

## Regularity of the Moments of the Solution of a Transport Equation

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Let  $u = u(x, v)$  satisfy the Transport Equation  $u + v \cdot \partial_x u = f$ ,  $x \in \mathbb{R}^N$ ,  $v \in \mathbb{R}^N$ , where  $f$  belongs to some space of type  $L^p(dx \otimes d\mu(v))$  (where  $\mu$  is a positive bounded measure on  $\mathbb{R}^N$ ). We study the resulting regularity of the moment  $\int u(x, v) d\mu(v)$  (in terms of fractional Sobolev spaces, for example). Counter-examples are given in order to test the optimality of our results. © 1988 Academic Press, Inc.

### I. INTRODUCTION

We are concerned with the regularity of the mean value (with respect to the velocity) of the solution of Transport Equations. Let  $u$  be the solution of

$$u + v \cdot \partial_x u = f, \quad x \in \mathbb{R}^N, v \in \mathbb{R}^N,$$

where  $f = f(x, v)$  is a given function. Assume that  $f$  belongs to some space of the type  $L^p(dx \otimes d\mu(v))$ , where  $\mu$  is a positive bounded measure on  $\mathbb{R}^N$ . The very fact we are examining in this paper is the following: generally speaking, the quantity  $\int u(x, v) d\mu(v)$  is more regular than  $u(\cdot, v)$  for any fixed  $v$ : integrating with respect to  $v$  brings some regularity in the  $x$ -dependence. This remark has already been formulated in terms of a compactness lemma [GPS] (see also [DL]). In this article, we present a more complete analysis of the following question: knowing the regularity of  $f$  (i.e., that  $f$  belongs to some  $L^p(dx \otimes d\mu(v))$ ), what is the resulting regularity for

$\int u(x, v) d\mu(v)$ ? In particular, the answer to this question will provide a generalization of the compactness lemma stated in [GPS].

The main result in this paper is the following

**THEOREM.** *Assume that there exists a positive constant  $C$  such that*

$$\text{Sup}_{e \in S^{N-1}} \text{ess } \mu(\{v \in \mathbb{R}^N / |v \cdot e| \leq \varepsilon\}) \leq C\varepsilon \quad \text{for all } \varepsilon > 0.$$

*Then, the operator  $f \mapsto \int u(x, v) d\mu(v)$  is continuous from  $L^2(dx \otimes d\mu(v))$  into  $H^{1/2}(\mathbb{R}^N)$ .*

The proof relies on a Fourier analysis of the cancellation of singularities for the operator  $f \rightarrow \int u(x, v) d\mu(v)$ , very similar to the one already used in [GPS]. The very principle of this proof seems to be the following.

Let  $\xi$  denote the Fourier variable dual to  $x$ , and let  $C(v)$  be a revolution cone in the  $\xi$ -space, centered of  $v \in \mathbb{R}^N \setminus \{0\}$ . We can produce an estimate of  $|\mathcal{F}(u(\cdot, v))|_{H^1(C(v))}$  in terms of  $\text{Sup}_{\xi \in \mathbb{R}^N} \mu(\{v \text{ s.t. } \xi \notin C(v)\})$ .

From our analysis of the  $L^2$ -case, we shall derive the “general” case (i.e., when  $f \in L^p(dx \otimes d\mu(v))$  with  $1 < p < +\infty$ ) by interpolation. Severe pathologies arise when  $p = 1$  or  $p = +\infty$ . However, we are able to produce weak compactness results when  $f$  belongs to some space of type  $L^1(dx; L^p(d\mu(v)))$  or  $L^1(d\mu(v); L^p(dx))$ , with  $p > 1$ , by solving the Transport Equation for  $\int u(x, v) d\mu(v)$  in terms of  $f$ , integrating  $u$  along the characteristics.

Generalizations of these results may be of some help to understand approximations of kinetic equations [BGPS].

The outline of this paper is as follows: in Section 2, we prove various generalizations of the above theorem; in Section 3, we study weak compactness results; and Section 4 is devoted to counterexamples, and the special case where  $x$  lives in a one-dimensional space.

## II. REGULARITY RESULTS

### 1. The Case of $\mathbb{R}^N$

Let  $\mu$  be a positive measure on  $\mathbb{R}^N$  satisfying the condition

$$\text{There exist two constants } C > 0 \text{ and } 0 < \gamma < 2 \text{ such that} \\ \text{Sup}_{e \in S^{N-1}} \text{ess } \mu(\{v \in \mathbb{R}^N / |v \cdot e| \leq \varepsilon\}) \leq C\varepsilon^\gamma \quad \text{for all } \varepsilon > 0. \tag{2.1}$$

Throughout this article, we use the notation

$$\tilde{f} = \int f(v) d\mu(v),$$

for any  $f \in L^1(d\mu(v))$ . In the sequel, we shall denote by  $C$  various positive constants. Our main result is

**THEOREM 1.** *Assume (2.1). Let  $u = u(x, v)$  be such that  $u$  and  $v \cdot \partial_x u$  both belong to  $L^2(dx \otimes d\mu(v))$ . Then the moment  $\tilde{u}(x) = \int u(x, v) d\mu(v)$  belongs to  $H^{\gamma/2}$ , and we have the inequality*

$$\begin{aligned} & \left( \iint |\tilde{u}(x) - \tilde{u}(y)|^2 |x - y|^{N+\gamma} dx dy \right)^{1/2} \\ & \leq C(\|u\|_{L^2})^{1-\gamma/2} \cdot (\|v \cdot \partial_x u\|_{L^2})^{\gamma/2}. \end{aligned} \quad (2.2)$$

*Proof of Theorem 1.* Let  $\xi$  denote the Fourier variable dual to  $x$ ; we define  $\varphi(\cdot, v)$  as the Fourier transform (with respect to  $x$ ) of  $u(\cdot, v)$ . The assumption on  $u$  may be formulated as

$$\varphi \text{ and } (v \cdot \xi) \varphi \text{ belong to } L^2(d\xi \otimes d\mu(v)). \quad (2.3)$$

In the sequel of the proof, we shall use the following lemma:

**LEMMA.** *Let  $\nu$  be a positive bounded measure on  $\mathbb{R}$  satisfying the condition*

$$\nu([- \varepsilon, \varepsilon]) \leq C\varepsilon^\gamma. \quad (2.4)$$

*Then we have*

$$\int_x^\infty d\nu(x)/x^2 \leq C\alpha^{\gamma-2}. \quad (2.5)$$

*Proof of the Lemma.* Integrating by parts we have

$$\int_x^A d\nu(s)/s^2 = [\psi(s)/(s^2)]_x^A + 2 \int_x^A \psi(s) ds/s^3.$$

In the above equality, we then use the fact that  $\psi(s) \leq Cs^\gamma$  (by (2.4)), and we let  $A$  go to  $+\infty$ : thus we obtain (2.5). ■

Let us go back to the proof of Theorem 1. We obviously have

$$\int |\xi|^\gamma \left| \int \varphi(\xi, v) d\mu(v) \right|^2 d\xi \leq 2 \int \left| \int_{|v \cdot \xi| \geq \alpha} |\xi|^{\gamma/2} \varphi(\xi, v) d\mu(v) \right|^2 d\xi \\ + 2 \int \left| \int_{|v \cdot \xi| < \alpha} |\xi|^{\gamma/2} \varphi(\xi, v) d\mu(v) \right|^2 d\xi. \quad (2.6)$$

By Cauchy–Schwarz inequality, we obtain

$$\left| \int_{|v \cdot \xi| \geq \alpha} |\xi|^{\gamma/2} \varphi(\xi, v) d\mu(v) \right|^2 \\ \leq \left( \int_{|v \cdot \xi| \geq \alpha} |\xi|^\gamma / |v \cdot \xi|^2 d\mu(v) \right) \cdot \left( \int |v \cdot \xi|^2 |\varphi(\xi, v)|^2 d\mu(v) \right). \quad (2.7)$$

Moreover, we can write

$$\int_{|v \cdot \xi| \geq \alpha} |\xi|^\gamma / |v \cdot \xi|^2 d\mu(v) = |\xi|^{\gamma-2} \int_{|v \cdot \xi| \geq \alpha} d\mu(v) / (v \cdot \xi / |\xi|)^2 \\ \leq C |\xi|^{\gamma-2} (\alpha / |\xi|)^{\gamma-2} = C \alpha^{\gamma-2},$$

according to (2.5) applied to the image of  $\mu$  by the orthogonal projection on the direction  $\xi$  (see assumption (2.1)). Substituting this into (2.7) yields

$$\left| \int_{|v \cdot \xi| \geq \alpha} |\xi|^{\gamma/2} \varphi d\mu(v) \right|^2 \leq C \alpha^{\gamma-2} \int |v \cdot \xi|^2 |\varphi|^2 d\mu(v). \quad (2.8)$$

Applying again Cauchy–Schwarz inequality, we have

$$\left| \int_{|v \cdot \xi| < \alpha} |\xi|^{\gamma/2} \varphi(\xi, v) d\mu(v) \right|^2 \\ \leq \left( \int_{|v \cdot \xi| < \alpha} |\xi|^\gamma d\mu(v) \right) \cdot \left( \int_{|v \cdot \xi| < \alpha} |\varphi|^2 d\mu(v) \right), \quad (2.9)$$

and

$$\int_{|v \cdot \xi| < \alpha} |\xi|^\gamma d\mu(v) = |\xi|^\gamma \int_{|v \cdot \xi| < \alpha} d\mu(v) \leq C \alpha^\gamma$$

by (2.1). Therefore

$$\left| \int_{|v \cdot \xi| < \alpha} |\xi|^{\gamma/2} \varphi(\xi, v) d\mu(v) \right|^2 \leq C \alpha^\gamma \int |\varphi|^2 d\mu(v). \quad (2.10)$$

Hence, with (2.8) and (2.10), (2.6) yields

$$\int |\xi|^\gamma \left| \int \varphi(\xi, v) d\mu(v) \right|^2 d\xi \leq C\alpha^{\gamma-2} \iint |v \cdot \xi|^2 |\varphi(\xi, v)|^2 d\mu(v) d\xi + C\alpha^\gamma \iint |\varphi(\xi, v)|^2 d\mu(v) d\xi. \tag{2.11}$$

This is valid for any positive  $\alpha$ . Therefore, by choosing

$$\alpha = \left( \iint |v \cdot \xi|^2 |\varphi(\xi, v)|^2 d\mu(v) d\xi \right)^{1/2} \left( \iint |\varphi(\xi, v)|^2 d\mu(v) d\xi \right)^{-1/2},$$

we obtain

$$\int |\xi|^\gamma |\varphi(\xi)|^2 d\xi \leq C \left( \iint |v \cdot \xi|^2 |\varphi(\xi, v)|^2 d\mu(v) d\xi \right)^{\gamma/2} \times \left( \iint |\varphi(\xi, v)|^2 d\mu(v) d\xi \right)^{1-\gamma/2}$$

Finally, using Plancherel's identity, and the (classical) inequality

$$\iint |\tilde{u}(x) - \tilde{u}(y)|^2 |x - y|^{N+\gamma} dx dy \leq C \int |\xi|^\gamma |\varphi(\xi)|^2 d\xi,$$

we deduce (2.2) from the above inequality. ■

We can generalize Theorem 1 to any  $L^p$  space, with  $1 < p < \infty$ , as follows.

**THEOREM 2.** *Assume (2.1). Let  $u = u(x, v)$  be such that  $u$  and  $v \cdot \partial_x u$  both belong to  $L^p(dx \otimes d\mu(v))$ , with  $1 < p < \infty$ . Then the moment  $u(x) = \int u(x, v) d\mu(v)$  belongs to  $W^{s,p}$  for any  $s$  satisfying  $0 < s < \inf(1/p, 1 - 1/p) \gamma$ , and we have the inequality*

$$\left( \iint |\tilde{u}(x) - \tilde{u}(y)|^p |x - y|^{N+sp} dx dy \right)^{1/p} \leq C \|u\|_{L^p}^{1-s} \cdot \|v \cdot \partial_x u\|_{L^p}^s. \tag{2.12}$$

*Proof of Theorem 2.* For any  $1 \leq p \leq +\infty$ , we can define a bounded linear operator  $T$ , from  $L^p(dx \otimes d\mu(v))$  into  $L^p(dx)$ , by  $Tf = \tilde{u}$ , where  $u$  is the unique solution in  $L^p(dx \otimes d\mu(v))$  of the Transport Equation

$$u + v \cdot \partial_x u = f, \quad x \in \mathbb{R}^N, v \in \mathbb{R}^N.$$

According to Theorem 1,  $T$  is continuous from  $L^2(dx \otimes d\mu(v))$  into  $H^{\gamma/2}$ .

Therefore, by a classical interpolation result,  $T$  is also continuous from  $L^p(dx \otimes d\mu(v))$  into  $W^{s,p}$ , for any  $0 < s < \inf(1/p, 1 - 1/p) \gamma$  (see [BL, Tr]). In particular, we have the inequality

$$\left( \iint |\tilde{u}(x) - \tilde{u}(y)|^p |x - y|^{N+sp} dx dy \right)^{1/p} \leq C(\|u\|_{L^p} + \|v \cdot \partial_x u\|_{L^p}).$$

We apply this inequality to  $u_\lambda(x, v) = u(\lambda x, v)$  for any  $\lambda > 0$ ; after the change of variable  $x \mapsto \lambda x$ , we obtain

$$\left( \iint |u(x) - u(y)|^p |x - y|^{N+sp} dx dy \right)^{1/p} \leq C(\lambda^{-s} \|u\|_{L^p} + \lambda^{1-s} \|v \cdot \partial_x u\|_{L^p}),$$

which holds for any  $\lambda > 0$ . By choosing  $\lambda = \|u\|_{L^p} / \|v \cdot \partial_x u\|_{L^p}$ , we obtain (2.12). ■

Our method can also be applied to the  $L^1$  case, and yields the following result.

**PROPOSITION 3.** *Define the operator  $T$  from  $L^1(dx \otimes d\mu(v))$  into  $L^1(dx)$  by  $Tf = \tilde{u}$ , where  $u$  is the unique solution in  $L^1(dx \otimes d\mu(v))$  of the Transport Equation*

$$u + v \cdot \partial_x u = f, \quad x \in \mathbb{R}^N, v \in \mathbb{R}^N.$$

*If  $K \subset L^1(dx \otimes d\mu(v))$  is bounded and uniformly integrable, then  $T(K)$  is compact in  $L^1_{\text{loc}}(dx)$ .*

*Proof of Proposition 3.* Let  $R$  be the resolvent  $(1 + v \cdot \partial_x)^{-1}$  of the Transport operator in  $L^1(dx \otimes d\mu(v))$ . For any  $f \in K$ , and  $\alpha > 0$ , we define

$$\chi_{\alpha, f} = 1_{\{(x, v); |f(x, v)| < \alpha\}}, \quad \omega_{\alpha, f} = 1 - \chi_{\alpha, f}.$$

We can write

$$u = Rf = \varphi + \psi,$$

where  $\varphi = R(f \cdot \omega_{\alpha, f})$  and  $\psi = R(f \cdot \chi_{\alpha, f})$ . Clearly, we have

$$\int |\tilde{\varphi}(x+h) - \tilde{\varphi}(x)| dx \leq 2 \int |f \cdot \omega_{\alpha, f}| dx d\mu(v).$$

Since  $K$  is uniformly integrable, for any  $\varepsilon > 0$ , there exists  $\alpha > 0$  such that

$$\int |\tilde{\varphi}(x+h) - \tilde{\varphi}(x)| dx < \varepsilon,$$

for any  $f \in K$ , and any  $h \in \mathbb{R}^N$ . With the  $\alpha$  chosen above, and since  $K$  is bounded in  $L^1(dx \otimes d\mu(v))$ , the set  $\{f \cdot \chi_{\alpha, f}; f \in K\}$  is bounded in  $L^2(dx \otimes d\mu(v))$ . According to Theorem 1, the set  $\{\psi \text{ s.t. } \psi = R(f \cdot \chi_{\alpha, f}); f \in K\}$  is bounded in  $H^{7/2}$ . In particular, for any bounded set  $S$  of  $\mathbb{R}^N$ ,

$$\int_S |\tilde{\psi}(x+h) - \tilde{\psi}(x)| dx \rightarrow 0$$

when  $h \rightarrow 0$ , uniformly with respect to  $f \in K$ . By coupling this with the above analogous result on  $\varphi$ , we obtain that  $T(K)$  is compact in  $L^1_{\text{loc}}(dx)$ . In particular, if  $K$  is weakly compact in  $L^1(dx \otimes d\mu(v))$ , then  $T(K)$  is compact in  $L^1_{\text{loc}}(dx)$ . However, the operator  $T$  is not weakly compact (see Section IV).

### 2. Bounded Domains

Until now, we were dealing with functions  $u$  defined on the whole  $x$ -space  $\mathbb{R}^N$ . Here is a localized version of the above results.

Let  $X$  be a regular bounded convex open set in  $\mathbb{R}^N$ . We denote by  $d\Sigma$  the surface measure on  $\partial X$ , and by  $n(x)$  the unit outward normal vector to  $X$  at  $x \in \partial X$ . We define

$$\begin{aligned} \Gamma &= \partial X \times \mathbb{R}^N, \\ \Gamma_+ &= \{(x, v) \in \Gamma; n(x) \cdot v > 0\}, \\ \Gamma_- &= \{(x, v) \in \Gamma; n(x) \cdot v < 0\}, \\ \Gamma_0 &= \{(x, v) \in \Gamma; n(x) \cdot v = 0\}. \end{aligned}$$

Assumption (2.1) ensures the usual condition on the characteristic set  $\Gamma_0$ ,

$$\int_{\Gamma_0} d\Sigma(x) d\mu(v) = 0.$$

Let us denote by  $d\sigma$  the measure

$$d\sigma = |v \cdot n(x)| d\Sigma(x) d\mu(v).$$

We are now ready to give a result similar to Theorems 1 and 2, but for functions a priori defined in  $X$ .

**THEOREM 4.** *Assume (2.1). Let  $u = u(x, v)$  be such that  $u$  and  $v \cdot \partial_x u$  both belong to  $L^p(X \times \mathbb{R}^N; dx \otimes d\mu(v))$ , and  $u|_{\Gamma_-}$  belongs to  $L^p(\Gamma_-; d\sigma)$ , for  $1 < p < +\infty$ . Then the moment  $\int u(x, v) d\mu(v)$  belongs to  $W^{s,p}(X)$  with*

$s = \gamma/2$  if  $p = 2$  and  $0 < s < \inf(1/p, 1 - 1/p) \gamma$  if  $p \neq 2$ . Moreover, we have the inequality

$$\left| \int u(\cdot, v) d\mu(v) \right|_{W^{s,p}} \leq C \left( \|u\|_{L^p} + \|v \cdot \partial_x u\|_{L^p} + \left( \int_{\Gamma_-} |u|^p d\sigma \right)^{1/p} \right). \quad (2.13)$$

*Proof of Theorem 4.* It consists in proving that such a function  $u$  is the restriction to  $X$  of a function to which Theorems 1 and 2 can be applied. The key of the proof relies in the following lemma.

LEMMA. (Extension of  $u$ ). *Let us define for  $1 < p < +\infty$*

$$W^p(X) = \{u(x, v) \text{ s.t. } u \text{ and } v \cdot \partial_x u \text{ both belong to } L^p(X \times \mathbb{R}^N; dx \otimes d\mu(v))\};$$

$$W_-^p(X) = \{u \in W^p(X) \text{ s.t. } u|_{\Gamma_-} \in L^p(\Gamma_-; d\sigma)\}.$$

*There exists a continuous extension operator*

$$\Pi: W_-^p(X) \rightarrow W^p(\mathbb{R}^N) \quad (\text{i.e., } (\Pi u)|_X = u).$$

*Proof of the Extension Lemma.* Let  $u = u(x, v)$  be defined as the solution of

$$u + v \cdot \partial_x u = f \quad \text{in } X; \quad u|_{\Gamma_-} = g;$$

we know that  $u \in W_-^p(X)$  iff  $f \in L^p(X \times \mathbb{R}^N; dx \otimes d\mu(v))$  and  $g \in L^p(\Gamma_-; d\sigma)$ . Let us first extend  $f$  to  $\mathbb{R}^N \times \mathbb{R}^N$ :

$$F(x, v) = \begin{cases} f(x, v) & \text{if } x \in X, \\ 0 & \text{otherwise.} \end{cases}$$

We begin by defining a distribution  $U = U(x, v)$  on  $\mathbb{R}_x^N \times \mathbb{R}_v^N$  as follows. If  $(x, v) \notin \{(y + tv, v), y \in X, t \in \mathbb{R}^+\}$ , then  $U(x, v) = 0$ . Otherwise, there exists a unique  $\tau_{x,v} \in \mathbb{R}^+$  such that  $(x - \tau_{x,v}v, v) \in \Gamma_-$ . In this case, we set

$$U(x, v) = e^{-\tau_{x,v}} g(x - \tau_{x,v}v, v) + \int_0^{\tau_{x,v}} e^{-t} F(x - tv, v) dt.$$

Clearly,  $U$  satisfies

$$U + v \cdot \partial_x U = F$$

in the sense of distributions. Now, we shall define  $\Pi u$  as a smooth truncation of  $U$ , the truncation being dependent on  $X$ . The domain  $X$  being

bounded, let us pick a positive  $R$  such that  $X \subset B_R$ . Let us define  $\varphi \in \mathcal{D}(\mathbb{R}^N)$  in the following way:

$$\begin{aligned} \varphi &= 0 && \text{outside } B_{2R}; \\ \varphi &= 1 && \text{in } B_R; \\ 0 &\leq \varphi \leq 1. \end{aligned}$$

Then we define

$$(\Pi u)(x, v) = \varphi(x) U(x, v).$$

It is easy to check that  $\Pi$  is a continuous extension operator from  $W_-^p(X)$  into  $W^p(\mathbb{R}^N)$ . To prove Theorem 4, it is enough to apply Theorems 1–2 to  $\Pi u$ . ■

*Remarks.* (1) The extension lemma that we present here was first proved by Cessenat, following previous results on the trace spaces associated to  $W^p(X)$  (see [C, DL]); we have given here a self-contained proof for the sake of completeness.

(2) In Theorem 4, we could have prescribed  $u|_{\Gamma_+}$  to be in  $L^p(\Gamma_+; d\sigma)$  instead of the same condition on  $u|_{\Gamma_-}$ . Anyway, according to [C], if

$$W_+^p(X) = \{u \in W^p(X) \text{ s.t. } u|_{\Gamma_+} \in L^p(\Gamma_+; d\sigma)\},$$

we know that  $W_-^p(X) = W_+^p(X)$ .

From Theorems 1–4, we derive the following compactness result:

**COROLLARY 5.** *Assume (2.1). Then, for any  $p$  such that  $1 < p < \infty$ , the operator  $T$  defined by*

$$Tu = \int u(\cdot, v) d\mu(v)$$

*is compact from  $W_{\pm}^p(X)$  into  $L^p(X; dx)$ , and from  $W^p(X)$  into  $L^p(\omega; dx)$ , for any  $\omega$  such that  $\omega \subset X$ .*

*Proof of Corollary 5.* To prove that  $T$  is compact from  $W_{\pm}^p(X)$  into  $L^p(X; dx)$ , we only have to apply the Rellich–Kondrachov theorem [A]. Then if  $\omega$  is such that  $\omega \subset X$ , we can define  $\psi \in \mathcal{D}(\mathbb{R}_x^N)$  such that

$$\begin{aligned} \psi &= 1 && \text{in } \omega; \\ \psi &= 0 && \text{outside } X; \\ 0 &\leq \psi \leq 1. \end{aligned}$$

We then define  $u(x, v) = \psi(x) u(x, v)$ ,  $u \in W^p(\mathbb{R}^N)$ , so that we can apply Theorems 1–2 to  $u$ . We conclude with the Rellich–Kondrachov theorem. ■

*Remarks.* (1) In the case of space dimension 1 (i.e.,  $N = 1$ ), the regularity results presented in Theorems 1–4 are far from being optimal. This will be discussed in Section IV.

(2) So far, the Transport operators considered in Theorems 1–4 were “stationary.” But evolution Transport operators can also be treated within the same framework. Indeed, we only have to notice that for  $u = u(t, x, v)$ , where  $x$  and  $v$  belong to  $\mathbb{R}^N$  and  $t$  belongs to  $\mathbb{R}$ , the condition

$$(\partial_t + v \cdot \partial_x) u \in L^p(dt \otimes dx \otimes d\mu(v))$$

is equivalent to

$$v' \cdot \partial_{x'} u \in L^p(dx' \otimes d\mu'(v')),$$

where  $x' = (t, x)$ ,  $v' = (v'', v)$  with  $v'' \in \mathbb{R}$ , and  $d\mu'(v') = \delta_1 \otimes d\mu(v)$ . Therefore, Theorems 1–4 can be applied to evolution Transport operators mutatis mutandis. In particular, we have to check that the measure  $\mu'$  satisfies (2.1), which means that  $\mu$  itself has to satisfy

$$\sup_{\substack{e \in \mathbb{R}^N, e'' \in \mathbb{R} \\ |e|^2 + e''^2 = 1}} \text{ess } \mu\{v \in \mathbb{R}^N; -e'' - \varepsilon \leq v \cdot e \leq -e'' + \varepsilon\} \leq C\varepsilon^\gamma. \quad (2.14)$$

(3) In Theorem 4, it is not necessary to assume that  $u$  belongs to  $L^p(X \times \mathbb{R}^N; dx \otimes d\mu(v))$ . Indeed

$$W_{\pm}^p(X) = \{u(x, v) \text{ s.t. } v \cdot \partial_x u \in L^p(X \times \mathbb{R}^N; dx \otimes d\mu(v)) \text{ and } u|_{\Gamma_{\pm}} \in L^p(\Gamma_{\pm}; d\sigma)\}$$

for any bounded convex open set in  $\mathbb{R}^N$ .

### III. WEAK COMPACTNESS RESULTS

Now, we try to extend the above regularity and compactness results to cases where the above ideas, namely coupling the use of a Fourier analysis of singularities with standard interpolation theorems, can no longer be applied. In this section, we shall always assume that  $N > 1$ ,  $\Omega$  will denote an arbitrary vector of  $S^{N-1}$ , and  $M^p$  will denote the Marcinkiewicz space (see [BB, BL]).<sup>1</sup>

<sup>1</sup>  $M^p = L^{p, \infty}$  in terms of Lorentz spaces.

For any  $f \in L^1(dx \otimes d\mu(v))$ , there is a unique solution  $u_f \in L^1(dx \otimes d\mu(v))$  of

$$u + v \cdot \partial_x u = f \tag{3.1}$$

which can be represented as

$$u_f(x, v) = \int_0^\infty e^{-s} f(x - sv, v) ds. \tag{3.1}'$$

PROPOSITION 6. Assume that  $d\mu(|v|\Omega) \leq dv(|v|\Omega) \otimes d\Omega$ , where  $d\Omega$  is the uniform surface measure on  $S^{N-1}$ , and  $v$  a positive bounded measure on  $\mathbb{R}^+$  such that

$$\int_0^\infty dv(x)/x < +\infty.$$

Then for any  $p, 1 < p \leq +\infty$ , the operator  $T: f \rightarrow \int u_f(\cdot, v) d\mu(v)$  is continuous from  $L^1(dx; L^p(d\mu(v)))$  into  $M^r$ , where

$$r = Np/(1 + (N - 1) p).$$

*Proof of Proposition 6.* First assume that  $p = +\infty$ . We have

$$\int u_f(x, v) d\mu(v) \leq \int dv(|v|) \int_0^\infty \int_{S^{N-1}} e^{-s} f(x - s|v|\Omega, |v|\Omega) d\Omega ds. \tag{3.2}$$

Now, we make the change of variables

$$(s, |v|, \Omega) \rightarrow (s, y); \quad y = x - s|v|\Omega;$$

(3.2) is then transformed into

$$\begin{aligned} & \int u_f(x, v) d\mu(v) \\ & \leq \int dv(|v|)/|v| \int f(y, |v|(x - y)/|x - y|) e^{-|x - y|/|v|} |x - y|^{N-1} dy. \end{aligned} \tag{3.3}$$

Define

$$k(y) = 1/|y|^{N-1}. \tag{3.4}$$

From (3.3), we notice that

$$\left| \int u_f(x, v) d\mu(v) \right| \leq Ck * \sup_{v \in \mathbb{R}^N} |f(\cdot, v)|. \tag{3.5}$$

Recall that  $k \in M^{N/(N-1)}$ . Since we have assumed that  $f \in L^1(dx; L^\infty(d\mu(v)))$ , we obtain the announced result from (3.5) (see [BB]). By interpolation, we obtain the general case where  $1 < p \leq +\infty$  (see [BL]). ■

*Remarks.* (1) As a consequence of Proposition 6, the operator  $T$  is weakly compact from  $f \in L^1(dx; L^p(d\mu(v)))$  into  $L^s_{loc}(dx)$ , for  $p > 1$  and  $s < r = Np/(1 + (N-1)p)$ .

(2) It is easy to see that the operator  $f \mapsto \int_0^\infty e^{-s} f(x - sv, v) ds$  is continuous from  $L^1(d\mu(v); L^p(dx))$  into  $L^p(dx; L^1(d\mu(v)))$ , for  $1 \leq p \leq +\infty$ . In particular,  $T$  is weakly compact from  $L^1(d\mu(v); L^p(dx))$  into  $L^s_{loc}(dx)$  for  $1 < p \leq +\infty$  and  $1 \leq s < p$ .

Finally, let us notice that the following embedding inequalities are impossible:

$$\|\tilde{u}\|_{L^p(\mathbb{R}^N)} \leq C[\|v \cdot \partial_x u\|_{L^x(dx; L^1(d\mu(v)))} + \|u\|_{L^q(dx; L^1(d\mu(v)))}] \tag{3.6}$$

and

$$\|\tilde{u}\|_{L^p(\mathbb{R}^N)} \leq C[\|v \cdot \partial_x u\|_{L^1(d\mu(v); L^x(dx))} + \|u\|_{L^q(dx; L^1(d\mu(v)))}], \tag{3.6}'$$

where  $1 \leq q < p$ . Indeed, assume that (3.6) holds; we apply it to  $u_\lambda(x, v) = u(\lambda x, v)$  for any  $\lambda > 0$ , and we make the change of variables  $x \rightarrow \lambda x$ ; with an adequate choice of  $\lambda$ , as in the proof of Theorem 2, we obtain that the following inequality holds:

$$\begin{aligned} \|\tilde{u}\|_{L^p(\mathbb{R}^N)} &\leq C(\|v \cdot \partial_x u\|_{L^x(dx; L^1(d\mu(v)))})^{qN(p-q)/(p(N+q))} \\ &\quad \cdot (\|u\|_{L^q(dx; L^1(d\mu(v)))})^{q(N+p)/(p(N+q))}. \end{aligned} \tag{3.7}$$

If we assume that (3.6)' holds, we obtain in the same way that

$$\begin{aligned} \|\tilde{u}\|_{L^p(\mathbb{R}^N)} &\leq C(\|v \cdot \partial_x u\|_{L^1(d\mu(v); L^x(dx))})^{qN(p-q)/(p(N+q))} \\ &\quad \cdot (\|u\|_{L^q(dx; L^1(d\mu(v)))})^{q(N+p)/(p(N+q))}. \end{aligned} \tag{3.7}'$$

Choose  $u_\varepsilon(x, v) = g_\varepsilon(v) f(x_1) \varphi(x')$  for  $v \in S^{N-1}$  and  $x = (x_1, x')$ ,  $x_1 \in \mathbb{R}$ ,  $x' \in \mathbb{R}^{N-1}$ , with  $g_\varepsilon(v) \geq 0$ ,  $\int_{S^{N-1}} g_\varepsilon(v) dv = 1$ ,  $g_\varepsilon \rightarrow \delta_p$ .  $P$  is the north pole, i.e.,  $P = (0, \dots, 0, 1)$  and  $f \in \mathcal{D}(\mathbb{R}^{N-1})$ . Then

$$\begin{aligned} \|v \cdot \partial_x u_\varepsilon\|_{L^\infty_x(L^1_v)} &\rightarrow \|f\|_{L^\infty} \|\partial\varphi/\partial x_N\|_{L^x} && \text{as } \varepsilon \rightarrow 0, \\ \|v \cdot \partial_x u_\varepsilon\|_{L^1_x(L^\infty_v)} &\rightarrow \|f\|_{L^\infty} \|\partial\varphi/\partial x_N\|_{L^x} && \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Fixing  $\varphi \geq 0$ ,  $\varphi \neq 0$ , (3.7) and (3.7)' give

$$\|f\|_{L^p} \leq C \|f\|_{L^\infty}^{q(N-p)/(p(N+q))} \|f\|_{L^q}^{q(N+p)/(p(N+q))}, \quad \forall f \in \mathcal{D}(\mathbb{R}), \quad (3.8)$$

and (3.8) is wrong by a dimensional argument, thus contradicting (3.6) and (3.6)'.

#### IV. SPECIAL CASES AND COUNTEREXAMPLES

##### 1. The Case of Space Dimension 1

In this paragraph, we keep assumption (2.1) on the measure  $\mu$ , and we assume that  $N=1$ . Then, the regularity results in Theorems 1-3 are no longer optimal. Assumption (2.1) is translated here as

$$\begin{aligned} &\mu \text{ is a bounded positive measure on } \mathbb{R} \text{ such that} \\ &\mu([- \varepsilon, \varepsilon]) \leq C\varepsilon^\gamma, \text{ for } 0 < \gamma < 1, \text{ for all } \varepsilon > 0. \end{aligned} \quad (4.1)$$

(We have eliminated the cases where  $\gamma > 1$ , which are obvious; see the remark below.)

LEMMA 7. Assume (4.1). Let  $u = u(x, v)$  be such that both  $u$  and  $v \cdot \partial_x u$  belong to  $L^\infty(dx \otimes d\mu(v))$ . Then the moment  $\tilde{u}(x) = \int u(x, v) d\mu(v)$  belongs to  $C^{0,\gamma}$ , with the inequality

$$\sup_{x \neq y} |\tilde{u}(x) - \tilde{u}(y)|/|x - y|^\gamma \leq C_\gamma \|u\|_{L^\infty}^{1-\gamma} \cdot \|v \cdot \partial_x u\|_{L^\infty}^\gamma \quad (4.2)$$

( $C_\gamma$  being a universal positive constant depending on  $\gamma$ ).

*Proof of Lemma 7.* Proceeding as in the lemma in the proof of Theorem 1, we obtain

$$\int_{|v| > \alpha} d\mu(v)/|v| \leq (2C\gamma)/(1 - \gamma) \alpha^{\gamma-1}, \quad \alpha > 0.$$

Then, we write

$$\begin{aligned} &\left| \int u(x, v) d\mu(v) - \int u(y, v) d\mu(v) \right| \\ &\leq \int_{|v| < \alpha} |u(x, v) - u(y, v)| d\mu(v) + \int_{|v| > \alpha} d\mu(v)/|v| \int_x^y |v \cdot \partial_x u(s, v)| ds \\ &\leq 2C \|u\|_{L^\infty} \alpha^\gamma + 2C\gamma/(1 - \gamma) \|v \cdot \partial_x u\|_{L^\infty} |x - y| \alpha^{\gamma-1} \end{aligned} \quad (4.3)$$

for any  $\alpha > 0$ . Then we choose

$$\alpha = \gamma / (1 - \gamma) \|v \partial_x u\|_{L^\alpha} |x - y| \|u\|_{L^x}^{-1}$$

and (4.3) implies inequality (4.2). ■

*Remarks.* (1) If  $\gamma = 1$  in (4.1), then, under the assumptions of Lemma 7,  $\int u(x, v) d\mu(v)$  does not live in any nice space (as it may be seen from carrying the above proof in this special case); we thus have to say that  $\mu$  satisfies (4.1) for any  $0 < \gamma < 1$ , with the resulting inequality (4.2).

(2) In the case where  $\gamma > 1$ , we obviously have

$$\int d\mu(v)/|v| < +\infty.$$

Therefore, under the assumptions of Lemma 7,  $\int u(x, v) d\mu(v)$  belongs to  $W^{1,\infty}(\mathbb{R}_x)$ , with the inequality

$$\left| \int u(\cdot, v) d\mu(v) \right|_{W^{1,x}} \leq C(\|u\|_{L^x} + \|v \cdot \partial_x u\|_{L^x}). \tag{4.4}$$

This is of course optimal: take  $\mu = \delta_1$ .

Lemma 7 and these two remarks obviously allow improvements on the regularity results for  $\int u(x, v) d\mu(v)$  which were given in Theorems 1–3. We refer to [BL, Tr] for the appropriate interpolation results. Results analogous to those of Section III can also be obtained in quite the same way; but we will not bother to do so. Let us rather consider the following case.

**LEMMA 8.** *Keep assumption (4.1). Let us consider a sequence  $u^n = u^n(x, v)$  such that  $u^n$  is bounded in  $L^p(d\mu(v); L^1(dx))$  for some  $p > 1$  and  $v \cdot \partial_x u^n$  is bounded in  $L^1(dx \otimes d\mu(v))$ . Then, the sequence  $\int u^n(x, v) d\mu(v)$  is compact in  $L^1_{loc}(dx)$ .*

The proof of Lemma 8 follows exactly the same arguments as in the proof of Lemma 7 and the classical compactness criterion in  $L^1_{loc}(dx)$  (see [A]).

## 2. Counterexamples

In this section, we show that our results are, in some sense, optimal by giving counterexamples in the limit cases.

**EXAMPLE 1.** In our first example, we consider the solution  $u_f$  of (3.1). If  $f$  belongs to  $K$ , a bounded subset of  $L^1(dx \otimes d\mu(v))$ , we show that the family  $(\tilde{u}_f)_{f \in K}$  is not necessarily weakly compact in  $L^1(dx)$ .

Thus consider a sequence  $f_n(x, v)$ ,  $x \in \mathbb{R}^N$ ,  $v \in \mathbb{R}^N$ ,  $0 \leq |v| \leq 1$ ,  $f_n \rightarrow \delta$  weakly, as  $n \rightarrow \infty$ , where  $\delta$  denotes the Dirac mass at  $x=0$ ,  $v = v_0 \in \{0 < |v| < 1\}$ . We choose for  $\mu$  the uniform measure on  $\{v; 0 \leq |v| \leq 1\}$ . The corresponding solution  $u_n$ , of (3.1), satisfies

$$\int u_n(x, v) dv = \iint_0^\infty f_n(x - tv, v) e^{-t} dt dv.$$

Thus

$$\int u_n(x, v) \varphi(x) dv dx \xrightarrow{n \rightarrow \infty} \int_0^\infty e^{-t} \varphi(tv_0) dt,$$

for any  $\varphi \in \mathcal{D}(\mathbb{R}^N)$ , and this proves our claim, since the weak limit of  $\tilde{u}_n$  is a measure with support on the line  $\{tv_0, t \in \mathbb{R}^+\}$ .

EXAMPLE 2. Our second example deals with the case where  $f$  is bounded in  $L^\infty(dx \otimes d\mu(v))$  and we prove that  $\tilde{u}_f$  need not be equicontinuous.

Now, we choose for  $\mu$  the uniform measure on  $S^{N-1}$ . Consider the solution  $u$  of (3.1) for  $f(x, v) = g(x)k(v)$ ,  $g \in L^\infty(\mathbb{R}^N)$ ,  $k \in L^\infty(S^{N-1})$ . Then, we have

$$\int u(x, v) dv = \int_{\mathbb{R}^N} g(y) e^{-|x-y|} |x-y|^{N-1} k((x-y)/|x-y|) dy.$$

Setting  $\Gamma_k(z) = e^{-|z|} |z|^{N-1} k(z/|z|)$ , the family  $\int u(x, v) dv$  is equicontinuous for all  $g$  such that  $\|g\|_{L^\infty} \leq 1$  if and only if there exists a modulus of continuity  $\rho$  such that

$$\|\Gamma_k(x+h) - \Gamma_k(x)\|_{L^1(\mathbb{R}_x^N)} \leq \rho(h).$$

This does not hold uniformly for  $\|k\|_{L^\infty} \leq 1$ , with the same  $\rho$ . Therefore,  $\tilde{u}_f$  is not equicontinuous for  $\|g\|_{L^\infty} \leq 1$ ,  $\|k\|_{L^\infty} \leq 1$  and our claim is proved.

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