

Fig. 1. The geometry of collisions in the center of mass reference frame

Set $V = v_1 - v \neq 0$; the map

$$\mathbf{S}^2 \ni \omega \to \sigma = \frac{V}{|V|} - 2\frac{V}{|V|} \cdot \omega\,\omega \in \mathbf{S}^2$$

is a double cover, and the image of the measure $|\cos(V, \omega)|d\omega$ under this map is precisely $2d\sigma$ (i.e. twice the uniform measure on $\mathbf{S}^2$). In other words, given a pair of colliding particles with pre-collision velocities v and v_1, $d\sigma_{v,v_1}(\omega)$ represents the element of solid angle around their post-collision velocities in the center of mass reference frame. In particular, the term $|\cos(V, \omega)|$ has an intrinsic meaning in the representation (1.3) of the collision relations in terms of the parameter ω. We decided to use this representation in order to be consistent with most of the references quoted in the present work.

Observe that

$$d\sigma_{v,v_1}(\omega) = d\sigma_{v_1,v}(\omega)\,. \tag{1.7}$$

Since the map $(v, v_1) \mapsto (v', v_1')$ is a linear isometry of $\mathbf{R}^3 \times \mathbf{R}^3$ for each $\omega \in \mathbf{S}^2$, one has

$$d\omega dv dv_1 = d\omega dv' dv_1' \quad \text{and} \quad d\sigma_{v,v_1}(\omega) dv dv_1 = d\sigma_{v',v_1'}(\omega) dv' dv_1'\,. \tag{1.8}$$

The collision kernel $b \equiv b(z, \omega)$ is in general an a.e. positive function defined on $\mathbf{R}^3 \times \mathbf{S}^2$ that encodes whichever features of the molecular interaction are relevant in kinetic theory; it satisfies the symmetries

$$b(v - v_1, \omega) = b(v_1 - v, \omega) = b(v' - v_1', \omega)\,, \tag{1.9}$$

for a.e. $(v, v_1, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$. It is also assumed to satisfy the condition

$$(1 + |v|)^{-2} \int_{|v_1|<R} \int_{\mathbf{S}^2} b(v - v_1, \omega) d\sigma_{v,v_1}(\omega) dv_1 \to 0 \text{ as } |v| \to +\infty \tag{1.10}$$

for all $R > 0$. This estimate holds for all physically relevant potentials satisfying Grad's angular cutoff assumption (see [32] and [18] pp. 74–79 for more details). These properties of the collision kernel b, especially the symmetries (1.8) and (1.9), imply that the relation

$$\begin{aligned}
&\iiint (f'f_1' - ff_1)\varphi(v, v_1, v', v_1')b(v - v_1, \omega)d\sigma_{v,v_1}(\omega)dv_1 dv \\
&\quad = \iiint (f'f_1' - ff_1)\phi(v, v_1, v', v_1')b(v - v_1, \omega)d\sigma_{v,v_1}(\omega)dv_1 dv \\
&\quad = \iiint (f'f_1' - ff_1)\Phi(v, v_1, v', v_1')b(v - v_1, \omega)d\sigma_{v,v_1}(\omega)dv_1 dv
\end{aligned} \tag{1.11}$$

with

$$\begin{aligned}
\phi(v, v_1, v', v_1') &= \tfrac{1}{2}(\varphi(v, v_1, v', v_1') + \varphi(v_1, v, v_1', v')) \\
\Phi(v, v_1, v', v_1') &= \tfrac{1}{4}(\varphi(v, v_1, v', v_1') + \varphi(v_1, v, v_1', v') \\
&\qquad - \varphi(v', v_1', v, v_1) - \varphi(v_1', v', v_1, v))
\end{aligned}$$

holds whenever these integrals make sense, for example if $f \in C_c(\mathbf{R}^3)$ and if $\varphi \in C(\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3)$.

The equilibrium states for the Boltzmann collision integral, in other words the number densities $E \equiv E(v)$ such that $\mathcal{B}(E, E) = 0$, are the Maxwellians, i.e. the distribution functions of the form

$$M_{(\rho,u,\theta)}(v) = \frac{\rho}{(2\pi\theta)^{3/2}} e^{-\frac{|v-u|^2}{2\theta}} \tag{1.12}$$

for some $\rho > 0$, $\theta > 0$ and $u \in \mathbf{R}^3$. Below, we shall always use the notation M to designate $M_{(1,0,1)}$.

In the sequel, we are concerned with solutions to the Boltzmann equation which converge to some Maxwellian state as $|x| \to +\infty$; without loss of generality, we assume that this Maxwellian is precisely M. Consider the following scaled variant of (1.1):

$$(1.13)\qquad \begin{aligned} &\epsilon\partial_t F_\epsilon + v\cdot\nabla_x F_\epsilon = \frac{1}{\epsilon}\mathcal{B}(F_\epsilon, F_\epsilon)\,, \quad t>0\,,\ x,v\in\mathbf{R}^3\,,\\ &F_\epsilon(t,x,v)\to M(v)\,,\ \text{as } |x|\to+\infty\,,\\ &F_\epsilon(0,x,v) = F_\epsilon^{in}(x,v)\,, \quad x,v\in\mathbf{R}^3\,, \end{aligned}$$

where $\epsilon > 0$ designates the common order of magnitude of the Knudsen and Mach numbers (see the introduction in [7] for a detailed discussion on these scalings), and where $F_\epsilon^{in} \geq 0$ a.e. is a family of measurable functions such that

$$(1.14)\qquad \sup_{\epsilon>0}\frac{1}{\epsilon^2}\iint\left[F_\epsilon^{in}\log\left(\frac{F_\epsilon^{in}}{M}\right) - F_\epsilon^{in} + M\right]dvdx < +\infty\,.$$

For any pair of measurable functions f and g defined a.e. on $\mathbf{R}^3\times\mathbf{R}^3$ and satisfying $f\geq 0$ and $g>0$ a.e., we use the following notation for the relative entropy

$$(1.15)\qquad H(f|g) = \iint\left[f\log\left(\frac{f}{g}\right) - f + g\right]dvdx \in [0,+\infty]\,.$$

A *renormalized solution relative to* M of (1.13) is a nonnegative function F_ϵ that belongs to $C(\mathbf{R}_+; L^1_{loc}(\mathbf{R}^3; L^1(\mathbf{R}^3)))$, satisfies

$$(1.16)\qquad \Gamma'\left(\frac{F_\epsilon}{M}\right)\mathcal{B}(F_\epsilon, F_\epsilon) \in L^1_{loc}\big(\mathbf{R}_+\times\mathbf{R}^3\times\mathbf{R}^3\big)$$

for all $\Gamma\in C^1(\mathbf{R}_+)$ such that

$$(1.17)\qquad \Gamma(0)=0 \text{ and } z\mapsto(1+z)\Gamma'(z) \text{ is bounded on } \mathbf{R}_+\,,$$

has finite relative entropy for all positive times:

$$(1.18)\qquad H(F_\epsilon(t,\cdot,\cdot)|M) < +\infty\,, \quad t>0\,,$$

and finally satisfies

$$(1.19)\qquad \begin{aligned} &\int_0^{+\infty}\iint\Gamma\left(\frac{F_\epsilon}{M}\right)\left(\partial_t\chi + \frac{1}{\epsilon}v\cdot\nabla_x\chi\right)Mdvdxdt\\ &+\iint\Gamma\left(\frac{F_\epsilon^{in}}{M}\right)\chi(0,x,v)Mdvdx\\ &+\frac{1}{\epsilon^2}\int_0^{+\infty}\iint\Gamma'\left(\frac{F_\epsilon}{M}\right)\mathcal{B}(F_\epsilon,F_\epsilon)\chi dvdxdt = 0 \end{aligned}$$

for each test function $\chi\in C_c^\infty(\mathbf{R}_+\times\mathbf{R}^3\times\mathbf{R}^3)$.

The methods due to R. DiPerna and P.-L. Lions [23] – and their extension by P.-L. Lions [43] – lead to the following global existence result.

Theorem 1.1. *Let $\epsilon > 0$ and $F_\epsilon^{in} \equiv F_\epsilon^{in}(x, v)$ be an a.e. nonnegative, measurable function defined on $\mathbf{R}^3 \times \mathbf{R}^3$ such that $H(F_\epsilon^{in}|M) < +\infty$. Then there exists a renormalized solution of (1.13) relative to M which satisfies*

- *the local conservation of mass in the sense of distributions*

$$\partial_t \int F_\epsilon dv + \nabla_x \cdot \frac{1}{\epsilon} \int v F_\epsilon dv = 0\,, \quad t > 0\,,\ x \in \mathbf{R}^3\,, \tag{1.20}$$

- *and the relative entropy inequality*

$$H(F_\epsilon(t, \cdot, \cdot)|M) + \frac{1}{\epsilon^2} \int_0^t \iint D(F_\epsilon) dv dx ds \leq H(F_\epsilon^{in}|M) \tag{1.21}$$

for all $t > 0$, where the dissipation term $D(f)$ is defined for all positive measurable functions $f \equiv f(v)$ by

$$D(f) = \tfrac{1}{4} \iint (f' f_1' - f f_1) \log\left(\frac{f' f_1'}{f f_1}\right) b(v - v_1, \omega) d\sigma_{v, v_1}(\omega) dv_1\,. \tag{1.22}$$

Whether the local conservation of momentum

$$\partial_t \int v F_\epsilon dv + \nabla_x \cdot \frac{1}{\epsilon} \int v \otimes v F_\epsilon dv = 0 \tag{1.23}$$

holds in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3$ is still unknown, unless F_ϵ is a classical solution of the Boltzmann equation. (P.-L. Lions and N. Masmoudi made an interesting additional observation on this particular point in [51]). This is one of the difficulties in rigorously deriving hydrodynamic models from the Boltzmann equation.

Likewise, it is still unknown whether (1.21) is an equality unless F_ϵ is a classical solution; the relation (1.21) with an equal sign is one of the most important formal properties of the Boltzmann equation, known as *Boltzmann's H Theorem.*

Remark. The notion of renormalized solution relative to M of the Boltzmann equation (1.13) slightly differs from the original notion of renormalized solution defined in [23] and [43]. By Theorem IV.1 of [43], for each $\epsilon > 0$ and each $F_\epsilon^{in} \geq 0$ a.e. such that $H(F_\epsilon^{in}|M) < +\infty$, there exists a *renormalized solution*[2] of the Cauchy problem (1.13), i.e. a function $F_\epsilon \in C([0, +\infty); L^1_{loc}(dx; L^1(dv)))$ such that

$$\frac{\mathcal{B}(F_\epsilon, F_\epsilon)}{1 + F_\epsilon} \in L^1_{loc}(dtdxdv)\,, \tag{1.24}$$

[2] Theorem IV.1 in [43] gives in fact the existence of a *weak solution* of the Boltzmann equation, a notion defined on pp. 548–549 of [47] that is stronger than the notion of renormalized solution: see [47], p. 551.

which satisfies (1.18) and is such that

$$(\epsilon\partial_t + v\cdot\nabla_x)\log(1+F_\epsilon) = \frac{1}{\epsilon}\frac{\mathcal{B}(F_\epsilon,F_\epsilon)}{1+F_\epsilon}$$

holds in the sense of distributions on $\mathbf{R}_+\times\mathbf{R}^3\times\mathbf{R}^3$ with initial condition

$$\log(1+F_\epsilon)\Big|_{t=0} = \log\left(1+F_\epsilon^{in}\right).$$

Observe that

$$0 \le (1+F_\epsilon)\left|\Gamma'\left(\frac{F_\epsilon}{M}\right)\right| \le (1+M)\sup_{z\ge 0}\left((1+z)|\Gamma'(z)|\right);$$

hence the bound (1.24) implies the bound (1.16). Based on this bound, an elementary argument shows that the renormalized solutions in the usual sense constructed in Theorem IV.1 of [43] are in fact renormalized solutions relative to M as defined above.

1.2. The Navier–Stokes equations. The Navier–Stokes equations govern the velocity field $u \equiv u(t,x) \in \mathbf{R}^3$ of an incompressible fluid. They are

$$\begin{aligned} \nabla_x\cdot u &= 0, & t>0,\ x\in\mathbf{R}^3,\\ \partial_t u + u\cdot\nabla_x u + \nabla_x p &= \nu\Delta_x u, & t>0,\ x\in\mathbf{R}^3, \end{aligned} \tag{1.25}$$

where $\nu > 0$ is the kinematic viscosity of the fluid. The first equality in (1.25) says that the fluid motion preserves the volume, and is referred to as the incompressibility condition; the second equality expresses Newton's second law of motion for any infinitesimal volume of fluid.

Consider the function spaces

$$\mathcal{H} = \left\{u\in L^2(\mathbf{R}^3;\mathbf{R}^3)\,\middle|\,\nabla_x\cdot u = 0\right\},\quad \mathcal{V} = \mathcal{H}\cap H^1(\mathbf{R}^3;\mathbf{R}^3).$$

In particular, $\mathcal{H}$ is the space of three-dimensional incompressible velocity fields with finite kinetic energy $\frac{1}{2}\int|u|^2dx$.

Let $u^{in}\in\mathcal{H}$, and consider the Cauchy problem for (1.25) with initial data

$$u(0,x) = u^{in}(x),\quad x\in\mathbf{R}^3. \tag{1.26}$$

A *weak solution* to the Cauchy problem (1.25)–(1.26) is an element $u \in C(\mathbf{R}_+; w\text{-}\mathcal{H})\cap L^2(\mathbf{R}_+;\mathcal{V})$ that satisfies the relation

(1.27)

$$\begin{aligned} \int_0^{+\infty}\!\!\int u(t,x)\cdot\partial_t\chi(t,x)dxdt + \int_0^{+\infty}\!\!\int u^{\otimes 2}(t,x):\nabla_x\chi(t,x)dxdt\\ + \int u^{in}(x)\cdot\chi(0,x) = \nu\int_0^{+\infty}\!\!\int \nabla_x u(s,x):\nabla_x\chi(s,x)dxds \end{aligned}$$

for each divergence free test vector field $\chi \in C_c^\infty(\mathbf{R}_+ \times \mathbf{R}^3; \mathbf{R}^3)$. We recall that the only global existence theorem known to hold for the Cauchy problem (1.25)–(1.26) without restriction on the size of the initial data in the class $\mathcal{H}$ of three-dimensional divergence-free velocity fields with finite kinetic energy is the following[3]

Theorem 1.2 (J. Leray [40]). *For each $u^{in} \in \mathcal{H}$, there exists at least one weak solution of (1.25)–(1.26) satisfying the energy inequality*

$$(1.28) \qquad \tfrac{1}{2}\int |u(t,x)|^2 dx + \nu \int_0^t \int |\nabla_x u(s,x)|^2 dx ds \le \tfrac{1}{2}\int |u^{in}(x)|^2 dx$$

for all $t > 0$.

A weak solution of (1.25)–(1.26) that satisfies in addition the energy inequality (1.25) for all $t > 0$ is referred to as *a Leray solution*. It is unknown whether there is a unique Leray solution of (1.25)–(1.26). However, if the system (1.25)–(1.26) has a classical solution with bounded x-derivatives, this solution is unique within the class of Leray solutions of (1.25)–(1.26). It remains unknown whether equality holds in (1.28), unless u is a classical solution of (1.25)–(1.26), much in the same way that it is unknown whether equality holds in (1.21) unless F_ϵ is a classical solution of (1.13).

The Leray energy inequality (1.28) and the DiPerna-Lions entropy inequality (1.21) are similar objects. More precisely, it was proved by C. Bardos, F. Golse and C.D. Levermore in [7] that the Leray energy inequality (1.28) is the limiting form of the DiPerna-Lions entropy inequality (1.21). This confirms the view expressed by P.-L. Lions (see [46], p. 432): "[...] the global existence result of [renormalized] solutions [...] can be seen as the analogue for Boltzmann's equation to the pioneering work on the Navier–Stokes equations by J. Leray".

1.3. The Navier–Stokes–Fourier system. The Navier–Stokes–Fourier system is an extension of the Navier–Stokes equations which governs both the velocity field $u \equiv u(t,x) \in \mathbf{R}^3$ and the (fluctuations of) temperature field $\theta \equiv \theta(t,x) \in \mathbf{R}$ in an incompressible fluid. In the setting considered below, the temperature field is just advected by the velocity field u and diffuses according to Fourier's law. More complicated effects such as viscous heating for example (see [10] or [44] p. 10, fla. (1.41)) do not appear in the scaling

[3] There is a theory of global existence and uniqueness of classical solutions to the Navier–Stokes equations (1.25) for all divergence free initial velocity fields that depend on two space variables only and belong to $H^2(\mathbf{R}^2, \mathbf{R}^3)$: for a concise presentation of these results, see [44] pp. 83 and 151. However, these solutions do not have finite kinetic energy (when considered as three-dimensional velocity fields) and thus are not covered by the discussion in the present paper: see Sect. 9.

considered in the present work. The Navier–Stokes–Fourier system reads

$$
\begin{aligned}
\nabla_x \cdot u &= 0\,, && t>0\,,\ x\in\mathbf{R}^3\,,\\
\partial_t u + u\cdot\nabla_x u + \nabla_x p &= \nu\Delta_x u\,, && t>0\,,\ x\in\mathbf{R}^3\,,\\
\partial_t\theta + u\cdot\nabla_x\theta &= \kappa\Delta_x\theta\,, && t>0\,,\ x\in\mathbf{R}^3\,,
\end{aligned}
\tag{1.29}
$$

where $\kappa > 0$ designates the heat conduction coefficient.

Consider the Cauchy problem for (1.29) with initial data

$$
u(0,x) = u^{in}(x)\,,\quad \theta(0,x) = \theta^{in}(x)\,,\quad x\in\mathbf{R}^3\,,
\tag{1.30}
$$

where $u^{in} \in \mathcal{H}$ and $\theta^{in} \in L^2(\mathbf{R}^3)$. A *weak solution* of the Cauchy problem (1.29)–(1.30) is a couple (u,θ) where u is a weak solution of the Navier–Stokes equation and θ a solution in the sense of distributions of the drift-diffusion Cauchy problem

$$
\begin{aligned}
\partial_t\theta + \nabla_x\cdot(u\theta) &= \kappa\Delta_x\theta\,, && t>0\,,\ x\in\mathbf{R}^3\,,\\
\theta(0,x) &= \theta^{in}(x)\,, && x\in\mathbf{R}^3\,.
\end{aligned}
\tag{1.31}
$$

Theorem 1.3. *For each $u^{in} \in \mathcal{H}$ and $\theta^{in} \in L^2(\mathbf{R}^3)$, there exists at least one weak solution (u,θ) of (1.29)–(1.30) that satisfies the energy inequality*

$$
\begin{aligned}
&\tfrac12\int\left(|u(t,x)|^2 + \tfrac52\theta(t,x)^2\right)dx\\
&\qquad + \int_0^t\!\!\int \left[\nu|\nabla_x u(s,x)|^2 + \tfrac52\kappa|\nabla_x\theta(s,x)|^2\right]dxds\\
&\qquad \le \tfrac12\int\left(|u^{in}(x)|^2 + \tfrac52\theta^{in}(x)^2\right)dx
\end{aligned}
\tag{1.32}
$$

for all $t > 0$.

A weak solution of (1.29)–(1.30) that also satisfies (1.32) is also referred to as a *Leray solution* in the sequel (although J. Leray himself did not study thermal effects, the theorem above is a straightforward extension of his fundamental paper [40]).

However, there is a certain arbitrariness in considering the Lyapunov functional

$$
\tfrac12\int\left(|u(t,x)|^2 + \tfrac52\theta(t,x)^2\right)dx
\tag{1.33}
$$

in the energy inequality (1.32). A similar existence theorem holds with the coefficient $\frac52$ multiplying the temperature replaced by any positive number. The reason for using specifically the coefficient $\frac52$ in the theorem above is that the quantity (1.33) is the leading order of the relative entropy (1.18) in the Navier–Stokes limit as $\epsilon \to 0$ in space dimension 3. In spite of the fact that this Lyapunov functional reduces to the kinetic energy in the case $\theta \equiv 0$, it does *not* coincide with the total (kinetic plus internal) energy in the general case: see [44] p. 110 and ff. for a detailed description of models involving the temperature in incompressible fluids.

1.4. Main results. The Navier–Stokes (or Navier–Stokes–Fourier) limit of the Boltzmann equation considers fluctuations of the number density about an absolute Maxwellian. The following notations for such fluctuations are taken from [7].

1.4.1. The Boltzmann equation near a uniform Maxwellian. First, the relative number density and the fluctuations of number density are denoted respectively by

$$G_\epsilon = \frac{F_\epsilon}{M}, \quad g_\epsilon = \frac{F_\epsilon - M}{\epsilon M}, \tag{1.34}$$

while the scaled collision integrand is

$$q_\epsilon = \frac{1}{\epsilon^2}(G'_\epsilon G'_{\epsilon 1} - G_\epsilon G_{\epsilon 1}). \tag{1.35}$$

The integral for the unit measure Mdv is denoted

$$\langle g \rangle = \int g(v)M(v)dv, \quad \text{for all } g \in L^1(Mdv). \tag{1.36}$$

Without loss of generality – i.e. after normalization if needed – one can assume that the measure

$$d\mu(v, v_1, \omega) = b(v - v_1, \omega)d\sigma_{v,v_1}(\omega)M_1 dv_1 M dv \tag{1.37}$$

also is a unit measure; the integral for this unit measure is denoted

$$\langle\langle q \rangle\rangle = \iiint q(v, v_1, \omega)d\mu(v, v_1, \omega), \quad \text{for all } q \in L^1(d\mu). \tag{1.38}$$

We shall need (minus) the linearized collision operator

$$\mathcal{L}g = \iint (g + g_1 - g' - g'_1)b(v - v_1, \omega)d\sigma_{v,v_1}(\omega)M_1 dv_1, \tag{1.39}$$

as well as (half) the Hessian of the collision integral at the Maxwellian M which is denoted by

$$\mathcal{Q}(g, g) = \iint (g'g'_1 - gg_1)b(v - v_1, \omega)d\sigma_{v,v_1}(\omega)M_1 dv_1. \tag{1.40}$$

Assume from now on that the Boltzmann collision kernel b satisfies the assumption

$$\frac{1}{b_\infty} \le b(z, \omega) \le b_\infty, \ z \in \mathbf{R}^3, \ \omega \in \mathbf{S}^2, \quad \text{for some } b_\infty > 0. \tag{H1}$$

The main properties of the linearized collision operator $\mathcal{L}$ were proved by H. Grad [32] and are recalled below.

Proposition 1.4. *For any collision kernel b satisfying (H1), $\mathcal{L}$ is a bounded nonnegative self-adjoint Fredholm operator on $L^2(Mdv)$ with nullspace*

$$\ker \mathcal{L} = \operatorname{span}\left\{1, v_1, v_2, v_3, |v|^2\right\}. \tag{1.41}$$

Because each entry of the tensor $v^{\otimes 2} - \frac{1}{3}|v|^2 I$ and of the vector $\frac{1}{2}v(|v|^2 - 5)$ is orthogonal to $\ker \mathcal{L}$, there exist a unique tensor A and a unique vector B such that

$$\begin{aligned} \mathcal{L}A &= v^{\otimes 2} - \tfrac{1}{3}|v|^2 I\,, \quad && A \in (\ker \mathcal{L})^\perp \subset L^2(Mdv)\,, \\ \mathcal{L}B &= \tfrac{1}{2}v(|v|^2 - 5)\,, \quad && B \in (\ker \mathcal{L})^\perp \subset L^2(Mdv)\,. \end{aligned} \tag{1.42}$$

The main properties of the bilinear operator $\mathcal{Q}$ used in the present work are collected in the next proposition.

Proposition 1.5. *For any collision kernel b satisfying (H1) and all $p \in [1, \infty]$, $\mathcal{Q}$ defines by polarization a continuous, symmetric bilinear operator (still denoted by $\mathcal{Q}$) from $L^p(Mdv) \times L^p(Mdv)$ to $L^p(Mdv)$. Further,*

$$\mathcal{Q}(g, g) = \tfrac{1}{2}\mathcal{L}(g^2)\,, \quad \textit{for all } g \in \ker \mathcal{L}\,. \tag{1.43}$$

Sketch of proof. The proof of (1.43) can be found in [6] or in [15] (Lemma 2.5, p. 74). As for the continuity property, pick $p \in [1, \infty)$ and $f \in L^p(Mdv)$; by (H1), the symmetry property (1.9) and Jensen's inequality (observing that the measure $d\sigma_{v,v_1}(\omega)M_1 dv_1$ has total mass 1)

$$\begin{aligned} \int \left| \iint f' f_1' b d\sigma_{v,v_1}(\omega) M_1 dv_1 \right|^p Mdv \qquad & \\ \le b_\infty^p \iiint |f'|^p |f_1'|^p d\sigma_{v,v_1}(\omega) M_1 dv_1 Mdv & \\ = b_\infty^p \iiint |f|^p |f_1|^p d\sigma_{v,v_1}(\omega) M_1 dv_1 Mdv & \\ = b_\infty^p \iint |f|^p |f_1|^p M_1 dv_1 Mdv = \|f\|_{L^p(Mdv)}^{2p}\,, & \end{aligned}$$

where the first equality follows from the change of variables $(v, v_1) \mapsto (v', v_1')$, the second equality in (1.8), and the last relation in (1.9). □

Within the class of collision kernels satisfying (H1), we further restrict our attention to those for which

$$\frac{|A(v)| + |B(v)|}{1 + |v|^p} \in L_v^\infty \text{ for some } p \ge 0\,. \tag{H2}$$

The class of collision kernels satisfying both (H1) and (H2) is not empty since it contains at least all collision kernels of the form $b(z, \omega) = \mathbf{b}(|\cos(z, \omega)|)$ satisfying (H1). These collision kernels correspond to cutoff Maxwellian molecules and satisfy (H2) with $p = 3$ (see [18], pp. 82–87).

1.4.2. The limit theorems. From now on, we denote by P the Leray projection, i.e. the orthogonal projection on the space of divergence free vector fields in $L^2(\mathbf{R}^3)$ – in particular, for any $p \in H^1(\mathbf{R}^3)$, one has $P(\nabla_x p) = 0$. The operator P so defined coincides with a classical pseudo-differential operator of order 0 on $\mathbf{R}^3$, and therefore has a natural extension to tempered distributions on $\mathbf{R}^3$.

Theorem 1.6 (Weak Navier–Stokes–Fourier limit). *Let b satisfy (H1)–(H2), and let F_ϵ^{in} be a family of a.e. nonnegative, measurable functions on $\mathbf{R}^3 \times \mathbf{R}^3$ satisfying the bound*

$$H\big(F_\epsilon^{in}\big|M\big) \leq C^{in}\epsilon^2 \tag{1.44}$$

for some $C^{in} > 0$ and all $\epsilon > 0$, as well as the convergence properties

$$\begin{aligned} P\left(\frac{1}{\epsilon}\int vF_\epsilon^{in}dv\right) &\to u^{in} \text{ in } w\text{-}L^2(\mathbf{R}^3) \text{ as } \epsilon \to 0\,,\\ \frac{1}{\epsilon}\int \big(\tfrac{1}{5}|v|^2 - 1\big)\big(F_\epsilon^{in} - M\big)dv &\to \theta^{in} \text{ in } w\text{-}L^2(\mathbf{R}^3) \text{ as } \epsilon \to 0\,. \end{aligned} \tag{1.45}$$

Let F_ϵ be a family of renormalized solutions to (1.13). Then the family

$$\left(\frac{1}{\epsilon}\int vF_\epsilon dv\,,\ \frac{1}{\epsilon}\int \big(\tfrac{1}{3}|v|^2 - 1\big)F_\epsilon dv\right)$$

is relatively compact in w-$L^1_{loc}(dtdx)$ and each of its limit points as $\epsilon \to 0$ is a weak solution of (1.29)–(1.30) with viscosity and heat diffusion coefficient given by the formulas

$$\nu = \tfrac{1}{10}\int A : (\mathcal{L}A)M dv\,, \quad \kappa = \tfrac{2}{15}\int B \cdot (\mathcal{L}B)M dv\,. \tag{1.46}$$

For well-prepared initial data, the weak-compactness result above leads to Leray instead of weak solutions. First we recall from [7] the following definition:

Definition 1.7. *A family $g_\epsilon \equiv g_\epsilon(x, v)$ of $L^1_{loc}(Mdvdx)$ converges to $g \equiv g(x, v)$ entropically at rate ϵ as $\epsilon \to 0$ if*

- *for each ϵ, $1 + \epsilon g_\epsilon \geq 0$ a.e. on $\mathbf{R}^3 \times \mathbf{R}^3$,*
- *$g_\epsilon \to g$ in w-$L^1_{loc}(Mdvdx)$ as $\epsilon \to 0$,*
- *and*

$$\frac{1}{\epsilon^2}H(M(1 + \epsilon g_\epsilon)|M) \to \tfrac{1}{2}\int \langle g^2\rangle dx$$

as $\epsilon \to 0$.

This notion of convergence is the natural one in the context of the Navier–Stokes(–Fourier) limit of the Boltzmann equation, as was shown in [7]. Specifically, this reference proved that the Leray energy inequality (1.32) is the limiting form of the DiPerna-Lions entropy inequality (1.21) by establishing the inequality recalled as (11.7) in Appendix B below. By using this inequality together with Theorem 1.6, one obtains the following convergence statement relating DiPerna-Lions solutions of the Boltzmann equation to Leray solutions of the Navier–Stokes–Fourier system.

Corollary 1.8 (Well-prepared data). *Under the same assumptions as in Theorem 1.6, assume that*

$$\frac{1}{\epsilon}\frac{F_\epsilon^{in}(x,v)-M(v)}{M(v)} \to u^{in}(x)\cdot v+\theta^{in}(x)\tfrac{1}{2}(|v|^2-5)$$

entropically at rate ϵ as $\epsilon \to 0$, where $u^{in} \in \mathcal{H}$. Then all limit points of the family

$$\left(\frac{1}{\epsilon}\int vF_\epsilon dv\,,\ \frac{1}{\epsilon}\int\left(\tfrac{1}{3}|v|^2-1\right)F_\epsilon dv\right)$$

as $\epsilon \to 0$ are Leray solutions of (1.29)–(1.30) with viscosity and heat diffusion coefficient given by the formulas (1.46).

Further, if the limiting initial data u^{in} is smooth and such that (1.29)–(1.30) has a (unique) smooth solution u, the weak compactness result above can be strengthened into a strong convergence result as shown below.

Theorem 1.9 (Strong Navier–Stokes–Fourier limit). *Under the same assumptions as in Theorem 1.6, assume that*

$$\frac{1}{\epsilon}\frac{F_\epsilon^{in}(x,v)-M(v)}{M(v)} \to u^{in}(x)\cdot v+\theta^{in}(x)\tfrac{1}{2}(|v|^2-5)$$

entropically at rate ϵ as $\epsilon \to 0$, where u^{in} is a divergence-free vector field such that the Navier–Stokes equations (1.25)–(1.26) with ν given by (1.46) have a strong solution u (see [20], Chaps. 9 and 10). Let θ be the solution of the drift-diffusion equation (1.31) with κ given by (1.46).

Then, for all $t \geq 0$,

$$\frac{1}{\epsilon}\frac{F_\epsilon(t,x,v)-M(v)}{M(v)} \to u(t,x)\cdot v+\theta(t,x)\tfrac{1}{2}(|v|^2-5)$$

entropically at rate ϵ as $\epsilon \to 0$.

Theorem 1.9 is a straightforward consequence of Theorem 1.6 and of a squeezing argument based on the fact that the inequality (1.32) becomes an equality in the case of strong solutions. Its proof closely follows the proofs of Theorems 6.2 and 7.4 in [7], and of Proposition 6.1 in [25]. In fact Theorem 1.9 holds true provided that (1.29)–(1.30) has a unique weak solution (u, θ) for which equality holds in (1.32), since its proof does not use consequences of the regularity of (u, θ) other than the uniqueness of the solution and the energy equality.

This discussion shows that the convergence proofs in the present paper would not become obsolete should one eventually prove that the Navier–Stokes equations posed in $\mathbf{R}^3$ have global smooth solutions for arbitrary smooth initial data. We shall return to this in Sect. 9.

1.5. The state of the art for the Navier–Stokes limit. The only previously existing results on the Navier–Stokes limit – without restrictions on the size or regularity of the initial data, i.e. starting from renormalized solutions of the Boltzmann equation – are due to C. Bardos, F. Golse and D. Levermore [7] for the steady problem and to P.-L. Lions and N. Masmoudi [50] for the time-dependent problem. Both are based on two assumptions recalled below:

- first, the family of renormalized solutions F_ϵ of (1.13) considered in the limit as $\epsilon \to 0$ satisfies local conservation of momentum, i.e.

$$\text{(A1)} \qquad \epsilon\partial_t \int vF_\epsilon dv + \nabla_x \cdot \int v^{\otimes 2} F_\epsilon dv = 0$$

  in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3$;
- in addition, the family F_ϵ is such that

  (A2)
$$\frac{(1+|v|^2)(F_\epsilon - M)^2}{\epsilon^2 M(F_\epsilon + M)} \text{ is relatively compact in } w\text{-}L^1_{loc}(dtdx; L^1(Mdv)) \,.$$

As mentioned above, whether renormalized solutions of the Boltzmann equation satisfy (A1) remains a major open problem. Likewise, the global conservation of energy is not guaranteed by the DiPerna-Lions theory in its present state; only the inequality

$$\text{(1.47)} \qquad \iint \tfrac{1}{2}|v|^2 F_\epsilon(t,x,v)dxdv \le \iint \tfrac{1}{2}|v|^2 F_\epsilon^{in}(x,v)dxdv$$

is known to hold for all $t > 0$ in the case of bounded domains. For this reason, only the Navier–Stokes equations, and not the Navier–Stokes–Fourier system, were derived in both references [7] and [50] under assumptions (A1) and (A2).

On the other hand, whether assumption (A2) is satisfied by renormalized solutions of the Boltzmann equation (1.13) in the Navier–Stokes scaling also remains unknown. It was proved by C. Bardos, F. Golse and C.D. Levermore – see Proposition 3.3 of [7] – that the quantity considered in (A2) is of order $O(|\log \epsilon|)$ in $L^\infty_t(L^1(Mdvdx))$. Such a control suffices to establish all hydrodynamic limits leading to linearized macroscopic models such as the acoustic limit in [8], [9] and [25], the Stokes limit in [7], [8] and [51] or the Stokes-Fourier limit in [25]). However, this control is not sufficient to obtain the Navier–Stokes limit.

It was noticed for the first time in [8,9] that the local conservation laws of momentum and energy could be established in the limit as $\epsilon \to 0$ by using the $O(|\log \epsilon|)$ estimate proved in Proposition 3.3 of [7] for the quantity appearing in (A2) and in the Stokes scaling. The method used in these works applied only to bounded collision kernels; however these papers made it clear that local conservation laws needed to be established in the hydrodynamic limit only. This was done for the first time by F. Golse and C.D. Levermore in [25], for general collision kernels (including in particular all hard cutoff potentials and Maxwell molecules), using both the v-v_1 and the (v, v_1)-(v', v_1') symmetries of the Boltzmann collision integral (1.8) and (1.9) with Young's inequality and some of its variants described in Appendix A below. More recently, C.D. Levermore and N. Masmoudi [41] announced a proof of these local conservation laws in the hydrodynamic limit and for the Navier–Stokes scaling, this time under an assumption which, although slightly weaker than (A2), also remained unverified.

Hence, verifying (A2) remained the main obstruction to deriving the Navier–Stokes equations from the Boltzmann equation.

1.6. Method of proof. In the present paper, we follow the method initiated in [25] and establish the local conservation laws of both momentum and energy in the limit as $\epsilon \to 0$: see Sect. 4 below. This step is based on two nonlinear estimates weaker than (A2). These estimates are stated in Proposition 3.4 and Corollary 3.5 in Sect. 3. In doing so, we bypass assumptions (A1)–(A2) that remain unverified.

In addition, the nonlinear controls in Proposition 3.4 and Corollary 3.5, together with other controls stated in Propositions 2.7, 3.8 and Corollary 3.9 allow one to take limits in some appropriately renormalized form of the Boltzmann equation (1.13) integrated against v and $\frac{1}{5}|v|^2 - 1$: this is done in Sect. 5, closely following the methods initiated in [8], [50] and [25].

Hence, the key points in this work are the new nonlinear controls stated

- in Proposition 3.4 and Corollary 3.5,
- in Propositions 2.7, 3.8 and Corollary 3.9.

These new nonlinear controls are combined with earlier techniques, such as:

- the entropy controls for fluctuations and velocity averaging estimates leading to relative L^1 compactness of appropriate moments of the fluctuations of number density already established in [7],
- the control of standing acoustic oscillations as in [50],
- the vanishing of conservation defects proved along the lines of [25].

The tools leading to the new nonlinear controls mentioned above are:

- two decompositions of the fluctuations of number density (the first being based on relative entropy controls while the second is based on the entropy production controls): see Sect. 2, and
- a refined velocity averaging result in L^1 (based on improved v-regularity estimates): see Sect. 3.

As regards this last item, we recall the basic principle of velocity averaging: let $f_n \equiv f_n(x, v)$ be a bounded sequence in $L^2_{loc}(\mathbf{R}^D \times \mathbf{R}^D)$ such that $v \cdot \nabla_x f_n$ is also bounded in $L^2_{loc}(\mathbf{R}^D \times \mathbf{R}^D)$. Then, given any $\xi^* \neq 0$, $f_n(\cdot, v)$ is microlocally bounded in H^1 in the direction ξ^* near each point $x \in \mathbf{R}^D$ and for each $v \in \mathbf{R}^D \setminus (\mathbf{R}\xi^*)^\perp$. Since $(\mathbf{R}\xi^*)^\perp$ is a set of (Lebesgue) measure zero, the integral of f_n on a small conical neighborhood of $(\mathbf{R}^*_\xi)^\perp$ is shown to vanish uniformly in n as the neighborhood shrinks to $(\mathbf{R}^*_\xi)^\perp$ by applying the Cauchy–Schwarz inequality. Following essentially this line of reasoning, F. Golse, B. Perthame and R. Sentis [28] proved[4] that, for any test function $\phi \in C_c(\mathbf{R}^D)$, the sequence of v-averages

$$\rho_n(x) = \int f_n(x, v)\phi(v)dv$$

is relatively compact in $L^2_{loc}(\mathbf{R}^D)$. However, if the sequences f_n and $v \cdot \nabla_x f_n$ are bounded in $L^1_{loc}(\mathbf{R}^D \times \mathbf{R}^D)$, it may happen that the sequence of v-averages ρ_n is not even locally uniformly integrable in $\mathbf{R}^D$. The simple counterexample given in [27] (Example 1, pp. 123–124) is based on concentrations in v. Intuitively, if the f_n's concentrate in the variable v in a single direction $v^* \neq 0$, the v-average ρ_n simply reduces to $\phi(v^*)f_n$ – in other words, the benefit of averaging in v is lost. Unless $D = 1$, one cannot expect to prove compactness on ρ_n in $L^1_{loc}(\mathbf{R}^D)$ under the sole assumption that f_n and $v \cdot \nabla_x f_n$ are bounded in $L^1_{loc}(\mathbf{R}^D \times \mathbf{R}^D)$.

Yet if one excludes such concentrations by assuming additional regularity *in the variable v only* on the f_n's – assuming for example that the sequence f_n is also bounded in $L^1_{loc}(dx; L^\infty_{loc}(dv))$ – the sequence of v-averages ρ_n is indeed locally uniformly integrable in $\mathbf{R}^D$. This was observed for the first time by L. Saint-Raymond [55]. (An earlier remark in the same direction appears in Lemma 8 of [27], stating the relative compactness of ρ_n in $L^1_{loc}(\mathbf{R}^D)$ under the additional assumption of slab symmetry[5]).

Various extensions of this new observation are described in detail (especially in the time-dependent setting needed in the present study) in the second half of Sect. 3 – especially in Lemma 3.6. It is based on a new interpolation mechanism involving the dispersion properties of the advection operator $v \cdot \nabla_x$ presented in Lemmas 3.2 and 3.3.

[4] V. Agoshkov [1] quotes without proof a somewhat analogous result obtained independently.

[5] In other words, f_n depends on only one space variable, say x_1, and on all the velocity variables $v_1, \dots, v_D$.

Thus, applying this new velocity averaging result to the Navier–Stokes limit requires additional v-regularity on (fluctuations of) the phase-space density. This was done by L. Saint-Raymond in [56] for the BGK model; doing the same for the Boltzmann equation is much more involved for reasons that will be described below. Modulo various truncations too technical to be discussed at this stage, this extra v-regularity is implied

- by the dissipation (i.e. the entropy production) estimate, and
- by a smoothing property in the v-variable for the gain part of the collision integral.

The derivations in [7] or [50] did not make much use of the dissipation controls. This was clearly seen as a major shortcoming in both articles. Indeed, when F_ϵ is a local Maxwellian, an easy argument (see [55]) shows that the ratio in (A2) is bounded but not necessarily weakly relatively compact in $L^1_{loc}(dtdx; L^1(Mdv))$. This observation suggests

- that the dissipation estimate implied by (1.21) might help to improve the $O(|\log\epsilon|)$ bound on the ratio in (A2) established in [7]; but
- that this dissipation estimate is not enough to prove the compactness statement in (A2), since the ratio in (A2) is in general not weakly relatively compact in $L^1_{loc}(dtdx; L^1(Mdv))$ when F_ϵ is a local Maxwellian, in other words when the dissipation term $D(F_\epsilon)$ in (1.22) vanishes.

Both these remarks have the merit of showing the importance of the dissipation estimate when trying to establish (A2). Yet they are somewhat misleading, especially as regards the precise manner in which this dissipation estimate is to be used for that purpose. Indeed, the local Maxwellian associated to F_ϵ – i.e. with the same macroscopic density, bulk velocity and temperature as F_ϵ – plays essentially no role in the present work.

For various reasons specific to the Boltzmann equation (see a more detailed discussion on this in Sect. 2), the distance from the phase-space density F_ϵ to its local Maxwellian is not generally well controlled by the dissipation estimate. The key to the extra-regularity in v needed for the Navier–Stokes limit consists in choosing a substitute for the local Maxwellian of the phase space density F_ϵ – see (2.13) below. This local "pseudo-equilibrium" is not defined by a formula showing its regularity properties in v explicitly such as (1.12). Instead, this local pseudo-equilibrium is defined in terms of the gain part of an *artificial* collision operator in such a way that its distance to F_ϵ is controlled by the dissipation integral. That the gain part of the collision integral is more regular in v than F_ϵ itself was proved by P.-L. Lions in [45]. In fact, what is used is not exactly the main result in [45], but an earlier and linear variant of it due to H. Grad [32] and R. Caflisch [16].

The idea of using a fictitious collision operator is somewhat reminiscent of the argument used by P.-L. Lions in [45] p. 423 to prove that any function $F \in L^1_v$ such that $\mathcal{B}(F, F) = 0$ is smooth – and therefore a Maxwellian.

It may also be interesting to compare the procedure described above for gaining compactness from the dissipation estimate with the work of L. Arkeryd and A. Nouri [2].

Both Sects. 2 and 3 describe the main steps leading to the nonlinear controls stated in Proposition 3.4, Corollary 3.5, and in Propositions 2.7, 3.8 and Corollary 3.9. However the complete proofs of these nonlinear controls are postponed to Sects. 6, 7 and 8 below.

Finally, Appendix B collects the main results established in [7] without using the unverified assumptions (A1) and (A2). Our proofs make occasional use of some of these results.

2. Analytical tools I: Decompositions of the number density based on entropy and dissipation

The present section is aimed at describing in detail the part of our argument that improves upon the use of dissipation controls that was made in [7] and [50].

Throughout the present section and the next, we consider F_ϵ, a family of renormalized solutions to (1.13) with initial data F_ϵ^{in} such that (1.14) holds, as well as the relative number densities G_ϵ and fluctuations g_ϵ defined in (1.34). The Boltzmann collision kernel b is assumed to satisfy (H1).

2.1. Entropy-based estimates

2.1.1. The Flat-Sharp decomposition. Most of the estimates in [7] were based on the relative entropy control

$$\frac{1}{\epsilon^2}\int \langle h(\epsilon g_\epsilon(t,x,\cdot)\rangle dx \leq C^{in} \tag{2.1}$$

inferred from (1.44) and the entropy inequality (1.21). We keep here the notations from [7] (see also Appendix A below) and denote the nonlinearity involved in the relative entropy by

$$h(z) = (1+z)\log(1+z) - z\,, \quad z > -1\,. \tag{2.2}$$

Since $h(z) \sim \frac{1}{2}z^2$ near $z = 0$, the entropy control (2.1) is as good as a L^2 control but only for the part of g_ϵ that does not exceed $1/\epsilon$ in size. This suggests splitting the fluctuation of relative number density g_ϵ as follows. First, we consider the class of bump functions

$$\Upsilon = \left\{\gamma : \mathbf{R}_+ \to [0,1] \,\middle|\, \gamma \in C^1\,,\ \gamma\left(\left[\tfrac{3}{4},\tfrac{5}{4}\right]\right) = \{1\}\,,\ \operatorname{supp}\gamma \subset \left[\tfrac{1}{2},\tfrac{3}{2}\right]\right\}. \tag{2.3}$$

We then present

The Flat-Sharp decomposition. Let $\gamma \in \Upsilon$; then

$$g_\epsilon = {}^\flat g_\epsilon + \epsilon\, {}^\sharp g_\epsilon \tag{2.4}$$

with

$$
{}^\flat g_\epsilon = \frac{1}{\epsilon}(G_\epsilon - 1)\gamma(G_\epsilon)\,, \qquad {}^\sharp g_\epsilon = \frac{1}{\epsilon^2}(G_\epsilon - 1)(1-\gamma(G_\epsilon))\,.
$$

While this new decomposition is not the same as the old one just before Corollary 3.2 of [7] (p. 696), it shares most of its key properties – and avoids one unpleasant feature that will be discussed after the statement of Proposition 3.4 below.

Proposition 2.1 (Entropy controls). *Assume that the bump function $\gamma \in \Upsilon$ as in (2.3). The relative fluctuation g_ϵ of number density satisfies the following estimates:*

- ${}^\flat g_\epsilon = O(1)$ *in* $L^\infty_t(L^2(Mdvdx))$;
- ${}^\sharp g_\epsilon = O(1)$ *in* $L^\infty_t(L^1(Mdvdx))$;
- *the family*

$$
\frac{1}{\epsilon}\langle g_\epsilon\rangle(1-\gamma(\langle G_\epsilon\rangle)) \text{ is of order } O(1) \text{ in } L^\infty_t\big(L^1_x\big)\,;
$$

- *for each compact subset E of $\mathbf{R}_+$,*

$$
g_\epsilon \mathbf{1}_E(G_\epsilon) = O(1) \text{ in } L^\infty_t(L^2(Mdvdx))\,;
$$

- *for each $\lambda > e$, and each $\epsilon > 0$, one has*

$$
\left\|\frac{1}{\epsilon}|g_\epsilon|\mathbf{1}_{G_\epsilon \geq \lambda}\right\|_{L^\infty_t(L^1(Mdvdx))} \leq C^{in}\frac{1}{\log\lambda - 1}\,.
$$

Proof. The control on ${}^\flat g_\epsilon$ relies on the entropy inequality (2.1) and on the following elementary inequality: there exists $c > 0$ such that

$$
h(z) \geq cz^2\,, \qquad |z| \leq \tfrac{1}{2}\,. \tag{2.5}
$$

Likewise the fourth control relies on the entropy inequality (2.1) and on the fact that, for each compact subset E of $\mathbf{R}_+$, there exists $c_E > 0$ such that

$$
h(z) \geq c_E z^2\,, \qquad z+1 \in E\,.
$$

The control on ${}^\sharp g_\epsilon$ relies on (2.1) and on the existence of $c' > 0$ such that

$$
h(z) \geq c'|z|\,, \qquad z \in \big[-1, -\tfrac{1}{4}\big] \cup \big[\tfrac{1}{4}, +\infty\big)\,. \tag{2.6}
$$

By Jensen's inequality and the convexity of h, one has

$$
\int h(\epsilon\langle g_\epsilon\rangle)dx \leq \int \langle h(\epsilon g_\epsilon)\rangle dx \leq C^{in}\epsilon^2
$$

because of (2.1). The third control follows from this estimate and (2.6).

Finally, observe that the function $z \mapsto \frac{h(z)}{z}$ is increasing on $\mathbf{R}_+$. Thus for each $\lambda > e$

$$h(\epsilon|g_\epsilon|) \geq \frac{h(\lambda-1)}{\lambda-1}\epsilon|g_\epsilon| \text{ wherever } G_\epsilon \geq \lambda\,.$$

(Notice that $G_\epsilon \geq \lambda$ implies that $g_\epsilon \geq e-1 > 0$). This and (10.1) imply that

$$\begin{aligned}\int \left\langle \frac{1}{\epsilon}|g_\epsilon|\mathbf{1}_{G_\epsilon \geq \lambda}\right\rangle dx &\leq \frac{\lambda-1}{h(\lambda-1)}\frac{1}{\epsilon^2}\int \langle h(\epsilon g_\epsilon)\rangle dx \\ &\leq C^{in}\frac{\lambda-1}{h(\lambda-1)} \leq \frac{C^{in}}{\log\lambda - 1}\,,\end{aligned}$$

which in turn establishes the last control. □

2.1.2. Pointwise estimates implied by the Flat-Sharp decomposition. The old decomposition introduced on p. 696 of [7] had one feature used repeatedly there and in [50], namely the fact that the L^1 part in this decomposition g_ϵ^2/N_ϵ controlled the square of the L^2 part g_ϵ/N_ϵ. The analogue with the Flat-Sharp decomposition (2.4) (namely, that $|{}^\flat g_\epsilon|^2 \leq |{}^\sharp g_\epsilon|$) is no longer true, but (2.4) leads to a precise localization of g_ϵ which has several useful implications.

Proposition 2.2 (Pointwise estimates). *Assume that the bump function $\gamma \in \Upsilon$ as in (2.3). The relative number density fluctuation g_ϵ satisfies the following estimates:*

- $\epsilon|{}^\flat g_\epsilon| \leq \frac{1}{2}$;
- $(1-\gamma(G_\epsilon)) \leq 4\epsilon^2|{}^\sharp g_\epsilon|$, *which implies that* $\frac{1}{\epsilon}(1-\gamma(G_\epsilon)) \leq 2|{}^\sharp g_\epsilon|^{1/2}$;
- $(1-\gamma(G_\epsilon))G_\epsilon \leq 5\epsilon^2|{}^\sharp g_\epsilon|$;
- *for* $k : \mathbf{R}_+ \to [0,1]$, *let* $C_k = \|z \mapsto zk(z)\|_{L^\infty}$; *then for any family* $V_\epsilon \to +\infty$ *as* $\epsilon \to 0$, *one has*[6]

$$\begin{aligned}&\int (k(G_\epsilon)|G_\epsilon - 1|)^\alpha M^\beta |v|^p \mathbf{1}_{|v|^2 \geq V_\epsilon} dv \\ &\qquad\qquad \lesssim \tfrac{4\pi}{\beta}(2\pi)^{-\frac{3\beta}{2}}(1+C_k)^\alpha V_\epsilon^{\frac{p+1}{2}} e^{-\frac{\beta}{2}V_\epsilon}\end{aligned}$$

as $\epsilon \to 0$.

Proof. Because γ is supported in $[\frac{1}{2}, \frac{3}{2}]$,

$$\epsilon|{}^\flat g_\epsilon| = |G_\epsilon - 1|\gamma(G_\epsilon) \leq |G_\epsilon - 1|\mathbf{1}_{|G_\epsilon - 1| \leq \frac{1}{2}} \leq \tfrac{1}{2}\,.$$

[6] The notation $a_\epsilon \lesssim b_\epsilon$ as $\epsilon \to 0$ means that there exists $c_\epsilon \sim b_\epsilon$ as $\epsilon \to 0$ such that $a_\epsilon \leq c_\epsilon$ for all ϵ small enough in $]0,1[$.

Because $(1-\gamma)$ vanishes identically on $[\frac{3}{4}, \frac{5}{4}]$,

$$\begin{aligned}\epsilon^2 |\,^\sharp g_\epsilon| &= |G_\epsilon - 1|(1-\gamma(G_\epsilon)) \\ &\geq |G_\epsilon - 1| \mathbf{1}_{|G_\epsilon - 1| \geq \frac{1}{4}} (1-\gamma(G_\epsilon)) \\ &\geq \tfrac{1}{4} \mathbf{1}_{|G_\epsilon - 1| \geq \frac{1}{4}} (1-\gamma(G_\epsilon)) \\ &= \tfrac{1}{4}(1-\gamma(G_\epsilon)) \,.\end{aligned}$$

Because $0 \leq 1-\gamma(G_\epsilon) \leq 1$, one also has

$$\epsilon^2 |\,^\sharp g_\epsilon| \geq \tfrac{1}{4}(1-\gamma(G_\epsilon))^2 \text{ and thus } \frac{1-\gamma(G_\epsilon)}{\epsilon} \leq 2 |\,^\sharp g_\epsilon|^{1/2} \,.$$

The third control is a direct consequence of the second because

$$(1-\gamma(G_\epsilon))G_\epsilon = \epsilon^2 \,^\sharp g_\epsilon + (1-\gamma(G_\epsilon)) \,.$$

The last control relies on the obvious estimate $k(G_\epsilon)|G_\epsilon - 1| \leq 1 + C_k$ and on the standard tail estimate for the Gaussian distribution:

$$\begin{aligned}\int M^\beta |v|^p \mathbf{1}_{|v|^2 \geq V_\epsilon} dv &= (2\pi)^{-\frac{3\beta}{2}} 4\pi \int_{\sqrt{V_\epsilon}}^{+\infty} e^{-\frac{\beta}{2}r^2} r^{p+2} dr \\ &\sim (2\pi)^{-\frac{3\beta}{2}} 4\pi \int_{\sqrt{V_\epsilon}}^{+\infty} -\frac{d}{dr}\left(\tfrac{1}{\beta} e^{-\frac{\beta}{2}r^2} r^{p+1}\right) dr \\ &= (2\pi)^{-\frac{3\beta}{2}} \frac{4\pi}{\beta} e^{-\frac{\beta}{2}V_\epsilon} V_\epsilon^{\frac{p+1}{2}}\end{aligned}$$

as $\epsilon \to 0$, since $V_\epsilon \to +\infty$. □

2.2. Dissipation-based estimates

2.2.1. Dissipation controls of the scaled collision integrand. While the fluctuations of relative number density g_ϵ satisfy the bound (2.1), the scaled collision integrand q_ϵ defined in (1.35) satisfies the dissipation bound

(2.7)
$$\begin{aligned}\frac{1}{4\epsilon^4} \int_0^{+\infty} \int \left\langle\!\!\left\langle r\left(\frac{\epsilon^2 q_\epsilon}{G_\epsilon G_{\epsilon 1}}\right) G_\epsilon G_{\epsilon 1} \right\rangle\!\!\right\rangle dx dt &= \frac{1}{\epsilon^4} \int_0^{+\infty} \iint D(F_\epsilon) dv dx dt \\ &\leq \frac{1}{\epsilon^2} H(F_\epsilon^{in} | M) \leq C^{in}\end{aligned}$$

inferred from (1.44) and the entropy inequality (1.21). Again we use the notations from [7] (see also Appendix A below) and denote the nonlinearity involved in the dissipation by

(2.8)
$$r(z) = z \log(1+z) \,, \quad z > -1 \,.$$

Since $r(z) \sim z^2$ near $z = 0$, the dissipation control is also as good as an L^2 control but again this is only true of the part of $q_\epsilon/(G_\epsilon G_{\epsilon 1})$ that does not exceed $1/\epsilon^2$ in size. This suggests decomposing the scaled collision integrand as

$$
\begin{aligned}
q_\epsilon &= {}^\flat q_\epsilon + \epsilon^2\, {}^\sharp q_\epsilon \text{ with}\\
{}^\flat q_\epsilon &= q_\epsilon \gamma\left(\frac{G'_\epsilon G'_{\epsilon 1}}{G_\epsilon G_{\epsilon 1}}\right) \text{ and}\\
{}^\sharp q_\epsilon &= \frac{1}{\epsilon^2} q_\epsilon \left(1 - \gamma\left(\frac{G'_\epsilon G'_{\epsilon 1}}{G_\epsilon G_{\epsilon 1}}\right)\right)
\end{aligned}
\tag{2.9}
$$

where γ is any element of Υ as in (2.3).

Proposition 2.3 (Dissipation controls). *The scaled collision integrand q_ϵ satisfies the following estimates*

- *for any function $\gamma \in \Upsilon$*

$$\iint \frac{{}^\flat q_\epsilon^2}{G_\epsilon G_{\epsilon 1}} b(v - v_1, \omega) d\sigma_{v,v_1}(\omega) M_1 dv_1 \leq \tfrac{4}{c} \frac{D(F_\epsilon)}{\epsilon^4 M}$$

where the constant c is defined in (2.5), so that

$$\left\langle\!\!\left\langle \frac{{}^\flat q_\epsilon^2}{G_\epsilon G_{\epsilon 1}} \right\rangle\!\!\right\rangle = O(1) \text{ in } L^1_{t,x}\,;$$

- *likewise*

$$\iint |\, {}^\sharp q_\epsilon | b(v - v_1, \omega) d\sigma_{v,v_1}(\omega) M_1 dv_1 \leq \tfrac{4}{c'} \frac{D(F_\epsilon)}{\epsilon^4 M}$$

where the constant c is defined in (2.6), so that

$$\langle\!\langle |\, {}^\sharp q_\epsilon | \rangle\!\rangle = O(1) \text{ in } L^1_{t,x}\,;$$

- *more generally, for any compact subset E of* $[-1, +\infty)$,

$$\left\langle\!\!\left\langle \frac{q_\epsilon^2}{G_\epsilon G_{\epsilon 1}} \mathbf{1}_E \left(\frac{G'_\epsilon G'_{\epsilon 1}}{G_\epsilon G_{\epsilon 1}} - 1\right) \right\rangle\!\!\right\rangle = O(1) \text{ in } L^1_{t,x}\,.$$

Proof. The elementary inequality $r(z) \geq h(z)$ for all $z > -1$ together with the inequalities (2.5) and (2.6) shows that, for the same positive constants c and c' as in these inequalities, one has

$$r(z) \geq cz^2\,, \qquad |z| \leq \tfrac{1}{2}\,, \tag{2.10}$$

and

$$r(z) \geq c'|z|\,, \qquad z \in \left(-1, -\tfrac{1}{4}\right] \cup \left[\tfrac{1}{4}, +\infty\right). \tag{2.11}$$

Then, the condition

$$\gamma\left(\frac{G'_\epsilon G'_{\epsilon 1}}{G_\epsilon G_{\epsilon 1}}\right) \neq 0 \text{ implies that } \left|\frac{G'_\epsilon G'_{\epsilon 1}}{G_\epsilon G_{\epsilon 1}} - 1\right| = \frac{\epsilon^2 |q_\epsilon|}{G_\epsilon G_{\epsilon 1}} \leq \tfrac{1}{2},$$

which shows that the first control follows from (2.10) and (2.7). Likewise, the condition

$$1-\gamma\left(\frac{G'_\epsilon G'_{\epsilon 1}}{G_\epsilon G_{\epsilon 1}}\right) \neq 0 \text{ implies that } \left|\frac{G'_\epsilon G'_{\epsilon 1}}{G_\epsilon G_{\epsilon 1}} - 1\right| = \frac{\epsilon^2 |q_\epsilon|}{G_\epsilon G_{\epsilon 1}} \geq \tfrac{1}{4},$$

which shows that the second control follows from (2.11) and (2.7).

As for the third control, it is obtained in exactly the same way as the first, because, for any compact $E \subset [-1, +\infty)$, there exists $c_E > 0$ such that $r(z) \geq c_E z^2$ for all $z \in E$. □

These estimates were apparently used for the first time in proving that conservation defects vanish as $\epsilon \to 0$ in the proof of the Stokes or Acoustic limits [25].

2.2.2. A second decomposition of g_ϵ. The Flat-Sharp decomposition (2.4) of g_ϵ is exclusively based on the level set of g_ϵ, i.e. on the size of the values taken by g_ϵ. Thus it cannot help in controlling the decay in v of g_ϵ, at least not beyond the $L^1((1+|v|^2)Mdv)$ control following from Young's inequality (see [7], Propositions 3.1 (1), 3.5 (1) and 3.3). This was the main reason for the unverified assumption (A2) from [7] recalled at the end of Sect. 1.

In the present work, we introduce a second decomposition of g_ϵ, based in particular on the dissipation controls above. An obvious idea would be to adapt the arguments used in the case of the BGK model (see [55], [56]) and decompose the number density as

$$F_\epsilon = (F_\epsilon - M_{F_\epsilon}) + (M_{F_\epsilon} - M) \tag{2.12}$$

where M_{F_ϵ} is the local Maxwellian with the same macroscopic density, bulk velocity and temperature as F_ϵ at every (t, x).

However, instead of using the local Maxwellian of F_ϵ in the decomposition (2.12) above, we propose to replace it with the following quantity.

The local pseudo-equilibrium. The substitute for M_{F_ϵ} considered in this work is

$$\frac{\mathcal{A}^+(M\tilde{G}_\epsilon, M\tilde{G}_\epsilon)}{\langle \tilde{G}_\epsilon \rangle} \tag{2.13}$$

with

$$\tilde{G}_\epsilon = 1 + \epsilon \, g^\flat_\epsilon = (1-\gamma(G_\epsilon)) + \gamma(G_\epsilon) G_\epsilon \tag{2.14}$$

and where $\mathcal{A}^+$ is the gain part of a fictitious collision operator:

$$\mathcal{A}^+(f,g) = \tfrac{1}{2}\iint (f'g_1' + f_1'g')d\sigma_{v,v_1}(\omega)dv_1 \,.$$

Using the gain term $\mathcal{A}^+$ in a decomposition like (2.12) is suggested by the inequality below (satisfied by any measurable, a.e. positive function F):

$$\begin{aligned} 4D(F) \geq \frac{R}{b_\infty}\left(\frac{\mathcal{A}^+(F,F)}{R} - F\right)\left(\log\left(\frac{\mathcal{A}^+(F,F)}{R}\right) - \log F\right) \\ \text{with } R = \int F dv \,, \end{aligned} \tag{2.15}$$

where $D(F)$ is the dissipation term defined in (1.22). This inequality (2.15) is obtained on account of (H1) by applying Jensen's inequality to the probability measure $F(v_1)d\sigma_{v,v_1}(\omega)dv_1/R$ and to the convex function $X \mapsto (X-Y)(\log X - \log Y)$ with $Y > 0$ fixed. It provides an explicit control of the distance between F and $\mathcal{A}^+(F,F)/R$ in terms of $D(F)$. If $\mathcal{A}^+(F,F)/R$ is replaced by the local Maxwellian M_F, the same control is known to be false: see for example the work of G. Toscani and C. Villani [61], eq. (25) and the discussion pp. 671–672. Replacing then $\mathcal{A}^+(MG_\epsilon, MG_\epsilon)/\langle G_\epsilon\rangle$ by (2.13) has three further advantages that are fully exploited below in Sect. 6:

- $\langle \tilde{G}_\epsilon \rangle > 0$,
- $\tilde{G}_\epsilon$ is L^∞-bounded and $L^1(Mdv)$-close to G_ϵ,
- $\tilde{G}_\epsilon$ is $L^2(Mdv)$-close to 1.

Let $k : \mathbf{R}_+ \to [0,1]$ and set $C_k = \|z \mapsto zk(z)\|_{L^\infty}$; then

$$\begin{aligned} k(G_\epsilon)Mg_\epsilon &= \frac{1}{\epsilon}k(G_\epsilon)\left(MG_\epsilon - \frac{\mathcal{A}^+(M\tilde{G}_\epsilon, M\tilde{G}_\epsilon)}{\langle \tilde{G}_\epsilon\rangle}\right) \\ &\quad + \frac{1}{\epsilon}k(G_\epsilon)\left(\frac{\mathcal{A}^+(M\tilde{G}_\epsilon, M\tilde{G}_\epsilon)}{\langle \tilde{G}_\epsilon\rangle} - M\right) \\ &= T_1 + T_2 \,. \end{aligned} \tag{2.16}$$

Using the last equality in (2.14) and the decomposition of the scaled collision integrand suggested by Proposition 2.3 to further decompose T_1 while decomposing T_2 as

$$\begin{aligned} T_2 = &\frac{k(G_\epsilon)}{\epsilon\langle \tilde{G}_\epsilon\rangle}\left(\mathcal{A}^+(M\tilde{G}_\epsilon, M\tilde{G}_\epsilon) - \mathcal{A}^+(M,M)\right) \\ &+ \frac{k(G_\epsilon)}{\epsilon}\mathcal{A}^+(M,M)\left(\frac{1}{\langle \tilde{G}_\epsilon\rangle} - 1\right) \end{aligned}$$

leads to the following crucial inequality.

The Relaxation-based decomposition. For each $\epsilon > 0$

$$
\begin{aligned}
k(G_\epsilon)M|g_\epsilon| \le\ & 4\sqrt{\tfrac{3b_\infty(c+c')}{cc'}}\sqrt{C_k+2}\ \epsilon\sqrt{M}\sqrt{\frac{D(F_\epsilon)}{\epsilon^4}} \\
& + 2M\left\langle |\flat g_\epsilon| + (C_k+3)\frac{1-\gamma(G_\epsilon)}{\epsilon}\right\rangle \\
& + \mathcal{A}^+\left(M\left[5|\flat g_\epsilon| + 9(C_k+1)\frac{1-\gamma(G_\epsilon)}{\epsilon}\right], M\right),
\end{aligned}
\tag{2.17}
$$

where $b_\infty > 0$ is the constant appearing in assumption (H1) on the collision kernel, while c and c' are the positive constants appearing in (2.5) and (2.6).

Deriving inequality (2.17) from the basic decomposition (2.16) requires nontrivial computations that will occupy most of Sect. 6 below (see in particular the Subsect. 6.1 there).

The benefits of using this new decomposition may not seem obvious at first sight. Observe however that the first term on the right-hand side of (2.17) is $O(\epsilon)$ in $L^2(M^{-1}dvdxdt)$, while the second is of the form $M \times O(1)_{L^\infty_t(L^2_x)}$. This has the following implications on the decay in the v-variable:

Proposition 2.4. *For any $p \ge 0$*

$$\int \epsilon\sqrt{M}\sqrt{\frac{D(F_\epsilon)}{\epsilon^4}}|v|^p dv = O(\epsilon) \ \ \text{in } L^2_{t,x}$$

and

$$\sup_{v\in\mathbf{R}^3}\left(|v|^p M\left\langle |\flat g_\epsilon| + (C_k+3)\frac{1-\gamma(G_\epsilon)}{\epsilon}\right\rangle\right) = O(1) \ \ \text{in } L^\infty_t\big(L^2_x\big).$$

The first estimate follows from the dissipation estimate (2.7) and the Cauchy–Schwarz inequality. The second estimate follows from the first and second entropy controls in Proposition 2.1, and the second pointwise estimate in Proposition 2.2.

It remains to see that the third term enjoys similar decay properties; this will result from classical estimates recalled below.

2.2.3. The Caflisch–Grad estimates. For all $p \in [1, +\infty]$ and all $s \ge 0$, define $L^{p,s}$ as the space of a.e.-defined measurable functions f on $\mathbf{R}^3$ such that $v \mapsto (1+|v|^s)f(v)$ belongs to $L^p(\mathbf{R}^3, dv)$ and set

$$\|f\|_{L^{p,s}} = \|(1+|v|^s)f\|_{L^p}.$$

Define further the linear operator

$$\mathcal{K}f = \frac{1}{\sqrt{M}}\mathcal{A}^+(\sqrt{M}f, M).$$

The properties of the operator $\mathcal{K}$ were studied in detail by H. Grad in his fundamental paper [32]. His estimates, later improved by R. Caflisch [16], are recalled below.

Proposition 2.5 (Caflisch–Grad estimates). *For all $s \geq 0$, the bilinear operator*

$$(f,g) \mapsto \mathcal{A}^+(f,g)$$

is a continuous map from $L^{1,s} \times L^{1,s}$ to $L^{1,s}$. Further

- *the operator $\mathcal{K}$ maps L^2 continuously into $L^{\infty,3/2}$;*
- *for each $s > 0$, $\mathcal{K}$ maps $L^{\infty,s}$ continuously into $L^{\infty,s+2}$;*
- *for each $s, \sigma > 0$, $\mathcal{K}$ maps $L^{2,s}$ continuously into $L^{\infty,s+\frac{3}{2}} + L^{2,\sigma}$.*

Proof. The second equality in (1.4) clearly implies that $\sup(|v|^2, |v_1|^2) \leq |v'|^2 + |v_1'|^2$. Thus for each $s > 0$ there exists $C_s > 1$ such that $|v|^s + |v_1|^s \leq C_s(|v'|^s + |v_1'|^s)$ for all v and $v_1 \in \mathbf{R}^3$ and all $\omega \in \mathbf{S}^2$. For each measurable f one therefore has

$$\begin{aligned}
\|\mathcal{A}^+(f,g)\|_{L^{1,s}} &\leq \tfrac{1}{2} \iiint (|f'||g_1'| + |f_1'||g'|)(1+|v|^s) d\sigma_{v,v_1}(\omega) dv_1 dv \\
&= \tfrac{1}{2} \iiint |f'||g_1'|(2 + |v|^s + |v_1|^s) d\sigma_{v,v_1}(\omega) dv_1 dv \\
&\leq \tfrac{1}{2} \iiint |f'||g_1'| C_s (2 + |v'|^s + |v_1'|^s) d\sigma_{v,v_1}(\omega) dv_1 dv \\
&\leq C_s \iiint |f'||g_1'|(1+|v'|^s)(1+|v_1'|^s) d\sigma_{v,v_1}(\omega) dv_1 dv \\
&= C_s \iiint |f||g_1|(1+|v|^s)(1+|v_1|^s) d\sigma_{v,v_1}(\omega) dv_1 dv \\
&= C_s \int |f|(1+|v|^s) dv \int |g|(1+|v|^s) dv\,,
\end{aligned}$$

which establishes the first statement. The first equality above comes from the v-v_1 symmetry in (1.8) and (1.4), while the penultimate equality results from the (v, v_1)-(v', v_1') symmetry in these same formulas (see (1.11)).

The second and third statements are much harder to prove: they are particular cases of estimates (6.1) and (6.2) of [16].

The fourth statement is an easy consequence of the first and second, as shown below. For all $f \in L^{2,s}$, write

$$\mathcal{K} f = a_1 + a_2$$

with a_1 defined by

$$\begin{aligned}
a_1 = \frac{1}{\sqrt{M}} \iint \tfrac{1}{2}\Big(&\sqrt{M'} f' \mathbf{1}_{|v|^2 \leq 2|v'|^2} M_1' \\
&+ \sqrt{M_1'} f_1' \mathbf{1}_{|v|^2 \leq 2|v_1'|^2} M'\Big) d\sigma_{v,v_1}(\omega) dv_1
\end{aligned}$$

while a_2 is defined by

$$a_2 = \frac{1}{\sqrt{M}} \iint \tfrac{1}{2}\Big(\sqrt{M'} f' M_1' (1 - \mathbf{1}_{|v|^2 \le 2|v'|^2}) + \sqrt{M_1'} f_1' M' (1 - \mathbf{1}_{|v|^2 \le 2|v_1'|^2})\Big) d\sigma_{v,v_1}(\omega) dv_1 \,.$$

Then

$$|v|^{\frac{3}{2}+s} |a_1| \le 2^{\frac{s}{2}} |v|^{\frac{3}{2}} \mathcal{K}(|v|^s |f|) \,. \tag{2.18}$$

Using again the second equality in (1.4) implies that, for all v, v_1 and ω, one has

$$|v'|^2 \ge \tfrac{1}{2}|v|^2 \quad \text{or} \quad |v_1'|^2 \ge \tfrac{1}{2}|v|^2 \,.$$

In other words, for all v, v_1 and ω,

$$1 \le \mathbf{1}_{|v|^2 \le 2|v'|^2} + \mathbf{1}_{|v|^2 \le 2|v_1'|^2} \,.$$

Therefore

$$\begin{aligned} |v|^\sigma |a_2| &\le \frac{1}{\sqrt{M}} \mathcal{A}^+\big(\sqrt{M}|f|, 2^{\frac{\sigma}{2}} |v|^\sigma M\big) \\ &= \iint \tfrac{1}{2}\Big(|f'|\sqrt{M_1'} 2^{\frac{\sigma}{2}} |v_1'|^\sigma + |f_1'|\sqrt{M'} 2^{\frac{\sigma}{2}} |v'|^\sigma\Big) \sqrt{M_1} d\sigma_{v,v_1}(\omega) dv_1 \\ &\le \left(\iint d\sigma_{v,v_1}(\omega) M_1 dv_1\right)^{\frac{1}{2}} \mathcal{A}^+(|f|^2, 2^\sigma |v|^{2\sigma} M)^{\frac{1}{2}} \\ &= \mathcal{A}^+(|f|^2, 2^\sigma |v|^{2\sigma} M)^{\frac{1}{2}} \,. \end{aligned} \tag{2.19}$$

The inequality (2.18) and the second continuity statement of Proposition 2.5 imply the existence of a positive constant C such that

$$\|a_1\|_{L^{\infty, \frac{3}{2}+s}} \le C \|f\|_{L^{2,s}} \,.$$

The inequality (2.19) and the first continuity statement of Proposition 2.5 imply the existence of a positive constant C' such that

$$\|a_2\|_{L^{2,\sigma}} \le C' \|f\|_{L^2} \le C' \|f\|_{L^{2,s}} \,.$$

These last two inequalities imply the fourth continuity statement of Proposition 2.5. □

2.2.4. Conclusion. The Caflisch–Grad estimates summarized in Proposition 2.5 above imply that the third term on the right-hand side of the inequality (2.17) also is uniformly bounded a.e. pointwise in v with good decay as $|v| \to +\infty$. Eventually this implies that, after truncating large values of g_ϵ, whatever remains of the fluctuations of relative number density is uniformly bounded a.e. pointwise in v and has good decay as $|v| \to +\infty$. Before going further, we need the following definition.

Definition 2.6. *Let f_ϵ a bounded family of $L^1_{loc}(\mathbf{R}^m_x \times \mathbf{R}^n_y)$. We say that f_ϵ is uniformly integrable in y if and only if, for each $\eta > 0$, there exists $\alpha > 0$ such that, for each measurable family $(A_x)_{x\in\mathbf{R}^m}$ of measurable sets in $\mathbf{R}^n_y$ satisfying*[7] $\sup_{x\in\mathbf{R}^m} |A_x| < \alpha$, *one has*

$$\int \left(\int_{A_x} |f_\epsilon(x, y)| dy \right) dx < \eta\,, \quad \textit{for each } \epsilon.$$

The family f_ϵ is said to be locally uniformly integrable in y if and only if $\mathbf{1}_K f_\epsilon$ is uniformly integrable in y for each compact $K \subset \mathbf{R}^m_x \times \mathbf{R}^n_y$.

The main consequences of the two decompositions introduced in this section are stated in the next proposition. Obviously, the proof of the Navier–Stokes–Fourier limit will also use many of the results already proved in [7], mainly from the analogue of the Flat-Sharp decomposition there. We have chosen to summarize these results in Appendix B below so as to avoid a cluttered presentation of the material that is genuinely new in the present paper.

Proposition 2.7 (**$^\flat g_\epsilon$ controls I).** *For any $\gamma \in \Upsilon$ defined by (2.3), the family $^\flat g_\epsilon$ has the following properties:*

- *for any sequence $\epsilon_n \to 0$, the associated sequence $M|{}^\flat g_{\epsilon_n}|^2$ is locally uniformly integrable in v;*
- $(1 + |v|^s)\, {}^\flat g_\epsilon = O(1)$ *in* $L^2_{loc}(dtdx; L^2(Mdv))$ *for all* $s \geq 0$.

In addition, one has

$$(1 + |v|^s) \frac{1 - \gamma(G_\epsilon)}{\epsilon^2} = O(1) \textit{ in } L^1_{loc}(dtdx; L^1(Mdv))$$

for all $s \geq 0$.

3. Analytical tools II: A new limiting case of velocity averaging in L^1

While the ideas presented in the previous section certainly help in controlling the v dependence of $^\flat g_\epsilon$, more is needed to estimate the $^\sharp g_\epsilon$ term. Indeed, in order to control $^\sharp g_\epsilon$

[7] For each measurable set $A \subset \mathbf{R}^d$, the notation $|A|$ designates the d-dimensional Lebesgue measure of A.

- because of the last entropy control (the fifth estimate in Proposition 2.1), it suffices to control the truncated family $^\sharp g_\epsilon \mathbf{1}_{G_\epsilon \le e^{1/\delta}}$ for some δ appropriately chosen in terms of ϵ;
- because of the last pointwise estimate (the fourth statement in Proposition 2.2), one can further truncate in v and consider finally the truncated family
$$^\sharp g_\epsilon \mathbf{1}_{G_\epsilon \le e^{1/\delta}} \mathbf{1}_{|v|^2 \le V_\epsilon}$$
for some well-chosen $V_\epsilon \to +\infty$.

Next, we split the expression above as

$$^\sharp g_\epsilon \mathbf{1}_{G_\epsilon \le e^{1/\delta}} \mathbf{1}_{|v|^2 \le V_\epsilon} = \left(\frac{1}{\epsilon} \mathbf{1}_{G_\epsilon \le e^{1/\delta}} g_\epsilon \right) \left((1 - \gamma(G_\epsilon)) \mathbf{1}_{|v|^2 \le V_\epsilon} \right) . \tag{3.1}$$

To control the first factor, one can think of using the Relaxation-based decomposition (2.17) with $k(z) = \mathbf{1}_{z \le e^{1/\delta}}$. This would lead to estimating

$$\begin{aligned} \frac{1}{\epsilon} \mathbf{1}_{G_\epsilon \le e^{1/\delta}} M g_\epsilon = \sqrt{M} O \left(\frac{e^{1/\delta}}{\epsilon} \right)_{L^2_{loc}(dtdx; L^\infty_v)} \\ + O(e^{1/2\delta})_{L^2(dtdx; L^{2,s})} \text{ for all } s \ge 0 ; \end{aligned} \tag{3.2}$$

– see estimate (7.14) below. This control would match nicely with an estimate of the second factor in (3.1)

$$(1 - \gamma(G_\epsilon)) \mathbf{1}_{|v|^2 \le V_\epsilon} \text{ in } L^\infty_{loc}(dtdx; L^1_v) ;$$

– see Lemma 7.2. This term is obviously $O(1)$ in $L^\infty_{t,x,v}$; because of the second pointwise estimate in Proposition 2.2 and the second entropy control in Proposition 2.1, it is also $O(\epsilon^2 e^{V_\epsilon/2})$ in $L^\infty_t(L^1_{x,v})$, which can be taken as $o(\epsilon)$ by a suitable choice of V_ϵ. This suggests trying the following classical argument.

Lemma 3.1. *Let $\chi \in L^1 \cap L^\infty(\mathbf{R}^m_x \times \mathbf{R}^n_y)$. There exists $\chi_1 \in L^\infty_x(L^1_y)$ and $\chi_2 \in L^1_x(L^\infty_y)$ such that $\chi = \chi_1 + \chi_2$ with*

$$|\chi_1| \le |\chi| \text{ and } |\chi_2| \le |\chi| \text{ a.e.,}$$

and

$$\|\chi_1\|_{L^\infty_x(L^1_y)} \le \|\chi\|^{1/2}_{L^1} \|\chi\|^{1/2}_{L^\infty} , \quad \|\chi_2\|_{L^1_x(L^\infty_y)} \le \|\chi\|^{1/2}_{L^1} \|\chi\|^{1/2}_{L^\infty} .$$

Proof. The decomposition is obtained as

$$\chi_1(x, y) = \mathbf{1}_E(x) \chi(x, y) , \quad \chi_2(x, y) = \mathbf{1}_{E^c}(x) \chi(x, y) ,$$

where, for some $\lambda > 0$,

$$E = \left\{ x \in \mathbf{R}^m \;\middle|\; \int |\chi(x, y)| dy \le \lambda \|\chi\|^{1/2}_{L^1_{x,y}} \right\} .$$

Indeed

$$\|\chi_1\|_{L^\infty_x(L^1_y)} = \sup_{x\in E}\int |\chi(x,y)|dy \le \lambda\|\chi\|^{1/2}_{L^1_{x,y}},$$

while

$$|E^c| \le \frac{1}{\lambda}\|\chi\|^{1/2}_{L^1_{x,y}}.$$

Thus

$$\|\chi_2\|_{L^1_x(L^\infty_y)} \le |E^c| \sup_{x\in E^c}\|\chi(x,\cdot)\|_{L^\infty_y} \le \frac{1}{\lambda}\|\chi\|^{1/2}_{L^1_{x,y}}\|\chi\|_{L^\infty_{x,y}}.$$

Picking $\lambda = \|\chi\|^{1/2}_{L^\infty_{x,y}}$ leads to the desired result. □

Applying the lemma to $\chi = (1-\gamma(G_\epsilon))\mathbf{1}_{|v|^2\le V_\epsilon}$ gives a χ_1 that can be paired with (3.2) as described above. Handling the χ_2 term requires a different idea presented below.

3.1. Advection/Dispersion bilinear interpolation. The Relaxation-based decomposition (2.17) improves the L^p-regularity in the variable v of suitable truncations of the number density fluctuations g_ϵ. It remains to transfer some of this extra L^p-regularity to the variable x. In the case where the spatial domain is the whole $\mathbf{R}^3$ space, doing so rests in particular on a dispersion argument originally due to C. Bardos and P. Degond [4] – see also [17]. We first recall this argument in the setting best adapted for future use in the present paper: see Proposition 1.11 of [15].

Lemma 3.2. *Let $\phi^0 \equiv \phi^0(t,x,v) \in L^\infty_t(L^p_x(L^q_v))$ for some $1 \le p < q \le +\infty$, and let $\phi \equiv \phi(\tau,t,x,v)$ be the solution of the Cauchy problem*

$$\begin{aligned}\partial_\tau\phi + \epsilon\partial_t\phi + v\cdot\nabla_x\phi = 0\,, &\quad \tau>0\,,\ t\in\mathbf{R}\,,\ x,v\in\mathbf{R}^3\,,\\ \phi(0,t,x,v) = \phi^0(t,x,v)\,, &\quad t\in\mathbf{R}\,,\ x,v\in\mathbf{R}^3\,.\end{aligned}\tag{3.3}$$

Then, for all $\tau\in\mathbf{R}^*$,

$$\|\phi(\tau,\cdot,\cdot,\cdot)\|_{L^\infty_t(L^q_x(L^p_v))} \le |\tau|^{-3\left(\frac{1}{p}-\frac{1}{q}\right)}\|\phi^0\|_{L^\infty_t(L^p_x(L^q_v))}.$$

In the Cauchy problem (3.3) the variable τ is not the physical time variable, but an independent, fictitious time variable used below as an interpolation parameter. In other words, let

$$U^\epsilon_\tau = e^{-\tau(\epsilon\partial_t + v\cdot\nabla_x)}$$

be the group generated by $-(\epsilon\partial_t + v\cdot\nabla_x)$. Then, any function $\phi^0\equiv\phi^0(t,x,v)$ is decomposed into

$$\phi^0(t,x,v) = \left(U^\epsilon_{\tau^*}\phi^0\right)(t,x,v) + \int_0^{\tau^*}(\epsilon\partial_t + v\cdot\nabla_x)\left(U^\epsilon_\tau\phi^0\right)(t,x,v)d\tau\,.\tag{3.4}$$

In this decomposition, the first term has enhanced L^p-regularity in the variable x while the second term is small with $|\tau^*|$ if ϕ^0 belongs to the domain of $\epsilon\partial_t + v \cdot \nabla_x$. Introducing this fictitious time variable might seem somewhat unusual; yet this is in complete analogy with the classical procedure due to J.-L. Lions (see [42]) relating interpolation spaces with spaces of traces (in the sense of restrictions to some boundary) of functions taking their values in Banach spaces. These former methods would typically use elliptic operators and the domains of their fractional powers as the means of gauging the smoothness of the functions under consideration. In the present work, we use instead precisely that advection operator which appears in the scaled Boltzmann equation (1.13) and are concerned with dispersion rather than regularity properties.

Second, we write and prove a formula that will be used together with Lemma 3.2.

Lemma 3.3. *Pick $\epsilon > 0$, $t^* > 0$, and $\tau^* > 0$, and consider*

$$\Omega = \{(\tau, t, x, v) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^3 \times \mathbf{R}^3 \,|\, 0 < \tau < \tau^*\,,\ 0 < t - \epsilon\tau < t^*\}\,.$$

Then, for all $f \in L^1_{loc}(dtdxdv)$ such that $(\epsilon\partial_t + v \cdot \nabla_x)f$ belongs to $L^1_{loc}(dtdxdv)$ and all compactly supported $\phi^0 \in L^\infty(dtdxdv)$,

(3.5)

$$\begin{aligned}\int_0^{t^*}\iint f\phi^0 dtdxdv = &\int_{\epsilon\tau^*}^{t^*+\epsilon\tau^*}\iint f\phi(\tau^*, \cdot, \cdot, \cdot)dtdxdv\\ &- \int_\Omega \phi(\tau, t, x, v)(\epsilon\partial_t + v \cdot \nabla_x) f(t, x, v) d\tau dtdxdv\,,\end{aligned}$$

where ϕ denotes the solution of (3.3).

The same identity (3.5) holds for any ϕ^0 in $L^\infty(dtdxdv)$ (not necessarily compactly supported) and all f in $L^1_{loc}(dt; L^1(dxdv))$.

Proof. Apply Green's formula to the integral

$$\int_\Omega f(t, x, v)(\partial_\tau + \epsilon\partial_t + v \cdot \nabla_x)\phi(\tau, t, x, v) d\tau dtdxdv = 0\,.$$

□

Set ϕ^0 to be the χ_2 term obtained by applying Lemma 3.1 to $\chi = (1 - \gamma(G_\epsilon))\mathbf{1}_{|v|^2 \le V_\epsilon}$; by Lemma 3.2, the resulting $\phi(\tau^*)$ enjoys for each $\tau^* > 0$ the same properties as the χ_1 term and is paired with (3.2) in the manner described above, while the streaming term is estimated by Young's inequality recalled in Appendix A and the third estimate in Proposition 2.1. This procedure eventually leads to the following crucial estimate.

Proposition 3.4 **($^\sharp g_\epsilon$ control).** *For each $\gamma \in \Upsilon$ defined by (2.3), the family $^\sharp g_\epsilon$ satisfies the estimate*

$$^\sharp g_\epsilon = O\left(\frac{1}{\log|\log\epsilon|}\right) \text{ in } L^1_{loc}(dtdx; L^1(Mdv))$$

as $\epsilon \to 0$.

This result shows a notable difference between the Flat-Sharp decomposition and the older decomposition in [7] (p. 696): here $^\sharp g_\epsilon \to 0$, which means that the whole of the hydrodynamic limit is contained in the L^2 part of the Flat-Sharp decomposition – namely $^\flat g_\epsilon$.

Proposition 3.4 leads to an amplification of the last statement in Proposition 2.7, stated below.

Corollary 3.5. *For each $\gamma \in \Upsilon$ defined by (2.3) and each $s \geq 0$, one has*

$$(1+|v|^s)\frac{1-\gamma(G_\epsilon)}{\epsilon^2} = O\left(\frac{1}{\sqrt{\log|\log\epsilon|}}\right) \text{ in } L^1_{loc}(dtdx; L^1(Mdv))$$

as $\epsilon \to 0$.

3.2. A limiting case of velocity averaging in L^1. The Advection/Dispersion bilinear interpolation procedure has another important application, leading to an improvement of the existing velocity averaging results in L^1 (namely, it extends the validity of Lemma 8 in [27] to any space dimension).

Lemma 3.6 **(Local uniform integrability by velocity averaging).** *Consider a bounded family f_ϵ of $L^\infty_{loc}(dt; L^1_{loc}(dxdv))$ indexed by $\epsilon \in [0,1]$ such that $(\epsilon\partial_t + v\cdot\nabla_x)f_\epsilon$ is bounded in $L^1_{loc}(dtdxdv)$. Suppose that f_ϵ is locally uniformly integrable in v. Then f_ϵ is locally uniformly integrable (in all variables t, x and v).*

Proof. Without loss of generality, one can assume that all the f_ϵ are supported in the ball of radius R centered at the origin in $\mathbf{R}_t \times \mathbf{R}^3_x \times \mathbf{R}^3_v$. Let B designate a measurable subset of that same ball.

For all $t \in \mathbf{R}$, call

$$B_t = \{(x,v) \in \mathbf{R}^3 \times \mathbf{R}^3 \mid (t,x,v) \in B\}$$

and

$$B_{t,x} = \{v \in \mathbf{R}^3 \mid (t,x,v) \in B\}\,.$$

Applying Lemma 3.1 twice to the indicator function of B leads to the decomposition

$$\mathbf{1}_B = \mathbf{1}_B\mathbf{1}_{B_1} + \mathbf{1}_B\mathbf{1}_{B_2}\mathbf{1}_{B^t_{21}} + \mathbf{1}_B\mathbf{1}_{B_2}\mathbf{1}_{B^t_{22}}\,,$$

where

$$B_1 = \left\{t \in \mathbf{R} \,\middle|\, |B_t| > |B|^{1/2}\right\}, \quad B_2 = \left\{t \in \mathbf{R} \,\middle|\, |B_t| \leq |B|^{1/2}\right\},$$

and, for all $t \in B_2$,

$$B_{21}^t = \left\{ x \in \mathbf{R}^3 \,\middle|\, |B_{t,x}| > |B|^{1/4} \right\}, \quad B_{22}^t = \left\{ x \in \mathbf{R}^3 \,\middle|\, |B_{t,x}| \le |B|^{1/4} \right\}.$$

Thus

$$\begin{aligned} \|\mathbf{1}_B \mathbf{1}_{B_1}\|_{L_t^1(L_{x,v}^\infty)} &\le |B|^{1/2}, \\ \|\mathbf{1}_B \mathbf{1}_{B_2} \mathbf{1}_{B_{21}^t}\|_{L_t^\infty(L_x^1(L_v^\infty))} &\le |B|^{1/4}, \\ \|\mathbf{1}_B \mathbf{1}_{B_2} \mathbf{1}_{B_{22}^t}\|_{L_{t,x}^\infty(L_v^1)} &\le |B|^{1/4}. \end{aligned} \tag{3.6}$$

Then

$$\iiint \mathbf{1}_B \mathbf{1}_{B_1} |f_\epsilon| dt dx dv \le \|f_\epsilon\|_{L_t^\infty(L_{x,v}^1)} \|\mathbf{1}_B \mathbf{1}_{B_1}\|_{L_t^1(L_{x,v}^\infty)} = O(|B|^{1/2}). \tag{3.7}$$

The definition of uniform integrability in v can be equivalently recast as follows: for each $\eta > 0$, there exists $\alpha > 0$ such that, for each measurable $\chi : \mathbf{R}_t \times \mathbf{R}_x^3 \times \mathbf{R}_v^3 \to \{0, 1\}$ such that $\|\chi\|_{L_{t,x}^\infty(L_v^1)} < \alpha$, one has

$$\iiint \chi |f_\epsilon| dt dx dv < \eta, \qquad \text{uniformly in } \epsilon. \tag{3.8}$$

Let $\eta > 0$ and α be so chosen; pick $0 < \tau^* < 1$ such that

$$\tau^* \|(\epsilon \partial_t + v \cdot \nabla_x) f\|_{L_{t,x,v}^1} < \eta. \tag{3.9}$$

Assume that B satisfies

$$|B|^{1/4} < \tau^{*3} \alpha < \alpha; \tag{3.10}$$

then, by using (3.8) for $\chi = \mathbf{1}_B \mathbf{1}_{B_2} \mathbf{1}_{B_{22}^t}$ together with the third inequality in (3.6),

$$\iiint \mathbf{1}_B \mathbf{1}_{B_2} \mathbf{1}_{B_{22}^t} |f_\epsilon| dt dx dv < \eta. \tag{3.11}$$

It remains to estimate

$$\iiint \mathbf{1}_B \mathbf{1}_{B_2} \mathbf{1}_{B_{21}^t} |f_\epsilon| dt dx dv.$$

Let $\phi^0 = \mathbf{1}_B \mathbf{1}_{B_2} \mathbf{1}_{B_{21}^t}$ and ϕ be the solution of the Cauchy problem (3.3). Notice that ϕ takes its values in $\{0, 1\}$. Lemma 3.2 and the second inequality in (3.6) imply that

$$\|\phi(\tau^*, \cdot, \cdot, \cdot)\|_{L_{t,x}^\infty(L_v^1)} \le \frac{1}{\tau^{*3}} \|\phi^0\|_{L_t^\infty(L_x^1(L_v^\infty))} \le \frac{|B|^{1/4}}{\tau^{*3}} < \alpha \tag{3.12}$$

by (3.10). Thus, the first term on the right-hand side of (3.5) satisfies

$$\int_{\epsilon\tau^*}^{t^*+\epsilon\tau^*}\iint |f_\epsilon(t,x,v)|\phi(\tau^*,t,x,v)dtdxdv < \eta$$

by (3.12) and (3.8). On account of this inequality, applying Lemma 3.3 with $|f_\epsilon|$ in place of f and τ^* as in (3.9) shows that

$$\iiint \mathbf{1}_B\mathbf{1}_{B_2}\mathbf{1}_{B_{21}^t}|f_\epsilon|dtdxdv \le \tau^*\|(\epsilon\partial_t + v\cdot\nabla_x)f\|_{L^1_{t,x,v}} + \eta < 2\eta\,. \tag{3.13}$$

Eventually, the three estimates (3.7), (3.11) and (3.13) show that

$$\iiint_B |f_\epsilon|dtdxdv < O(\alpha^2) + 3\eta$$

for each measurable $B \subset \mathbf{R}_t \times \mathbf{R}^3 \times \mathbf{R}^3$ satisfying (3.10), once τ^* has been chosen as in (3.9). This implies that the family f_ϵ is locally uniformly integrable on $\mathbf{R}_t \times \mathbf{R}^3 \times \mathbf{R}^3$ (in all variables t, x and v). □

One might object that the final result in Lemma 3.6 above is about f_ϵ itself and not about its moments in v, so that the reference to velocity averaging results as in [28] and [27] may seem inappropriate. However, the only nontrivial part in the proof of this result is the step appealing to Lemma 3.3, which essentially amounts to proving the local uniform integrability of $\int f_\epsilon dv$ (see [55] for the first observation in this direction). Besides, it has been previously noticed in various contexts that the velocity averaging method combined with additional regularity estimates in the velocity variable only gives compactness in *all* variables: see for instance [48], [22], and [13]. Our Lemma 3.6 can be viewed as an analogue in L^1 of these hypoellipticity results.

In fact, applying Lemma 3.3 a second time would show that, under the same assumptions on the family f_ϵ, the family of moments $\int f_\epsilon\chi(v)dv$ is "strongly compact in $L^1_{loc}(dtdx)$ relatively to the x variable" for each $\chi \in C_c(\mathbf{R}^3)$. We refer to [30] for a precise statement and a complete proof. However, in order to establish the strong compactness of moments, we use here a slightly different argument.

Lemma 3.7. *Let f_ϵ be a bounded family of $L^2_{loc}(dtdx; L^2(Mdv))$ indexed by $\epsilon \in [0,1]$ such that both families $|f_\epsilon|^2$ and $(\epsilon\partial_t + v\cdot\nabla_x)f_\epsilon$ are locally uniformly integrable with respect to the measure $Mdvdxdt$. Then, for each function $\phi \equiv \phi(v)$ in $L^2(Mdv)$, each $t^* > 0$ and each compact $K \subset \mathbf{R}^3$, there exists a function $\eta : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\lim_{z\to 0^+}\eta(z) = 0$ and*

$$\left\|\int f_\epsilon(t,x+y,v)\phi(v)M(v)dv - \int f_\epsilon(t,x,v)\phi(v)M(v)dv\right\|^2_{L^2([0,t^*]\times K)} \le \eta(|y|)$$

for each $y \in \mathbf{R}^3$ such that $|y| \le 1$, uniformly in $\epsilon \in [0,1]$.

This lemma is a minor amplification of Theorem 3 in [27] and its proof is deferred to Appendix C.

Finally, the first $\flat g_\epsilon$ controls (Proposition 2.7), the $\sharp g_\epsilon$ controls (Proposition 3.4) and both Lemmas 3.6 and 3.7 lead to the last piece of information needed about the $\flat g_\epsilon$ family, stated below.

Proposition 3.8 (**$\flat g_\epsilon$ controls II**). *Let $\gamma \in \Upsilon$ as in (2.3). Then the family $\flat g_\epsilon$ has the following properties*

- *for each compact $Q \subset \mathbf{R}_+ \times \mathbf{R}^3$ and each sequence $\epsilon_n \to 0$, the extracted sequence $(t, x, v) \mapsto \mathbf{1}_Q(t, x)|\flat g_{\epsilon_n}(t, x, v)|^2$ is uniformly integrable in $\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$ with respect to the measure $dtdxMdv$;*
- *for each sequence $\epsilon_n \to 0$, each function $\phi \equiv \phi(v)$ in $L^2(Mdv)$, each $t^* > 0$, and each compact $K \subset \mathbf{R}^3$, there exists a function $\eta : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\lim_{z \to 0^+} \eta(z) = 0$ and*

$$\int_0^{t^*} \int_K \left| \langle \flat g_{\epsilon_n} \phi \rangle (t, x + y) - \langle \flat g_{\epsilon_n} \phi \rangle (t, x) \right|^2 dxdt \le \eta(|y|)$$

for each $y \in \mathbf{R}^3$ such that $|y| \le 1$, uniformly in n.

We close this section with an amplification of the first statement of Proposition 3.8.

Corollary 3.9. *Let $\gamma \in \Upsilon$ as in (2.3). Then the family $\flat g_\epsilon$ has the following properties*

- *for each compact $Q \subset \mathbf{R}_+ \times \mathbf{R}^3$ and each sequence $\epsilon_n \to 0$, the extracted sequence $(t, x, v) \mapsto \mathbf{1}_Q(t, x)(\flat g_{\epsilon_n} \mathcal{L} \flat g_{\epsilon_n})(t, x, v)$ is uniformly integrable in $\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$ with respect to the measure $dtdxMdv$;*
- *for each compact $Q \subset \mathbf{R}_+ \times \mathbf{R}^3$ and each sequence $\epsilon_n \to 0$, the extracted sequence $(t, x, v) \mapsto \mathbf{1}_Q(t, x)\mathcal{Q}(\flat g_{\epsilon_n}, \flat g_{\epsilon_n})(t, x, v)$ is uniformly integrable in $\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$ with respect to the measure $dtdxMdv$.*

4. Estimating conservation defects

The controls stated in Sect. 2 and proved below establish the local conservation laws of momentum and energy in the limit as $\epsilon \to 0$, by essentially the same method as in the proof of the Stokes-Fourier limit by F. Golse and C.D. Levermore [25].

Before stating the main result of the present section, we need to introduce a new class of bump functions. For each $C > 0$, set

$$\Upsilon_C = \{\gamma \in \Upsilon \mid \|\gamma'\|_{L^\infty} \le C\}.$$

Consider the transformation $\mathcal{T}$ defined by $\mathcal{T}\gamma = 1 - (1 - \gamma)^2$; clearly $\mathcal{T}$ maps Υ_C into Υ_{2C}. Define

$$\tilde{\Upsilon} = \mathcal{T}\Upsilon_8 \subset \Upsilon_{16}; \quad \text{notice that } \tilde{\Upsilon} \neq \emptyset \text{ since } \Upsilon_8 \neq \emptyset. \tag{4.1}$$

For each $\gamma \in \tilde{\Upsilon}$, define

$$\hat{\gamma}(z) = \gamma(z) + (z-1)\frac{d\gamma}{dz}(z)\,. \tag{4.2}$$

Notice that

$$\operatorname{supp}\hat{\gamma} \subset \left[\tfrac{1}{2}, \tfrac{3}{2}\right], \quad \hat{\gamma}\left(\left[\tfrac{3}{4}, \tfrac{5}{4}\right]\right) = \{1\}\,. \tag{4.3}$$

On the other hand, let $\tilde{\gamma} \in \Upsilon_8$ be such that $\gamma = \mathcal{T}\tilde{\gamma}$ (the existence of $\tilde{\gamma}$ being guaranteed by the fact that $\gamma \in \tilde{\Upsilon}$). One has

$$1 - \hat{\gamma}(z) = (1-\tilde{\gamma}(z))\left[(1-\tilde{\gamma}(z)) - 2(z-1)\frac{d\tilde{\gamma}}{dz}(z)\right], \quad z \geq 0$$

so that

$$|1-\hat{\gamma}(z)| \leq 9(1-\tilde{\gamma}(z))\,, \quad z \geq 0\,. \tag{4.4}$$

Proposition 4.1 (Vanishing of conservation defects). *Let $\gamma \in \tilde{\Upsilon}$, and denote by $\xi \equiv \xi(v)$ any collision invariant (i.e. $\xi(v) = 1$ or $\xi(v) = v_1, \ldots, v_3$ or else $\xi(v) = |v|^2$) or any linear combination thereof. Then*

$$\partial_t \langle \overset{\flat}{g}_\epsilon \xi \rangle + \frac{1}{\epsilon}\nabla_x \cdot \langle v \overset{\flat}{g}_\epsilon \xi \rangle \to 0\,, \ \textit{in } L^1_{loc}(\mathbf{R}^+ \times \mathbf{R}^3)$$

as $\epsilon \to 0$.

Proof. We start from the renormalized form (1.19) of the Boltzmann equation (1.13) with $\Gamma(Z) = (Z-1)\gamma(Z)$

$$\left(\partial_t + \frac{1}{\epsilon}v\cdot\nabla_x\right)(M\overset{\flat}{g}_\epsilon)$$
$$= \frac{1}{\epsilon^3}\iint (F'_\epsilon F'_{\epsilon 1} - F_\epsilon F_{\epsilon 1})\left(\gamma(G_\epsilon) + (G_\epsilon - 1)\frac{d\gamma}{dz}(G_\epsilon)\right) b d\sigma_{v,v_1}(\omega) dv_1$$

from which we deduce that

$$\partial_t \langle \overset{\flat}{g}_\epsilon \xi \rangle + \frac{1}{\epsilon}\nabla_x \cdot \langle v \overset{\flat}{g}_\epsilon \xi \rangle = \frac{1}{\epsilon}\langle\!\langle q_\epsilon \hat{\gamma}_\epsilon \xi \rangle\!\rangle$$

with the notation

$$\hat{\gamma}_\epsilon = \hat{\gamma}(G_\epsilon)\,,$$

the function $\hat{\gamma}$ being defined in terms of γ by (4.2).

In order to estimate the L^1-norm of the conservation defects, we consider the decomposition

$$\frac{1}{\epsilon}\langle\!\langle q_\epsilon \hat{\gamma}_\epsilon \xi \rangle\!\rangle = \frac{1}{\epsilon}\langle\!\langle q_\epsilon \hat{\gamma}_\epsilon (1-\hat{\gamma}_{\epsilon 1})\xi \rangle\!\rangle + \frac{1}{\epsilon}\langle\!\langle q_\epsilon \hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1}\xi \rangle\!\rangle\,. \tag{4.5}$$

Because our choice of ξ satisfies $\xi + \xi_1 = \xi' + \xi_1'$, using the definition (1.35) and the collisional symmetry (1.11) with $f = MG_\epsilon$ and $\varphi = \hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1} \xi \mathbf{1}_{|v|^2+|v_1|^2>16|\log\epsilon|}$ or $\varphi = \hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1} \xi \mathbf{1}_{|v|^2+|v_1|^2\le 16|\log\epsilon|}$ implies that

$$
\begin{aligned}
&\frac{1}{\epsilon}\left\langle\!\left\langle q_\epsilon \hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1}\xi \right\rangle\!\right\rangle \\
&= \frac{1}{\epsilon}\left\langle\!\left\langle q_\epsilon \hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1}\xi \mathbf{1}_{|v|^2+|v_1|^2>16|\log\epsilon|}\right\rangle\!\right\rangle + \frac{1}{\epsilon}\left\langle\!\left\langle q_\epsilon \hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1}\xi \mathbf{1}_{|v|^2+|v_1|^2\le 16|\log\epsilon|}\right\rangle\!\right\rangle \\
&= \frac{1}{2\epsilon^3}\left\langle\!\left\langle (G_\epsilon' G_{\epsilon 1}' - G_\epsilon G_{\epsilon 1})\hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1}(\xi+\xi_1)\mathbf{1}_{|v|^2+|v_1|^2>16|\log\epsilon|}\right\rangle\!\right\rangle \\
&+ \frac{1}{4\epsilon^3}\left\langle\!\left\langle (G_\epsilon' G_{\epsilon 1}' - G_\epsilon G_{\epsilon 1})(\hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1} - \hat{\gamma}_\epsilon' \hat{\gamma}_{\epsilon 1}')(\xi+\xi_1)\mathbf{1}_{|v|^2+|v_1|^2\le 16|\log\epsilon|}\right\rangle\!\right\rangle .
\end{aligned}
$$

On the other hand, observe that

$$
(\hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1} - \hat{\gamma}_\epsilon' \hat{\gamma}_{\epsilon 1}') = (\hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1} - 1)\hat{\gamma}_\epsilon' \hat{\gamma}_{\epsilon 1}' - (\hat{\gamma}_\epsilon' \hat{\gamma}_{\epsilon 1}' - 1)\hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1}
$$

so that, using again (1.11) with $f = MG_\epsilon$ and this time with $\varphi = (\hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1} - 1)\hat{\gamma}_\epsilon' \hat{\gamma}_{\epsilon 1}'(\xi+\xi_1)\mathbf{1}_{|v|^2+|v_1|^2\le 16|\log\epsilon|}$, we have

$$
\begin{aligned}
&\frac{1}{4\epsilon^3}\left\langle\!\left\langle (G_\epsilon' G_{\epsilon 1}' - G_\epsilon G_{\epsilon 1})(\hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1} - \hat{\gamma}_\epsilon' \hat{\gamma}_{\epsilon 1}')(\xi+\xi_1)\mathbf{1}_{|v|^2+|v_1|^2\le 16|\log\epsilon|}\right\rangle\!\right\rangle \\
&= \frac{1}{2\epsilon^3}\left\langle\!\left\langle (G_\epsilon' G_{\epsilon 1}' - G_\epsilon G_{\epsilon 1})(\hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1} - 1)\hat{\gamma}_\epsilon' \hat{\gamma}_{\epsilon 1}'(\xi+\xi_1)\mathbf{1}_{|v|^2+|v_1|^2\le 16|\log\epsilon|}\right\rangle\!\right\rangle .
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\left|\partial_t \langle \overset{\flat}{g}_\epsilon \xi\rangle + \frac{1}{\epsilon}\nabla_x \cdot \langle v \overset{\flat}{g}_\epsilon \xi\rangle\right| \le \frac{1}{\epsilon}\left\langle\!\left\langle |q_\epsilon||\hat{\gamma}_\epsilon||1-\hat{\gamma}_{\epsilon 1}||\xi| \right\rangle\!\right\rangle \\
&\quad + \frac{1}{2\epsilon}\left\langle\!\left\langle |q_\epsilon||\hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1}|(|\xi|+|\xi_1|)\mathbf{1}_{|v|^2+|v_1|^2>16|\log\epsilon|}\right\rangle\!\right\rangle \\
&\quad + \frac{1}{2\epsilon}\left\langle\!\left\langle |q_\epsilon||\hat{\gamma}_\epsilon \hat{\gamma}_{\epsilon 1} - 1||\hat{\gamma}_\epsilon' \hat{\gamma}_{\epsilon 1}'|(|\xi|+|\xi_1|)\mathbf{1}_{|v|^2+|v_1|^2\le 16|\log\epsilon|}\right\rangle\!\right\rangle \\
&\quad = I_1 + I_2 + I_3 .
\end{aligned}
\tag{4.6}
$$

For any $\eta \ge \epsilon$, Young's inequality (10.4) together with the elementary properties of h and r recalled in (10.1) and (10.3) (see Appendix A below) imply that

$$
\begin{aligned}
\frac{1}{\epsilon}|q_\epsilon||\xi(v)| &\le \frac{4\eta}{\epsilon^4}G_\epsilon G_{\epsilon 1}\left[h\left(\frac{\epsilon^2 q_\epsilon}{G_\epsilon G_{\epsilon 1}}\right) + h^*\left(\frac{\epsilon}{4\eta}|\xi(v)|\right)\right] \\
&\le \frac{4\eta}{\epsilon^4}G_\epsilon G_{\epsilon 1} r\left(\frac{\epsilon^2 q_\epsilon}{G_\epsilon G_{\epsilon 1}}\right) + \frac{4}{\epsilon^2\eta}G_\epsilon G_{\epsilon 1}e^{|\xi(v)|/4} .
\end{aligned}
\tag{4.7}
$$

Then, using (2.7) and the elementary bounds

$$
|\hat{\gamma}_\epsilon| \le 9\,, \quad |1-\hat{\gamma}_\epsilon| \le 9\,, \quad 0 \le G_\epsilon|\hat{\gamma}_\epsilon| \le \tfrac{27}{2}
\tag{4.8}
$$

leads to

$$\begin{aligned}
\|I_1\|_{L^1([0,t^*]\times Q)} &\le 16\cdot 9^2\cdot\eta\int_0^{t^*}\int_Q\int \frac{D(F_\epsilon)}{\epsilon^4}dvdxdt\\
&+\frac{4}{\eta}\int_0^{t^*}\int_Q\left\langle\!\!\left\langle G_\epsilon G_{\epsilon 1}|\hat\gamma_\epsilon|\frac{|1-\hat\gamma_{\epsilon 1}|}{\epsilon^2}e^{|\xi(v)|/4}\right\rangle\!\!\right\rangle dxdt\\
&\le 16\cdot 9^2\cdot C^{in}\eta+\frac{4\cdot\frac{27}{2}\cdot b_\infty\langle e^{|\xi|/4}\rangle}{\eta}\int_0^{t^*}\int_Q\left\langle\frac{|1-\hat\gamma_\epsilon|}{\epsilon^2}G_\epsilon\right\rangle dxdt\\
&\le 16\cdot 9^2\cdot C^{in}\eta+\frac{4\cdot\frac{27}{2}\cdot 5\cdot b_\infty\langle e^{|\xi|/4}\rangle}{\eta}\int_0^{t^*}\int_Q\left\langle\frac{1}{\epsilon^2}|1-\hat\gamma_\epsilon||G_\epsilon-1|\right\rangle dxdt\,,
\end{aligned}$$

where the last inequality follows from the same argument as in the proof of the third pointwise control in Proposition 2.2. Because of the inequality (4.4), one has

$$\left\langle\frac{1}{\epsilon^2}|1-\hat\gamma_\epsilon||G_\epsilon-1|\right\rangle\le 9\left\langle\frac{1}{\epsilon^2}(1-\tilde\gamma_\epsilon)|G_\epsilon-1|\right\rangle$$

so that, by applying Proposition 3.4 with the bump function $\tilde\gamma\in\Upsilon_8$ and by choosing $\eta=1/\sqrt{\log|\log\epsilon|}$ we get

$$\|I_1\|_{L^1([0,t^*]\times Q)}\le\frac{C}{\sqrt{\log|\log\epsilon|}}\,. \tag{4.9}$$

Next we estimate the term I_2 defined in (4.6). Replacing $|\xi(v)|$ with $|\xi(v)|+|\xi(v_1)|$ in Young's inequality (4.7) above and using again the bounds (4.8) implies, by integrating first in ω, that

(4.10)

$$\begin{aligned}
\|I_2\|_{L^1([0,t^*]\times Q)} &\le 8\cdot 9^2\cdot C^{in}\eta+\frac{2}{\eta\epsilon^2}\\
&\quad\times\int_0^{t^*}\int_Q\left\langle\!\!\left\langle G_\epsilon G_{\epsilon 1}|\hat\gamma_\epsilon||\hat\gamma_{\epsilon 1}|e^{(|\xi|+|\xi_1|)/4}\mathbf{1}_{|v|^2+|v_1|^2\ge 16|\log\epsilon|}\right\rangle\!\!\right\rangle dxdt\\
&\le 8\cdot 9^2\cdot C^{in}\eta+\frac{2\cdot(\frac{27}{2})^2\cdot b_\infty}{\eta\epsilon^2}\\
&\quad\times\int_0^{t^*}\int_Q\iint e^{(|\xi|+|\xi_1|)/4}\mathbf{1}_{|v|^2+|v_1|^2\ge 16|\log\epsilon|}MM_1dvdv_1dxdt\\
&\le O(\eta)+\frac{1}{\eta\epsilon^2}O(\epsilon^4|\log\epsilon|^2)=O(\epsilon|\log\epsilon|^2)
\end{aligned}$$

by choosing this time $\eta=\epsilon$.

It remains to estimate the term I_3 in (4.6). Using the decomposition (2.9) of the scaled collision integrand and once again the bounds (4.8) shows that

$$
\begin{aligned}
I_3 \le{}& \frac{9^2 \cdot (1+9^2)}{2\epsilon^3} \left\langle\!\left\langle \epsilon^4 |\sharp q_\epsilon| (|\xi'| + |\xi_1'|) \mathbf{1}_{|v'|^2+|v_1'|^2 \le 16|\log\epsilon|} \right\rangle\!\right\rangle \\
&+ \frac{1}{2\epsilon^3} \left\langle\!\left\langle \epsilon^2 |\flat q_\epsilon| |\hat\gamma_\epsilon \hat\gamma_{\epsilon 1}| |1 - \hat\gamma'_\epsilon \hat\gamma'_{\epsilon 1}| (|\xi'| + |\xi_1'|) \mathbf{1}_{|v'|^2+|v_1'|^2 \le 16|\log\epsilon|} \right\rangle\!\right\rangle .
\end{aligned}
\tag{4.11}
$$

(Observe that we have exchanged the primed and unprimed variables in the integral defining I_3, on account of (1.9) and the second equality in (1.8)). Denote by I_3^1 and I_3^2 the two terms on the right-hand side of (4.11). The second statement in Proposition 2.3 shows that, for each $t^* > 0$ and each compact subset Q of $\mathbf{R}^3$, one has

$$
\left\| I_3^1 \right\|_{L^1([0,t^*]\times Q)} \le C\epsilon |\log\epsilon| . \tag{4.12}
$$

By the Cauchy–Schwarz inequality,

$$
\begin{aligned}
\left\| I_3^2 \right\|_{L^1([0,t^*]\times Q)} \le{}& \left\| \left\langle\!\left\langle \frac{|\flat q_\epsilon|^2}{G_\epsilon G_{\epsilon 1}} \right\rangle\!\right\rangle \right\|^{1/2}_{L^1([0,t^*]\times Q)} \left\| \frac{1}{4\epsilon^2} \left\langle\!\left\langle G_\epsilon G_{\epsilon 1} |\hat\gamma_\epsilon \hat\gamma_{\epsilon 1}|^2 \right.\right.\right. \\
&\left.\left.\left. \times |1 - \hat\gamma'_\epsilon \hat\gamma'_{\epsilon 1}|^2 (|\xi'| + |\xi_1'|)^2 \mathbf{1}_{|v'|^2+|v_1'|^2 \le 16|\log\epsilon|} \right\rangle\!\right\rangle \right\|^{1/2}_{L^1([0,t^*]\times Q)} .
\end{aligned}
$$

This inequality, together with the first estimate in Proposition 2.3, the third inequality in (4.8), and the formula

$$
1 - \hat\gamma' \hat\gamma_1' = 1 - \hat\gamma' + \hat\gamma'(1 - \hat\gamma_1')
$$

imply that

$$
\begin{aligned}
&\left\| I_3^2 \right\|_{L^1([0,t^*]\times Q)} \\
&\quad \le 9^2 \cdot \left(\tfrac{27}{2}\right)^2 \cdot \frac{2\sqrt{C^{in}}}{\sqrt{c}} \left\| \frac{1}{4\epsilon^2} \left\langle\!\left\langle (|1 - \hat\gamma'_\epsilon| + 9|1 - \hat\gamma'_{\epsilon 1}|)^2 (|\xi'| + |\xi_1'|)^2 \right\rangle\!\right\rangle \right\|^{1/2}_{L^1([0,t^*]\times Q)} \\
&\quad \le C \left\| \frac{1}{\epsilon^2} \langle |1 - \hat\gamma_\epsilon| (1 + |v|^4) \rangle \right\|^{1/2}_{L^1([0,t^*]\times Q)} \\
&\quad \le C \left\| \frac{9}{\epsilon^2} \langle (1 - \tilde\gamma(G_\epsilon))(1 + |v|^4) \rangle \right\|^{1/2}_{L^1([0,t^*]\times Q)} \le \frac{C}{(\log|\log\epsilon|)^{1/4}}
\end{aligned}
\tag{4.13}
$$

for some constant $C > 0$, where the penultimate inequality is based on (4.4) and the last inequality follows from Corollary 3.5 applied to the bump function $\tilde\gamma \in \Upsilon_8$. (The constant C^{in} is that which appears in (1.44), while the constant c is defined in (2.5)).

Combining estimates (4.9), (4.10), (4.12) and (4.13) gives the expected convergence. □

5. Proof of the weak Navier–Stokes limit theorem

Throughout this section, it is assumed that the bump function γ belongs to $\tilde{\Upsilon}$ (defined by (4.1)). Using Proposition 4.1, the classical Sobolev embedding theorems, and the continuity of pseudo-differential operators of order 0 on $W^{s,p}$ for $1 < p < +\infty$, one sees that, for all $s > 0$

(5.1)
$$\partial_t P\langle v \,\flat g_\epsilon\rangle + P\nabla_x \cdot \frac{1}{\epsilon}\left\langle\left(v^{\otimes 2} - \tfrac{1}{3}|v|^2 I\right) \flat g_\epsilon\right\rangle \to 0 \text{ in } L^1_{loc}\left(dt; W^{-s,1}_{loc}(\mathbf{R}^3)\right),$$

and

$$\partial_t\left\langle\left(\tfrac{1}{5}|v|^2 - 1\right) \flat g_\epsilon\right\rangle + \nabla_x \cdot \frac{1}{\epsilon}\left\langle v\left(\tfrac{1}{5}|v|^2 - 1\right) \flat g_\epsilon\right\rangle \to 0 \text{ in } L^1_{loc}(dtdx)\,, \tag{5.2}$$

as $\epsilon \to 0$. (We recall that P is the Leray projection, i.e. the $L^2(dx)$-orthogonal projection on the space of divergence-free vector fields).

By Proposition 2.1 and Theorem 11.1 in Appendix B below, pick any sequence $\epsilon_n \to 0$ such that

$$\begin{aligned} \flat g_{\epsilon_n} &\to g \text{ in } w^*\text{-}L^\infty_t(L^2(Mdvdx))\,,\\ \gamma_{\epsilon_n} q_{\epsilon_n} &\to q \text{ in } w\text{-}L^1_{loc}(dtdx; L^1((1+|v|^2)d\mu))\,. \end{aligned} \tag{5.3}$$

In this section, we deal exclusively with such extracted sequences, drop the index n and abuse the notations g_ϵ, $\flat g_\epsilon$, $\sharp g_\epsilon$, q_ϵ and so on to designate the subsequences g_{ϵ_n}, $\flat g_{\epsilon_n}$, $\sharp g_{\epsilon_n}$ and q_{ϵ_n}. Call u and θ the limiting (fluctuations of) velocity and temperature fields defined by

$$\langle v \,\flat g_\epsilon\rangle \to u\,, \quad \left\langle\left(\tfrac{1}{3}|v|^2 - 1\right) \flat g_\epsilon\right\rangle \to \theta \text{ in } w^*\text{-}L^\infty_t\left(L^2_x\right). \tag{5.4}$$

The second entropy control in Proposition 2.1 implies that $\flat g_\epsilon$ and g_ϵ have the same limit g in $w\text{-}L^1_{loc}(dtdx; L^1(Mdv))$; hence the Boussinesq relation (11.3) and the incompressibility condition (11.2) hold:

$$\nabla_x \cdot u = 0\,, \quad \theta + \langle g\rangle = 0\,; \tag{5.5}$$

(see Theorem 11.1 in Appendix B).

Denote by ζ either the tensor A or the vector B defined in (1.42). By Proposition 1.4, $\mathcal{L}$ is self-adjoint on $L^2(Mdv)$ so that

$$\begin{aligned} \frac{1}{\epsilon}\langle(\mathcal{L}\zeta)\,\flat g_\epsilon\rangle &= \frac{1}{\epsilon}\langle\zeta(\mathcal{L}\,\flat g_\epsilon)\rangle = \frac{1}{\epsilon}\langle\!\langle\zeta(\flat g_\epsilon + \flat g_{\epsilon 1} - \flat g'_\epsilon - \flat g'_{\epsilon 1})\rangle\!\rangle\\ &= \left\langle\!\!\left\langle \zeta\left[\frac{1}{\epsilon}(\flat g_\epsilon + \flat g_{\epsilon 1} - \flat g'_\epsilon - \flat g'_{\epsilon 1}) + (\flat g_\epsilon \flat g_{\epsilon 1} - \flat g'_\epsilon \flat g'_{\epsilon 1})\right]\right\rangle\!\!\right\rangle\\ &\quad + \langle\zeta\mathcal{Q}(\flat g_\epsilon, \flat g_\epsilon)\rangle\,. \end{aligned} \tag{5.6}$$

The first term in the (last) right hand side of (5.6) converges to the diffusion term while the second term converges to the convection term in the Navier–Stokes–Fourier system. These limits are analyzed in detail in the next subsections.

5.1. The diffusion term. The convergence to the diffusion term is obtained by an argument that closely follows [25], except that the present work uses the Flat-Sharp decomposition (2.4) instead of the decomposition introduced on p. 696 of [7] as in [25]. This apparently minor difference makes our analysis slightly more difficult than that in [25].

Proposition 5.1. *Define*

$$\nu = \tfrac{1}{10}\langle A : \mathcal{L}A\rangle\,, \qquad \kappa = \tfrac{2}{15}\langle B \cdot \mathcal{L}B\rangle\,.$$

Then, as $\epsilon \to 0$,

$$\frac{1}{\epsilon}\langle(\mathcal{L}A)\, \overset{\flat}{g}_\epsilon\rangle - \langle A\mathcal{Q}(\overset{\flat}{g}_\epsilon, \overset{\flat}{g}_\epsilon)\rangle \to -\nu\left(\nabla_x u + (\nabla_x u)^T\right);$$
$$\frac{1}{\epsilon}\langle(\mathcal{L}B)\, \overset{\flat}{g}_\epsilon\rangle - \langle B\mathcal{Q}(\overset{\flat}{g}_\epsilon, \overset{\flat}{g}_\epsilon)\rangle \to -\tfrac{5}{2}\kappa\nabla_x\theta$$

in w-$L^1_{loc}(dtdx)$.

Proof. The convergence to the diffusion term depends upon a careful analysis of the integrand appearing in the first term of the (last) right-hand side in (5.6) that involves the scaled collision integrand

$$\begin{aligned} q_\epsilon &= \frac{1}{\epsilon^2}(G'_\epsilon G'_{\epsilon1} - G_\epsilon G_{\epsilon1}) \\ &= \frac{1}{\epsilon}(g'_\epsilon + g'_{\epsilon1} - g_\epsilon - g_{\epsilon1}) + (g'_\epsilon g'_{\epsilon1} - g_\epsilon g_{\epsilon1})\,. \end{aligned}$$

Indeed consider the decomposition of the integrand in (5.6) as

$$\begin{aligned} &\left[\frac{1}{\epsilon}(\overset{\flat}{g}_\epsilon + \overset{\flat}{g}_{\epsilon1} - \overset{\flat}{g}{}'_\epsilon - \overset{\flat}{g}{}'_{\epsilon1}) + (\overset{\flat}{g}_\epsilon \overset{\flat}{g}_{\epsilon1} - \overset{\flat}{g}{}'_\epsilon \overset{\flat}{g}{}'_{\epsilon1})\right] \\ &\quad = -q_\epsilon\gamma_\epsilon\gamma_{\epsilon1}\gamma'_\epsilon\gamma'_{\epsilon1} \\ &\quad + \left[\overset{\flat}{g}_\epsilon \overset{\flat}{g}_{\epsilon1}(1 - \gamma'_\epsilon\gamma'_{\epsilon1}) - \overset{\flat}{g}{}'_\epsilon \overset{\flat}{g}{}'_{\epsilon1}(1 - \gamma_\epsilon\gamma_{\epsilon1})\right] \\ &\quad + \frac{1}{\epsilon}\left[\overset{\flat}{g}_\epsilon + \overset{\flat}{g}_{\epsilon1} - \overset{\flat}{g}{}'_\epsilon - \overset{\flat}{g}{}'_{\epsilon1} - \gamma_\epsilon\gamma_{\epsilon1}\gamma'_\epsilon\gamma'_{\epsilon1}(g_\epsilon + g_{\epsilon1} - g'_\epsilon - g'_{\epsilon1})\right] \\ &\quad = I_1 + I_2 + I_3\,. \end{aligned} \tag{5.7}$$

Notice that this decomposition is slightly more complicated than its analogue in [25] (formula (10.6) there). In particular, the analogue of the normalizing factor γ_ϵ in that work is an *affine* function of g_ϵ. This accounts for additional cancellations in (5.7).

Step 1: controlling I_1. By Theorem 11.1 in Appendix B below,

$$q_\epsilon\gamma_\epsilon \to q \text{ in } w\text{-}L^1_{loc}(dtdx; L^1(d\mu))\,.$$

On the other hand, each term in the product $\gamma'_\epsilon\gamma_{\epsilon 1}\gamma'_{\epsilon 1}$ is bounded by 1 in $L^\infty(dtdxd\mu)$. Hence the decomposition

$$1-\gamma'_\epsilon\gamma_{\epsilon 1}\gamma'_{\epsilon 1} = 1-\gamma'_\epsilon + \gamma'_\epsilon(1-\gamma'_{\epsilon 1}) + \gamma'_\epsilon\gamma'_{\epsilon 1}(1-\gamma_{\epsilon 1})$$

implies that

$$0 \le 1-\gamma'_\epsilon\gamma_{\epsilon 1}\gamma'_{\epsilon 1} \le (1-\gamma'_\epsilon)+(1-\gamma'_{\epsilon 1})+(1-\gamma_{\epsilon 1}) \le 4\epsilon^2\left(|\,^\sharp g'_\epsilon|+|\,^\sharp g'_{\epsilon 1}|+|\,^\sharp g_{\epsilon 1}|\right),$$

because of the second pointwise estimate in Proposition 2.2. Using the second entropy control in Proposition 2.1 with the symmetries (1.8) and (1.9) shows that

$$\langle\!\langle |1-\gamma'_\epsilon\gamma_{\epsilon 1}\gamma'_{\epsilon 1}|\rangle\!\rangle \le 4\epsilon^2\langle\!\langle |\,^\sharp g'_\epsilon|+|\,^\sharp g'_{\epsilon 1}|+|\,^\sharp g_{\epsilon 1}|\rangle\!\rangle \le 12\epsilon^2 b_\infty \|\,^\sharp g_\epsilon\|_{L^1(Mdv)} \to 0$$

in $L^1_{loc}(dtdx)$ as $\epsilon \to 0$. Hence the family $\gamma'_\epsilon\gamma_{\epsilon 1}\gamma'_{\epsilon 1}$ is bounded (by 1) in $L^\infty(dtdxd\mu)$ and converges in measure to 1 on compacts sets in $\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$ as $\epsilon \to 0$. By Lemma 14.1 in Appendix E below,

$$q_\epsilon\gamma_\epsilon\gamma_{\epsilon 1}\gamma'_\epsilon\gamma'_{\epsilon 1} \to q \text{ in } w\text{-}L^1_{loc}(dtdx; L^1(d\mu)). \tag{5.8}$$

We claim that in fact

$$q_\epsilon\gamma_\epsilon\gamma_{\epsilon 1}\gamma'_\epsilon\gamma'_{\epsilon 1} \to q \text{ in } w\text{-}L^2_{loc}(dtdx; L^2(d\mu)). \tag{5.9}$$

Indeed, unless $\gamma_\epsilon\gamma_{\epsilon 1}\gamma'_\epsilon\gamma'_{\epsilon 1} = 0$, one has

$$\tfrac{1}{4} \le G_\epsilon G_{\epsilon 1} \le \tfrac{9}{4}, \quad -\tfrac{8}{9} \le \frac{G'_\epsilon G'_{\epsilon 1}}{G_\epsilon G_{\epsilon 1}} - 1 \le 8;$$

then, the third statement of Proposition 2.3 with $E = [-\frac{8}{9}, 8]$ implies that

$$\langle\!\langle (q_\epsilon\gamma_\epsilon\gamma'_\epsilon\gamma_{\epsilon 1}\gamma'_{\epsilon 1})^2\rangle\!\rangle \le \tfrac{9}{4}\left\langle\!\!\left\langle \frac{q_\epsilon^2}{G_\epsilon G_{\epsilon 1}}\mathbf{1}_E\left(\frac{G'_\epsilon G'_{\epsilon 1}}{G_\epsilon G_{\epsilon 1}} - 1\right)\right\rangle\!\!\right\rangle = O(1)_{L^1_{t,x}}.$$

This bound together with the w-L^1 convergence (5.8) imply (5.9). In particular, $\zeta = A$ or $\zeta = B$ belongs to $L^2(Mdv)$ in either case (see (1.42) above), so that the w-L^2 convergence (5.9) implies that

$$\langle\!\langle \zeta q_\epsilon\gamma_\epsilon\gamma'_\epsilon\gamma_{\epsilon 1}\gamma'_{\epsilon 1}\rangle\!\rangle \to \langle\!\langle \zeta q\rangle\!\rangle = \langle \zeta\, v\cdot\nabla_x g\rangle \text{ in } w\text{-}L^2_{loc}(dtdx). \tag{5.10}$$

The last equality in (5.10) follows from the limiting Boltzmann equation (11.6) in Theorem 11.1 (see Appendix B).

Step 2: controlling I_2 and I_3. The first pointwise estimate in Proposition 2.2 and the obvious formula

$$\begin{aligned} 1-\gamma'_{\epsilon 1}\gamma'_\epsilon &= (1-\gamma'_\epsilon) + \gamma'_\epsilon(1-\gamma'_{\epsilon 1}) \\ &\le (1-\gamma'_\epsilon) + (1-\gamma'_{\epsilon 1}) \end{aligned}$$

imply that

$$\langle\!\langle |\zeta||\flat g_\epsilon \flat g_{\epsilon 1}|(1-\gamma'_\epsilon\gamma'_{\epsilon 1})\rangle\!\rangle \le \frac{1}{4\epsilon^2}\langle\!\langle |\zeta|(1-\gamma'_\epsilon + 1 - \gamma'_{\epsilon 1})\rangle\!\rangle \,.$$

By assumption (H2) and relation (1.4), there exists $p \ge 0$ and $C_p > 0$ such that, for all (v, v_1, ω) in $\mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2$

$$|\zeta(v)| \le C_p(1 + |v'|^p + |v'_1|^p) \text{ for each } (v, v_1, \omega) \in \mathbf{R}^3 \times \mathbf{R}^3 \times \mathbf{S}^2 \,.$$

Thus, by using the (v, v_1)-(v', v'_1) symmetry (1.8) and (1.9), we have

(5.11)

$$\begin{aligned}\langle\!\langle |\zeta||\flat g_\epsilon \flat g_{\epsilon 1}|(1-\gamma'_\epsilon\gamma'_{\epsilon 1})\rangle\!\rangle &\le \frac{C_p}{4\epsilon^2}\langle\!\langle (1 + |v'|^p + |v'_1|^p)(1-\gamma'_\epsilon + 1 - \gamma'_{\epsilon 1})\rangle\!\rangle \\ &\le \frac{b_\infty C_p}{2\epsilon^2}\langle 1 + |v|^p\rangle\langle (1+|v|^p)(1-\gamma_\epsilon)\rangle \,.\end{aligned}$$

The term I_3 in the right-hand side of (5.7) can be recast as

$$\begin{aligned}\frac{1}{\epsilon}\big(&\flat g_\epsilon(1-\gamma_{\epsilon 1}\gamma'_\epsilon\gamma'_{\epsilon 1}) + \flat g_{\epsilon 1}(1-\gamma_\epsilon\gamma'_\epsilon\gamma'_{\epsilon 1}) \\ &- \flat g'_\epsilon(1-\gamma_\epsilon\gamma_{\epsilon 1}\gamma'_{\epsilon 1}) - \flat g'_{\epsilon 1}(1-\gamma_\epsilon\gamma_{\epsilon 1}\gamma'_\epsilon)\big) \,.\end{aligned}$$

Then, proceeding as in (5.11) leads to

(5.12)

$$\begin{aligned}&\left|\left\langle\!\!\left\langle |\zeta|\frac{1}{\epsilon}|\flat g_\epsilon|(1-\gamma_{\epsilon 1}\gamma'_\epsilon\gamma'_{\epsilon 1})\right\rangle\!\!\right\rangle\right| \\ &\quad\le \frac{C_p}{2\epsilon^2}\langle\!\langle (1 + |v'|^p + |v'_1|^p)(1-\gamma'_\epsilon + 1 - \gamma'_{\epsilon 1})\rangle\!\rangle + \frac{1}{2\epsilon^2}\langle\!\langle |\zeta|(1-\gamma_{\epsilon 1})\rangle\!\rangle \\ &\quad\le \frac{b_\infty C_p}{\epsilon^2}\langle 1 + |v|^p\rangle\langle (1+|v|^p)(1-\gamma_\epsilon)\rangle + \frac{b_\infty}{2\epsilon^2}\langle |\zeta|\rangle\langle 1-\gamma_\epsilon\rangle \,.\end{aligned}$$

Both inequalities (5.11) and (5.12) (and those deduced from (5.11)–(5.12) by the v-v_1 and the (v, v_1)-(v', v'_1) symmetries (1.7), (1.8) and (1.9)) with Corollary 3.5 imply that

$$\langle\!\langle |I_2|\rangle\!\rangle \to 0 \text{ and } \langle\!\langle |I_3|\rangle\!\rangle \to 0 \text{ in } L^1_{loc}(dtdx) \text{ as } \epsilon \to 0 \,. \tag{5.13}$$

Conclusion. Using both convergences (5.10) and (5.13) in formula (5.7) implies that

$$\left\langle\!\!\left\langle \zeta\left[\frac{1}{\epsilon}(\flat g_\epsilon + \flat g_{\epsilon 1} - \flat g'_\epsilon - \flat g'_{\epsilon 1}) + (\flat g_\epsilon \flat g_{\epsilon 1} - \flat g'_\epsilon \flat g'_{\epsilon 1})\right]\right\rangle\!\!\right\rangle \to -\langle \zeta v\cdot\nabla_x g\rangle$$

in w-$L^1_{loc}(dtdx)$ as $\epsilon \to 0$. Because of (5.6) this is equivalent to

$$\frac{1}{\epsilon}\langle(\mathcal{L}\zeta)\flat g_\epsilon\rangle - \langle \zeta\mathcal{Q}(\flat g_\epsilon, \flat g_\epsilon)\rangle \to -\langle \zeta\, v\cdot\nabla_x g\rangle$$

in $w\text{-}L^1_{loc}(dtdx)$ as $\epsilon \to 0$. Using the limiting form of the number density fluctuation (see (11.1) below in Appendix B) leads precisely to the statement of Proposition 5.1. □

5.2. The convection term. The convection term is the nonlinear part of the limiting system and its convergence is therefore the most difficult to establish. The analysis below rests not only on all the previous sections but also on the arguments in [7] and [50].

Proposition 5.2. *The following convergences hold in the sense of distributions on* $\mathbf{R}_+^* \times \mathbf{R}^3$*:*

$$
\begin{aligned}
P\nabla_x \cdot \langle A\mathcal{Q}(\,^\flat g_\epsilon,\,^\flat g_\epsilon)\rangle &\to P\nabla_x \cdot u^{\otimes 2}\,,\\
\nabla_x \cdot \langle B\mathcal{Q}(\,^\flat g_\epsilon,\,^\flat g_\epsilon)\rangle &\to \tfrac{5}{2}\nabla_x \cdot (u\theta)\,,
\end{aligned}
$$

as $\epsilon \to 0$. *(We recall that* P *is the Leray projection, i.e. the* L^2*-orthogonal projection on the space of divergence-free vector fields).*

Define

$$
^\flat\rho_\epsilon = \langle\,^\flat g_\epsilon\rangle\,,\quad ^\flat u_\epsilon = \langle v\,^\flat g_\epsilon\rangle\,,\quad ^\flat\theta_\epsilon = \left\langle\left(\tfrac{1}{3}|v|^2 - 1\right)\,^\flat g_\epsilon\right\rangle .
$$

- First, we establish that

$$
\begin{aligned}
\nabla_x \cdot \langle A\mathcal{Q}(\,^\flat g_\epsilon,\,^\flat g_\epsilon)\rangle - \nabla_x \cdot \langle A\mathcal{Q}(\Pi\,^\flat g_\epsilon, \Pi\,^\flat g_\epsilon)\rangle \to 0\\
\nabla_x \cdot \langle B\mathcal{Q}(\,^\flat g_\epsilon,\,^\flat g_\epsilon)\rangle - \nabla_x \cdot \langle B\mathcal{Q}(\Pi\,^\flat g_\epsilon, \Pi\,^\flat g_\epsilon)\rangle \to 0
\end{aligned}
$$

  in some appropriate sense as $\epsilon \to 0$, where

$$
\Pi\,^\flat g_\epsilon = \,^\flat\rho_\epsilon + \,^\flat u_\epsilon \cdot v + \,^\flat\theta_\epsilon \tfrac{1}{2}(|v|^2 - 3)\,;
$$

  in other words, Π designates the orthogonal projection on the nullspace of $\mathcal{L}$ in $L^2(Mdv)$;
- Then we show that

$$
\begin{aligned}
P\nabla_x \cdot \langle A\mathcal{Q}(\Pi\,^\flat g_\epsilon, \Pi\,^\flat g_\epsilon)\rangle &= P\nabla_x \cdot \,^\flat u_\epsilon^{\otimes 2} \to P\nabla_x \cdot u^{\otimes 2}\,,\\
\nabla_x \cdot \langle B\mathcal{Q}(\Pi\,^\flat g_\epsilon, \Pi\,^\flat g_\epsilon)\rangle &= \tfrac{5}{2}\nabla_x \cdot (\,^\flat u_\epsilon\,^\flat\theta_\epsilon) \to \tfrac{5}{2}\nabla_x \cdot (u\theta)\,,
\end{aligned}
$$

  in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3$ as $\epsilon \to 0$.

Step 1. Relaxation to the local infinitesimal Maxwellian
The main result in this first step expresses that the distance from $^\flat g_\epsilon$ to the space of local infinitesimal Maxwellians – i.e. elements of $L^2_{loc}(dtdx) \otimes \ker\mathcal{L}$ – vanishes in the limit as $\epsilon \to 0$.

Lemma 5.3. *As* $\epsilon \to 0$

$$
^\flat g_\epsilon - \Pi\,^\flat g_\epsilon \to 0 \quad \textit{in } L^2_{loc}(dtdx; L^2(Mdv))\,. \tag{5.14}
$$

Proof. By Proposition 1.4, $\mathcal{L}$ is a nonnegative bounded Fredholm operator on $L^2(Mdv)$, and therefore is coercive on $(\ker \mathcal{L})^\perp$. Thus, in order to establish (5.14), it suffices to prove that

$$\langle \overset{\flat}{g}_\epsilon \mathcal{L} \overset{\flat}{g}_\epsilon \rangle \to 0 \text{ in } L^1_{loc}(dtdx)\,. \tag{5.15}$$

In order to do so, we pick an arbitrary compact $Q \subset \mathbf{R}_+ \times \mathbf{R}^3$ and split $\mathbf{1}_Q \langle \overset{\flat}{g}_\epsilon \mathcal{L} \overset{\flat}{g}_\epsilon \rangle$ as

$$\mathbf{1}_Q \langle \overset{\flat}{g}_\epsilon \mathcal{L} \overset{\flat}{g}_\epsilon \rangle = (\mathbf{1}_Q - \chi_\epsilon) \langle \overset{\flat}{g}_\epsilon \mathcal{L} \overset{\flat}{g}_\epsilon \rangle + \chi_\epsilon \langle \overset{\flat}{g}_\epsilon \mathcal{L} \overset{\flat}{g}_\epsilon \rangle \tag{5.16}$$

with

$$\chi_\epsilon = \frac{\mathbf{1}_Q}{1 + \frac{1}{3}\epsilon \langle |g_\epsilon| \rangle}\,. \tag{5.17}$$

By Corollary 3.9, the sequence of functions $\mathbf{1}_Q \overset{\flat}{g}_\epsilon \mathcal{L} \overset{\flat}{g}_\epsilon$ is uniformly integrable with respect to the measure $Mdvdxdt$ and therefore relatively compact in w-$L^1(Mdvdxdt)$ by Dunford–Pettis' theorem. On the other hand, $0 \le \mathbf{1}_Q - \chi_\epsilon \le 1$ and

$$\mathbf{1}_Q - \chi_\epsilon = \frac{\frac{1}{3}\epsilon \langle |g_\epsilon| \rangle \mathbf{1}_Q}{1 + \frac{1}{3}\epsilon \langle |g_\epsilon| \rangle} \le \tfrac{1}{3}\epsilon \langle |g_\epsilon| \rangle \mathbf{1}_Q\,.$$

By the first and second entropy controls in Proposition 2.1 $\mathbf{1}_Q - \chi_\epsilon$ converges to 0 in $L^1_{loc}(dtdx)$ and therefore in ($Mdvdxdt$-)measure on compact subsets of $\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$. Lemma 14.1 in Appendix E implies that

$$(\mathbf{1}_Q - \chi_\epsilon) \langle \overset{\flat}{g}_\epsilon \mathcal{L} \overset{\flat}{g}_\epsilon \rangle = (\mathbf{1}_Q - \chi_\epsilon) \mathbf{1}_Q \langle \overset{\flat}{g}_\epsilon \mathcal{L} \overset{\flat}{g}_\epsilon \rangle \to 0 \text{ in } L^1(dtdx)\,. \tag{5.18}$$

Then we prove that

$$\chi_\epsilon \langle \overset{\flat}{g}_\epsilon \mathcal{L} \overset{\flat}{g}_\epsilon \rangle \to 0 \text{ in } L^1(dtdx)\,. \tag{5.19}$$

This is done in two steps. In the first one, we consider the quantity

(5.20)
$$\begin{aligned} &\epsilon \chi_\epsilon \overset{\flat}{g}_\epsilon \left(\iint q_\epsilon b d\sigma_{v,v_1}(\omega) M_1 dv_1 + \frac{1}{\epsilon} \mathcal{L} \overset{\flat}{g}_\epsilon \right) \\ &\qquad = \epsilon \overset{\flat}{g}_\epsilon \big[- \chi_\epsilon \mathcal{L} \overset{\sharp}{g}_\epsilon + \chi_\epsilon \mathcal{Q}(\overset{\flat}{g}_\epsilon, \overset{\flat}{g}_\epsilon) + \mathcal{Q}(\overset{\sharp}{g}_\epsilon, \chi_\epsilon(2\epsilon \overset{\flat}{g}_\epsilon + \epsilon^2 \overset{\sharp}{g}_\epsilon)) \big]\,. \end{aligned}$$

Because $\mathcal{L} = -2\mathcal{Q}(1, \cdot)$, the continuity property stated in Proposition 1.5 implies that $\mathcal{L}$ is a bounded operator on $L^1(Mdv)$. Then, as $\epsilon \to 0$,

(5.21)
$$\|\epsilon \overset{\flat}{g}_\epsilon \, \chi_\epsilon \mathcal{L} \overset{\sharp}{g}_\epsilon \|_{L^1(dtdx; L^1(Mdv))} \le \tfrac{1}{2} \|\mathcal{L}\|_{L^1} \|\mathbf{1}_Q \overset{\sharp}{g}_\epsilon\|_{L^1(dtdx; L^1(Mdv))} \to 0$$

because of Proposition 3.4 and the first pointwise estimate in Proposition 2.2. Likewise the L^1-continuity of $\mathcal{Q}$ in Proposition 1.5 implies that

$$
\begin{aligned}
&\left\| \epsilon \, {}^\flat g_\epsilon \, \mathcal{Q}({}^\sharp g_\epsilon, \chi_\epsilon(2\epsilon \, {}^\flat g_\epsilon + \epsilon^2 \, {}^\sharp g_\epsilon)) \right\|_{L^1(dtdx;\,L^1(Mdv))} \\
&\quad = \left\| \epsilon \, {}^\flat g_\epsilon \, \mathcal{Q}(\mathbf{1}_Q \, {}^\sharp g_\epsilon, \chi_\epsilon(2\epsilon \, {}^\flat g_\epsilon + \epsilon^2 \, {}^\sharp g_\epsilon)) \right\|_{L^1(dtdx;\,L^1(Mdv))} \\
&\quad \le \tfrac{1}{2} \|\mathcal{Q}\|_{L^1} \|\mathbf{1}_Q \, {}^\sharp g_\epsilon\|_{L^1(dtdx;\,L^1(Mdv))} \\
&\quad \times \left(2\|\epsilon \, {}^\flat g_\epsilon\|_{L^\infty(dtdx;\,L^1(Mdv))} + \|\epsilon^2 \chi_\epsilon \, {}^\sharp g_\epsilon\|_{L^\infty(dtdx;\,L^1(Mdv))} \right) \to 0
\end{aligned}
\tag{5.22}
$$

again by Proposition 3.4, the first pointwise estimate in Proposition 2.2 and the elementary inequality

$$
|\epsilon^2 \chi_\epsilon \, {}^\sharp g_\epsilon| \le \frac{\epsilon |g_\epsilon|}{1 + \frac{1}{3}\epsilon \langle |g_\epsilon| \rangle} \text{ implying that } \|\epsilon^2 \chi_\epsilon \, {}^\sharp g_\epsilon\|_{L^1(Mdv)} \le 3 \, .
$$

Finally, by Corollary 3.9 $\mathbf{1}_Q \mathcal{Q}({}^\flat g_\epsilon, {}^\flat g_\epsilon)$ is uniformly integrable for the measure $Mdvdxdt$ and therefore relatively compact in w-$L^1(Mdvdxdt)$ (by Dunford–Pettis' theorem), while $\epsilon \, {}^\flat g_\epsilon$ is bounded in $L^\infty_{t,x,v}$ and $\epsilon \, {}^\flat g_\epsilon \to 0$ in $L^2_{loc}(Mdvdxdt)$ as $\epsilon \to 0$. Thus $\epsilon \, {}^\flat g_\epsilon$ converges to 0 in ($Mdvdxdt$-)measure on compact subsets of $\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$. By Lemma 14.1 in Appendix E

$$
\epsilon \, {}^\flat g_\epsilon \mathcal{Q}({}^\flat g_\epsilon, {}^\flat g_\epsilon) \to 0 \text{ in } L^1_{loc}(dtdx;\, L^1(Mdv)) \, . \tag{5.23}
$$

The convergence statements (5.21), (5.22), (5.23), above show that

$$
\left\| \epsilon \chi_\epsilon \, {}^\flat g_\epsilon \left(\iint q_\epsilon b d\sigma_{v,v_1}(\omega) M_1 dv_1 d\omega + \frac{1}{\epsilon} \mathcal{L} \, {}^\flat g_\epsilon \right) \right\|_{L^1(Mdvdxdt)} \to 0 \, . \tag{5.24}
$$

The second step in the proof of (5.19) is the following lemma that will also be used elsewhere.

Lemma 5.4. *Assume that $\gamma \in \Upsilon$. The sequence*

$$
\langle\!\langle |q_\epsilon \, {}^\flat g_\epsilon| \rangle\!\rangle \text{ is bounded in } L^1_{loc}\big(dt;\, L^1_x\big) \, . \tag{5.25}
$$

Postponing the proof of Lemma 5.4, observe that both estimates (5.25) and (5.24) imply that (5.19) holds. The limit (5.18) being already established, this proves that (5.15) holds, as announced. □

Proof of Lemma 5.4. Young's inequality (10.4) from Appendix A implies that, for all $t^* > 0$ and $\epsilon \in [0, 1]$

$$
\begin{aligned}
&\int_0^{t^*} \int \langle\!\langle \, |q_\epsilon \, {}^\flat g_\epsilon| \, \rangle\!\rangle \, dxdt \\
&\qquad \le \int_0^{t^*} \int \frac{1}{\epsilon^4} \left\langle\!\!\left\langle G_\epsilon G_{\epsilon 1} \left[r\left(\frac{\epsilon^2 q_\epsilon}{G_\epsilon G_{\epsilon 1}} \right) + h^*\big(\epsilon^2 \, {}^\flat g_\epsilon\big) \right] \right\rangle\!\!\right\rangle dxdt \\
&\qquad \le 4 \int_0^{t^*} \iint \frac{D(F_\epsilon)}{\epsilon^4} dvdxdt + h^*(1) \int_0^{t^*} \int \langle\!\langle G_\epsilon G_{\epsilon 1} \, {}^\flat g_\epsilon^2 \rangle\!\rangle \, dxdt \, .
\end{aligned}
$$

The first term on the right-hand side of this inequality is controlled by the entropy dissipation control (2.7). In order to control the second term, we first decompose it as

$$\left\langle\!\!\left\langle G_\epsilon G_{\epsilon 1} \overset{\flat}{g}{}_\epsilon^2 \right\rangle\!\!\right\rangle = \left\langle\!\!\left\langle \epsilon^2 G_\epsilon \overset{\flat}{g}{}_\epsilon^2 G_{\epsilon 1} \frac{1-\gamma_{\epsilon 1}}{\epsilon^2} \right\rangle\!\!\right\rangle + \left\langle\!\!\left\langle G_\epsilon \overset{\flat}{g}{}_\epsilon^2 G_{\epsilon 1} \gamma_{\epsilon 1} \right\rangle\!\!\right\rangle .$$

Using the bound (H1) on b, the first and the third pointwise controls in Proposition 2.2 lead to

$$\left| \left\langle\!\!\left\langle \epsilon^2 G_\epsilon \overset{\flat}{g}{}_\epsilon^2 G_{\epsilon 1} \frac{1-\gamma_{\epsilon 1}}{\epsilon^2} \right\rangle\!\!\right\rangle \right| \leq \tfrac{15}{8} b_\infty \langle | \overset{\sharp}{g}_\epsilon | \rangle$$

and

$$\left| \left\langle\!\!\left\langle G_\epsilon \overset{\flat}{g}{}_\epsilon^2 G_{\epsilon 1} \gamma_{\epsilon 1} \right\rangle\!\!\right\rangle \right| \leq \tfrac{9}{4} b_\infty \langle | \overset{\flat}{g}_\epsilon |^2 \rangle .$$

Eventually, the last three inequalities lead to

(5.26)

$$\begin{aligned} \int_0^{t^*} \int \left\langle\!\!\left\langle \left| q_\epsilon \overset{\flat}{g}_\epsilon \right| \right\rangle\!\!\right\rangle dx dt &\leq 4 \left\| \frac{D(F_\epsilon)}{\epsilon^4} \right\|_{L^1_{t,x,v}} \\ &+ t^* h^*(1) b_\infty \left(\tfrac{15}{8} \| \overset{\sharp}{g}_\epsilon \|_{L^\infty_t(L^1(Mdvdx))} + \tfrac{9}{4} \| \overset{\flat}{g}_\epsilon \|^2_{L^\infty_t(L^2(Mdvdx))} \right) . \end{aligned}$$

Therefore (5.25) holds (see Proposition 2.1). □

The following convergences are easy consequences of Lemma 5.3.

Corollary 5.5. *One has*

(5.27)
$$\begin{aligned} \langle A \mathcal{Q}(\overset{\flat}{g}_\epsilon, \overset{\flat}{g}_\epsilon) \rangle - \overset{\flat}{u}{}_\epsilon^{\otimes 2} + \tfrac{1}{3} | \overset{\flat}{u}_\epsilon |^2 I \to 0 \\ \langle B \mathcal{Q}(\overset{\flat}{g}_\epsilon, \overset{\flat}{g}_\epsilon) \rangle - \tfrac{5}{2} \overset{\flat}{u}_\epsilon \overset{\flat}{\theta}_\epsilon \to 0 \end{aligned}$$

in $L^1_{loc}(dtdx)$ *as* $\epsilon \to 0$.

Proof. We first recall from (1.42) that $\zeta \in L^2(Mdv)$. By the Cauchy–Schwarz inequality and the L^2 continuity of $\mathcal{Q}$ stated in Proposition 1.5, one has, for some constant $C > 0$

(5.28)

$$\begin{aligned} &| \langle \zeta \mathcal{Q}(\overset{\flat}{g}_\epsilon, \overset{\flat}{g}_\epsilon) \rangle - \langle \zeta \mathcal{Q}(\Pi \overset{\flat}{g}_\epsilon, \Pi \overset{\flat}{g}_\epsilon) \rangle | = | \langle \zeta \mathcal{Q}(\overset{\flat}{g}_\epsilon - \Pi \overset{\flat}{g}_\epsilon, \overset{\flat}{g}_\epsilon + \Pi \overset{\flat}{g}_\epsilon) \rangle | \\ &\quad \leq \| \zeta \|_{L^2(Mdv)} \| \mathcal{Q}(\overset{\flat}{g}_\epsilon - \Pi \overset{\flat}{g}_\epsilon, \overset{\flat}{g}_\epsilon + \Pi \overset{\flat}{g}_\epsilon) \|_{L^2(Mdv)} \\ &\quad \leq C \| \zeta \|_{L^2(Mdv)} \| \overset{\flat}{g}_\epsilon + \Pi \overset{\flat}{g}_\epsilon \|_{L^2(Mdv)} \| \overset{\flat}{g}_\epsilon - \Pi \overset{\flat}{g}_\epsilon \|_{L^2(Mdv)} \\ &\quad \leq 2C \| \zeta \|_{L^2(Mdv)} \| \overset{\flat}{g}_\epsilon \|_{L^2(Mdv)} \| \overset{\flat}{g}_\epsilon - \Pi \overset{\flat}{g}_\epsilon \|_{L^2(Mdv)} \\ &\quad = O(1)_{L^\infty_t(L^2_x)} \| \overset{\flat}{g}_\epsilon - \Pi \overset{\flat}{g}_\epsilon \|_{L^2(Mdv)} \to 0 \end{aligned}$$

in $L^1_{loc}(dtdx)$ as $\epsilon \to 0$, by the first entropy control in Proposition 2.1 and Lemma 5.3.

The symmetry relations

$$\langle (\mathcal{L}A) f(v)\rangle = \langle (\mathcal{L}A) p(|v|^2)\rangle = 0\,,$$
$$\langle (\mathcal{L}B) f(v)\rangle = \langle (\mathcal{L}B) p(|v|^2)\rangle = 0\,,$$

which hold for any $f \in \ker \mathcal{L}$ and any polynomial p, and formula (1.43) imply that

$$\begin{aligned}\langle \zeta \mathcal{Q}(\Pi\, {}^\flat g_\epsilon, \Pi\, {}^\flat g_\epsilon)\rangle &= \left\langle \zeta\, \tfrac12 \mathcal{L}\big((\Pi\, {}^\flat g_\epsilon)^2\big)\right\rangle \\ &= \tfrac12 \left\langle (\mathcal{L}\zeta)(\Pi\, {}^\flat g_\epsilon)^2\right\rangle \\ &= \tfrac12 \left\langle (\mathcal{L}\zeta)\big[\, {}^\flat u_\epsilon^{\otimes 2} : v^{\otimes 2} + {}^\flat\theta_\epsilon\, {}^\flat u_\epsilon \cdot v(|v|^2-3)\big]\right\rangle .\end{aligned} \tag{5.29}$$

Collecting the results from (5.28) and (5.29) and using the convergence (11.1) proves that

$$\langle \zeta \mathcal{Q}({}^\flat g_\epsilon, {}^\flat g_\epsilon)\rangle - \tfrac12 \left\langle (\mathcal{L}\zeta)\big[\, {}^\flat u_\epsilon^{\otimes 2} : v^{\otimes 2} + {}^\flat\theta_\epsilon\, {}^\flat u_\epsilon \cdot v(|v|^2-3)\big]\right\rangle \to 0$$

which in turn implies the convergences stated in (5.27). □

Step 2. Convergence of the nonlinear convection terms
Because the Navier–Stokes(–Fourier) system is nonlinear, weak convergences are not enough to take limits in the convection terms. First, in the next lemma we identify two quantities that converge strongly.

Lemma 5.6. *As $\epsilon \to 0$,*

$$P\, {}^\flat u_\epsilon \to u\,, \ \ \textit{and} \ \ \tfrac15 (3\, {}^\flat\theta_\epsilon - 2\, {}^\flat\rho_\epsilon) \to \theta \tag{5.30}$$

in $L^2_{loc}(dtdx)$.

Proof. Step 1: a priori estimates. By Proposition 4.1

$$\begin{aligned} &\partial_t P\, {}^\flat u_\epsilon + P\left(\nabla_x \cdot \frac{1}{\epsilon}\langle (\mathcal{L}A)\, {}^\flat g_\epsilon\rangle\right) \to 0 \\ &\partial_t\big(\tfrac32\, {}^\flat\theta_\epsilon - {}^\flat\rho_\epsilon\big) + \nabla_x \cdot \frac{1}{\epsilon}\langle (\mathcal{L}B)\, {}^\flat g_\epsilon\rangle \to 0 \end{aligned} \tag{5.31}$$

in $L^1_{loc}(dt; W^{-s,1}_{loc}(dx))$ for all $s > 0$ and in $L^1_{loc}(dtdx)$ respectively. By Proposition 5.1, for $\zeta = A$ or $\zeta = B$,

$$\frac{1}{\epsilon}\langle (\mathcal{L}\zeta)\, {}^\flat g_\epsilon\rangle - \langle \zeta \mathcal{Q}({}^\flat g_\epsilon, {}^\flat g_\epsilon)\rangle \text{ is bounded in } L^1_{loc}(dtdx)\,. \tag{5.32}$$

Because $\zeta \in L^2(Mdv)$ (see (1.42)), by the Cauchy–Schwarz inequality and the L^2 continuity of $\mathcal{Q}$ stated in Proposition 1.5, there exists $C > 0$ such that

$$(5.33)\qquad |\langle \zeta \mathcal{Q}(\overset{\flat}{g}_\epsilon, \overset{\flat}{g}_\epsilon)\rangle| \le C\|\zeta\|_{L^2(Mdv)}\|\overset{\flat}{g}_\epsilon\|^2_{L^2(Mdv)} = O(1)_{L^\infty_t(L^1_x)}$$

because of the first entropy control in Proposition 2.1. These last two controls imply that the family

$$(5.34)\qquad \frac{1}{\epsilon}\langle(\mathcal{L}\zeta)\overset{\flat}{g}_\epsilon\rangle \text{ is bounded in } L^1_{loc}(dtdx)\,.$$

Thus, the convergences (5.31) and the bound (5.34) imply that

$$(5.35)\qquad \begin{aligned} &\text{for all } s > 1\,, \quad \partial_t P\,{}^\flat u_\epsilon \text{ is bounded in } L^1_{loc}(dt; W^{-s,1}(\mathbf{R}^3))\,,\\ &\text{and } \partial_t\big(\tfrac{3}{2}\,{}^\flat\theta_\epsilon - \overset{\flat}{\rho}_\epsilon\big) \text{ is bounded in } L^1_{loc}(dt; W^{-1,1}(\mathbf{R}^3))\,. \end{aligned}$$

Further, the second statement of Proposition 3.8 with $\phi(v) = v$ or $\phi(v) = \frac{1}{2}(|v|^2 - 5)$, and the L^2-continuity and translation invariance of P imply that

$$(5.36)\qquad \int_0^{t^*}\int_Q |P\,{}^\flat u_\epsilon(t, x+y) - P\,{}^\flat u_\epsilon(t,x)|^2 dxdt \le \eta(|y|)$$

$$\text{and } \int_0^{t^*}\int_Q \Big|\big(\tfrac{3}{2}\,{}^\flat\theta_\epsilon - \overset{\flat}{\rho}_\epsilon\big)(t, x+y) - \big(\tfrac{3}{2}\,{}^\flat\theta_\epsilon - \overset{\flat}{\rho}_\epsilon\big)(t,x)\Big|^2 dxdt \le \eta(|y|)$$

for each $y \in \mathbf{R}^3$ such that $|y| \le 1$, uniformly in ϵ, where η is the modulus of continuity in the statement of Proposition 3.8.

Step 2: convergence of $\frac{3}{2}\,{}^\flat\theta_\epsilon - \overset{\flat}{\rho}_\epsilon$. The L^1 variant of Aubin's lemma (see [58], p. 84, Theorem 5), and both estimates (5.35) and (5.36) imply that the sequence $\frac{3}{2}\,{}^\flat\theta_\epsilon - \overset{\flat}{\rho}_\epsilon$ is relatively compact in $L^1_{loc}(dtdx)$. On the other hand, the sequence $(\frac{3}{2}\,{}^\flat\theta_\epsilon - \overset{\flat}{\rho}_\epsilon)^2$ is locally uniformly integrable on $\mathbf{R}_+ \times \mathbf{R}^3$ by the first assertion in Proposition 3.8. Hence the sequence $\frac{3}{2}\,{}^\flat\theta_\epsilon - \overset{\flat}{\rho}_\epsilon$ is relatively compact in $L^2_{loc}(dtdx)$. (Indeed, if a_n converges to a in measure and if a_n^2 is locally uniformly integrable, then $a_n \to a$ in L^2_{loc}). By (5.4), we already know that

$$\tfrac{3}{2}\,{}^\flat\theta_\epsilon - \overset{\flat}{\rho}_\epsilon \to \tfrac{3}{2}\theta - \langle g\rangle$$

in w-$L^2_{loc}(dtdx)$ as $\epsilon \to 0$, and hence this convergence holds in the strong topology of $L^2_{loc}(dtdx)$. The second relation in (5.5) (the Boussinesq relation) finally implies that

$$\tfrac{3}{2}\,{}^\flat\theta_\epsilon - \overset{\flat}{\rho}_\epsilon \to \tfrac{5}{2}\theta \text{ strongly in } L^2_{loc}(dtdx)$$

as $\epsilon \to 0$, and this is precisely the second convergence in (5.30).

Step 3: convergence of $P\,{}^\flat u_\epsilon$. Let $\xi \in C^\infty_c(\mathbf{R}^3)$ be such that $\xi(x) = 0$ whenever $|x| > 1$, $\xi \ge 0$ and $\int \xi(x)dx = 1$. For each $\delta > 0$ we define

$\xi_\delta(x) = \delta^{-3}\xi(x/\delta)$. The first entropy control in Proposition 2.1 and the first estimate in (5.35) imply that, for all $s > 0$ and all $\delta > 0$,

$$P\,{}^\flat u_\epsilon \star \xi_\delta \text{ is bounded in } L^\infty_t\big(H^s_x\big)\,,$$
$$\partial_t P\,{}^\flat u_\epsilon \star \xi_\delta \text{ is bounded in } L^1_{loc}\big(dt;\,H^s_x\big)\,.$$

By Theorem 1, p. 71, and Lemma 4, p. 77 of [58], the family $P\,{}^\flat u_\epsilon \star \xi_\delta$ is relatively compact in $L^2_{loc}(dtdx)$ for each $\delta > 0$. The first convergence in (5.4) and the L^2-continuity of P imply that

$$\begin{aligned} P\,{}^\flat u_\epsilon &\to Pu \in w\text{-}L^2_{loc}(dtdx) \text{ while}\\ P\,{}^\flat u_\epsilon \star \xi_\delta &\to Pu \star \xi_\delta \text{ strongly in } L^2_{loc}(dtdx)\,. \end{aligned} \tag{5.37}$$

Hence

$$P\,{}^\flat u_\epsilon \cdot (P\,{}^\flat u_\epsilon \star \xi_\delta) \to Pu \cdot (Pu \star \xi_\delta) \text{ in } L^1_{loc}(dtdx)$$

as $\epsilon \to 0$. On the other hand $Pu \star \xi_\delta \to Pu$ in $L^2_{loc}(dtdx)$ and

$$P\,{}^\flat u_\epsilon - P\,{}^\flat u_\epsilon \star \xi_\delta \to 0 \text{ in } L^2_{loc}(dtdx) \text{ uniformly in } \epsilon \text{ as } \delta \to 0$$

because of the first estimate in (5.36). Therefore

$$|P\,{}^\flat u_\epsilon|^2 \to |Pu|^2 \text{ in } L^1_{loc}(dtdx) \text{ as } \epsilon \to 0\,,$$

With the first convergence in (5.37), this implies that

$$P\,{}^\flat u_\epsilon \to Pu \text{ strongly in } L^2_{loc}(dtdx) \tag{5.38}$$

as $\epsilon \to 0$. The first relation in (5.5) (the incompressibility condition) implies that $Pu = u$, so that (5.38) coincides with the first convergence in (5.30). $\square$

As indicated above, the convergences

$${}^\flat u_\epsilon \to u\,, \qquad {}^\flat\theta_\epsilon \to \theta$$

coming from (5.4) hold in w-$L^2_{loc}(dt;\,L^2_x)$ only, and even the strong limits (5.30) do not imply that

$${}^\flat u_\epsilon^{\otimes 2} \to u^{\otimes 2}\,, \qquad {}^\flat u_\epsilon\,{}^\flat\theta_\epsilon \to u\theta$$

in the sense of distributions on $\mathbf{R}^*_+ \times \mathbf{R}^3$. Instead, one has the following convergences.

Corollary 5.7. *As $\epsilon \to 0$,*

$$\begin{aligned} P\nabla_x \cdot {}^\flat u_\epsilon^{\otimes 2} &\to P\nabla_x \cdot u^{\otimes 2}\\ \nabla_x \cdot ({}^\flat u_\epsilon\,{}^\flat\theta_\epsilon) &\to \nabla_x \cdot (u\theta) \end{aligned} \tag{5.39}$$

in the sense of distributions on $\mathbf{R}^*_+ \times \mathbf{R}^3$.

Proof. Denote $\nabla_x \pi_\epsilon = {}^\flat u_\epsilon - P\,{}^\flat u_\epsilon$ and $\beta_\epsilon = {}^\flat\rho_\epsilon + {}^\flat\theta_\epsilon$; one has

$$\nabla_x \pi_\epsilon \to 0 \text{ and } \tfrac{2}{5}\beta_\epsilon = {}^\flat\theta_\epsilon - \tfrac{2}{5}\left(\tfrac{3}{2}\,{}^\flat\theta_\epsilon - {}^\flat\rho_\epsilon\right) \to 0 \tag{5.40}$$

in w-$L^2_{loc}(dtdx)$, by the incompressibility and Boussinesq relations (see Theorem 11.1 in Appendix B). This and (5.30) imply that

$$\begin{aligned} P\nabla_x \cdot \left({}^\flat u_\epsilon^{\otimes 2}\right) - P\nabla_x \cdot \left((\nabla_x \pi_\epsilon)^{\otimes 2}\right) - P\nabla_x \cdot (u^{\otimes 2}) \to 0 \\ \nabla_x \cdot ({}^\flat u_\epsilon\, {}^\flat\theta_\epsilon) - \tfrac{2}{5}\nabla_x \cdot (\beta_\epsilon \nabla_x \pi_\epsilon) - \nabla_x \cdot (u\theta) \to 0 \end{aligned} \tag{5.41}$$

in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3$. (In other words, the cross-terms in the quadratic expressions $\nabla_x \cdot {}^\flat u_\epsilon^{\otimes 2}$ and $\nabla_x \cdot ({}^\flat u_\epsilon\, {}^\flat\theta_\epsilon)$ vanish with ϵ).

Because of (5.41), proving (5.39) reduces to proving that

$$P\nabla_x \cdot (\nabla_x \pi_\epsilon)^{\otimes 2} \to 0 \quad \text{and} \quad \nabla_x \cdot (\beta_\epsilon \nabla_x \pi_\epsilon) \to 0 \tag{5.42}$$

in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3$.

First we use a mollifier in the space variable, as follows: let $\xi \in C_c^\infty(\mathbf{R}^3)$ be such that $\xi(x) = 0$ whenever $|x| > 1$, $\xi \geq 0$ and $\int \xi(x)dx = 1$. For each $\delta > 0$ we define $\xi_\delta(x) = \delta^{-3}\xi(x/\delta)$ and the sequences $\pi_\epsilon^\delta = \pi_\epsilon * \xi_\delta$ and $\beta_\epsilon^\delta = ({}^\flat\rho_\epsilon + {}^\flat\theta_\epsilon) * \xi_\delta$.

Then, by Proposition 4.1 and (5.34)

$$\epsilon\partial_t \langle v\, {}^\flat g_\epsilon \rangle + \nabla_x \left\langle \tfrac{1}{3}|v|^2\, {}^\flat g_\epsilon \right\rangle = -\nabla_x \cdot \langle (\mathcal{L}A)\, {}^\flat g_\epsilon \rangle + \left\langle\!\left\langle v q_\epsilon \hat{\gamma}_\epsilon \right\rangle\!\right\rangle \to 0$$

$$\epsilon\partial_t \left\langle \tfrac{1}{3}|v|^2\, {}^\flat g_\epsilon \right\rangle + \nabla_x \cdot \left\langle \tfrac{5}{3} v\, {}^\flat g_\epsilon \right\rangle = -\nabla_x \cdot \left\langle \tfrac{2}{3}(\mathcal{L}B)\, {}^\flat g_\epsilon \right\rangle + \left\langle\!\left\langle \tfrac{1}{3}|v|^2 q_\epsilon \hat{\gamma}_\epsilon \right\rangle\!\right\rangle \to 0$$

in $L^1_{loc}(dt; W^{-1,1}_{loc}(\mathbf{R}^3))$. (In fact Proposition 4.1 and (5.34) show that these vanishing terms are of order $O(\epsilon)$, but that much information is not needed here). In other words,

$$\epsilon\partial_t\, {}^\flat u_\epsilon + \nabla_x({}^\flat\rho_\epsilon + {}^\flat\theta_\epsilon) \to 0$$

$$\epsilon\partial_t({}^\flat\rho_\epsilon + {}^\flat\theta_\epsilon) + \tfrac{5}{3}\nabla_x \cdot {}^\flat u_\epsilon \to 0$$

in $L^1_{loc}(dt; W^{-1,1}_{loc}(\mathbf{R}^3))$. In particular, applying $I - P$ to the first equation results in

$$\epsilon\partial_t \nabla_x \pi_\epsilon + \nabla_x \beta_\epsilon \to 0$$

$$\epsilon\partial_t \beta_\epsilon + \tfrac{5}{3}\Delta_x \pi_\epsilon \to 0$$

in $L^1_{loc}(dt; W^{-1-s,1}_{loc}(\mathbf{R}^3))$ for all $s > 0$. Then applying the mollifier ξ_δ leads to

$$\epsilon\partial_t \nabla_x \pi_\epsilon^\delta + \nabla_x \beta_\epsilon^\delta \to 0$$

$$\epsilon\partial_t \beta_\epsilon^\delta + \tfrac{5}{3}\Delta_x \pi_\epsilon^\delta \to 0$$

in $L^1_{loc}(dt; H^s_{loc}(\mathbf{R}^3))$ for any $s > 0$, while the families β^δ_ϵ and $\nabla_x \pi^\delta_\epsilon$ are bounded in $L^\infty_t(L^2_x)$ by the first control of Proposition 2.1. The local argument for the incompressible limit due to P.-L. Lions and N. Masmoudi [49] (recalled in Lemma 13.1 of Appendix D below) shows that, for all $\delta > 0$

$$P\nabla_x \cdot \left(\left(\nabla_x \pi^\delta_\epsilon\right)^{\otimes 2}\right) \to 0\,, \quad \nabla_x \cdot \left(\beta^\delta_\epsilon \nabla_x \pi^\delta_\epsilon\right) \to 0 \tag{5.43}$$

in the sense of distributions on $\mathbf{R}^*_+ \times \mathbf{R}^3$ as $\epsilon \to 0$.

It remains to remove the mollifier ξ_δ in (5.43). In order to do so, observe that by the last statement in Proposition 3.8, for each $t^* > 0$ and each bounded $Q \subset \mathbf{R}^3$, there exists an increasing function $\eta : \mathbf{R}_+ \to \mathbf{R}_+$ such that $\lim_{z\to 0^+} \eta(z) = 0$ and

$$\|\nabla_x \pi_\epsilon(t, x+y) - \nabla_x \pi_\epsilon(t,x)\|_{L^2([0,t^*]\times Q)} \le \eta(|y|)\,,$$
$$\|\beta_\epsilon(t, x+y) - \beta_\epsilon(t,x)\|_{L^2([0,t^*]\times Q)} \le \eta(|y|)\,,$$

for all $y \in \mathbf{R}^3$ such that $|y| \le 1$, uniformly in $\epsilon > 0$. This implies that

$$\left\|\nabla_x \pi^\delta_\epsilon - \nabla_x \pi_\epsilon\right\|_{L^2([0,t^*]\times Q)} \le \eta(\delta)\,, \quad \left\|\beta^\delta_\epsilon - \beta_\epsilon\right\|_{L^2([0,t^*]\times Q)} \le \eta(\delta) \tag{5.44}$$

uniformly in ϵ. Hence

$$\begin{aligned} &P\nabla_x \cdot \left((\nabla_x \pi_\epsilon)^{\otimes 2}\right) - P\nabla_x \cdot \left(\left(\nabla_x \pi^\delta_\epsilon\right)^{\otimes 2}\right) \to 0\,, \\ &\tfrac{2}{5}\nabla_x \cdot (({}^\flat\!\rho_\epsilon + {}^\flat\theta_\epsilon)\nabla_x \pi_\epsilon) - \tfrac{2}{5}\nabla_x \cdot \left(\beta^\delta_\epsilon \nabla_x \pi^\delta_\epsilon\right) \to 0 \end{aligned} \tag{5.45}$$

in the sense of distributions on $\mathbf{R}^*_+ \times \mathbf{R}^3$, uniformly in ϵ as $\delta \to 0$.

The limit (5.43) and the uniform convergence in (5.45) eventually imply (5.42), which in turn establishes (5.39). □

Finally, Corollaries 5.5 and 5.7 imply Proposition 5.2.

5.3. Proof of the weak Navier–Stokes–Fourier limit. We conclude this section by showing how the results from the two previous subsections and those from [7] that are recalled in Appendix B eventually imply the Navier–Stokes–Fourier limit (Theorem 1.6).

We recall that g_ϵ and q_ϵ in fact designate subsequences satisfying (5.3) and (5.4). Then we take limits in all the terms appearing in (5.1) as well as in (5.2):

$$P\,{}^\flat u_\epsilon \to u\,, \quad \tfrac{1}{5}(3\,{}^\flat\theta_\epsilon - 2\,{}^\flat\!\rho_\epsilon) \to \theta \text{ in } L^2_{loc}(dtdx)$$

by Lemma 5.6, while, by Propositions 5.1 and 5.2,

$$P\left(\nabla_x \cdot \frac{1}{\epsilon}\langle(\mathcal{L}A)\,{}^\flat\! g_\epsilon\rangle\right) \to P\nabla_x \cdot (u \otimes u) - \nu\Delta_x u\,,$$
$$\nabla_x \cdot \frac{1}{\epsilon}\langle(\mathcal{L}B)\,{}^\flat\! g_\epsilon\rangle \to \tfrac{5}{2}\nabla_x \cdot (u\theta) - \tfrac{5}{2}\kappa\Delta_x\theta\,,$$

in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3$. This shows that (u, θ) satisfies the Navier–Stokes–Fourier system (1.29) in the sense of distributions on $\mathbf{R}_+^* \times \mathbf{R}^3$.

On the other hand, the controls (5.35) and Appendix C of [44] imply that the quantities

$$P\, {}^\flat u_\epsilon \to u\,, \quad \tfrac{1}{5}(3\, {}^\flat\theta_\epsilon - 2\, {}^\flat\rho_\epsilon) \to \theta \text{ in } C\big(\mathbf{R}_+; w\text{-}L^2_{loc}(dx)\big) \text{ as } \epsilon \to 0\,.$$

In particular, at $t = 0$ one has

$$P\langle v\, {}^\flat g_\epsilon(0)\rangle \to u^{in}\,, \quad \tfrac{1}{5}\langle(|v|^2 - 5)\, {}^\flat g_\epsilon(0)\rangle \to \theta^{in}$$

in $w\text{-}L^2_{loc}(dx)$. This proves that (u, θ) satisfies the initial condition (1.30).

Finally

$$\langle v g_\epsilon\rangle = {}^\flat u_\epsilon + \epsilon\langle v\, {}^\sharp g_\epsilon\rangle \to u$$
$$\big\langle\big(\tfrac{1}{3}|v|^2 - 1\big) g_\epsilon\big\rangle = \big\langle\big(\tfrac{1}{3}|v|^2 - 1\big)\, {}^\flat g_\epsilon\big\rangle + \epsilon\big\langle\big(\tfrac{1}{3}|v|^2 - 1\big)\, {}^\sharp g_\epsilon\big\rangle \to \theta$$

in $w\text{-}L^1_{loc}(dtdx)$ because of (5.4) and the second entropy estimate in Proposition 2.1. This completes the proof of Theorem 1.6.

6. Proving Proposition 2.7

In this and the next two sections, we prove the results announced in Sects. 2 and 3 and subsequently used in Sects. 4 and 5.

We begin with the results stated in Sect. 2, where the Relaxation-based decomposition and Proposition 2.7 were stated without proof.

6.1. Proof of the Relaxation-based decomposition. Returning to the equality (2.16), we see that

(6.1)
$$k(G_\epsilon) M g_\epsilon = T_1 + T_2$$

with $k : \mathbf{R}_+ \to [0, 1]$ such that $\|z \mapsto zk(z)\|_{L^\infty} = C_k$ and

$$\begin{aligned} T_1 &= \frac{k(G_\epsilon)}{\epsilon\langle\tilde{G}_\epsilon\rangle}[MG_\epsilon\langle\tilde{G}_\epsilon\rangle - \mathcal{A}^+(M\tilde{G}_\epsilon, M\tilde{G}_\epsilon)] \\ &= \frac{k(G_\epsilon)}{\epsilon\langle\tilde{G}_\epsilon\rangle} M \iint (G_\epsilon\tilde{G}_{\epsilon 1} - \tilde{G}'_\epsilon\tilde{G}'_{\epsilon 1}) d\sigma_{v,v_1}(\omega) M_1 dv_1\,, \end{aligned}$$

where the last equality uses the classical relation $M'M'_1 = MM_1$ that follows from the formula (1.12) defining M and the microscopic conservation laws (1.4). Expanding each term of the form $\tilde{G}_\epsilon = (1-\gamma_\epsilon) + \gamma_\epsilon G_\epsilon$ in the formula above leads to

(6.2)

$$\begin{aligned}
T_1 &= \frac{k(G_\epsilon)}{\epsilon\langle\tilde{G}_\epsilon\rangle} M \iint (1-\gamma_{\epsilon 1})(G_\epsilon - \tilde{G}'_\epsilon \tilde{G}'_{\epsilon 1}) d\sigma_{v,v_1}(\omega) M_1 dv_1 \\
&+ \frac{k(G_\epsilon)}{\epsilon\langle\tilde{G}_\epsilon\rangle} M \iint \gamma_{\epsilon 1}(1-\gamma'_\epsilon)(1-\gamma'_{\epsilon 1})(G_\epsilon G_{\epsilon 1} - 1) d\sigma_{v,v_1}(\omega) M_1 dv_1 \\
&+ \frac{k(G_\epsilon)}{\epsilon\langle\tilde{G}_\epsilon\rangle} M \iint \gamma_{\epsilon 1}\gamma'_\epsilon(1-\gamma'_{\epsilon 1})(G_\epsilon G_{\epsilon 1} - G'_\epsilon) d\sigma_{v,v_1}(\omega) M_1 dv_1 \\
&+ \frac{k(G_\epsilon)}{\epsilon\langle\tilde{G}_\epsilon\rangle} M \iint \gamma_{\epsilon 1}(1-\gamma'_\epsilon)\gamma'_{\epsilon 1}(G_\epsilon G_{\epsilon 1} - G'_{\epsilon 1}) d\sigma_{v,v_1}(\omega) M_1 dv_1 \\
&- \frac{\epsilon k(G_\epsilon)}{\langle\tilde{G}_\epsilon\rangle} M \iint \gamma_{\epsilon 1}\gamma'_\epsilon\gamma'_{\epsilon 1}\, {}^\flat q_\epsilon d\sigma_{v,v_1}(\omega) M_1 dv_1 \\
&- \frac{\epsilon^3 k(G_\epsilon)}{\langle\tilde{G}_\epsilon\rangle} M \iint \gamma_{\epsilon 1}\gamma'_\epsilon\gamma'_{\epsilon 1}\, {}^\sharp q_\epsilon d\sigma_{v,v_1}(\omega) M_1 dv_1 \\
&= T_{11} + T_{12} + T_{13} + T_{14} + T_{15} + T_{16}\,.
\end{aligned}$$

In this decomposition of T_1, the first four terms result from using the truncated density $\tilde{G}_\epsilon$ (instead of G_ϵ) in the definition of the local pseudo-equilibrium (2.13).

The first statement in Proposition 2.2 implies that

$$\begin{aligned}
&0 \le \tfrac{1}{2}\gamma(G_\epsilon) \le \gamma(G_\epsilon)G_\epsilon \le \tfrac{3}{2}\gamma(G_\epsilon) \le \tfrac{3}{2}\,; \\
&\text{thus } \tfrac{1}{2} \le \tilde{G}_\epsilon \le \tfrac{3}{2} \text{ and } \tfrac{1}{2} \le \langle\tilde{G}_\epsilon\rangle \le \tfrac{3}{2} \text{ a.e.}
\end{aligned} \tag{6.3}$$

while the assumptions on k imply that

$$0 \le k(G_\epsilon) \le 1 \text{ and } 0 \le k(G_\epsilon)G_\epsilon \le C_k\,. \tag{6.4}$$

These two bounds immediately imply that

$$|T_{11}| \le 2\big(C_k + \tfrac{9}{4}\big) M \left\langle \frac{1-\gamma(G_\epsilon)}{\epsilon} \right\rangle . \tag{6.5}$$

Then

(6.6)

$$\begin{aligned}
|T_{13}| + |T_{14}| &\le (3C_k + 3) M \iint \left(\frac{1-\gamma'_{\epsilon 1}}{\epsilon} + \frac{1-\gamma'_\epsilon}{\epsilon} \right) d\sigma_{v,v_1}(\omega) M_1 dv_1 \\
&= (6C_k + 6)\mathcal{A}^+ \left(M \frac{1-\gamma(G_\epsilon)}{\epsilon}, M \right) .
\end{aligned}$$

using again the relation $M'M_1' = MM_1$ recalled above as well as the bounds (6.3) and (6.4). Likewise

$$
\begin{aligned}
|T_{12}| &\leq (3C_k+2)M \iint \frac{(1-\gamma_\epsilon')(1-\gamma_{\epsilon 1}')}{\epsilon} d\sigma_{v,v_1}(\omega) M_1 dv_1 \\
&\leq (3C_k+2)M \iint \left(\frac{1-\gamma_\epsilon'}{2\epsilon} + \frac{1-\gamma_{\epsilon 1}'}{2\epsilon}\right) d\sigma_{v,v_1}(\omega) M_1 dv_1 \qquad (6.7)\\
&= (3C_k+2)\mathcal{A}^+\left(M\frac{1-\gamma(G_\epsilon)}{\epsilon}, M\right),
\end{aligned}
$$

where the second inequality in (6.7) come from the elementary estimate $\alpha\beta \leq \frac{1}{2}(\alpha+\beta)$ which holds for all $\alpha, \beta \in [0,1]$. Since γ and k are bounded by 1 while the collision kernel b satisfies the assumption (H1), one has

$$
\begin{aligned}
|T_{15}| &\leq 2M\epsilon \left(\iint k(G_\epsilon)\gamma_{\epsilon 1} G_\epsilon G_{\epsilon 1} d\sigma_{v,v_1}(\omega) M_1 dv_1\right)^{1/2} \\
&\quad \times \left(\iint \frac{\flat q_\epsilon^2}{G_\epsilon G_{\epsilon 1}} d\sigma_{v,v_1}(\omega) M_1 dv_1\right)^{1/2} \qquad (6.8)\\
&\leq 2\epsilon \sqrt{\tfrac{6b_\infty}{c}} \sqrt{C_k} \sqrt{M} \sqrt{\frac{D(F_\epsilon)}{\epsilon^4}}
\end{aligned}
$$

by the Cauchy–Schwarz inequality and the first statement in Proposition 2.3. Likewise

$$
|T_{16}| \leq 2M\epsilon^3 \iint |{}^\sharp q_\epsilon| d\sigma_{v,v_1}(\omega) M_1 dv_1 \leq \tfrac{8b_\infty}{c'} \frac{D(F_\epsilon)}{\epsilon} \qquad (6.9)
$$

by the second statement in Proposition 2.3. On the other hand, the definition (2.9) implies that $|{}^\sharp q_\epsilon| \leq \frac{1}{\epsilon^2}|q_\epsilon|$; thus

(6.10)

$$
|T_{16}| \leq 2M\epsilon^3 \iint \frac{1}{\epsilon^2} k(G_\epsilon)\gamma_{\epsilon 1}\gamma_\epsilon'\gamma_{\epsilon 1}' |q_\epsilon| d\sigma_{v,v_1}(\omega) M_1 dv_1 \leq \left(3C_k + \tfrac{9}{2}\right)\frac{1}{\epsilon} M,
$$

since the bounds (6.3) and (6.4) imply that

$$
\epsilon^2 k(G_\epsilon)\gamma_{\epsilon 1}\gamma_\epsilon'\gamma_{\epsilon 1}'|q_\epsilon| \leq k(G_\epsilon)\gamma_{\epsilon 1}\gamma_\epsilon'\gamma_{\epsilon 1}'(G_\epsilon G_{\epsilon 1} + G_\epsilon' G_{\epsilon 1}') \leq \tfrac{3}{2}C_k + \tfrac{9}{4}.
$$

Because of the elementary inequality $\min(a,b) \leq \sqrt{ab}$ which holds for all positive a and b, the inequalities (6.9) and (6.10) imply

$$
|T_{16}| \leq 2\sqrt{\tfrac{6b_\infty}{c'}} \sqrt{C_k + \tfrac{3}{2}}\, \epsilon \sqrt{M} \sqrt{\frac{D(F_\epsilon)}{\epsilon^4}}. \qquad (6.11)
$$

Estimating T_2 is simpler: the decomposition of it written in Sect. 2 and the obvious relation $\mathcal{A}^+(M,M) = M$ (that follows from the equality $M'M'_1 = MM_1$ recalled above) lead to

$$
\begin{aligned}
T_2 &= \frac{k(G_\epsilon)}{\epsilon\langle\tilde{G}_\epsilon\rangle}\big(\mathcal{A}^+(M\tilde{G}_\epsilon, M\tilde{G}_\epsilon) - \mathcal{A}^+(M,M)\big)\\
&\quad + \frac{k(G_\epsilon)}{\epsilon}\mathcal{A}^+(M,M)\left(\frac{1}{\langle\tilde{G}_\epsilon\rangle} - 1\right)\\
&= \frac{k(G_\epsilon)}{\langle\tilde{G}_\epsilon\rangle}\mathcal{A}^+(M\,{}^\flat g_\epsilon, M(\tilde{G}_\epsilon + 1)) - \frac{k(G_\epsilon)}{\langle\tilde{G}_\epsilon\rangle}M\langle{}^\flat g_\epsilon\rangle\\
&= T_{21} + T_{22}\,.
\end{aligned} \tag{6.12}
$$

By (6.3),

$$
|T_{21}| \le 2\mathcal{A}^+\big(M|{}^\flat g_\epsilon|, \tfrac{5}{2}M\big) = 5\mathcal{A}^+(M|{}^\flat g_\epsilon|, M)\,; \tag{6.13}
$$

for the same reason

$$
|T_{22}| \le 2M\langle|{}^\flat g_\epsilon|\rangle\,. \tag{6.14}
$$

Using the decomposition (6.2) with the bounds (6.5), (6.6), (6.7), (6.8) and (6.11) and the decomposition (6.12) with the bounds (6.13) and (6.14) leads to the inequality (2.17).

6.2. Proof of Proposition 2.7

6.2.1. Proof of the first statement. Consider the inequality (2.17) with the choice $k = \gamma$, where γ is the function involved in the Flat-Sharp decomposition (2.4). There, $C_k = \frac{3}{2}$ so that (2.17) and the second inequality in Proposition 2.2 imply that, for some positive constant C (depending only on b_∞),

$$
\begin{aligned}
\sqrt{M}|{}^\flat g_\epsilon| &\le C\epsilon\sqrt{\frac{D(F_\epsilon)}{\epsilon^4}} + C\sqrt{M}\left\langle|{}^\flat g_\epsilon| + |{}^\sharp g_\epsilon|^{1/2}\right\rangle\\
&\quad + \frac{C}{\sqrt{M}}\mathcal{A}^+\big(M\big[|{}^\flat g_\epsilon| + |{}^\sharp g_\epsilon|^{1/2}\big], M\big).
\end{aligned}
$$

Therefore, using the Caflisch–Grad estimates (particularly the second continuity statement in Proposition 2.5) shows that, for some positive constant C' (depending only on b_∞)

$$
\begin{aligned}
M|{}^\flat g_\epsilon|^2 &\le C'\epsilon^2\frac{D(F_\epsilon)}{\epsilon^4}\\
&\quad + C'\big(\|{}^\flat g_\epsilon\|^2_{L^2(Mdv)} + \|{}^\sharp g_\epsilon\|_{L^1(Mdv)}\big)\left(M + (1+|v|)^{-3}\right)\,.
\end{aligned}
$$

Because of the dissipation control (2.7) and the first and second statements in Proposition 2.1, this last inequality shows that

$$M|\flat g_\epsilon|^2 = a_\epsilon + b_\epsilon \text{ with } a_\epsilon = O(\epsilon^2)_{L^1_{t,x,v}} \text{ and } b_\epsilon = O(1)_{L^\infty_t(L^1_x(L^\infty_v))} .$$

Let ϵ_n be any sequence converging to 0, let $Q = Q_1 \times Q_2$ be a compact subset of $\mathbf{R}_+ \times \mathbf{R}^3$ and let $\eta > 0$ be chosen arbitrarily small. Pick $N > 0$ such that, for all $n \geq N$,

$$\|a_{\epsilon_n}\|_{L^1_{t,x,v}} < \eta . \tag{6.15}$$

Then, pick $\alpha > 0$ satisfying

$$\alpha |Q_1| \sup_{n\geq 0} \|b_{\epsilon_n}\|_{L^\infty_t(L^1_x(L^\infty_v))} < \eta \tag{6.16}$$

and such that, for each measurable set $B \subset \mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$ satisfying $|B| < \alpha |Q|$,

$$\sup_{1\leq n\leq N} \iiint_B |a_{\epsilon_n}| dtdxdv < \eta . \tag{6.17}$$

Consider a measurable family $(A_{t,x})_{(t,x)\in Q}$ of measurable subsets of $\mathbf{R}^3_v$ such that

$$\sup_{(t,x)\in Q} |A_{t,x}| < \alpha . \tag{6.18}$$

Then, by using (6.15), (6.16), (6.18) and (6.17) with

$$B = \{(t, x, v) \mid (t, x) \in Q \, , \ v \in A_{t,x}\} ,$$

one has

$$\iint \mathbf{1}_Q \left(\int_{A_{t,x}} M|\flat g_{\epsilon_n}|^2 dv \right) dtdx \leq \eta + \alpha |Q_1| \|b_\epsilon\|_{L^\infty_t(L^1_x(L^\infty_v))} < 2\eta .$$

This immediately entails the first statement in Proposition 2.7.

6.2.2. Proof of the second statement. Set $k(z) = \min(1, \frac{3}{z})$ for $z \in \mathbf{R}^*_+$; then $C_k = 3$. Observe that

$$\begin{aligned} k(G_\epsilon)|g_\epsilon| &= |g_\epsilon| \mathbf{1}_{G_\epsilon \leq 3} + \frac{3|g_\epsilon|}{G_\epsilon} \mathbf{1}_{G_\epsilon > 3} \\ &\leq |g_\epsilon| \mathbf{1}_{G_\epsilon \leq 3} + \frac{3|g_\epsilon|}{\frac{2}{3} + \frac{1}{3} G_\epsilon} \end{aligned}$$

so that, by the fourth statement in Proposition 2.1 and the last statement of Theorem 11.1 (see below in Appendix B), the family

$$k(G_\epsilon) g_\epsilon \text{ is bounded in } L^\infty_t(L^2(Mdvdx)) . \tag{6.19}$$

With the choice of k above, one sees immediately that, for each $z \in \mathbf{R}_+^*$,

(6.20)
$$\gamma(z) \leq k(z)\,, \quad \tfrac{1}{4}(1-\gamma(z)) \leq k(z)|z-1|(1-\gamma(z)) \leq k(z)|z-1|\,.$$

Therefore
$$|{}^\flat g_\epsilon| + \frac{1-\gamma(G_\epsilon)}{\epsilon} \leq 5k(G_\epsilon)|g_\epsilon|$$

so that (2.17) and the second pointwise control in Proposition 2.2 imply that

$$\begin{aligned} k(G_\epsilon)\sqrt{M}|g_\epsilon| &\leq k(G_\epsilon)\sqrt{M}|g_\epsilon|\mathbf{1}_{|v|^2>V_\epsilon} + C\epsilon\sqrt{\frac{D(F_\epsilon)}{\epsilon^4}}\mathbf{1}_{|v|^2\leq V_\epsilon} \\ &\quad + C\sqrt{M}\left\langle|{}^\flat g_\epsilon| + |{}^\sharp g_\epsilon|^{1/2}\right\rangle \\ &\quad + \frac{C}{\sqrt{M}}\mathcal{A}^+\left(Mk(G_\epsilon)|g_\epsilon|, M\right) \end{aligned} \tag{6.21}$$

for some constant $C > 0$.

By the last statement in Proposition 2.2

$$\begin{aligned} k(G_\epsilon)\sqrt{M}|g_\epsilon|\mathbf{1}_{|v|^2>V_\epsilon} &= \frac{1}{\epsilon}k(G_\epsilon)|G_\epsilon - 1|(\sqrt{M}\mathbf{1}_{|v|^2>V_\epsilon}) \\ &= O\left(\frac{1}{\epsilon}e^{-\frac{1}{4}V_\epsilon}V_\epsilon^{\frac{p}{2}+\frac{1}{4}}\right)_{L^\infty_{t,x}(L^{2,p}_v)} \end{aligned}$$

for all $p \geq 0$; next, the dissipation bound (2.7) implies that

$$\begin{aligned} \epsilon\sqrt{\frac{D(F_\epsilon)}{\epsilon^4}}\mathbf{1}_{|v|^2\leq V_\epsilon} &= \sqrt{\frac{D(F_\epsilon)}{\epsilon^4}}\left(\epsilon(1+|v|)^p\mathbf{1}_{|v|^2\leq V_\epsilon}\right)(1+|v|)^{-p} \\ &= O(1)_{L^2_{t,x,v}}O\left(\epsilon V_\epsilon^{\frac{p}{2}}\right)(1+|v|)^{-p}\,. \end{aligned}$$

Finally, the first and second entropy controls in Proposition 2.1 imply that

$$\sqrt{M}\left\langle|{}^\flat g_\epsilon| + |{}^\sharp g_\epsilon|^{1/2}\right\rangle = O(1)_{L^\infty_t(L^2_x(L^{2,p}_v)))}$$

for all $p \geq 0$.

Using the last three estimates with the choice $V_\epsilon = 10|\log\epsilon|$ in the inequality (6.21) implies that

$$k(G_\epsilon)\sqrt{M}|g_\epsilon| \leq O(1)_{L^2_{loc}(dtdx;L^{2,p}_v)} + \frac{C}{\sqrt{M}}\mathcal{A}^+\left(Mk(G_\epsilon)|g_\epsilon|, M\right) \tag{6.22}$$

for all $p \geq 0$. Pick such a $p > 1$; because of (6.19) and the second continuity statement in Proposition 2.5, one has

$$k(G_\epsilon)\sqrt{M}|g_\epsilon| = O(1)_{L^2_{loc}(dtdx;L^{2,p}_v)} + O(1)_{L^2_{loc}(dtdx;L^{\infty,3/2}_v)}\,.$$

In (6.22), using the third and fourth continuity statements in Proposition 2.5 with $\sigma = p$, one gets the control

$$k(G_\epsilon)\sqrt{M}|g_\epsilon| = O(1)_{L^2_{loc}(dtdx;L^{2,p}_v)} + O(1)_{L^2_{loc}(dtdx;L^{\infty,p+3/2}_v)} + O(1)_{L^2_{loc}(dtdx;L^{\infty,3/2+2N}_v)}$$

by induction on $N \geq 1$. For $N > p/2$ this implies that

$$\begin{aligned} k(G_\epsilon)\sqrt{M}|g_\epsilon| &= O(1)_{L^2_{loc}(dtdx;L^{2,p}_v)} + O(1)_{L^2_{loc}(dtdx;L^{\infty,p+3/2}_v)} \\ &= O(1)_{L^2_{loc}(dtdx;L^{2,p-1}_v)} \end{aligned}$$

because $L^{\infty,p+3/2}(\mathbf{R}^3) \subset L^{2,p-1}(\mathbf{R}^3)$. Since $p > 1$ was arbitrary, this implies that, for all $p \geq 0$

$$k(G_\epsilon)\sqrt{M}|g_\epsilon| = O(1)_{L^2_{loc}(dtdx;L^{2,p}_v)} . \tag{6.23}$$

This control and the first inequality in (6.20) establish the second statement in Proposition 2.7.

6.2.3. Proof of the third statement. Let $t^* > 0$ and Q be a compact subset of $\mathbf{R}^3$. By the Cauchy–Schwarz inequality

$$\begin{aligned} &\left\| \frac{k(G_\epsilon)}{\epsilon} M|g_\epsilon|(1-\gamma(G_\epsilon))(1+|v|)^p \right\|_{L^1([0,t^*]\times Q;L^1_v)} \\ &\leq \left\| k(G_\epsilon)\sqrt{M}|g_\epsilon|(1+|v|)^p \right\|_{L^2([0,t^*]\times Q;L^2_v)} \left\| \frac{1-\gamma(G_\epsilon)}{\epsilon} \right\|_{L^2([0,t^*]\times Q;L^2(Mdv))} \\ &\leq \left\| k(G_\epsilon)\sqrt{M}|g_\epsilon|(1+|v|)^p \right\|_{L^2([0,t^*]\times Q;L^2_v)} \left\| 2|{}^\sharp g_\epsilon|^{1/2} \right\|_{L^2([0,t^*]\times Q;L^2(Mdv))} , \end{aligned}$$

where the last inequality follows from the second pointwise estimate in Proposition 2.2. Thus, because of (6.23) and the second statement in Proposition 2.1

$$\frac{k(G_\epsilon)}{\epsilon^2} M|G_\epsilon - 1|(1-\gamma(G_\epsilon))(1+|v|)^p = O(1)_{L^1_{loc}(dtdx;L^1_v)} ;$$

this and the second inequality in (6.20) imply the third statement in Proposition 2.7.

7. Proving Proposition 3.4 and Corollary 3.5

The proof follows the argument sketched in Sect. 3.

7.1. Step 1: truncating large values of G_ϵ. Choose $\delta \in]0,1[$ and a C^1 function $k^\delta : \mathbf{R}_+ \to [0,1]$ such that $k^\delta \equiv 1$ on $[0, e^{1/\delta}]$, $k^\delta \equiv 0$ on $[2e^{1/\delta} + \infty)$ and $\|(k^\delta)'\|_{L^\infty} \leq 2e^{-1/\delta}$. Thus $C_{k^\delta} \leq 2e^{1/\delta}$. We use the notation k^δ_ϵ below to designate $k^\delta(G_\epsilon)$.

We start from the decomposition

$$\text{(7.1)} \qquad {}^\sharp g_\epsilon = k^\delta_\epsilon \, {}^\sharp g_\epsilon + \left(1 - k^\delta_\epsilon\right) {}^\sharp g_\epsilon$$

and use the last statement of Proposition 2.1 which shows that

$$\text{(7.2)} \qquad \begin{aligned} \left\| \left(1 - k^\delta_\epsilon\right) {}^\sharp g_\epsilon \right\|_{L^\infty_t(L^1(Mdvdx))} &\leq \left\| \frac{1}{\epsilon} g_\epsilon \mathbf{1}_{G_\epsilon \geq e^{1/\delta}} \right\|_{L^\infty_t(L^1(Mdvdx))} \\ &\leq C^{in} \frac{1}{\frac{1}{\delta} - 1} = C^{in} \frac{\delta}{1-\delta} \,. \end{aligned}$$

7.2. Step 2: truncating large velocities. For any family $V_\epsilon > 0$ and each $p \geq 0$, one has

$$\begin{aligned} k^\delta_\epsilon M |{}^\sharp g_\epsilon| \mathbf{1}_{|v|^2 > V_\epsilon} &\leq k^\delta_\epsilon |G_\epsilon - 1| V_\epsilon^{-p/2} \left(M \frac{1 - \gamma(G_\epsilon)}{\epsilon^2} |v|^p \mathbf{1}_{|v|^2 > V_\epsilon} \right) \\ &\leq 2e^{1/\delta} V_\epsilon^{-p/2} O(1)_{L^1_{loc}(dtdx; L^1_v))} \end{aligned}$$

because of the last statement in Proposition 2.7. Therefore, for each $p \geq 0$, each $t^* > 0$, and each compact $Q \subset \mathbf{R}^3$

$$\text{(7.3)} \qquad \left\| k^\delta_\epsilon \, {}^\sharp g_\epsilon \mathbf{1}_{|v|^2 > V_\epsilon} \right\|_{L^1([0,t^*] \times Q; L^1(Mdv))} = O\left(\frac{e^{1/\delta}}{V_\epsilon^{p/2}} \right) .$$

7.3. Step 3: $L^1 - L^\infty$ controls. First we recall that, because of the second pointwise estimate in Proposition 2.2 and the second entropy control in Proposition 2.1, one has

$$\text{(7.4)} \qquad \|M(1 - \gamma(G_\epsilon))\|_{L^\infty_t(L^1_{x,v})} = O(\epsilon^2) \,, \text{ while } \|1 - \gamma(G_\epsilon)\|_{L^\infty_{t,x,v}} \leq 1 \,.$$

Therefore

$$\text{(7.5)} \qquad \begin{aligned} &\|(1 - \gamma(G_\epsilon))\mathbf{1}_{|v|^2 \leq V_\epsilon}\|_{L^\infty_t(L^1_{x,v})} \\ &\qquad \leq \|\mathbf{1}_{|v|^2 \leq V_\epsilon} M^{-1}\|_{L^\infty_v} \|M(1 - \gamma(G_\epsilon))\|_{L^\infty_t(L^1_{x,v})} \\ &\qquad = O\left(e^{\frac{1}{2} V_\epsilon} \epsilon^2 \right) . \end{aligned}$$

Applying Lemma 3.1 and taking into account the second equality in (7.4) and (7.5) implies the existence of ϕ_ϵ^0 and ϕ_ϵ^1 such that

$$0 \le \phi_\epsilon^j \le (1-\gamma(G_\epsilon))\mathbf{1}_{|v|^2\le V_\epsilon}\,, \quad j=0,1\,, \tag{7.6}$$

$$\begin{aligned} \phi_\epsilon^0 &= O\Big(\epsilon e^{\frac{1}{4}V_\epsilon}\Big)_{L_t^\infty(L_x^1(L_v^\infty))} \\ \phi_\epsilon^1 &= O\Big(\epsilon e^{\frac{1}{4}V_\epsilon}\Big)_{L_{t,x}^\infty(L_v^1)} \end{aligned} \tag{7.7}$$

and

$$(1-\gamma(G_\epsilon))\mathbf{1}_{|v|^2\le V_\epsilon} = \phi_\epsilon^0 + \phi_\epsilon^1\,. \tag{7.8}$$

7.4. Step 4: Applying the Advection/Dispersion interpolation

7.4.1. Controlling the advection term. Set

$$\tilde{k}_\delta^\epsilon = (|Z-1|k^\delta(Z))'_{|Z=G_\epsilon} = \operatorname{sgn}(G_\epsilon - 1)k^\delta(G_\epsilon) + |G_\epsilon - 1|(k^\delta)'(G_\epsilon)\,.$$

Lemma 7.1. *The family of relative fluctuations g_ϵ satisfies the estimate*

$$\left|(\epsilon\partial_t + v\cdot\nabla_x)\left(\frac{k_\delta^\epsilon}{\epsilon}|g_\epsilon|\right)\right| \le O\left(e^{1/\delta}\right)_{L^1_{loc}(dt;L^1(Mdvdx))} + O\left(\frac{e^{1/\delta}}{\epsilon^2}\right)_{L^\infty_{t,x,v}}\,.$$

Proof. Because F_ϵ is a renormalized solution relative to M of (1.13), using the renormalized formulation (1.19) with $\Gamma(Z) = |Z-1|k^\delta(Z)$ leads to

$$(\epsilon\partial_t + v\cdot\nabla_x)\left(\frac{k_\delta^\epsilon}{\epsilon}|g_\epsilon|\right) = \iint \frac{\tilde{k}_\delta^\epsilon}{\epsilon} q_\epsilon b d\sigma_{v,v_1}(\omega) M_1 dv_1\,.$$

Young's inequality (10.4) in Appendix A below implies that

$$\begin{aligned} &\iint \frac{|\tilde{k}_\delta^\epsilon|}{\epsilon}|q_\epsilon| b d\sigma_{v,v_1}(\omega) M_1 dv_1 \\ &\qquad = \iint \frac{|\tilde{k}_\delta^\epsilon|}{\epsilon^4}\frac{\epsilon^2|q_\epsilon|}{G_\epsilon G_{\epsilon 1}}\epsilon G_\epsilon G_{\epsilon 1} b d\sigma_{v,v_1}(\omega) M_1 dv_1 \\ &\qquad \le \frac{|\tilde{k}_\delta^\epsilon|}{\epsilon^4}\iint r\left(\frac{\epsilon^2|q_\epsilon|}{G_\epsilon G_{\epsilon 1}}\right) G_\epsilon G_{\epsilon 1} b d\sigma_{v,v_1}(\omega) M_1 dv_1 \\ &\qquad + \frac{|\tilde{k}_\delta^\epsilon|}{\epsilon^4} h^*(\epsilon) G_\epsilon \iint G_{\epsilon 1} b d\sigma_{v,v_1}(\omega) M_1 dv_1 \\ &\qquad \le \frac{4|\tilde{k}_\delta^\epsilon|}{\epsilon^4}\frac{D(F_\epsilon)}{M} + \frac{h^*(1) b_\infty |\tilde{k}_\delta^\epsilon|}{\epsilon^2} G_\epsilon \langle G_\epsilon\rangle\,, \end{aligned}$$

where the last inequality comes from the superquadratic property of h^* (see (10.3) in Appendix A below), the equality in (2.7) and the bound on the collision kernel implied by (H1). Since $|\tilde{k}_\delta^\epsilon| \leq 1 + 2e^{-1/\delta} \cdot 2e^{1/\delta} \leq 5$ and $\tilde{k}_\delta^\epsilon \equiv 0$ whenever $G_\epsilon \geq 2e^{1/\delta}$, one has

$$\begin{aligned}
\iint \frac{|\tilde{k}_\delta^\epsilon|}{\epsilon} |q_\epsilon| b M_1 dv_1 d\sigma_{v,v_1}(\omega) \leq\ & \frac{20}{\epsilon^4} \frac{D(F_\epsilon)}{M} \\
& + \frac{h^*(1) b_\infty 10 e^{1/\delta}}{\epsilon^2} |\langle G_\epsilon - 1\rangle| \mathbf{1}_{|\langle G_\epsilon\rangle - 1| > 1/2} \\
& + \frac{h^*(1) b_\infty 10 e^{1/\delta}}{\epsilon^2} \left(1 + |\langle G_\epsilon - 1\rangle| \mathbf{1}_{|\langle G_\epsilon\rangle - 1| \leq 1/2}\right) \\
& \leq I_1 + I_2 \,,
\end{aligned} \tag{7.9}$$

with

$$I_1 = \frac{20}{\epsilon^4} \frac{D(F_\epsilon)}{M} + \frac{h^*(1) b_\infty 10 e^{1/\delta}}{\epsilon^2} |\langle G_\epsilon - 1\rangle| \mathbf{1}_{|\langle G_\epsilon\rangle - 1| > 1/2} \tag{7.10}$$

and

$$I_2 = \frac{h^*(1) b_\infty 10 e^{1/\delta}}{\epsilon^2} \left(1 + |\langle G_\epsilon - 1\rangle| \mathbf{1}_{|\langle G_\epsilon\rangle - 1| \leq 1/2}\right) . \tag{7.11}$$

Because of the elementary inequality

$$|\langle G_\epsilon - 1\rangle| \mathbf{1}_{|\langle G_\epsilon - 1\rangle| > 1/2} \leq \epsilon |\langle g_\epsilon\rangle| (1 - \gamma(\langle G_\epsilon\rangle))$$

the dissipation bound (2.7) and the third entropy control in Proposition 2.1 imply that

$$I_1 = O\left(e^{1/\delta}\right)_{L^1_{loc}(dt; L^1(M dv dx))} . \tag{7.12}$$

On the other hand

$$I_2 = O\left(\frac{e^{1/\delta}}{\epsilon^2}\right)_{L^\infty_{t,x,v}} . \tag{7.13}$$

Using the inequality (7.9) with (7.12) and (7.13) leads to the announced estimate. □

7.4.2. Using the L^∞ bound in v. We state in the following lemma an important auxiliary estimate which is used below in two different ways.

Lemma 7.2. *Let $\phi \equiv \phi(t,x,v)$ be a measurable function defined a.e. on $\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$. Then, for each $t^* > 0$ the family of relative number density fluctuations g_ϵ satisfies the bound*

$$\left\| \frac{k_\delta^\epsilon}{\epsilon} M |g_\epsilon| |\phi| \right\|_{L^1([0,t^*];L^1_{x,v})} \leq C e^{1/2\delta} \|M\phi\|^{1/2}_{L^\infty_t(L^1_{x,v})} \|\phi\|^{1/2}_{L^\infty_{t,x,v}} \left\| \frac{D(F_\epsilon)}{\epsilon^4} \right\|^{1/2}_{L^1_{t,x,v}}$$
$$+ C \frac{e^{1/\delta}}{\epsilon} \|M\phi\|^{1/2}_{L^\infty_t(L^1_{x,v})} \|\phi\|^{1/2}_{L^\infty_{t,x}(L^1_v)} \left(\| {}^\flat g_\epsilon \|_{L^\infty_t(L^2(Mdvdx))} + \| {}^\sharp g_\epsilon \|^{1/2}_{L^\infty_t(L^1(Mdvdx))} \right).$$

Proof. Because of the inequality (2.17) used here with $C_k = 2e^{1/\delta}$

$$\frac{k_\delta^\epsilon}{\epsilon} M |g_\epsilon| |\phi| \leq 4 \sqrt{4 e^{1/\delta} \tfrac{3 b_\infty (c+c')}{cc'}} \sqrt{M} |\phi| \sqrt{\frac{D(F_\epsilon)}{\epsilon^4}}$$
$$+ 2 \frac{M}{\epsilon} |\phi| \left\langle | {}^\flat g_\epsilon | + 5 e^{1/\delta} \frac{1-\gamma(G_\epsilon)}{\epsilon} \right\rangle$$
$$+ \frac{1}{\epsilon} |\phi| \mathcal{A}^+ \left(M \left[5 | {}^\flat g_\epsilon | + 27 e^{1/\delta} \frac{1-\gamma(G_\epsilon)}{\epsilon} \right], M \right).$$

Using the second pointwise estimate in Proposition 2.2 and the second Caflisch–Grad estimate in Proposition 2.5 shows that, for some constant $C > 0$

(7.14)
$$\frac{k_\delta^\epsilon}{\epsilon} M |g_\epsilon| |\phi| \leq C e^{1/2\delta} \sqrt{M} |\phi| \sqrt{\frac{D(F_\epsilon)}{\epsilon^4}}$$
$$+ C e^{1/\delta} \frac{\sqrt{M}}{\epsilon} |\phi| \left((1+|v|)^{-3/2} + \sqrt{M} \right) \left(\| {}^\flat g_\epsilon \|_{L^2(Mdv)} + \| {}^\sharp g_\epsilon \|^{1/2}_{L^1(Mdv)} \right)$$

which implies the announced estimate by using the Cauchy–Schwarz inequality. □

7.4.3. The Advection/Dispersion interpolation estimate. First apply Lemma 7.2 with $\phi = \phi^1_\epsilon$. Because of estimates (7.4), (7.6) and (7.7), this leads, for each $t^* > 0$, to

$$
\begin{aligned}
\left\| \frac{k_\delta^\epsilon}{\epsilon} M |g_\epsilon| |\phi_\epsilon^1| \right\|_{L^1([0,t^*];L^1_{x,v})} &\le C e^{1/2\delta} O(\epsilon) \left\| \frac{D(F_\epsilon)}{\epsilon^4} \right\|^{1/2}_{L^1_{t,x,v}} \\
&\quad + C \frac{e^{1/\delta}}{\epsilon} O(\epsilon) O\left(\sqrt{\epsilon} e^{\frac{1}{8}V_\epsilon}\right) \\
&\quad \times \left(\| {}^\flat g_\epsilon \|_{L^\infty_t(L^2(Mdvdx))} + \| {}^\sharp g_\epsilon \|^{1/2}_{L^\infty_t(L^1(Mdvdx))} \right) \\
&\le C e^{1/2\delta} O(\epsilon) + C e^{1/\delta} O\left(\sqrt{\epsilon} e^{\frac{1}{8}V_\epsilon}\right).
\end{aligned} \tag{7.15}
$$

Now we use the Advection/Dispersion interpolation technique, as explained in Lemma 3.3. Let $t^* > 0$ and $\tau^* > 0$. Below, t^* is fixed while τ is the fictitious time used as interpolation parameter. Define $\phi_\epsilon = \phi_\epsilon(\tau, t, x, v)$ as the solution of the free transport equation (3.3) with initial data $\mathbf{1}_{[0,t^*]}(t)\phi_\epsilon^0(t, x, v)$. Observe that, for each $\tau \in [0, \tau^*]$,

$$
\begin{aligned}
\|\phi_\epsilon(\tau, \cdot, \cdot, \cdot)\|_{L^\infty_{t,x,v}} &= 1\,, \\
\|\phi_\epsilon(\tau, \cdot, \cdot, \cdot)\|_{L^\infty_t(L^1(Mdvdx))} &= O(\epsilon^2)\,,
\end{aligned} \tag{7.16}
$$

since the estimates on ϕ_ϵ that follow from (7.4) and (7.6) are obviously propagated by the free transport equation (3.3). Because of Lemma 3.2 and the first estimate in (7.7)

$$
\|\phi_\epsilon(\tau^*, \cdot, \cdot, \cdot)\|_{L^\infty_{t,x}(L^1_v)} = \frac{1}{\tau^{*\,3}} O\left(\epsilon e^{\frac{1}{4}V_\epsilon}\right). \tag{7.17}
$$

Using (7.16), (7.17), and applying Lemma 7.2 with $\phi = \phi_\epsilon(\tau^*, \cdot, \cdot, \cdot)$ shows that

$$
\begin{aligned}
\left\| \frac{k_\delta^\epsilon}{\epsilon} M |g_\epsilon| |\phi_\epsilon(\tau^*)| \right\|_{L^1([0,t^*];L^1_{x,v})} &\le C e^{1/2\delta} O(\epsilon) \left\| \frac{D(F_\epsilon)}{\epsilon^4} \right\|^{1/2}_{L^1_{t,x,v}} \\
&\quad + C \frac{e^{1/\delta}}{\epsilon} O(\epsilon) \frac{1}{\tau^{*\,3/2}} O\left(\sqrt{\epsilon} e^{\frac{1}{8}V_\epsilon}\right) \\
&\quad \times \left(\| {}^\flat g_\epsilon \|_{L^\infty_t(L^2(Mdvdx))} + \| {}^\sharp g_\epsilon \|^{1/2}_{L^\infty_t(L^1(Mdvdx))} \right) \\
&\le C e^{1/2\delta} O(\epsilon) + \frac{C e^{1/\delta}}{\tau^{*\,3/2}} O\left(\sqrt{\epsilon} e^{\frac{1}{8}V_\epsilon}\right).
\end{aligned} \tag{7.18}
$$

Finally, we apply Lemma 7.1 which, together with the first estimate in (7.16) implies that

$$
\begin{aligned}
&\int_\Omega |\phi_\epsilon| \left| (\epsilon \partial_t + v \cdot \nabla_x) \left(\frac{k_\delta^\epsilon}{\epsilon} |g_\epsilon| \right) \right| M dv dx dt d\tau \\
&\le \tau^* O\left(e^{1/\delta}\right) \|\phi_\epsilon\|_{L^\infty_{\tau,t,x,v}} + \tau^* O\left(\frac{e^{1/\delta}}{\epsilon^2}\right) \|\phi_\epsilon\|_{L^\infty_{\tau,t}(L^1(Mdvdx))} \\
&\le \tau^* O\left(e^{1/\delta}\right)
\end{aligned} \tag{7.19}
$$

where Ω is as in the statement of Lemma 3.3, i.e.

$$\Omega = \{(\tau, t, x, v) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^3 \times \mathbf{R}^3 \mid 0 < \tau < \tau^* ,\ 0 < t - \epsilon\tau < t^*\} .$$

Eventually, we use all three estimates (7.15), (7.18) and (7.19), as explained in Sect. 3 before Proposition 3.4. This leads to

$$\begin{aligned}
&\int_0^{t^*} \int \left\langle \frac{k_\delta^\epsilon}{\epsilon} |g_\epsilon| (1 - \gamma(G_\epsilon)) \mathbf{1}_{|v|^2 \le V_\epsilon} \right\rangle dxdt \\
&\quad \le \int_0^{t^*} \int \left\langle \frac{k_\delta^\epsilon}{\epsilon} |g_\epsilon| \phi_\epsilon^1 \right\rangle dxdt + \int_0^{t^*} \int \left\langle \frac{k_\delta^\epsilon}{\epsilon} |g_\epsilon| \phi_\epsilon^0 \right\rangle dxdt \\
&\quad \le \int_0^{t^*} \int \left\langle \frac{k_\delta^\epsilon}{\epsilon} |g_\epsilon| \phi_\epsilon^1 \right\rangle dxdt + \int_{\epsilon\tau^*}^{t^*+\epsilon\tau^*} \int \left\langle \frac{k_\delta^\epsilon}{\epsilon} |g_\epsilon| \phi_\epsilon(\tau^*) \right\rangle dxdt \\
&\quad + \int_\Omega |\phi_\epsilon| \left| (\epsilon\partial_t + v \cdot \nabla_x) \left(\frac{k_\delta^\epsilon}{\epsilon} |g_\epsilon| \right) \right| M dv dx dt d\tau \\
&\quad \le C e^{1/2\delta} O(\epsilon) + C e^{1/\delta} O\left(\sqrt{\epsilon} e^{\frac{1}{8} V_\epsilon} \right) \\
&\quad + C e^{1/2\delta} O(\epsilon) + \frac{C e^{1/\delta}}{\tau^{*\,3/2}} O\left(\sqrt{\epsilon} e^{\frac{1}{8} V_\epsilon} \right) + \tau^* O\left(e^{1/\delta} \right) .
\end{aligned} \tag{7.20}$$

The second inequality in (7.20) follows from applying Lemma 3.3 to $f = \frac{1}{\epsilon} |k_\delta^\epsilon| g_\epsilon$ and $\phi = \phi_\epsilon$.

7.5. Step 5: Final estimate. We conclude the proof of Proposition 3.4 by using the three estimates (7.2), (7.3) and finally (7.20). Let $t^* > 0$ and Q be a compact subset of $\mathbf{R}^3$; one has

$$\begin{aligned}
\int_0^{t^*} \int_Q \langle |\,^\sharp g_\epsilon| \rangle dxdt &\le \int_0^{t^*} \int_Q \left\langle \frac{1 - k_\delta^\epsilon}{\epsilon} |g_\epsilon| (1 - \gamma(G_\epsilon)) \right\rangle dxdt \\
&+ \int_0^{t^*} \int_Q \left\langle \frac{k_\delta^\epsilon}{\epsilon} |g_\epsilon| (1 - \gamma(G_\epsilon)) \mathbf{1}_{|v|^2 > V_\epsilon} \right\rangle dxdt \\
&+ \int_0^{t^*} \int_Q \left\langle \frac{k_\delta^\epsilon}{\epsilon} |g_\epsilon| (1 - \gamma(G_\epsilon)) \mathbf{1}_{|v|^2 \le V_\epsilon} \right\rangle dxdt \\
&\le C^{in} \frac{\delta t^*}{1 - \delta} + O\left(\frac{e^{1/\delta}}{V_\epsilon^{p/2}} \right) \\
&+ C e^{1/2\delta} O(\epsilon) + C e^{1/\delta} O\left(\sqrt{\epsilon} e^{\frac{1}{8} V_\epsilon} \right) \\
&+ C e^{1/2\delta} O(\epsilon) + \frac{C e^{1/\delta}}{\tau^{*\,3/2}} O\left(\sqrt{\epsilon} e^{\frac{1}{8} V_\epsilon} \right) + \tau^* O\left(e^{1/\delta} \right) .
\end{aligned} \tag{7.21}$$

In this last estimate, pick

$$\delta = \frac{1}{\log|\log\epsilon|} \text{ so that } e^{1/\delta} = |\log\epsilon|\,,$$

with

$$V_\epsilon = 2|\log\epsilon| \text{ so that } e^{\frac{1}{8}V_\epsilon} = \left(\frac{1}{\epsilon}\right)^{1/4}\;;$$

finally pick

$$\tau^* = \epsilon^{1/5}e^{\frac{1}{20}V_\epsilon} = \epsilon^{1/10}$$

and set $p > 2$. Substituting these values in (7.21) results in

$$\|\,^\sharp g_\epsilon\|_{L^1([0,t^*]\times Q;L^1(Mdv))} \le C\left(\frac{1}{\log|\log\epsilon|} + |\log\epsilon|^{1-p/2} + \epsilon^{1/10}|\log\epsilon|\right)$$

which concludes the proof of Proposition 3.4.

7.6. Proof of Corollary 3.5. Because of Proposition 3.4 and the second pointwise estimate in Proposition 2.2, one has

$$\frac{1-\gamma(G_\epsilon)}{\epsilon^2} = O\left(\frac{1}{\log|\log\epsilon|}\right) \text{ in } L^1_{loc}(dtdx;\, L^1(Mdv))\,.$$

On the other hand, by the last statement of Proposition 2.7

$$\frac{1-\gamma(G_\epsilon)}{\epsilon^2}(1+|v|)^{2s} = O(1) \text{ in } L^1_{loc}(dtdx;\, L^1(Mdv))$$

for all $s > 0$. The Cauchy–Schwarz inequality and these last two controls imply that

$$\frac{1-\gamma(G_\epsilon)}{\epsilon^2}(1+|v|)^{s} = O\left(\frac{1}{\sqrt{\log|\log\epsilon|}}\right) \text{ in } L^1_{loc}(dtdx;\, L^1(Mdv))\,,$$

as announced.

8. Proving Proposition 3.8

8.1. Uniform integrability of $|\,^\flat g_{\epsilon_n}|^2$. Pick $\gamma \in \Upsilon$. Since F_ϵ is a renormalized solution of (1.13) relatively to M, using the nonlinear function $\Gamma(Z) = (Z-1)^2\gamma(Z)^2$ in the renormalized formulation (1.19) gives

$$(8.1)\qquad (\epsilon\partial_t + v\cdot\nabla_x)\,^\flat g_\epsilon^2 = 2\iint q_\epsilon\,^\flat g_\epsilon \hat\gamma_\epsilon b\,d\sigma_{v,v_1}(\omega)M_1dv_1\,,$$

with $\hat{\gamma}$ defined in terms of the truncation γ by (4.2). By Lemma 5.4, for each $t^* > 0$,

$$\begin{aligned}(8.2)\qquad &\|(\epsilon\partial_t + v\cdot\nabla_x)\,\flat g_\epsilon^2\|_{L^1([0,t^*];L^1(Mdvdx))}\\ &\qquad\le 2\|\hat{\gamma}\|_{L^\infty}\|q_\epsilon\,\flat g_\epsilon\|_{L^1([0,t^*];L^1(d\mu dx))} = O(1)\,.\end{aligned}$$

On the other hand, the first statement of Proposition 2.7 expresses that, for each sequence $\epsilon_n \to 0$, the extracted sequence $M|\flat g_{\epsilon_n}|^2$ is locally uniformly integrable in v. This and (8.2) imply that this extracted sequence is in fact locally uniformly integrable on $\mathbf{R}_+\times\mathbf{R}^3\times\mathbf{R}^3$ in all variables t, x and v, by Lemma 3.6.

Further $(1+|v|)^s\,\flat g_\epsilon$ is bounded in $L^2_{loc}(dtdx; L^2(Mdv))$ for all $s\ge 0$ according to the second statement in Proposition 2.7. This and the local uniform integrability of the sequence $M|\flat g_{\epsilon_n}|^2$ with respect to the Lebesgue measure on $\mathbf{R}_+\times\mathbf{R}^3\times\mathbf{R}^3$ implies that, for each compact $Q\subset\mathbf{R}_+\times\mathbf{R}^3$, the sequence $\mathbf{1}_Q(t,x)|\flat g_{\epsilon_n}(t,x,v)|^2$ is uniformly integrable on $\mathbf{R}_+\times\mathbf{R}^3\times\mathbf{R}^3$ with respect to the measure $dtdxMdv$, which is precisely the first statement in Proposition 3.8.

8.2. Strong compactness of moments in the x-variable. Because F_ϵ is a renormalized solution of (1.13) relative to M, using this time the nonlinear function $\Gamma(Z) = (Z-1)\gamma(Z)$ in the renormalized formulation (1.19) gives

$$(8.3)\qquad (\epsilon\partial_t + v\cdot\nabla_x)\,\flat g_\epsilon = \iint q_\epsilon\hat{\gamma}_\epsilon b d\sigma_{v,v_1}(\omega)M_1dv_1\,.$$

with $\hat{\gamma}$ defined in terms of γ by (4.2). Observe that, denoting $N_\epsilon = \frac{2}{3}+\frac{1}{3}G_\epsilon$

$$(8.4)\qquad |q_\epsilon\hat{\gamma}_\epsilon| = \left(\tfrac{2}{3}+\tfrac{1}{3}G_\epsilon\right)|\hat{\gamma}(G_\epsilon)|\,\frac{|q_\epsilon|}{N_\epsilon} \le \tfrac{7}{6}\|\hat{\gamma}\|_{L^\infty}\frac{|q_\epsilon|}{N_\epsilon}$$

since $\hat{\gamma}$ is supported in $[\frac{1}{2},\frac{3}{2}]$. As recalled in the last statement in Theorem 11.1 (see Appendix B below), the family q_ϵ/N_ϵ is relatively compact in w-$L^1_{loc}(dtdx; L^1((1+|v|^2)d\mu))$. This and the inequality (8.4) imply that the family $q_\epsilon\hat{\gamma}_\epsilon$ also is relatively compact in w-$L^1_{loc}(dtdx; L^1(d\mu))$, which implies in turn that the family

$$\iint q_\epsilon\hat{\gamma}_\epsilon b d\sigma_{v,v_1}(\omega)M_1dv_1 \text{ is relatively compact in } w\text{-}L^1_{loc}(dtdx; L^1(Mdv)).$$

By Dunford–Pettis' theorem, this and (8.3) eventually imply that

$$\begin{gathered}(\epsilon_n\partial_t + v\cdot\nabla_x)\,\flat g_{\epsilon_n} \text{ is locally uniformly integrable in } [0,t^*]\times\mathbf{R}^3\times\mathbf{R}^3\\ \text{with respect to the measure } dtdxMdv\,.\end{gathered}$$

This and the first statement in Proposition 3.8 entail the second statement in that same proposition by applying Lemma 3.7.

8.3. Proof of Corollary 3.9. Consider the operator $|\mathcal{L}|$ defined by[8]

$$|\mathcal{L}|\phi = \iint (\phi + \phi_1 + \phi' + \phi_1')b(v - v_1, \omega)M_1 d\sigma_{v,v_1}(\omega)dv_1 \,.$$

Using the elementary inequality $ab \leq \frac{1}{2}(a^2 + b^2)$ repeatedly, one sees that

$$(8.5) \qquad |{}^\flat g_\epsilon \mathcal{L}\, {}^\flat g_\epsilon| \leq 2b_\infty \,{}^\flat g_\epsilon^2 + \tfrac{1}{2}|\mathcal{L}|({}^\flat g_\epsilon^2)\,, \quad |\mathcal{Q}({}^\flat g_\epsilon, {}^\flat g_\epsilon)| \leq \tfrac{1}{2}|\mathcal{L}|({}^\flat g_\epsilon^2)\,.$$

Pick a compact set $Q \subset \mathbf{R}_+ \times \mathbf{R}^3$ and a sequence $\epsilon_n \to 0$. By the first part of Proposition 3.8, the sequence $(t, x, v) \mapsto \mathbf{1}_Q(t, x)\, {}^\flat g_{\epsilon_n}(t, x, v)^2$ (henceforth denoted $\mathbf{1}_Q\, {}^\flat g_{\epsilon_n}^2$) is uniformly integrable in $\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$ with respect to the measure $dtdxMdv$. By Dunford–Pettis' theorem it is relatively compact in w-$L^1(dtdxMdv)$. On the other hand, the same argument as in the proof of Proposition 1.5 shows that the linear operator $|\mathcal{L}|$ is continuous in $L^1(Mdv)$. Therefore the sequence $\mathbf{1}_Q|\mathcal{L}|({}^\flat g_{\epsilon_n}^2)$ is also relatively compact in w-$L^1(dtdxMdv)$. Applying Dunford–Pettis' theorem again implies that this sequence is uniformly integrable in $\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3$ with respect to the measure $dtdxMdv$. With the inequalities (8.5), this implies the result in Corollary 3.9.

9. Conclusions

The proof of the Navier–Stokes limit of the Boltzmann equation presented in this work can be extended in a number of ways.

First, the case of periodic flows (i.e. of the spatial domain $\mathbf{T}^3 = \mathbf{R}^3/\mathbf{Z}^3$ as in [7]) may be treated after only slight modifications. Indeed, the compactness results based on dispersion estimates in the present analysis concern *local* L^p spaces.

It is possible to handle a more general class of collision kernels than considered by following the same general strategy with, however, very significant modifications. The case of a hard-sphere gas is of particular importance, since it is so far the only case for which the Boltzmann equation has been rigorously derived from Newtonian mechanics – by O. Lanford [39]. In various applications, it would also be important to extend the convergence results in the present paper to the case of both hard and soft cutoff potentials proposed by H. Grad [32] (see also [18]). In general, neither hard nor soft cutoff potentials satisfy our assumption (H1); this can be remedied at the expense of rather technical modifications of our arguments in Sects. 4-8. As for assumption (H2), its role in the proof of Proposition 5.1 is crucial. As mentioned above, (H2) is thus far known to hold with $p = 3$ for cutoff Maxwell molecules only. Whether this assumption (H2) is satisfied by hard-sphere and more general cutoff potentials remains an open

[8] This is not to be confused with the common usage for this notation, i.e. the nonnegative self-adjoint operator in the polar decomposition of $\mathcal{L}$ which is of no interest here since $\mathcal{L}$ is itself a nonnegative self-adjoint operator.

problem as of now. An alternative to establishing the validity of (H2) for all cutoff potentials would consist of trying to merge the methods leading to Propositions 2.7, 3.4 and 3.8 with those used by C.D. Levermore and N. Masmoudi [41] to treat conservation defects. This however remains an open problem, as the analysis in [41] makes essential use of a variant of the nonlinear control (A2), slightly weaker than (A2) but which we still do not know how to establish by the methods of the present paper, except possibly in the case of soft cutoff potentials

Another natural extension of the present work would be to treat boundary value problems when the spatial domain is a smooth open set in $\mathbf{R}^3$. This issue has been rather systematically studied at the formal level by using Hilbert's expansion, for instance by Y. Sone [60]. In view of the importance of this problem for applications to the dynamics of rarefied gases, it would be extremely desirable to confirm these formal asymptotic results by mathematical proofs. This problem can certainly be solved by the methods presented in this paper combined with those developed by N. Masmoudi and L. Saint-Raymond in [52].

Still another natural extension of our results would be to treat the case of two-dimensional flows (more precisely, 2 and 1/2-dimensional flows in the terminology of [44], p. 151). By this, we mean the case where the number density F_ϵ is a function of two space variables only but of all three velocity variables: e.g. $F_\epsilon \equiv F(t, x_1, x_2, v_1, v_2, v_3)$. This is not a particular case of the theory presented here, since such densities have infinite relative entropy with respect to any uniform Maxwellian state. However, our method carries over to this case by considering as relative entropy the quantity

$$\frac{1}{\epsilon^2}\iint \left[F_\epsilon \log\left(\frac{F_\epsilon}{M}\right) - F_\epsilon + M \right] dv_1 dv_2 dv_3 dx_1 dx_2$$

instead of (1.15) where the integral bears on *all* space variables. The Navier–Stokes limit in this setting leads to velocity fields that have three components but depend on only two space variables, i.e. 2 and 1/2-dimensional flows. What is usually known as a two-dimensional flow is the particular case of a 2 and 1/2-dimensional flow where the third component of the velocity field is identically zero initially, and therefore remains so for all subsequent times.

This case is interesting because global classical solutions to the two-dimensional Navier–Stokes equations are known to exist without restrictions on the size of the initial data, at variance with the three-dimensional case: see for instance [44], p. 83. Moreover, weak solutions of the two-dimensional Navier–Stokes equations are known to be uniquely determined by their initial data and satisfy the energy equality – the inequality (1.28) becomes an equality for weak solutions of the two-dimensional Navier–Stokes equations (see for instance Theorem 3.1 in [44], p. 81).

The special properties of two-dimensional flows have interesting consequences on the hydrodynamic limit:

- in the two-dimensional case, the analogue of Theorem 1.6 shows that the family $\frac{1}{\epsilon}\int vF_\epsilon dv$ converges in w-$L^1_{loc}(\mathbf{R}_+ \times \mathbf{R}^2))$ to the (unique) weak solution of the two-dimensional Navier–Stokes equations with initial data u^{in} defined in (1.45); likewise
- in the two-dimensional case, the analogue of Theorem 1.9 holds without regularity assumptions on the limiting velocity field, provided that the initial data u^{in} defined in (1.45) is smooth – indeed the two-dimensional Navier–Stokes equations propagate the smoothness of the initial data.

In any case, two dimensional flows can be derived in this way from the Boltzmann equation, in contrast with the derivation of the Navier–Stokes equations from the lattice gas considered by J. Quastel and H.-T. Yau [54]. Whether this is a spurious feature of their particle model, or is due to the fact that some fluctuations at the level of the particle dynamics are discarded in the description by the Boltzmann equation is not completely clear.

Finally, a few comments on the role of weak solutions in this work are in order.

It has been repeatedly asserted that both renormalized solutions of the Boltzmann equation and Leray solutions of the Navier–Stokes equations are physically unsatisfying, because to this date, they are not known to be uniquely determined by their initial data and thus have no predictive value. While this might cast doubts on the soundness of the program outlined in [7], we insist that the reason for considering weak solutions in this program is *not* that they are the only ones known to exist for all time and initial data of arbitrary size. A more crucial reason is that the only a priori estimates known to this date on the scaled Boltzmann equation (1.13) that are uniform as the Knudsen number ϵ tends to 0 come from the DiPerna-Lions entropy inequality (1.21). This inequality holds for all renormalized solutions of (1.13) and yields the Leray energy inequality (1.28) in the limit as $\epsilon \to 0$. Hence, Leray solutions of the Navier–Stokes system and renormalized solutions of the Boltzmann equation should be viewed as the natural objects to which uniform a priori estimates apply rather than a source of spurious technical difficulties caused by our lack of knowledge on the regularity of solutions to the Navier–Stokes or Boltzmann equations in three space dimensions.

To illustrate this, suppose we are given a family F_ϵ^{in} of initial data leading for each $\epsilon > 0$ to a classical solution F_ϵ of the scaled Boltzmann equation (1.13) that satisfies the local conservation laws of momentum and energy. Suppose in addition that F_ϵ^{in} converges in the strongest possible sense – and at least entropically at rate ϵ – to $M_{(1,u^{in},1)}$ with u^{in} a smooth, divergence-free initial velocity field leading to a classical solution u of the Navier–Stokes equations.

In order to prove for instance that

$$\frac{1}{\epsilon}\int vF_\epsilon dv \to u$$

in the weakest possible sense – at least in the sense of distributions – as $\epsilon \to 0$, it does not seem that one can use the local conservation law of momentum satisfied by F_ϵ for each $\epsilon > 0$. Indeed, passing to the limit as $\epsilon \to 0$ in the momentum flux

$$\frac{1}{\epsilon^2} \int \left(v^{\otimes 2} - \tfrac{1}{3}|v|^2 I\right) F_\epsilon dv$$

involves in particular a term of the form

$$\int \left(v^{\otimes 2} - \tfrac{1}{3}|v|^2 I\right) \mathcal{Q}(g_\epsilon, g_\epsilon) M dv$$

(where g_ϵ is the relative fluctuation of F_ϵ about M defined in (1.34)). All that we know about the family g_ϵ from the entropy inequality (1.21) is that g_ϵ is relatively compact in $w\text{-}L^1_{loc}(dtdx; w\text{-}L^1(Mdv))$, not in $w\text{-}L^2_{loc}(dtdx; w\text{-}L^1(Mdv))$, and this is not enough to guarantee that

$$\int \left(v^{\otimes 2} - \tfrac{1}{3}|v|^2 I\right) \mathcal{Q}(g_\epsilon, g_\epsilon) M dv \to u \otimes u - \tfrac{1}{3}|u|^2 I$$

in the sense of distributions as $\epsilon \to 0$. Apparently, the only way around this consists of replacing g_ϵ with its L^2 part $^\flat g_\epsilon$ in the Flat-Sharp decomposition (2.4). At this point, the benefit of knowing that F_ϵ satisfies the local conservation laws of momentum and energy is lost. Therefore, one has to deal with exactly the same conservation defects as in Sect. 4; the proof of convergence must follow essentially the same steps as in Sect. 5 and Sects. 6 and 7, and so the main technical burden in the present paper cannot be dispensed with. Likewise, although the global weak solutions to the BGK model constructed by B. Perthame [53] satisfy the local conservation law of momentum, the only derivation known to this date of the Navier–Stokes equations from that model in [56] uses a renormalized form of the BGK equation – in other words, renormalization is used in taking the hydrodynamic limit even though it is not needed to define the solution of the kinetic model. This leads to estimating conservation defects in the same manner as in the present paper.

Similar difficulties arise if one tries to use Boltzmann's H Theorem and apply instead Yau's relative entropy method; the proof of convergence would also require all the controls in Propositions 3.4 and 3.8; see the work of F. Golse, C.D. Levermore and L. Saint-Raymond [26].

Thus, to summarize this discussion, it seems doubtful that dealing with classical (instead of weak) solutions would simplify in any significant way the proof of the Navier–Stokes limit of the Boltzmann equation.

10. Appendix A. Young's inequality

The functions $h : [-1, +\infty[\to \mathbf{R}_+$ and $r :]-1, +\infty[\to \mathbf{R}_+$ in (2.2), (2.8) are both strictly convex, and satisfy, for all $z > -1$,

(10.1) $$h(|z|) \leq h(z)\,, \quad r(|z|) \leq r(z)\,, \quad h(z) \leq r(z)\,.$$

The Legendre transform of h is defined for all $p \in \mathbf{R}$ by

$$h^*(p) = \sup_{z>-1} (pz - h(z)) = e^p - p - 1\,; \tag{10.2}$$

that of r is also defined for all $p \in \mathbf{R}$ by the implicit relation

$$r^*(p) = \sup_{z>-1} (pz - r(z)) = \frac{z^2}{1+z}\,, \text{ with } \log(1+z) + \frac{z}{1+z} = p\,.$$

Further, the Legendre transform h^* is super-quadratic, i.e.

$$h^*(\eta p) \leq \eta^2 h^*(p)\,, \quad p \in \mathbf{R}_+\,,\ \eta \in [0,1]\,. \tag{10.3}$$

Finally Young's inequality states that, for all $p \in \mathbf{R}$, $z > -1$ and $\eta \in [0, 1]$,

$$p|z| \leq \frac{1}{\eta} h(z) + \eta h^*(p) \leq \frac{1}{\eta} r(z) + \eta h^*(p)\,. \tag{10.4}$$

11. Appendix B. A compendium of results from [7]

Some of the result in [7] were established in the greatest possible generality, and in particular did not use any of the assumptions left unverified there. We have recorded them below without proof; they are used in various places in the present work. While these statements were established in the case of a spatial domain equal to the torus, the proofs from [7] can be adapted to the spatial domain $\mathbf{R}^3$.

Theorem 11.1. *Under assumptions (H1)–(H2), let F_ϵ be a family of renormalized solutions to (1.13) with initial data F_ϵ^{in} satisfying (1.14), and define the associated family of fluctuations by*

$$g_\epsilon = \frac{F_\epsilon - M}{\epsilon M}\,.$$

Then

- g_ϵ *is relatively compact in w-$L^1_{loc}(dtdx; L^1((1+|v|^2)Mdv))$ and all its limit points as $\epsilon \to 0$ are local infinitesimal Maxwellians*

  $$g(t,x,v) = \rho(t,x) + u(t,x)\cdot v + \theta(t,x)\tfrac{1}{2}(|v|^2 - 3) \tag{11.1}$$

  where the velocity field u satisfies the incompressibility condition

  $$\nabla_x \cdot u = 0 \tag{11.2}$$

  while the fluctuations of macroscopic density and temperature satisfy the Boussinesq relation

  $$\nabla_x(\rho + \theta) = 0 \text{ which implies that } \rho + \theta = 0\,; \tag{11.3}$$

- *the rescaled collision integrands*

$$q_\epsilon = \frac{1}{\epsilon^2}(G'_\epsilon G'_{\epsilon 1} - G_\epsilon G_{\epsilon 1}) \tag{11.4}$$

are such that the renormalized family $\gamma(G_\epsilon)q_\epsilon$ *is relatively compact in* $w\text{-}L^1_{loc}(dtdx; L^1((1+|v|^2)d\mu))$*; further, any of the limit points* q *of* $\gamma(G_\epsilon)q_\epsilon$ *as* $\epsilon \to 0$ *satisfies the* $d\mu$*-symmetry relations*

$$\langle\!\langle \phi(v) q \rangle\!\rangle = \langle\!\langle \tfrac{1}{4}(\phi + \phi_1 - \phi' - \phi'_1) q \rangle\!\rangle ; \tag{11.5}$$

- *for any subsequence* $\epsilon_n \to 0$ *such that*

$$g_{\epsilon_n} \to g \text{ and } \gamma(G_{\epsilon_n})q_{\epsilon_n} \to q$$

in $w\text{-}L^1_{loc}(dtdx; L^1((1+|v|^2)Mdv))$ *and in* $w\text{-}L^1_{loc}(dtdx; L^1((1+|v|^2)d\mu))$ *respectively, the limits* g *and* q *satisfy the limiting Boltzmann equation*

$$v \cdot \nabla_x g = \iint q b(v - v_1, \omega) d\sigma_{v,v_1}(\omega) M_1 dv_1 ; \tag{11.6}$$

- *for any limit point* g *(of the form (11.1)) of the family of fluctuations* g_ϵ *as* $\epsilon \to 0$*, and for each* $t > 0$*, one has*

$$\begin{aligned} &\tfrac{1}{2}\int (\rho(t,x)^2 + |u(t,x)|^2 + \tfrac{3}{2}\theta(t,x)^2)dx \\ &\quad + \tfrac{1}{2}\int_0^t \int (\nu|\nabla_x u + (\nabla_x u)^T|^2 + 5\kappa|\nabla_x\theta|^2)dxds \\ &\quad \le \varliminf_{\epsilon\to 0} \int \left\langle \frac{1}{\epsilon^2} h(\epsilon g^{in}_\epsilon)\right\rangle dx . \end{aligned} \tag{11.7}$$

- *denoting* $N_\epsilon = \frac{2}{3} + \frac{1}{3}G_\epsilon$, g_ϵ/N_ϵ *is bounded in* $L^\infty_t(L^2(Mdvdx))$ *and* q_ϵ/N_ϵ *is relatively compact in* $w\text{-}L^1_{loc}(dtdx; L^1((1+|v|^2)d\mu))$.

12. Appendix C. Velocity averaging

Proposition 12.1. *Let* ϕ_ϵ *be a bounded family of* $L^2_{loc}(dtdx; L^2(Mdv))$ *indexed by* $\epsilon \in [0,1]$ *such that both families* $|\phi_\epsilon|^2$ *and* $(\epsilon\partial_t + v\cdot\nabla_x)\phi_\epsilon$ *are locally uniformly integrable with respect to the measure* $Mdvdxdt$*. Then, for each function* $\psi \equiv \psi(v)$ *in* $L^2(Mdv)$*, each* $t^* > 0$ *and each compact* $Q \subset \mathbf{R}^3$*, there exists a function* $\eta : \mathbf{R}_+ \to \mathbf{R}_+$ *such that* $\lim_{z\to 0^+}\eta(z) = 0$

$$\left\| \int \phi_\epsilon(t, x+y, v)\psi(v)M(v)dv - \int \phi_\epsilon(t,x,v)\psi(v)M(v)dv \right\|_{L^2([0,t^*]\times Q)} \le \eta(|y|)$$

for each $y \in \mathbf{R}^3$ *such that* $|y| \le 1$*, uniformly in* $\epsilon \in [0,1]$.

Proof. Since $C_c(\mathbf{R}^3)$ is dense in $L^2(Mdv)$, there exists a sequence $\psi_n \in C_c(\mathbf{R}^3)$ such that $\|\psi - \psi_n\|_{L^2(Mdv)} \to 0$ as $n \to +\infty$. Since the family ϕ_ϵ is bounded in $L^2_{loc}(dtdx; L^2(Mdv))$, for each $t^* > 0$ and each compact $Q \subset \mathbf{R}^3$,

$$\left\| \int \phi_\epsilon(t,x)(\psi(v) - \psi_n(v))Mdv \right\|_{L^2([0,t^*]\times Q)} \leq \sup_{\epsilon\in[0,1]} \|\phi_\epsilon\|_{L^2([0,t^*]\times Q; L^2(Mdv))} \|\psi - \psi_n\|_{L^2(Mdv)} \to 0$$

uniformly in $\epsilon \in [0,1]$ as $n \to +\infty$. Thus we can assume without loss of generality that $\psi \in C_c(\mathbf{R}^3)$ and that all the ϕ_ϵ are supported in some compact set $K \subset \mathbf{R}^*_+ \times \mathbf{R}^3 \times \mathbf{R}^3$; thus we henceforth consider ϕ_ϵ as defined on $\mathbf{R} \times \mathbf{R}^3 \times \mathbf{R}^3$. Let $\lambda > 0$; define

$$\phi_\epsilon + (\epsilon\partial_t + v\cdot\nabla_x)\phi_\epsilon = \Phi_\epsilon\,.$$

The assumptions made on ϕ_ϵ guarantee that the family Φ_ϵ is uniformly integrable with respect to the measure $Mdvdxdt$. Proceeding as in the proof of Theorem 3 of [27] (see pp. 115–116), one shows first that

(12.1)
$$\left\| \int \phi_\epsilon(t,x+y,v)\psi(v)M(v)dv - \int \phi_\epsilon(t,x,v)\psi(v)M(v)dv \right\|_{L^1_{t,x}} \to 0$$

uniformly in $\epsilon \in [0,1]$ as $|y| \to 0$.

It remains to prove that the same convergence holds in $L^2_{t,x}$. We therefore split

(12.2)
$$\begin{aligned}
&\iint \left| \int [\phi_\epsilon(t,x+y,v) - \phi_\epsilon(t,x,v)]\psi(v)M(v)dv \right|^2 dxdt \\
&\leq \iint \rho_\epsilon^<(t,x+y) \left| \int [\phi_\epsilon(t,x+y,v) - \phi_\epsilon(t,x,v)]\psi(v)M(v)dv \right| dxdt \\
&+ \iint \rho_\epsilon^<(t,x) \left| \int [\phi_\epsilon(t,x+y,v) - \phi_\epsilon(t,x,v)]\psi(v)M(v)dv \right| dxdt \\
&+ \iint \rho_\epsilon^>(t,x+y) \left(\int [|\phi_\epsilon(t,x+y,v)| + |\phi_\epsilon(t,x,v)|]\psi(v)M(v)dv \right) dxdt \\
&+ \iint \rho_\epsilon^>(t,x) \left(\int [|\phi_\epsilon(t,x+y,v)| + |\phi_\epsilon(t,x,v)|]\psi(v)M(v)dv \right) dxdt
\end{aligned}$$

with the notations

$$\rho_\epsilon^<(t,x) = \int \left|\phi_\epsilon \mathbf{1}_{|\phi_\epsilon|\leq\lambda}\right|(t,x,v)|\psi|Mdv$$

$$\rho_\epsilon^>(t,x) = \int \left|\phi_\epsilon \mathbf{1}_{|\phi_\epsilon|>\lambda}\right|(t,x,v)|\psi|Mdv\,.$$

The first and second integrals on the right-hand side of the inequality (12.2) are less than

(12.3)
$$\lambda\langle|\psi|\rangle \left\| \int \phi_\epsilon(t,x+y,v)\psi(v)M(v)dv - \int \phi_\epsilon(t,x,v)\psi(v)M(v)dv \right\|_{L^1_{t,x}} ,$$

while the third and the fourth are less than

$$2\langle|\psi|^2\rangle \|M^{1/2}\phi_\epsilon \mathbf{1}_{|\phi_\epsilon|>\lambda}\|_{L^2_{t,x,v}} \|M^{1/2}\phi_\epsilon\|_{L^2_{t,x,v}} .$$

This term vanishes as $\lambda \to +\infty$ uniformly in $\epsilon \in [0,1]$ since the family $|\phi_\epsilon|^2$ is uniformly integrable with respect to the measure $Mdvdxdt$. On the other hand, for any fixed $\lambda > 0$, the first and second integrals in (12.2) vanish as $|y| \to 0$ uniformly in $\epsilon \in [0,1]$ by (12.3) and (12.1). This concludes the proof. □

13. Appendix D. Compensated compactness for acoustic waves

We recall below the elegant argument proposed by P.-L. Lions and N. Masmoudi to establish the incompressible limit of the compressible Navier–Stokes equations [49]. A similar result had been obtained earlier by a somewhat different method: see [33] and [57].

Lemma 13.1. *Let $c \neq 0$. Consider two families ϕ_ϵ and ψ_ϵ bounded in $L^\infty_{loc}(dt; L^2_{loc}(dx))$ and in $L^\infty_{loc}(dt; H^1_{loc}(\mathbf{R}^3))$ respectively, such that*

$$\partial_t\phi_\epsilon + \frac{1}{\epsilon}\Delta_x\psi_\epsilon = \frac{1}{\epsilon}F_\epsilon ,$$
$$\partial_t\nabla_x\psi_\epsilon + \frac{c^2}{\epsilon}\nabla_x\phi_\epsilon = \frac{1}{\epsilon}G_\epsilon ,$$

where $F_\epsilon \to 0$ and $G_\epsilon \to 0$ in $L^1_{loc}(dt; L^2_{loc}(dx))$. Then

$$P\nabla_x\cdot\left((\nabla_x\psi_\epsilon)^{\otimes 2}\right) \to 0 , \text{ and } \nabla_x\cdot(\phi_\epsilon\nabla_x\psi_\epsilon) \to 0$$

in the sense of distributions on $\mathbf{R}^*_+ \times \mathbf{R}^3$.

Proof. By elementary computations,

$$\begin{aligned}\nabla_x\cdot\left((\nabla_x\psi_\epsilon)^{\otimes 2}\right) &= \tfrac{1}{2}\nabla_x\left(|\nabla_x\psi_\epsilon|^2\right) + \nabla_x\psi_\epsilon\Delta_x\psi_\epsilon \\ &= \tfrac{1}{2}\nabla_x\left(|\nabla_x\psi_\epsilon|^2 - c^2|\phi_\epsilon|^2\right) - \partial_t\left(\epsilon\phi_\epsilon\nabla_x\psi_\epsilon\right) \\ &\quad + F_\epsilon\nabla_x\psi_\epsilon + \phi_\epsilon G_\epsilon ,\end{aligned}$$

and, likewise

$$\begin{aligned}c^2\nabla_x\cdot(\phi_\epsilon\nabla_x\psi_\epsilon) &= c^2\phi_\epsilon\Delta\psi_\epsilon + c^2\nabla_x\phi_\epsilon\cdot\nabla_x\psi_\epsilon \\ &= -\tfrac{1}{2}\partial_t\left(\epsilon(c^2|\phi_\epsilon|^2 + |\nabla_x\psi_\epsilon|^2)\right) + c^2\phi_\epsilon F_\epsilon + G_\epsilon\cdot\nabla_x\psi_\epsilon .\end{aligned}$$

The bounds assumed on ϕ_ϵ and ψ_ϵ ensure that the families $\epsilon\phi_\epsilon\nabla_x\psi_\epsilon$, $\epsilon(c^2|\phi_\epsilon|^2+|\nabla_x\psi_\epsilon|^2)$, $F_\epsilon\nabla_x\psi_\epsilon$, $G_\epsilon\cdot\nabla_x\psi_\epsilon$, $\phi_\epsilon F_\epsilon$ and $\phi_\epsilon G_\epsilon$ all vanish with ϵ in $L^1_{loc}(dtdx)$, which, together with the two elementary formulas above, implies the announced convergence. □

14. Appendix E. Consequences of the Dunford–Pettis theorem

Let $(X,\mathcal{M})$ be a measurable space and μ be a positive measure on X such that $\mu(X)<+\infty$.

In [24] (Theorem 3.2.1, p. 376), N. Dunford and B.J. Pettis gave a criterion for subsets of $L^1(X,\mu)$ to be weakly relatively (sequentially) compact. In the following lemma we apply their result to study the weak L^1-continuity of certain bilinear expressions.

Lemma 14.1. *Let a_n and b_n be two sequences of real-valued measurable functions defined (a.e.) on X, such that a_n is bounded in $L^\infty(X,\mu)$ and $b_n\to b$ in w-$L^1(X,\mu)$ as $n\to+\infty$.*

- *Assume that $a_n\to a$ in measure as $n\to+\infty$; then $a_nb_n\to ab$ in w-$L^1(X,\mu)$ as $n\to+\infty$.*
- *Assume that $a_n\to 0$ in measure as $n\to+\infty$; then $a_nb_n\to 0$ strongly in $L^1(X,\mu)$ as $n\to+\infty$.*

Proof. We first prove the second assertion. For each $\epsilon>0$ and each $n\in\mathbf{N}$, let $A(n,\epsilon)=\{x\in X\mid |a_n(x)|>\epsilon\}$. Then

$$\begin{aligned}&\int_X|a_n(x)b_n(x)|d\mu(x)\\&\quad=\int_{A(n,\epsilon)}|a_n(x)b_n(x)|d\mu(x)+\int_{A(n,\epsilon)^c}|a_n(x)b_n(x)|d\mu(x)\\&\quad\le\sup_{k\ge0}\|a_k\|_{L^\infty}\int_{A(n,\epsilon)}|b_n(x)|d\mu(x)+\epsilon\sup_{k\ge0}\|b_k\|_{L^1}\end{aligned}$$

Since a_n converges to 0 in measure, $\mu(A(n,\epsilon))\to0$ as $n\to+\infty$. On the other hand, the sequence b_n converges weakly in w-$L^1(X,\mu)$, and thus is uniformly integrable on X by the Dunford–Pettis theorem. Therefore,

$$\int_{A(n,\epsilon)}|b_n(x)|d\mu(x)\to0$$

as $n\to+\infty$. This and the previous inequality imply that

$$\overline{\lim_{n\to+\infty}}\int_X|a_n(x)b_n(x)|d\mu(x)\le\epsilon\sup_{k\ge0}\|b_k\|_{L^1}\,.$$

Since b_n is a weakly convergent sequence in $L^1(X,\mu)$, it is bounded in $L^1(X,\mu)$. The inequality above, which holds for each $\epsilon>0$, shows that $\|a_nb_n\|_{L^1}\to0$ as $n\to+\infty$.

Next we prove the first assertion. Write $a_n b_n = (a_n - a)b_n + ab_n$. Observe that $a \in L^\infty(X, \mu)$: indeed the sequence a_n is bounded in $L^\infty(X, \mu)$ and there exists a subsequence a_{n_k} of a_n that converges to a a.e. on X. By the second assertion $(a_n - a)b_n \to 0$ in $L^1(X, \mu)$; on the other hand, $ab_n \to ab$ in w-$L^1(X, \mu)$ as $n \to +\infty$ since $a \in L^\infty(X, \mu)$ and $b_n \to b$ in w-$L^1(X, \mu)$. Hence $a_n b_n \to ab$ in w-$L^1(X, \mu)$ as $n \to +\infty$. □

Lemma 14.1 is a slight amplification of Appendix B in [7] – with the notion of convergence in measure replacing that of a.e. convergence. The first assertion in this Lemma is also a consequence of Proposition 1 of [34], p. 222.

Acknowledgements. We express our gratitude to L. Desvillettes and C. Villani who read a first version of this article in detail, and to C. Bardos, C.D. Levermore, N. Masmoudi and the referee, whose suggestions helped us clarify some of our arguments. We also thank H.S. Dumas for correcting our English. Finally we both acknowledge the support of the European Research Training Network HyKE (contract HPRN-CT-2002-00282).

References

1. Agoshkov, V.: Space of functions with differential difference characteristics and smoothness of solutions of the transport equation. Dokl. Akad. Nauk SSSR **276**, 1289–1293 (1984)
2. Arkeryd, L., Nouri, A.: A compactness result related to the stationary Boltzmann equation in a slab with applications to the existence theory. Indiana Univ. Math. J. **44**, 815–839 (1995)
3. Asano, K.: Conference at the 4th International Workshop on Mathematical Aspects of Fluid and Plasma Dynamics. Kyoto 1991
4. Bardos, C., Degond, P.: Global existence for the Vlasov-Poisson equation in 3 space variables with small initial data. Ann. Inst. H. Poincaré, Anal. Non Linéaire **2**, 101–118 (1985)
5. Bardos, C., Golse, F., Levermore, D.: Sur les limites asymptotiques de la théorie cinétique conduisant à la dynamique des fluides incompressibles. C. R. Acad. Sci., Paris **309**, 727–732 (1989)
6. Bardos, C., Golse, F., Levermore, D.: Fluid Dynamic Limits of Kinetic Equations I: Formal Derivations. J. Stat. Phys. **63**, 323–344 (1991)
7. Bardos, C., Golse, F., Levermore, C.D.: Fluid Dynamic Limits of Kinetic Equations II: Convergence Proofs for the Boltzmann Equation. Commun. Pure Appl. Math. **46**, 667–753 (1993)
8. Bardos, C., Golse, F., Levermore, C.D.: Acoustic and Stokes Limits for the Boltzmann Equation. C. R. Acad. Sci., Paris **327**, 323–328 (1999)
9. Bardos, C., Golse, F., Levermore, C.D.: The Acoustic Limit for the Boltzmann Equation. Arch. Ration. Mech. Anal. **153**, 177–204 (2000)
10. Bardos, C., Levermore, C.D.: Kinetic Equations and an Incompressible Fluid Dynamical Limit that Recovers Viscous Heating. In preparation (2003)
11. Bardos, C., Ukai, S.: The Classical Incompressible Navier–Stokes Limit of the Boltzmann Equation. Math. Models Methods Appl. Sci. **1**, 235–257 (1991)
12. Lagha-Benabdallah, A.: Limites des équations d'un fluide compressible lorsque la compressibilité tend vers zéro. (French) Fluid dynamics (Varenna, 1982), 139–165. Lecture Notes Math. **1047**. Berlin: Springer 1984
13. Bouchut, F.: Hypoelliptic regularity in kinetic equations. J. Math. Pures Appl. **9**, 313–327 (2002)

14. Bouchut, F., Desvillettes, L.: A proof of the smoothing properties of the positive part of Boltzmann's kernel. Rev. Mat. Iberoam. **14**, 47–61 (1998)
15. Bouchut, F., Golse, F., Pulvirenti, M.: Kinetic Equations and Asymptotic Theory, ed. by B. Perthame, L. Desvillettes, Series in Applied Mathematics **4**. Paris: Gauthier-Villars 2000
16. Caflisch, R.: The Boltzmann equation with a soft potential. I. Linear, spatially-homogeneous. Commun. Math. Phys. **74**, 71–95 (1980)
17. Castella, F., Perthame, B.: Estimations de Strichartz pour les équations de transport cinétique. C. R. Acad. Sci., Paris, Sér. I, Math. **322**, 535–540 (1996)
18. Cercignani, C.: Mathematical Methods in Kinetic Theory. New York: Plenum Press 1990
19. Cercignani, C., Illner, R., Pulvirenti, M.: The Mathematical Theory of Dilute Gases. New York: Springer 1994
20. Constantin, P., Foias, C.: Navier–Stokes Equations. Chicago Lectures in Mathematics. Chicago: The University of Chicago Press 1988
21. DeMasi, A., Esposito, R., Lebowitz, J.: Incompressible Navier–Stokes and Euler Limits of the Boltzmann Equation. Commun. Pure Appl. Math. **42**, 1189–1214 (1990)
22. Desvillettes, L., Golse, F.: On a model Boltzmann equation without angular cutoff. Differ. Integral Equ. **13**, 567–594 (2000)
23. DiPerna, R.J., Lions, P.-L.: On the Cauchy Problem for the Boltzmann Equation: Global Existence and Weak Stability Results. Ann. Math. **130**, 321–366 (1990)
24. Dunford, N., Pettis, B.J.: Linear Operations on Summable Functions. Trans. Am. Math. Soc. **47**, 323–392 (1940)
25. Golse, F., Levermore, C.D.: The Stokes-Fourier and Acoustic Limits for the Boltzmann Equation. Commun. Pure Appl. Math. **55**, 336–393 (2002)
26. Golse, F., Levermore, C.D., Saint-Raymond, L.: La méthode de l'entropie relative pour les limites hydrodynamiques de modèles cinétiques. Séminaire: Equations aux Dérivées Partielles, 1999–2000, Exp. No. XIX, 23 pp., Ecole Polytech., Palaiseau 2000
27. Golse, F., Lions, P.-L., Perthame, B., Sentis, R.: Regularity of the Moments of the Solution of a Transport Equation. J. Funct. Anal. **76**, 110–125 (1988)
28. Golse, F., Perthame, B., Sentis, R.: Un résultat de compacité pour les équations de transport et application au calcul de la limite de la valeur propre principale de l'opérateur de transport. C. R. Acad. Sci., Paris **301**, 341–344 (1985)
29. Golse, F., Saint-Raymond, L.: The Navier–Stokes limit for the Boltzmann equation. C. R. Acad. Sci., Paris **333**, 897–902 (2001)
30. Golse, F., Saint-Raymond, L.: Velocity averaging in L^1 for the transport equation. C. R. Acad. Sci., Paris **334**, 557–562 (2002)
31. Golse, F., Saint-Raymond, L.: Work in preparation
32. Grad, H.: Asymptotic theory of the Boltzmann equation. II. 1963 Rarefied Gas Dynamics (Proc. 3rd Internat. Sympos., Palais de l'UNESCO, Paris, 1962), Vol. I, pp. 26–59
33. Grenier, E.: Fluides en rotation et ondes d'inertie. C. R. Acad. Sci., Paris **321**, 711–714 (1995)
34. Grothendieck, A.: Topological vector spaces. New York: Gordon Breach 1973
35. Hilbert, D.: Mathematical Problems. International Congress of Mathematicians, Paris 1900. Translated and reprinted in Bull. Am. Math. Soc. **37**, 407–436 (2000)
36. Klainerman, S., Majda, A.: Singular limits of quasilinear hyperbolic systems with large parameters and the incompressible limit of compressible fluids. Commun. Pure Appl. Math. **34**, 481–524 (1981)
37. Lachowicz, M.: On the initial layer and the existence theorem for the nonlinear Boltzmann equation. Math. Methods Appl. Sci. **9**, 342–366 (1987)
38. Landau, L., Lifshitz, E.: Course of theoretical physics. Vol. 6. Fluid mechanics. Oxford: Pergamon Press 1987
39. Lanford, O.: Time evolution of large classical systems. In: "Dynamical systems, theory and applications" (Rencontres, Battelle Res. Inst., Seattle, Wash., 1974), pp. 1–111. Lecture Notes Phys. **38**. Berlin: Springer 1975

40. Leray, J.: Sur le mouvement d'un fluide visqueux emplissant l'espace. Acta Math. **63**, 193–248 (1934)
41. Levermore, C.D., Masmoudi, N.: From the Boltzmann Equation to an Incompressible Navier–Stokes–Fourier System. Work in preparation
42. Lions, J.-L.: Théorèmes de trace et d'interpolation I, II. Ann. Sc. Norm. Super. Pisa, Cl. Sci., IV. Ser. **13**, 389–403 (1959); **14**, 317–331 (1960)
43. Lions, P.-L.: Conditions at infinity for Boltzmann's equation. Commun. Partial. Differ. Equations **19**, 335–367 (1994)
44. Lions, P.-L.: Mathematical Topics in Fluid Mechanics, Vol. 1: Incompressible Models. Oxford Lecture Series in Mathematics and its Applications, 3. Oxford Science Publications. New York: The Clarendon Press, Oxford University Press 1996
45. Lions, P.-L.: Compactness in Boltzmann's equation via Fourier integral operators and applications I. J. Math. Kyoto Univ. **34**, 391–427 (1994)
46. Lions, P.-L.: Compactness in Boltzmann's equation via Fourier integral operators and applications II. J. Math. Kyoto Univ. **34**, 429–461 (1994)
47. Lions, P.-L.: Compactness in Boltzmann's equation via Fourier integral operators and applications III. J. Math. Kyoto Univ. **34**, 539–584 (1994)
48. Lions, P.-L.: On Boltzmann and Landau equations. Philos. Trans. R. Soc. Lond., Ser. A, Math. Phys. Eng. Sci. **346**, 191–204 (1994)
49. Lions, P.-L., Masmoudi, N.: Une approche locale de la limite incompressible. (French) C. R. Acad. Sci., Paris, Sér. I, Math. **329**, 387–392 (1999)
50. Lions, P.-L., Masmoudi, N.: From Boltzmann Equations to Navier–Stokes Equations I. Arch. Ration. Mech. Anal. **158**, 173–193 (2001)
51. Lions, P.-L., Masmoudi, N.: From Boltzmann Equations to the Stokes and Euler Equations II. Arch. Ration. Mech. Anal. **158**, 195–211 (2001)
52. Masmoudi, N., Saint-Raymond, L.: From the Boltzmann equation to the Stokes-Fourier system in a bounded domain. To appear in Commun. Pure Appl. Math.
53. Perthame, B.: Global Existence to the BGK Model of the Boltzmann Equation. J. Differ. Equations **82**, 191–205 (1989)
54. Quastel, J., Yau, H.-T.: Lattice gases, large deviations, and the incompressible Navier–Stokes equations. Ann. Math. (2) **148**, 51–108 (1998)
55. Saint-Raymond, L.: Discrete time Navier–Stokes limit for the BGK Boltzmann equation. Commun. Partial Differ. Equations **27**, 149–184 (2002)
56. Saint-Raymond, L.: From the Boltzmann BGK Equation to the Navier–Stokes System. Ann. Sci. Éc. Norm. Supér., IV. Sér. **36**, 271–317 (2003)
57. Schochet, S.: Fast singular limits of hyperbolic PDEs. J. Differ. Equations **114**, 476–512 (1994)
58. Simon, J.: Compact sets in the space $L^p(0, T; B)$. Ann. Mat. Pura Appl., IV. Ser. **146**, 65–96 (1987)
59. Sone, Y.: Asymptotic Theory of Flow of a Rarefied Gas over a Smooth Boundary II. In: Rarefied Gas Dynamics. Vol. II, pp. 737–749, ed. by D. Dini. Pisa: Editrice Tecnico Scientifica 1971
60. Sone, Y.: Kinetic Theory and Fluid Mechanics. Boston: Birkhäuser 2002
61. Toscani, G., Villani, C.: Sharp entropy dissipation bounds and explicit rate of trend to equilibrium for the spatially homogeneous Boltzmann equation. Commun. Math. Phys. **203**, 667–706 (1999)
62. Villani, C.: Limites hydrodynamiques de l'équation de Boltzmann [d'après C. Bardos, F. Golse, D. Levermore, Lions, P.-L., N. Masmoudi, L. Saint-Raymond]. Séminaire Bourbaki, vol. 2000–2001, Exp. 893