Mean Field Kinetic Equations

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Contents

Chapter 1

A Gallery of Models

In physics, the term “plasma” refers to a state of matter analogous to a gas, except that a significant fraction of its microscopic constituents are not electrically neutral. Typically, a plasma will contain a non negligible quantity of ions and electrons, in addition to atoms and molecules.

Plasma dynamics is a branch of physics which is of paramount importance in several contexts, such as astrophysics, astronomy or atmospheric sciences. All stars are made of a plasma; the interplanetary, or interstellar or even intergalactic medium is a plasma. The solar wind is an important example of plasma. Several atmospheric phenomena (lightning, aurorae borealis) involve plasmas. Plasmas are also encoutered in a great variety of industrial applications, such as plasma displays, ion thrusters, discharges, chemical vapor deposition, controlled thermonuclear fusion...

As a consequence of this great variety of physical contexts, plasma dynamics is extremely intricate, and involves many different mathematical models. The present chapter will introduce only a few of them.

1.1 Kinetic Formalism

Consider a system of identical point particles. If the total number of such particles per unit volume is large enough, the state of the system at time t can be described statistically in the single particle phase space, by considering the distribution function $f \equiv f(t, x, v)$ that is the number density of particles which are located at the position x and have velocity v at time t. Henceforth we assume that $x, v \in \mathbf{R}^n$, the cases of dimension $n = 2$ or 3 being of practical interest.

If Ω and $\mathcal{V}$ are (measurable) subsets of the Euclidian space $\mathbf{R}^n$, the total number $\mathcal{N}_{\Omega,\mathcal{V}}(t)$ of particles to be found in Ω with velocities in $\mathcal{V}$ at time t is

$$\mathcal{N}_{\Omega,\mathcal{V}}(t) = \iint_{\Omega\times\mathcal{V}} f(t, x, v)\,dx\,dv\,.$$

This relation can be viewed as a definition of f.

More generally, if $\phi(x,v)$ is an additive physical quantity for a particle located at $x \in \mathbf{R}^n$ with velocity $v \in \mathbf{R}^n$, the corresponding quantity for the portion of the particle system to be found in Ω at time t is

$$\Phi_\Omega(t) = \iint_{\Omega\times\mathbf{R}^n} \phi(x,v) f(t,x,v) dx dv\,.$$

For instance, denoting by m the mass of one particle, the momentum of a particle with velocity v is $\phi(v) = mv$, so that the total momentum of the portion of the particle system located in Ω at time t is

$$P_\Omega(t) = \iint_{\Omega\times\mathbf{R}^n} mvf(t,x,v) dx dv\,.$$

Likewise, the kinetic energy of a particle with velocity v is $\phi(v) = \frac{1}{2}m|v|^2$ so that the total energy of the portion of the particle system located in Ω at time t is

$$\mathcal{E}_\Omega(t) = \iint_{\Omega\times\mathbf{R}^n} \tfrac{1}{2}m|v|^2 f(t,x,v) dx dv\,.$$

Another example is the angular momentum about the origin for a particle with velocity v located at the position x, that is $\phi(x,v) = x \times (mv)$ (with the usual notation $a \times b$ for the cross product of two vectors $a, b \in \mathbf{R}^3$); in that case, the total angular momentum of the portion of the particle system located in Ω at time t is

$$L_\Omega(t) = \iint_{\Omega\times\mathbf{R}^n} x \times mvf(t,x,v) dx dv\,.$$

All these quantities $P_\Omega(t)$, $\mathcal{E}_\Omega(t)$, $L_\Omega(t)$ are referred to as "macroscopic observables" defined by the distribution function f that is a statistical quantity at the microscopic scale. The macroscopic observables are the quantities of physical interest, which can be measured in practice, unlike the distribution function itself.

On the other hand, the evolution of the distribution function is usually rather well known and follows from theoretical considerations.

Assume that each particle in the system under consideration is subject to an acceleration field $a \equiv a(t,x,v) \in \mathbf{R}^n$. In other words, particle trajectories are solutions of the differential system

$$\begin{cases} \dot{X}(t) = V(t)\,, \\ \dot{V}(t) = a(t, X(t), V(t))\,. \end{cases}$$

Henceforth, we denote by

$$t \mapsto (X(t,t_0,x_0,v_0), V(t,t_0,x_0,v_0))$$

the solution of the differential system above satisfying

$$X(t_0,t_0,x_0,v_0) = x_0\,, \quad V(t_0,t_0,x_0,v_0) = v_0\,.$$

Assume for simplicity that the transformation

$$(x_0, v_0) \mapsto (X(t, t_0, x_0, v_0), V(t, t_0, x_0, v_0))$$

preserves the Lebesgue measure in $\mathbf{R}^n \times \mathbf{R}^n$ — i.e. is volume-preserving in the single particle phase space $\mathbf{R}^n \times \mathbf{R}^n$.

We shall also assume that the effect of collisions between particles can be neglected.

For each open subset $\mathcal{O}_{t_0}$ of $\mathbf{R}^n \times \mathbf{R}^n$, consider

$$\mathcal{O}_t := \{(X(t, t_0, x_0, v_0), V(t, t_0, x_0, v_0)) \text{ s.t. } (x_0, v_0) \in \mathcal{O}_{t_0}\} ,$$

for all $t \geq t_0$.

During the evolution of the particle system under consideration, there is no destruction or creation of particles; therefore, the total number of particles to be found in the portion $\mathcal{O}_{t_0}$ of phase space at time t_0 is equal to the total number of particles to be found in $\mathcal{O}_t$ at time $t \geq t_0$. In other words

$$\iint_{\mathcal{O}_{t_0}} f(t_0, x_0, v_0) dx_0 dv_0 = \iint_{\mathcal{O}_t} f(t, x, v) dx dv , \quad t \geq t_0 .$$

In the integral on the right hand side of this equality, substitute $X(t, t_0, x_0, v_0)$ to x and $V(t, t_0, x_0, v_0)$ to v:

$$\iint_{\mathcal{O}_{t_0}} f(t_0, x_0, v_0) dx_0 dv_0 = \iint_{\mathcal{O}_{t_0}} f(t, X(t, t_0, x_0, v_0), V(t, t_0, x_0, v_0)) dx_0 dv_0$$

— notice that the Jacobian of this transformation, assumed to be volume-preserving in the single particle phase space $\mathbf{R}^n \times \mathbf{R}^n$, is ± 1 .

Since this equality is assumed to hold for each open subset $\mathcal{O}_{t_0}$ of $\mathbf{R}^n \times \mathbf{R}^n$, we conclude that

$$f(t_0, x_0, v_0) = f(t, X(t, t_0, x_0, v_0), V(t, t_0, x_0, v_0))$$

for all $t \geq t_0$, and all $x_0, v_0 \in \mathbf{R}^n$, assuming that both sides of the equality above are continuous in (x_0, v_0).

If moreover f is a smooth (C^1 being enough) function of its arguments, and if the acceleration field $a \equiv a(t, x, v)$ is of class C^1, one has

$$\begin{cases} X(t, t_0, x_0, v_0) = x_0 + \Delta t \, v_0 + o(\Delta t) , \\ V(t, t_0, x_0, v_0) = v_0 + \Delta t \, a(t_0, x_0, v_0) + o(\Delta t) \end{cases}$$

as $\Delta t \to 0$, with the notation

$$t - t_0 = \Delta t .$$

Therefore

$$\begin{aligned} f(t_0, x_0, v_0) = f(t_0 + \Delta t, x_0 + \Delta t \, v_0, v_0 + \Delta t \, a(t_0, x_0, v_0)) = f(t_0, x_0, v_0) \\ + \Delta t \, (\partial_t f(t_0, x_0, v_0) + v_0 \cdot \nabla_x f(t_0, x_0, v_0) + a(t_0, x_0, v_0) \cdot \nabla_v f(t_0, x_0, v_0)) \\ + o(\Delta t) , \end{aligned}$$

and since must be true for all Δt, all $x_0, v_0 \in \mathbf{R}^n$ and all t_0, we conclude that the distribution function f must satisfy the partial differential equation

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) + a(t,x,v) \cdot \nabla_v f(t,x,v) = 0\,.$$

This equation is usually referred to as the "Liouville equation" in the Hamiltonian case — i.e. when the vector field $(v, a(t,x,v))$ driving the particle system is Hamiltonian — and as the "Vlasov equation" when it is coupled to another equation — or system of equations — modeling the effect of the particle system on the acceleration field a.

Specifically, Vlasov models for particle systems are kinetic models where each particle is subject to the acceleration field created by all the other particles in the system. In other words, Vlasov models are models where the acceleration field $a(t,x,v)$ is a functional of the unknown particle distribution function f itself.

In the sequel, we give some examples of Vlasov type models, mostly in the context of plasma dynamics.

1.2 The Vlasov-Poisson Model

Consider a gas of identical charged particles, with mass m and charge q in space dimension 3. The electrostatic force exerted at time t on a particle with charge q located at the position x by a distribution of point charges $q\rho \equiv q\rho(t,y)$ is

$$F(t,x) = \frac{q^2}{4\pi\epsilon_0} \int_{\mathbf{R}^3} \frac{x-y}{|x-y|^3} \rho(t,y)dy\,,$$

where ϵ_0 is the vacuum permittivity. Notice that this force is repulsive between charges of identical signs.

The electrostatic force can be recast as

$$F(t,x) = qE(t,x)\,,$$

where

$$E(t,x) = \frac{q}{4\pi\epsilon_0} \int_{\mathbf{R}^3} \frac{x-y}{|x-y|^3} \rho(t,y)dy\,.$$

Since $\frac{x}{|x|^3} = -\nabla \frac{1}{|x|}$, one has

$$E(t,x) = -\nabla_x \phi(t,x)\,,$$

where

$$\phi(t,x) = \frac{q}{4\pi\epsilon_0} \int_{\mathbf{R}^3} \frac{1}{|x-y|} \rho(t,y)dy\,.$$

The scalar $\phi(t,x)$ is the electrostatic potential created at the position x and at time t by the distribution of charges $q\rho$.

In space dimension $n = 3$, the function

$$G(x) := \tfrac{1}{4\pi}\frac{1}{|x|}$$

is the unique fundamental solution of the operator $-\Delta$ converging to 0 at infinity (see Theorem 8.3.1 in [6]):

$$\begin{cases} -\Delta G = \delta_0\,, \\ G(x) \to 0 \text{ as } |x| \to \infty\,. \end{cases}$$

Thus, the Vlasov-Poisson system takes the form

$$\begin{cases} \partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) - \tfrac{q}{m}\nabla_x \phi(t,x) \cdot \nabla_v f(t,x,v) = 0\,, \\ \\ -\Delta\phi(t,x) = \tfrac{1}{\epsilon_0} q \rho_f(t,x)\,, \\ \\ \rho_f(t,x) = \displaystyle\int_{\mathbf{R}^3} f(t,x,v)dv\,. \end{cases}$$

In writing this system, one assumes implicitly that the force exerted on one of the charged particles at the position x at time t by all the other particles is

$$F(t,x) = \frac{q^2}{4\pi\epsilon_0}\int_{\mathbf{R}^3} \frac{x-y}{|x-y|^3} f(t,y,v)dydv\,.$$

In other words, the distribution function of the system of the charged particles other than the particle at the position x subject to the force F is approximated by the distribution function of the total particle system. This approximation, which is typical of all mean field models, is equivalent to assuming that the effect of each individual particle is negligible when compared to the collective effect of the whole particle system. Indeed, all the models considered here are valid only for systems involving a large number of particles — large enough so that the methods of statistical mechanics can be applied.

The Vlasov-Poisson model written above is somewhat unrealistic since all the particles have the same charge q, so that the particle system under consideration is not electrically neutral, which is physically irrealistic. In practice, one considers systems of charged particles of different species, indexed by $\alpha \in A$. Assuming that the particles of species α have mass q_α and mass m_α, and denoting by f_α the distribution function of particles of species α, the electrostatic potential created by the total system of charged particles is

$$\phi(t,x) = \frac{1}{\epsilon_0}\sum_{\alpha\in A} q_\alpha \iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{f_\alpha(t,y,v)dydv}{|x-y|}\,.$$

The Vlasov-Poisson system for this system of charged particles takes the form

$$\begin{cases} \partial_t f_\alpha(t,x,v) + v\cdot\nabla_x f_\alpha(t,x,v) - \tfrac{q_\alpha}{m_\alpha}\nabla_x\phi(t,x)\cdot\nabla_v f_\alpha(t,x,v) = 0\,, \quad \alpha\in A, \\ \\ -\Delta\phi(t,x) = \tfrac{1}{\epsilon_0}\sum_{\alpha\in A} q_\alpha\rho_\alpha(t,x)\,, \\ \\ \rho_\alpha(t,x) = \int_{\mathbf{R}^3} f_\alpha(t,x,v)dv\,. \end{cases}$$

In other words, there is a Vlasov equation for each species of particles, coupled to a single field equation — the Poisson equation — for the electrostatic potential. Notice that each Vlasov equation takes the form

$$\partial_t f_\alpha + \mathrm{div}_x(vf_\alpha) - \tfrac{q_\alpha}{m_\alpha}\,\mathrm{div}_v(f_\alpha\nabla_x\phi) = 0\,.$$

Therefore, if f_α and ϕ are both of class C^1 and decay rapidly enough as $(x,v)\to\infty$, one has

$$\begin{aligned} \frac{d}{dt}\iint_{\mathbf{R}^3\times\mathbf{R}^3} f_\alpha(t,x,v)dxdv = &-\iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathrm{div}_x(vf_\alpha)(t,x,v)dxdv \\ &+\tfrac{q_\alpha}{m_\alpha}\iint_{\mathbf{R}^3\times\mathbf{R}^3} \mathrm{div}_v(f_\alpha\nabla\phi)(t,x,v)dxdv = 0 \end{aligned}$$

so that

$$\iint_{\mathbf{R}^3\times\mathbf{R}^3} f_\alpha(t,x,v)dxdv = \mathrm{Const.}\,, \quad \alpha\in A\,.$$

In particular, if the particle system is globally neutral at time $t=0$, i.e. if

$$\sum_{\alpha\in A} q_\alpha\iint_{\mathbf{R}^3\times\mathbf{R}^3} f_\alpha(0,x,v)dxdv = 0\,,$$

it remains globally neutral for all $t\geq 0$, since

$$\sum_{\alpha\in A} q_\alpha\iint_{\mathbf{R}^3\times\mathbf{R}^3} f_\alpha(t,x,v)dxdv = \sum_{\alpha\in A} q_\alpha\iint_{\mathbf{R}^3\times\mathbf{R}^3} f_\alpha(0,x,v)dxdv = 0\,.$$

There is a variant of the Vlasov-Poisson model that appears in cosmology. In that case, $f(t,x,v)$ is the distribution function of a system of identical, electrically neutral point particles with mass m. Each particle is subject to the gravitation force field created by all the other particles. The force exerted on a particle with mass m at the position x and at time t by a the population of all the other particles with distribution function f is

$$F(t,x) = -\Gamma m^2\iint_{\mathbf{R}^3\times\mathbf{R}^3}\frac{x-y}{|x-y|^3}f(t,y,v)dydv\,,$$

where Γ is Newton's gravitation constant. Notice that, at variance with the case of the electrostatic force, this interaction is attractive.

Arguing as above, this force field is recast as

$$F(t,x) = m\nabla_x \Phi(t,x)$$

where

$$\Phi(t,x) = \Gamma m \iint_{\mathbf{R}^3\times\mathbf{R}^3} \frac{f(t,y,v)dydv}{|x-y|} = 4\pi\Gamma m \int_{\mathbf{R}^3} G(x-y)\rho_f(t,y)\,,$$

with

$$\rho_f(t,x) = \int_{\mathbf{R}^3} f(t,x,v)dv\,,$$

and where G denotes as above the fundamental solution of the Laplacian converging to 0 at infinity.

Therefore, the gravitational Vlasov-Poisson system (sometimes referred to as the "Liouville-Newton system") takes the form

$$\begin{cases} \partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) + \frac{1}{m}\nabla_x\Phi(t,x)\cdot\nabla_v f(t,x,v) = 0\,, \\ \\ -\Delta\Phi(t,x) = 4\pi\Gamma m\rho_f(t,x)\,, \\ \\ \rho_f(t,x) = \int_{\mathbf{R}^3} f(t,x,v)dv\,. \end{cases}$$

As in the electrostatic case, there are also variants of this model involving different species of particles with different masses. However, there is obviously no analogue of the global neutrality condition in this context.

The mathematical theory of the Vlasov-Poisson system is slightly more involved in the gravitational case than in the electrostatic case, because the nonlinearity in the gravitational case is associated to an attractive interaction, promoting mass concentration.

1.3 The Vlasov-Maxwell Model

Charged particles at rest or in motion generate an electric field; charged particles in motion also generate a magnetic field. When the typical speed of charged particles in the system under consideration is sufficiently large, the contribution of the magnetic field to the electromagnetic force must be taken into account.

We recall that the Lorentz force exerted by an electric field E and a magnetic field B on a particle with charge q located at the position x and moving with a velocity v at time t is

$$F(t,x,v) = q(E(t,x) + v\times B(t,x))\,.$$

The electromagnetic field (E, B) generated by a system of moving identical charged particles with electric charge q is governed by the system of Maxwell's equations recalled below:

$$\left\{\begin{array}{ll}\operatorname{div}_x B = 0\,, & \text{(no magnetic charges)}\\ \operatorname{curl}_x E = -\partial_t B\,, & \text{(Faraday's equation)}\\ \operatorname{div}_x E = \frac{1}{\epsilon_0} q\rho\,, & \text{(Gauss' equation)}\\ \operatorname{curl}_x B = \mu_0 q j + \frac{1}{c^2}\partial_t E\,, & \text{(Ampère's equation)}\end{array}\right.$$

In these equations, the constants ϵ_0 and μ_0 are respectively the vacuum permittivity and magnetic permeability, while

$$c^2 = \frac{1}{\epsilon_0 \mu_0}$$

is the speed of light in vacuum. The source terms in the right hand sides of the Gauss and the Ampère equations are the charge density $q\rho$ and the current density qj.

Combining the Gauss and Ampère equations, one finds that

$$\begin{aligned}\partial_t(q\rho) + \operatorname{div}_x(qj) &= \epsilon_0 \partial_t \operatorname{div}_x E + \frac{1}{\mu_0} \operatorname{div}_x(\operatorname{curl}_x B - \tfrac{1}{c^2}\partial_t E)\\ &= \epsilon_0 \partial_t \operatorname{div}_x E - \frac{1}{\mu_0 c^2} \operatorname{div}_x \partial_t E = 0\end{aligned}$$

which is the local conservation of electric charge. The local conservation of charge can be viewed as a necessary compatibility condition for ρ and j to be the source terms in Maxwell's system of equations.

In fact, the equation written by Ampère related the magnetic field B created by an electric current density qj in a permanent regime, so that

$$\operatorname{curl}_x B = \mu_0 q j\,.$$

Maxwell's idea is that, in general, one must add to the current density qj a displacement current $-\epsilon_0 \partial_t E$, so that the last equation in Maxwell's system takes the form

$$\operatorname{curl}_x B = \mu_0 \left(qj + \epsilon_0 \partial_t E\right)\,.$$

Therefore, the Vlasov-Maxwell system for the particle distribution function

$f \equiv f(t, x, v)$ and the electromagnetic field $(E, B) \equiv (E, B)(t, x)$ is

$$\begin{cases} \partial_t f + v \cdot \nabla_x f + \dfrac{q}{m}(E + v \times B) \cdot \nabla_v f = 0\,, \\ \operatorname{div}_x B = 0\,, \\ \partial_t B + \operatorname{curl}_x E = 0\,, \\ \operatorname{div}_x E = \frac{1}{\epsilon_0} q \rho_f\,, \\ \frac{1}{c^2}\partial_t E - \operatorname{curl}_x B = -\mu_0 q j_f\,, \\ \rho_f = \displaystyle\int_{\mathbf{R}^3} f dv\,, \\ j_f = \displaystyle\int_{\mathbf{R}^3} v f dv\,. \end{cases}$$

Observe that

$$\operatorname{div}_v(v \times B) = (\operatorname{curl}_v v) \cdot B = \operatorname{curl}_v(\nabla_v \tfrac{1}{2}|v|^2) = 0\,,$$

so that the Vlasov equation can be put in the form

$$\partial_t f + \operatorname{div}_x(vf) = -\frac{q}{m} \operatorname{div}_v((E + v \times B)f)\,.$$

Integrating in v both sides of this equality, assuming that f, E and B are of class C^1 and that f decays rapidly enough as $|v| \to \infty$, we see that

$$\partial_t \rho_f + \operatorname{div}_x j_f = 0\,,$$

so that the local conservation of charge is a consequence of the Vlasov equation — as well as of the Maxwell system. In other words, the Vlasov equation implies the local conservation of charge, which is a necessary compatibility condition to be satisfied by the charge and current densities in order for the Maxwell system to have a solution.

On principle, the speed or massive particles should not exceed the speed of light, but this is not guaranteed in the model above. There is another variant of the Vlasov-Maxwell system, referred to as the "relativistic Vlasov-Maxwell system" in which the particle speed is less than the speed of light.

In this model, the particle distribution function is given in terms of the momentum variable $\xi \in \mathbf{R}^3$, instead of the velocity variable v. In other words, the distribution function $f \equiv f(t, x, \xi)$ is the density of particles at the position x with momentum ξ at time t. The energy of a particle with mass m and momentum ξ is

$$e(\xi) := \sqrt{m^2c^4 + c^2|\xi|^2}\,,$$

and the corresponding velocity is

$$v(\xi) := \nabla e(\xi) = \frac{c^2\xi}{\sqrt{m^2c^4 + c^2|\xi|^2}}\,.$$

Observe that $|v(\xi)| < c$ if $m > 0$.

With this notation, the relativistic Vlasov-Maxwell system takes the form

$$\begin{cases} \partial_t f + v(\xi)\cdot\nabla_x f + q(E + v(\xi)\times B)\cdot\nabla_\xi f = 0\,, \\ \operatorname{div}_x B = 0\,, \\ \partial_t B + \operatorname{curl}_x E = 0\,, \\ \operatorname{div}_x E = \frac{1}{\epsilon_0} q\rho_f\,, \\ \frac{1}{c^2}\partial_t E - \operatorname{curl}_x B = -\mu_0 q j_f\,, \\ \rho_f = \displaystyle\int_{\mathbf{R}^3} f d\xi\,, \\ j_f = \displaystyle\int_{\mathbf{R}^3} v(\xi) f d\xi\,. \end{cases}$$

Here again, the local conservation of charge is a consequence of the Vlasov equation. Indeed

$$\operatorname{div}_\xi(v(\xi)\times B) = (\operatorname{curl}_\xi v(\xi))\cdot B = (\operatorname{curl}_\xi \nabla e(\xi))\cdot B = 0\,.$$

Therefore, the Vlasov equation takes the form

$$\partial_t f + \operatorname{div}_x(v(\xi)f) = -q\operatorname{div}_\xi((E + v(\xi)\times B)f)$$

and, assuming that f, E and B are of class C^1 and that f decays rapidly enough as $|\xi|\to\infty$, we conclude that

$$\partial_t\rho_f + \operatorname{div}_x j_f = \partial_t\int_{\mathbf{R}^3} f d\xi + \operatorname{div}_x\int_{\mathbf{R}^3} v(\xi) f d\xi = 0\,.$$

In this case again, one should consider different species of particles, labelled by $\alpha\in A$, with distribution functions f_α. Denoting by m_α and q_α respectively the mass and electric charge of particles of species α, the relativistic Vlasov-Maxwell system takes the form

$$\begin{cases} \partial_t f_\alpha + v_\alpha(\xi)\cdot\nabla_x f_\alpha + q_\alpha(E + v_\alpha(\xi)\times B)\cdot\nabla_\xi f_\alpha = 0\,, \\ \operatorname{div}_x B = 0\,, \\ \partial_t B + \operatorname{curl}_x E = 0\,, \\ \operatorname{div}_x E = \frac{1}{\epsilon_0}\displaystyle\sum_{\alpha\in A} q_\alpha\rho_\alpha\,, \\ \frac{1}{c^2}\partial_t E - \operatorname{curl}_x B = -\mu_0\displaystyle\sum_{\alpha\in A} q_\alpha j_\alpha\,, \\ \rho_\alpha = \displaystyle\int_{\mathbf{R}^3} f_\alpha d\xi\,, \\ j_\alpha = \displaystyle\int_{\mathbf{R}^3} v_\alpha(\xi) f_\alpha d\xi\,, \end{cases}$$

where

$$v_\alpha(\xi) := \frac{c^2\xi}{\sqrt{m_\alpha^2 c^4 + c^2|\xi|^2}} \,.$$

Here again, the Vlasov equation implies that

$$\partial_t \rho_\alpha + \operatorname{div}_x j_\alpha = 0 \,,$$

by the same argument as in the single species case. Combining all these identities shows that

$$\partial_t \sum_{\alpha \in A} q_\alpha \rho_\alpha + \operatorname{div}_x \sum_{\alpha \in A} q_\alpha j_\alpha = 0 \,,$$

that is the local conservation of charge to be satisfied in order for the Maxwell system to have a solution. (This verification is left to the reader as an easy exercise.)

1.4 The Vlasov-Darwin Model

If the typical speed of charged particles is very small when compared to the speed of light, the Maxwell system of equations reduces to the equations of electrostatics, and the Vlasov-Maxwell system to the much simpler Vlasov-Poisson system. Unfortunately, magnetic effects disappear in this approximation.

There is however another approximation of the Vlasov-Maxwell system, the Vlasov-Darwin system, that retains the magnetic part of the particle interaction. In the Vlasov-Darwin system however, the potential created by each charged particle is transmitted instantaneously to all the other particles as in the Vlasov-Poisson model. In other words, the field equation is a Poisson equation, instead of Maxwell's system of equations — which is equivalent to a wave equation.

Given an electromagnetic field (E, B), write the Helmholtz decomposition of the electric field

$$E = E_{sol} + E_{irr} \,, \qquad \text{with } \operatorname{div}_x E_{sol} = 0 \text{ and } \operatorname{curl}_x E_{irr} = 0 \,.$$

The Darwin approximation of electromagnetism is based on the assumption

$$(HD) \qquad |\partial_t E_{sol}| \ll |\partial_t E_{irr}| \,.$$

Under this assumption, the Ampère equation in Maxwell's system takes the form

$$\operatorname{curl}_x B = \mu_0 q j + \tfrac{1}{c^2}\partial_t E_{irr} \,.$$

In other words, the Darwin system of equations for the electromagnetic field takes the form

$$\begin{cases} \operatorname{div}_x B = 0 \,, \\ \operatorname{curl}_x E = -\partial_t B \,, \\ \operatorname{div}_x E = \frac{1}{\epsilon_0} q\rho \,, \\ \operatorname{curl}_x B = \mu_0 q j + \frac{1}{c^2}\partial_t E_{irr} \,. \end{cases}$$

As in the case of the original Maxwell system, using the relation

$$\epsilon_0\mu_0 = \frac{1}{c^2}$$

shows that

$$\begin{aligned}\partial_t(q\rho) + \operatorname{div}_x(qj) &= \epsilon_0\partial_t \operatorname{div}_x E + \frac{1}{\mu_0}\operatorname{div}_x(\operatorname{curl}_x B - \tfrac{1}{c^2}\partial_t E_{irr})\\ &= \epsilon_0(\partial_t \operatorname{div}_x E - \operatorname{div}_x \partial_t E_{irr}) = 0\,,\end{aligned}$$

since

$$\operatorname{div}_x E = \operatorname{div}_x E_{irr}\,.$$

In other words, the local conservation of charge is verified by ρ and j also in the case of the Darwin system. As in the case of the original Maxwell system, the local conservation of charge is a necessary compatibility condition to be satisfied by the source terms ρ and j in the Dariwn system.

Then

$$\begin{aligned}-\Delta_x B &= \operatorname{curl}_x(\operatorname{curl}_x B) - \nabla_x(\operatorname{div}_x B)\\ &= \operatorname{curl}_x(\operatorname{curl}_x B) = \mu_0 q \operatorname{curl}_x j + \tfrac{1}{c^2}\operatorname{curl}_x \partial_t E_{irr} = \mu_0 q \operatorname{curl}_x j\,.\end{aligned}$$

Going back to the Helmholtz decomposition of the electric field, we first observe that

$$-\Delta_x E_{irr} = \operatorname{curl}_x(\operatorname{curl}_x E_{irr}) - \nabla_x(\operatorname{div}_x E_{irr}) = -\nabla_x(\operatorname{div}_x E_{irr}) = -\tfrac{1}{\epsilon_0}q\nabla_x\rho\,.$$

On the other hand

$$\begin{aligned}-\Delta_x E_{sol} &= \operatorname{curl}_x(\operatorname{curl}_x E_{sol}) - \nabla_x(\operatorname{div}_x E_{sol})\\ &= \operatorname{curl}_x(\operatorname{curl}_x E_{sol}) = \operatorname{curl}_x(\operatorname{curl}_x(E - E_{irr}))\\ &= \operatorname{curl}_x(\operatorname{curl}_x E) = -\operatorname{curl}_x(\partial_t B)\\ &= -\partial_t \operatorname{curl}_x B = -\mu_0 q\partial_t j - \tfrac{1}{c^2}\partial_t^2 E_{irr}\end{aligned}$$

Now

$$E_{irr} = \tfrac{1}{\epsilon_0}q\Delta_x^{-1}\nabla_x\rho$$

and the relation $\frac{1}{c^2} = \epsilon_0\mu_0$ shows that

$$\tfrac{1}{c^2}\partial_t E_{irr} = \mu_0 q\nabla_x\Delta_x^{-1}\partial_t\rho = -\mu_0 q\nabla_x\Delta_x^{-1}\operatorname{div}_x j\,.$$

Here we have used the local conservation of charge in the last equality above.

Therefore

$$-\Delta_x E_{sol} = -\mu_0 q\partial_t j - \tfrac{1}{c^2}\partial_t^2 E_{irr} = -\mu_0 q\partial_t(j - \nabla_x\Delta_x^{-1}\operatorname{div}_x j)\,.$$

In the end, the Darwin system can be put in the form

$$\left\{\begin{aligned}-\Delta_x B &= \mu_0 q \operatorname{curl}_x j\,,\\ -\Delta_x E_{irr} &= -\tfrac{1}{\epsilon_0}q\nabla_x\rho\,,\\ -\Delta_x E_{sol} &= \mu_0 q\partial_t\Delta_x^{-1}\operatorname{curl}_x(\operatorname{curl}_x j)\,.\end{aligned}\right.$$

This formulation of the Darwin system makes it clear that it is an elliptic system of equations, at variance with the original Maxwell system, which is known to be hyperbolic. (In fact, the Maxwell system can be reduced to a system of wave equations, and is a fundamental example of an hyperbolic system of PDEs.)

The Vlasov-Darwin system is therefore

$$
\left\{
\begin{array}{l}
\partial_t f + v \cdot \nabla_x f + \dfrac{q}{m}(E + v \times B) \cdot \nabla_v f = 0\,, \\
\operatorname{div}_x B = 0\,, \\
\operatorname{curl}_x E = -\partial_t B\,, \\
\operatorname{div}_x E = \frac{1}{\epsilon_0} q\rho\,, \\
\operatorname{curl}_x B = \mu_0 q j + \frac{1}{c^2}\partial_t E_{irr}\,.
\end{array}
\right.
$$

As explained in the presentation of the Vlasov-Maxwell system, the Vlasov equation implies the relation

$$\partial_t \rho_f + \operatorname{div}_x j_f = 0$$

which is a necessary compatibility condition to be satisfied by the source terms in order for the Darwin system to have a solution, as observed earlier in this section.

Chapter 2

Transport Equations

All the kinetic models considered in this course involve transport equations. Transport equations are linear PDEs of order 1. An important feature of 1st order PDEs is the method of characteristics, which reduces their study to that of ODE systems. The present chapter discusses the method of characteristics for transport equations.

2.1 Transport Equations with Constant Coefficients

As a warm-up, we first treat the constant coefficient case.

Let $v \in \mathbf{R}^N \setminus \{0\}$ be given. The transport equation is

$$\partial_t f + v \cdot \nabla_x f = 0\,,$$

where the unknown is the function $f \equiv f(t,x) \in \mathbf{R}$, defined for all (t,x) in $\mathbf{R} \times \mathbf{R}^N$.

Definition 2.1.1 *The characteristic curve of the transport operator $\partial_t + v \cdot \nabla_x$ passing through $y \in \mathbf{R}^N$ at time $t = 0$ is the set*

$$\{(t, \gamma(t)) \,|\, t \in \mathbf{R}\}$$

where γ is the solution of the differential system (systems of characteristics associated to the transport operator $\partial_t + v \cdot \nabla_x$)

$$\begin{cases} \dot{\gamma}(t) = v\,, \\ \gamma(0) = y\,. \end{cases}$$

In the constant coefficient case considered here, one has obviously

$$\gamma(t) = y + tv$$

so that the set

$$\{(t,\gamma(t))\,|\,t\in\mathbf{R}\}=\{(t,y+tv)\,|\,t\in\mathbf{R}\}$$

is the straight line in the affine space $\mathbf{R}^{N+1}$ with direction defined by the vector $(1,v)\in\mathbf{R}^{N+1}$ and passing through the point $(0,y)$. (As we shall see later, in the variable coefficient case, characteristic curves are no longer straight lines in general, so that the terminology "characteristic curves" is justified.)

The interest of the notion of characteristic curve for solving the free transport equation is explained by the following observation.

Let $f\in C^1(\mathbf{R}_+\times\mathbf{R}^N)$ be a solution of the free transport equation, and let γ be a solution of the system of characteristics associated to the transport operator $\partial_t+v\cdot\nabla_x$. The map $t\mapsto f(t,\gamma(t))$ is of class C^1 on $\mathbf{R}_+$, being the composition of the maps f and $t\mapsto(t,\gamma(t))$, which are both of class C^1. By the chain rule

$$\begin{aligned}\frac{d}{dt}f(t,\gamma(t))&=\partial_t f(t,\gamma(t))+\sum_{k=1}^N\partial_{x_k}f(t,\gamma(t))\dot\gamma_k(t)\\&=\partial_t f(t,\gamma(t))+\sum_{k=1}^N v_k\partial_{x_k}f(t,\gamma(t))\\&=(\partial_t f+v\cdot\nabla_x f)(t,\gamma(t))=0\,.\end{aligned}$$

Therefore

$$f(t,\gamma(t))=\text{Const.}$$

In other words, each C^1 solution of the free transport equation is constant along all characteristic curves of the transport operator $\partial_t+v\cdot\nabla_x$. This observation leads to the following existence and uniqueness theorem.

Theorem 2.1.2 *For each $f^{in}\in C^1(\mathbf{R}^N)$, the Cauchy problem for the transport equation*

$$\begin{cases}\partial_t f(t,x)+v\cdot\nabla_x f(t,x)=0\,, & x\in\mathbf{R}^N\,,\ t>0\,,\\ f(0,x)=f^{in}(x)\,,\end{cases}$$

has a unique solution $f\in C^1(\mathbf{R}_+\times\mathbf{R}^N)$. This solution is given by the explicit formula

$$f(t,x)=f^{in}(x-tv)\,,\qquad x\in\mathbf{R}^N\,,\ t\geq 0\,.$$

Proof. Following the argument before Theorem 2.1.2, if $f\in C^1(\mathbf{R}_+\times\mathbf{R}^N)$ is a solution of the free transport equation above, then the map $t\mapsto f(t,y+tv)$ is constant on $\mathbf{R}_+$ for each $y\in\mathbf{R}^N$. Therefore

$$f(t,y+tv)=f(0,y)=f^{in}(y)\quad\text{for all } y\in\mathbf{R}^N \text{ and } t\geq 0\,.$$

With the substitution $x=y+tv$, i.e. $y=x-tv$, one finds that

$$f(t,x)=f^{in}(x-tv)\,,\quad x\in\mathbf{R}^N\,,\ t\geq 0\,.$$

This proves the uniqueness part of the theorem.

Conversely, consider the function

$$f : \mathbf{R} \times \mathbf{R}^N \ni (t,x) \mapsto f^{in}(x - tv) \in \mathbf{R}\,.$$

This function is of class C^1 on $\mathbf{R} \times \mathbf{R}^N$, being the composition of f^{in} and $(t,x) \mapsto x - tv$ which are both of class C^1. By the chain rule

$$\partial_t f(t,x) = \nabla f^{in}(x - tv) \cdot (-v)\,, \qquad \text{and } \nabla_x f(t,x) = \nabla f^{in}(x - tv)\,,$$

so that

$$\partial_t f(t,x) + v \cdot \nabla_x f(t,x) = 0\,, \quad x \in \mathbf{R}^N\,,\ t \in \mathbf{R}\,.$$

On the other hand, the initial condition $f\big|_{t=0} = f^{in}$ is an obvious consequence of the explicit formula giving f in terms of f^{in}. This proves the existence part of the theorem. ■

Exercise 2.1 *Let $a \in C^1(\mathbf{R}_+ \times \mathbf{R}^N)$, $S \in C^1(\mathbf{R}_+ \times \mathbf{R}^N)$ and $f^{in} \in C^1(\mathbf{R}^N)$. Solve the Cauchy problem for the transport equation with amplification or damping rate a and source S:*

$$\begin{cases} \partial_t f(t,x) + v \cdot \nabla_x f(t,x) + a(t,x) f(t,x) = S(t,x)\,, \quad x \in \mathbf{R}^N\,,\ t > 0\,, \\ f(0,x) = f^{in}(x)\,. \end{cases}$$

Prove the existence and uniqueness of a solution $f \in C^1(\mathbf{R}_+ \times \mathbf{R}^N)$, that is given by the explicit formula

$$\begin{aligned} f(t,x) &= f^{in}(x - tv) \exp\left(-\int_0^t a(s, x_t v + sv)ds\right) \\ &\quad + \int_0^t S(s, x - tv + sv) \exp\left(-\int_s^t a(\tau, x_t v + \tau v)d\tau\right) ds\,, \end{aligned}$$

for all $x \in \mathbf{R}^N$ and $t \geq 0$.

2.2 Transport Equations with Variable Coefficients

Let $V \equiv V(t,x) \in \mathbf{R}^N$ be a time dependent vector field defined on $[0,T] \times \mathbf{R}^N$ for some $T > 0$. We are concerned with the Cauchy problem

$$\begin{cases} \partial_t f(t,x) + V(t,x) \cdot \nabla_x f(t,x) = 0\,, \quad x \in \mathbf{R}^N\,,\ 0 < t < T\,, \\ f(0,x) = f^{in}(x)\,, \end{cases}$$

where $f^{in} \equiv f^{in}(x) \in \mathbf{R}$ is given while $f \equiv f(t,x) \in \mathbf{R}$ is the unknown.

We shall assume that the vector field V satisfies the following conditions: first, each component V_i of the vector field V has partial derivatives with respect to the variables x_j for $j = 1, \dots, N$, and

$$(H1) \qquad V \in C([0,T] \times \mathbf{R}^N; \mathbf{R}^N) \quad \text{and } \nabla_x V \in C([0,T] \times \mathbf{R}^N; M_N(\mathbf{R})) .$$

Moreover, we assume that there exists $\kappa > 0$ such that

$$|V(t,x)| \le \kappa(1 + |x|) \quad \text{for all } (t,x) \in [0,T] \times \mathbf{R}^N .$$

Definition 2.2.1 *Let γ be the solution of the differential system*

$$\begin{cases} \dot{\gamma}(s) = V(s, \gamma(s)) \\ \gamma(t) = x . \end{cases}$$

The set

$$\{(s, \gamma(s)) \mid s \in [0,T]\}$$

is called the characteristic curve of the transport operator $\partial_t + V(t,x) \cdot \nabla_x$ passing through x at time $s = t$.

The method of characteristics for the transport equation with variable coefficients is split in two steps:
(a) defining the flow associated to the ODE system of characteristic curves;
(b) using this flow to solve the transport equation.

2.2.1 The Characteristic Flow

The existence, uniqueness and regularity of the solution of the differential system of characteristics is summarized in the following statement.

Theorem 2.2.2 *Assume that the vector field V satisfies the conditions (H1)-(H2). Then, for each $t \in [0,T]$ and each $x \in \mathbf{R}^N$, the ODE system*

$$\begin{cases} \dot{\gamma}(s) = V(s, \gamma(s)) \\ \gamma(t) = x . \end{cases}$$

has a unique solution $s \mapsto \gamma(s)$ that is of class C^1 on $[0,T]$.

This solution is henceforth denoted by

$$\gamma(s) =: X(s,t,x) .$$

The map X satisfies the following properties
(a) $X \in C^1([0,T] \times [0,T] \times \mathbf{R}^N; \mathbf{R}^N)$;
(b) the cross partial derivatives $\partial_s \partial_{x_j} X(s,t,x)$ and $\partial_{x_j} \partial_s X(s,t,x)$ exist for all $(s,t,x) \in [0,T] \times [0,T] \times \mathbf{R}^N$ and all $j = 1, \dots, N$, and for all $j = 1, \dots, N$,

$$\partial_s \partial_{x_j} X(s,t,x) = \partial_{x_j} \partial_s X(s,t,x) \quad \text{for all } (s,t,x) \in [0,T] \times [0,T] \times \mathbf{R}^N .$$

Besides $\partial_s \partial_{x_j} X \in C([0,T] \times [0,T] \times \mathbf{R}^N; \mathbf{R}^N)$*;*
(c) finally, if V *satisfies the additional condition*

$$(H3) \qquad V \in C^k([0,T] \times \mathbf{R}^N; \mathbf{R}^N) \qquad \textit{and } \nabla_x V \in C^k([0,T] \times \mathbf{R}^N; M_N(\mathbf{R}))\,,$$

for some $k \geq 1$*, then one has*

$$X \in C^{k+1}([0,T] \times [0,T] \times \mathbf{R}^N; \mathbf{R}^N)\,.$$

Proof. Since V satisfies the condition (H1), it satisfies the assumptions of the Cauchy-Lipschitz theorem. Therefore, the differential system of characteristics has a unique C^1 maximal solution γ that is defined on some open interval $I(t,x) \subset [0,T]$ such that $t \in I(t,x)$.

For all $s \in I(t,x)$, one has

$$|\gamma(s)| \leq |x| + \left| \int_t^s |V(\tau, \gamma(\tau))| d\tau \right| \leq |x| + \kappa \left| \int_t^s (1 + |\gamma(\tau)|) d\tau \right|$$

so that, by Gronwall's inequality

$$|\gamma(s)| \leq (|x| + \kappa T) e^{\kappa T}\,, \quad s \in I(t,x)\,.$$

Thus

$$\sup_{s \in I(t,x)} |\gamma(s)| < \infty\,, \qquad \text{and therefore } \overline{I(t,x)} = [0,T]\,.$$

Properties (a) and (b) of the map X follow from the differentiability properties of the solution of a differential equation with respect to the initial data or the parameters in the equation. Property (c) is obtained by applying properties (a) and (b) to any partial derivative of order k of X, i.e. $\partial^\alpha_{(t,x)} X$ with $\alpha \in \mathbf{N}^{N+1}$ and $|\alpha| = k$. ■

As suggested by the proof above, assumption (H2) is essential so that the differential system of characteristics has global solutions. If the vector field V fails to satisfy (H2), it may happen that the characteristic curves are not defined on the same time interval, as shown by the following exercise.

Exercise 2.2 *Pick* $N = 1$ *and set* $V(t,x) := x^2$*. Prove that the Cauchy problem for the Riccati equation*

$$\begin{cases} \dot{\gamma}(s) = \gamma(s)^2\,, \\ \gamma(t) = x\,, \end{cases}$$

has a unique maximal solution, given by the formula

$$\gamma(s) := \frac{x}{1 - (s-t)x}$$

for

$$\begin{cases} s < t + \dfrac{1}{x} & \textit{if } x > 0 \\ s > t + \dfrac{1}{x} & \textit{if } x < 0 \end{cases}$$

As explained in the proof of Theorem 2.1.2, in the constant coefficient case, solving the transport equation by the method of characteristics involves the substitution (change of variables) $\gamma(0) \mapsto \gamma(t)$. While this substitution is trivial in the constant coefficient case, its analogue in the variable coefficient case is not, and is based on the following theorem.

Theorem 2.2.3 *Assume that the vector field V satisfies the conditions (H1) and (H2). Then*
(a) the map X satisfies the flow property:

$$X(t_3, t_2, X(t_2, t_1, x)) = X(t_3, t_1, x) \text{ for all } x \in \mathbf{R}^N \text{ and } t_1, t_2, t_3 \in [0, T]\,;$$

(b) for each $s, t \in [0, T]$, the map $\mathbf{R}^N \ni x \mapsto X(s, t, x) \in \mathbf{R}^N$, denoted by $X(s, t, \cdot)$, is a C^1-diffeomorphism of $\mathbf{R}^N$ onto itself;
(c) set $J(s, t, x) := det(D_x X(s, t, x))$; then J is the solution of the Cauchy problem

$$\begin{cases} \partial_s J(s, t, x) = \operatorname{div}_x V(s, X(s, t, x)) J(s, t, x)\,, & x \in \mathbf{R}^N\,,\ s, t \in [0, T]\,, \\ J(t, t, x) = 1\,; \end{cases}$$

(d) for each $s, t \in [0, T]$, the diffeomorphism $X(s, t, \cdot)$ is orientation preserving. Moreover, if

$$\operatorname{div}_x V(t, x) = 0 \text{ for all } x \in \mathbf{R}^N \text{ and } t \in [0, T]\,,$$

then the diffeomorphism $X(s, t, \cdot)$ leaves the Lebesgue measure $\mathscr{L}^N$ of $\mathbf{R}^N$ invariant. In other words, for each $\phi \in C_c(\mathbf{R}^N)$, one has

$$\int_{\mathbf{R}^N} \phi(X(s, t, x)) dx = \int_{\mathbf{R}^N} \phi(x) dx\,, \quad \text{for all } s, t \in [0, T]\,.$$

Proof. By definition of the map X, observe that both maps

$$t_3 \mapsto X(t_3, t_2, X(t_2, t_1, x))$$

and

$$t_3 \mapsto X(t_3, t_1, x)$$

are integral curves of the vector field V passing through $X(t_2, t_1, x)$ for $t_3 = t_2$. Since V satisfies the assumptions of the Cauchy-Lipschitz theorem because of the condition (H1), there exists at most one solution of the Cauchy problem for the differential system defined by the vector field V, so that both integral curves above must coincide. This proves (a).

Statement (a) implies that

$$X(t, s, \cdot) \circ X(s, t, \cdot) = X(s, t, \cdot) \circ X(t, s, \cdot) = id_{\mathbf{R}^N}$$

for each $s, t \in [0, T]$, so that $X(s, t, \cdot)$ is one-to-one and onto. Since $X(s, t, \cdot) \in C^1(\mathbf{R}^N; \mathbf{R}^N)$ for all $s, t \in [0, T]$ by statement (a) in Theorem 2.2.2, we conclude that $X(s, t, \cdot)$ is a C^1-diffeomorphism from $\mathbf{R}^N$ to $\mathbf{R}^N$, with inverse $X(s, t, \cdot)^{-1} = X(t, s, \cdot)$. This proves (b).

By statements (a) and (b) of Theorem 2.2.2,

$$\partial_s D_x X = D_x \partial_s X \in C([0,T]\times[0,T]\times \mathbf{R}^N; M_N(\mathbf{R}))$$

and

$$\begin{aligned}\partial_s J(s,t,x) &= \partial_s \det(D_x X(s,t,x))\\ &= \det(D_x X(s,t,x))\operatorname{trace}(D_x X(s,t,x)^{-1}\partial_s D_x X(s,t,x))\,.\end{aligned}$$

(Indeed, we recall the classical formula

$$(D\det)(A)\cdot B = \det(A)\operatorname{trace}(A^{-1}B)\,,$$

for all $A \in GL_N(\mathbf{C})$ and $B \in M_N(\mathbf{C})$.) Thus

$$\begin{aligned}\partial_s J(s,t,x) &= \partial_s \det(D_x X(s,t,x))\\ &= J(s,t,x)\operatorname{trace}(D_x X(s,t,x)^{-1} D_x \partial_s X(s,t,x))\\ &= J(s,t,x)\operatorname{trace}(D_x X(s,t,x)^{-1} D_x(V(s,X(s,t,x))))\\ &= J(s,t,x)\operatorname{trace}(D_x X(s,t,x)^{-1}(D_x V)(s,X(s,t,x))D_x X(s,t,x))\\ &= J(s,t,x)\operatorname{trace}((D_x V)(s,X(s,t,x))D_x X(s,t,x) D_x X(s,t,x)^{-1})\\ &= J(s,t,x)\operatorname{trace}((D_x V)(s,X(s,t,x)))\\ &= J(s,t,x)\operatorname{div}_x V(s,X(s,t,x))\,,\end{aligned}$$

since $\operatorname{trace}(AB) = \operatorname{trace}(BA)$ for all $A, B \in M_N(\mathbf{R})$. Finally $X(t,t,x) = x$ so that $D_x X(t,t,x) = I$. Therefore $J(t,t,x) = 1$, and this completes the proof of (c).

By statement (c), one has

$$J(s,t,x) = \exp\left(\int_t^s \operatorname{div}_x V(\tau, X(\tau,t,x))d\tau\right) > 0\,,$$

so that the diffeomorphism $X(s,t,\cdot)$ preserves the orientation. Besides

$$\operatorname{div}_x V \equiv 0 \Rightarrow J(s,t,x) = 1 \text{ for all } s,t\in[0,T] \text{ and } x \in \mathbf{R}^N\,.$$

In this case, the diffeomorphism $X(s,t,\cdot)$ preserves the Lebesgue measure $\mathscr{L}^N$ on $\mathbf{R}^N$, by the usual change of variables formula. ■

2.2.2 Solving the Transport Equation

With the properties of the flow X associated to the vector field V obtained in the previous section, we can state our main result on the solution of the transport equation.

Theorem 2.2.4 *Assume that the vector field V satisfies the conditions (H1) and (H2), and let $f^{in} \in C^1(\mathbf{R}^N)$. Then the Cauchy problem for the transport equation*

$$\begin{cases}\partial_t f(t,x) + V(t,x)\cdot\nabla_x f(t,x) = 0\,, \quad x\in\mathbf{R}^N\,,\ 0<t<T\,,\\ f(0,x) = f^{in}(x)\,,\end{cases}$$

has a unique solution $f \in C^1([0,T]\times\mathbf{R}^N)$. *This solution is given by the formula*

$$f(t,x) = f^{in}(X(0,t,x))\,, \quad \textit{for all } t \in [0,T] \textit{ and } x \in \mathbf{R}^N\,.$$

Proof. The proof of this result closely follows the argument for the constant coefficients case.

Step 1: uniqueness. If $f \in C(\mathbf{R}_+ \times \mathbf{R}^N)$, the map

$$[0,T] \ni t \mapsto f(t,X(t,0,y)) \in \mathbf{R} \text{ is of class } C^1$$

being the composition of the C^1 maps f and $X(\cdot,0,y)$. By the chain rule

$$\begin{aligned}\frac{d}{dt}f(t,X(t,0,y)) &= \partial_t f(t,X(t,0,y)) + \nabla_x f(t,X(t,0,y))\cdot\partial_s X(t,0,y)\\ &= \partial_t f(t,X(t,0,y)) + \nabla_x f(t,X(t,0,y))\cdot V(t,X(t,0,y))\\ &= (\partial_t f + V\cdot\nabla_x f)(t,X(t,0,y)) = 0\,.\end{aligned}$$

Therefore the map $t \mapsto f(t,X(t,0,y))$ is constant on $[0,T]$, so that

$$f(t,X(t,0,y)) = f(0,y) = f^{in}(y)\,.$$

Next we set

$$x := X(t,0,y)\,, \quad \text{so that } y = X(0,t,x)$$

by statement (b) in Theorem 2.2.3. Thus

$$f(t,x) = f^{in}(X(0,t,x))\,, \quad \text{for each } (t,x) \in [0,T]\times\mathbf{R}^N\,.$$

Step 2: existence. First, we check that the formula

$$f(t,x) = f^{in}(X(0,t,x))$$

defines an element of $C^1([0,T]\times\mathbf{R}^N)$. Indeed, the function f so defined is the composition of the maps f^{in} and $(t,x) \mapsto X(0,t,x)$ which are both of class C^1 — see statement (a) in Theorem 2.2.2.

Obviously this function satisfies $f\big|_{t=0} = f^{in}$.

It remains to prove that the formula above defines a solution of the transport equation. This is much less obvious than in the constant coefficient case. The key observation is the following lemma.

Lemma 2.2.5 *One has*

$$\partial_t X(s,t,x) + (V(t,x)\cdot\nabla_x)X(s,t,x) = 0\,, \quad x \in \mathbf{R}^N\,,\ s,t \in (0,T)\,.$$

Taking Lemma 2.2.5 for granted, consider the inner product of each side of the identity above with $\nabla f^{in}(X(0,t,x))$. Observe that, by the chain rule

$$\nabla f^{in}(X(0,t,x))\cdot\partial_t X(0,t,x) = \partial_t(f^{in}(X(0,t,x)))\,,$$

while

$$\begin{aligned}\nabla f^{in}(X(0,t,x)) \cdot (V(t,x)\cdot\nabla_x)X(0,t,x) \\ &= \langle df^{in}(X(0,t,x)), D_xX(0,t,x)V(t,x)\rangle \\ &= D_x(f^{in}(X(0,t,x)))V(t,x) \\ &= V(t,x)\cdot\nabla_x(f^{in}(X(0,t,x)))\,.\end{aligned}$$

Hence

$$\begin{aligned}&(\partial_t + V(t,x)\cdot\nabla_x)(f^{in}(X(0,t,x))) \\ &= \nabla f^{in}(X(0,t,x))\cdot(\partial_t X(0,t,x) + (V(t,x)\cdot\nabla_x)X(0,t,x) = 0\,.\end{aligned}$$

(In fact, one could also deduce the first equality above directly from the chain rule, by considering $(\partial_t + V(t,x)\cdot\nabla_x)$ as a first order linear differential operator, i.e. the derivation along the vector field $(1, V(t,x)) \in \mathbf{R}^{N+1}$.) ∎

Proof of Lemma 2.2.5. Start from the flow property in statement (a) of Theorem 2.2.3:

$$X(t_3, t_2, X(t_2, t_1, x)) = X(t_3, t_1, x)$$

and differentiate both sides of this identity with respect to the variable t_2. Since the right hand side is independent of t_2, one finds that

$$\begin{aligned}&\partial_t X(t_3, t_2, X(t_2, t_1, x)) + D_xX(t_3, t_2, X(t_2, t_1, x))\partial_s X(t_2, t_1, x) \\ &= \partial_t X(t_3, t_2, X(t_2, t_1, x)) + D_xX(t_3, t_2, X(t_2, t_1, x))V(t_2, X(t_2, t_1, x)) = 0\,.\end{aligned}$$

Set $t_2 = t_1 = t$ and $t_3 = s$ in the identity above leads to

$$\partial_t X(s,t,x) + D_xX(s,t,x)V(t,x) = 0\,,$$

which is precisely the desired equality. ∎

2.3 Conservative Transport and Weak Solutions

We first recall the notion of transportation of measures.

Definition 2.3.1 *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces (meaning that $\mathcal{A}$ and $\mathcal{B}$ are σ-algebras of subsets of X and Y respectively). Let $T : X \to Y$ be an $(\mathcal{A}, \mathcal{B})$-measurable map, and let μ be a positive measure on $(X, \mathcal{A})$. The formula*

$$\nu(B) := \mu(T^{-1}(B))$$

defines a positive measure on $(Y, \mathcal{B})$, denoted

$$\nu =: T\#\mu\,,$$

and referred to as "the push-forward of the measure μ under the map T".

The definition of $\nu = T\#\mu$ can be equivalently recast as follows:

$$\int_Y \mathbf{1}_B(y)\nu(dy) = \int_X \mathbf{1}_{T^{-1}(B)}(x)\mu(dx) = \int_X \mathbf{1}_B(T(x))\mu(dx)$$

since

$$\mathbf{1}_{T^{-1}(B)} = \mathbf{1}_B \circ T\,.$$

This formula is easily generalized in the next proposition.

Proposition 2.3.2 *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be measurable spaces, let $T : X \to Y$ be an $(\mathcal{A}, \mathcal{B})$-measurable map, and let μ be a positive measure on $(X, \mathcal{A})$. Set $\nu := T\#\mu$. Then*

$$\phi \in L^1(Y, \nu) \Rightarrow \phi \circ T \in L^1(X, \mu)$$

and

$$\int_Y \phi(y)\nu(dy) = \int_X \phi(T(x))\mu(dx)\,.$$

Proof. We already know that the sought formula is true whenever $\phi = \mathbf{1}_B$ with $B \in \mathcal{B}$. By linearity, it is also true whenever ϕ is a linear combination of finitely many indicator functions of the form $\mathbf{1}_{B_i}$ for $i = 1, \ldots, N$. By density of the linear span of integrable indicator functions in $L^1(Y, \nu)$, the formula holds for all $\phi \in L^1(Y, \nu)$. ■

Exercise 2.3 *Let $f \in L^1(\mathbf{R}^N)$ satisfy $f \geq 0$ a.e. on $\mathbf{R}^N$, and $T : \mathbf{R}^N \to \mathbf{R}^N$ be a C^1-diffeomorphism. Compute the push-forward measure $T\#(f\mathscr{L}^N)$. (Answer: $f \circ T^{-1}|\,det(DT \circ T^{-1})|^{-1}\mathscr{L}^N$. Hint: use the change-of-variables formula.)*

In the sequel, we shall often need the notion of "weak solution" of 1st order PDEs equations. The notion of weak solution of a PDE is most easily defined within the theory of distributions. However, in the case of 1st order PDEs, and especially in the context of statistical mechanics, it is most convenient to consider weak solutions that are measures, instead of more general distributions. One reason for this is that the solutions or 1st order PDEs that we are interested in are distributions functions of particles systems, which are nonnegative by definition. Since a positive distribution is of order 0, and thus is identified with a Radon measure (see for instance Theorem 3.2.11 in [6]), solutions of Liouville type equations of physical interest in statistical mechanics cannot be more singular than positive measures.

Definition 2.3.3 *Let $V \in C([0, T] \times \mathbf{R}^N)$, and let $\mu^{in} \in \mathcal{M}(\mathbf{R}^N)$. A weak solution of the Cauchy problem for the conservative transport equation*

$$\begin{cases} \partial_t \mu + \operatorname{div}_x(\mu V) = 0\,, \\ \mu\big|_{t=0} = \mu^{in}\,, \end{cases}$$

is an element of μ *of*[1] $C([0,T]; w-\mathcal{M}(\mathbf{R}^N))$ *that satisfies the initial condition and the equality*

$$\int_0^T \int_{\mathbf{R}^N} (\partial_t \phi(t,x) + V(t,x) \cdot \nabla_x \phi(t,x)) \mu(t,dx) dt = 0$$

for each $\phi \in C_c^1((0,T) \times \mathbf{R}^N)$.

This notion of weak solution and the classical notion of solution are related as follows.

Proposition 2.3.4 *Let* $f \in C^1([0,T] \times \mathbf{R}^N)$. *Then*

$$\partial_t f + \operatorname{div}_x(fV) = 0 \text{ on } [0,T] \times \mathbf{R}^N$$

if and only if

$$\int_0^T \int_{\mathbf{R}^N} (\partial_t \phi(t,x) + V(t,x) \cdot \nabla_x \phi(t,x)) f(t,x) dx dt = 0$$

for each $\phi \in C_c^1((0,T) \times \mathbf{R}^N)$.

Proof. Consider the vector field

$$W : (t,x) \mapsto (\phi f, \phi f V)(t,x)$$

defined on $[0,T] \times \mathbf{R}^N$ with values in $\mathbf{R} \times \mathbf{R}^N$. By construction, $W \in C_c^1((0,T) \times \mathbf{R}^N; \mathbf{R} \times \mathbf{R}^N)$, so that, by Green's formula applied in the domain $(0,T) \times \mathbf{R}^N$, one has

$$\begin{aligned} 0 &= \iint_{(0,T)\times\mathbf{R}^N} \operatorname{div}_{t,x}(f\phi, f\phi V)(t,x) dx dt \\ &= \iint_{(0,T)\times\mathbf{R}^N} (\partial_t \phi(t,x) + V(t,x) \cdot \nabla_x \phi(t,x)) f(t,x) dx dt \\ &+ \iint_{(0,T)\times\mathbf{R}^N} \phi(t,x)(\partial_t f(t,x) + V(t,x) \cdot \nabla_x f(t,x)) dx dt \,. \end{aligned}$$

Therefore

$$\begin{aligned} &\iint_{(0,T)\times\mathbf{R}^N} (\partial_t \phi(t,x) + V(t,x) \cdot \nabla_x \phi(t,x)) f(t,x) dx dt \\ &= -\iint_{(0,T)\times\mathbf{R}^N} \phi(t,x)(\partial_t f(t,x) + \operatorname{div}_x(f(t,x)V(t,x))) dx dt \end{aligned}$$

[1]The notation $w-\mathcal{M}(\mathbf{R}^N)$ designates the set of Radon measures on $\mathbf{R}^N$ equipped with its weak topology, i.e. the topology defined by the family of seminorms

$$C_c(\mathbf{R}^N) \ni \phi \mapsto \left| \int_{\mathbf{R}^N} \phi(x) \mu(dx) \right| .$$

for each $\phi \in C_c^1((0,T) \times \mathbf{R}^N)$.

Thus, if the integral on the left hand side of the inequality above is 0 for all $\phi \in C_c^1((0,T) \times \mathbf{R}^N)$, we conclude that the continuous function $\partial_t f + \mathrm{div}_x(fV)$ satisfies

$$\iint_{(0,T)\times\mathbf{R}^N} \phi(t,x)(\partial_t f(t,x) + \mathrm{div}_x(f(t,x)V(t,x)))dxdt = 0$$

for each $\phi \in C_c^1((0,T) \times \mathbf{R}^N)$, so that

$$\partial_t f + \mathrm{div}_x(fV) = 0 \quad \text{ on } (0,T) \times \mathbf{R}^N .$$

Conversely, if $\partial_t f + \mathrm{div}_x(fV) = 0$ on $(0,T) \times \mathbf{R}^N$, the integral on the right hand side is zero for all $\phi \in C_c^1((0,T) \times \mathbf{R}^N)$, so that

$$\iint_{(0,T)\times\mathbf{R}^N} (\partial_t \phi(t,x) + V(t,x) \cdot \nabla_x \phi(t,x))f(t,x)dxdt = 0$$

for each $\phi \in C_c^1((0,T) \times \mathbf{R}^N)$, which means precisely that f is a weak solution of the transport equation above in $(0,T) \times \mathbf{R}^N$. ■

Notice that we consider here the 1st order PDE

$$\partial_t \mu + \mathrm{div}_x(\mu V) = 0 ,$$

instead of

$$\partial_t \mu + V \cdot \nabla_x \mu = 0$$

as in the previous section. The former PDE is referred to as being "in conservative form" for reasons that will be explained below.

Theorem 2.3.5 *Let $V \equiv V(t,x) \in \mathbf{R}^N$ satisfy assumptions (H1)-(H2), and let $\mu^{in} \in \mathcal{M}^+(\mathbf{R}^N)$ (the set of positive Radon measures on $\mathbf{R}^N$). Then the Cauchy problem*

$$\begin{cases} \partial_t \mu + \mathrm{div}_x(\mu V) = 0 , \\ \mu\big|_{t=0} = \mu^{in} , \end{cases}$$

has a unique weak solution μ. This weak solution is given by the formula

$$\mu(t) = X(t,0,\cdot)\#\mu^{in} , \quad t \in [0,T].$$

In particular, $\mu(t) \in \mathcal{M}^+(\mathbf{R}^N)$ for all $t \in [0,T]$, and if μ^{in} is a bounded measure, then $\mu(t)$ is a bounded measure for all $t \in [0,T]$ and one has

$$\int_{\mathbf{R}^N} \mu(t,dx) = \int_{\mathbf{R}^N} \mu^{in}(dx) .$$

Proof. As in the proof of Theorem 2.2.4, we split the proof in two steps.

Step 1: existence. Let $\phi \equiv \phi(t,x)$ be an element of $C^1_c((0;T) \times \mathbf{R}^N)$; by statement (a) in Theorem 2.2.2, the function

$$t \mapsto \int_{\mathbf{R}^N} \phi(t, X(t,0,y))\mu^{in}(dy)$$

is of class C^1 on $[0,T]$. Denoting $\mu(t) := X(t,0,\cdot)\#\mu^{in}$, one has

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbf{R}^N} \phi(t, X(t,0,y))\mu^{in}(dy)\\
&\qquad = \int_{\mathbf{R}^N} (\partial_t \phi(t, X(0,t,y)) + \nabla_x \phi(t, X(t,0,y)) \cdot \partial_t X(t,0,y))\mu^{in}(dy)\\
&\qquad = \int_{\mathbf{R}^N} (\partial_t \phi(t, X(0,t,y)) + \nabla_x \phi(t, X(t,0,y)) \cdot V(t, X(t,0,y)))\mu^{in}(dy)\\
&\qquad = \int_{\mathbf{R}^N} (\partial_t \phi(t,x) + V(t,x) \cdot \nabla_x \phi(t,x))\mu(t,dx)\,,
\end{aligned}$$

after substituting x to $X(0,t,y)$. Integrating both sides of the equality above with respect to the variable t, one finds that

$$\begin{aligned}
0 &= \left[\int_{\mathbf{R}^N} \phi(t, X(t,0,y))\mu^{in}(dy)\right]_{t=0}^{t=T}\\
&= \int_0^T \left(\frac{d}{dt}\int_{\mathbf{R}^N} \phi(t, X(t,0,y))\mu^{in}(dy)\right) dt\\
&= \int_0^T \int_{\mathbf{R}^N} (\partial_t \phi(t,x) + V(t,x) \cdot \nabla \phi(t,x))\mu(t,dx)dt\,,
\end{aligned}$$

so that μ is a weak solution of the conservative transport equation. Since it obviously satisfies the initial condition, μ is a weak solution of the Cauchy problem.

Step 2: uniqueness. Let μ be a weak solution of the Cauchy problem, and let $\psi \in C^1_c(\mathbf{R}^N)$. Set

$$\nu(t) := X(0,t,\cdot)\#\mu(t)\,.$$

We compute

$$\frac{d}{dt}\int_{\mathbf{R}^N} \psi(x)\nu(t,dx) \text{ in } \mathcal{D}'((0,T))\,.$$

Let $\chi \in C^\infty_c((0,T))$; then

$$-\int_0^T \chi'(t)\left(\int_{\mathbf{R}^N} \psi(x)\nu(t,dx)\right) dt = -\int_0^T \int_{\mathbf{R}^N} \chi'(t)\psi(X(0,t,y))\mu(t,dy)dt\,.$$

By Theorem 2.2.4, we already know that the map $(t,x) \mapsto \psi(X(0,t,y)$ is of class C^1 on $[0,T] \times \mathbf{R}^N$ and satisfies

$$(\partial_t + V(t,x) \cdot \nabla_x)\psi(X(0,t,x)) = 0\,, \qquad \text{for all } (t,x) \in [0,T] \times \mathbf{R}^N\,.$$

Consider then the function Ψ defined by the formula $\Psi(t,y)=\chi(t)\psi(X(0,t,y))$; obviously $\Psi\in C^1([0,T]\times\mathbf{R}^N)$ and

$$(\partial_t+V(t,y)\cdot\nabla_y)\Psi(t,y)=\chi'(t)\psi(X(0,t,y))\,,\qquad\text{for all } (t,y)\in[0,T]\times\mathbf{R}^N\,.$$

If we knew that $\operatorname{supp}(\Psi)$ is compact in $(0,T)\times\mathbf{R}^N$, we would conclude that

$$\begin{aligned}&-\int_0^T\chi'(t)\left(\int_{\mathbf{R}^N}\psi(x)\nu(t,dx)\right)dt\\ =&-\int_0^T\int_{\mathbf{R}^N}(\partial_t+V(t,y)\cdot\nabla_x)\Psi(t,y)\mu(t,dy)dt=0\,,\end{aligned}$$

since μ is a weak solution of the transport equation.Therefore, the continuous function

$$[0,T]\ni t\mapsto\int_{\mathbf{R}^N}\psi(x)\nu(t,dx)$$

satisfies

$$\frac{d}{dt}\int_{\mathbf{R}^N}\psi(x)\nu(t,dx)=0\text{ in }\mathcal{D}'((0,T))\,.$$

This function is therefore a constant on $[0,T]$ so that

$$\int_{\mathbf{R}^N}\psi(x)\nu(t,dx)=\int_{\mathbf{R}^N}\psi(x)\nu(0,dx)=\int_{\mathbf{R}^N}\psi(x)\mu(0,dx)=\int_{\mathbf{R}^N}\psi(x)\mu^{in}(dx)\,.$$

Since this identity holds for each $\psi\in C_c^1(\mathbf{R}^N)$, we conclude that

$$\nu(t)=X(0,t,\cdot)\#\mu(t)=\mu^{in}\,,$$

so that

$$\mu(t)=X(t,0,\cdot)\#\mu^{in}\,,\qquad\text{for all } t\in[0,T]\,.$$

It remains to prove that $\operatorname{supp}(\Psi)$ is compact in $(0,T)\times\mathbf{R}^N$. As already observed in the proof of Theorem 2.2.2, the condition (H2) on the vector field V implies that

$$|X(s,t,y)|\le(|y|+\kappa T)e^{\kappa T}\,,\qquad\text{for all } y\in\mathbf{R}^N\text{ and } s,t,\in[0,T]\,.$$

Thus

$$\operatorname{supp}(\psi)\subset B(0,R)\Rightarrow\operatorname{supp}(\psi\circ X(0,t,\cdot))\subset B(0,(R+\kappa T)e^{\kappa T})$$

for all $t\in[0,T]$. Since χ has support in $[\epsilon,T-\epsilon]$ for some $\epsilon>0$, we conclude that

$$\operatorname{supp}(\Psi)\subset[\epsilon,T-\epsilon]\times B(0,(R+\kappa T)e^{\kappa T})\,,$$

which concludes the proof. ■

Finally, we specialize Theorem 2.3.5 to the case of C^1 initial data.

Theorem 2.3.6 *Let $V \equiv V(t,x) \in \mathbf{R}^N$ satisfy assumptions (H1), (H2) and (H3) with $k = 1$, and let $f^{in} \in C^1(\mathbf{R}^N)$. Then the Cauchy problem*

$$\begin{cases} \partial_t f(t,x) + \operatorname{div}_x(f(t,x)V(t,x)) = 0\,, \\ f\big|_{t=0} = f^{in}\,, \end{cases}$$

has a unique solution $f \in C^1([0,T]\times\mathbf{R}^N)$. This solution is given by the formula

$$f(t,x) = f^{in}(X(0,t,x))J(0,t,x)\,, \quad (t,x) \in [0,T]\times\mathbf{R}^N.$$

In particular, $f(t,\cdot) \in L^1(\mathbf{R}^N)$ for all $t \in [0,T]$ if $f^{in} \in L^1(\mathbf{R}^N)$, and one has

$$\int_{\mathbf{R}^N} f(t,x)dx = \int_{\mathbf{R}^N} f^{in}(x)dx\,.$$

Proof. As above, the proof is split in two steps.

Step 1: uniqueness

Since the equation is linear, the uniqueness of the solution reduces to the following statement: let $g \in C^1([0,T]\times\mathbf{R}^N)$ satisfy

$$\begin{cases} \partial_g f(t,x) + \operatorname{div}_x(g(t,x)V(t,x)) = 0\,, \\ g\big|_{t=0} = 0\,. \end{cases}$$

Then $g = 0$.

Expanding the second term on the left hand side of the transport equation, one has

$$(\partial_t + V(t,x)\cdot\nabla_x)g(t,x) = -g(t,x)\operatorname{div}_x V(t,x)\,.$$

Let $(s,t,y) \mapsto X(s,t,y)$ be the characteristic flow of the transport operator $\partial_t + V(t,x)\cdot\nabla_x$ above. The function defined as follows

$$(0,T)\times\mathbf{R}^N \ni (t,y) \mapsto g(t,X(t,0,y))$$

is of class C^1 and satisfies

$$\begin{cases} \dfrac{d}{dt}g(t,X(t,0,y)) = (\partial_t g + V\cdot\nabla_x g)(t,X(t,0,y) \\ \qquad\qquad\qquad\qquad = -g(t,X(t,0,y))(\operatorname{div}_x V)(t,X(t,0,y))\,, \\ g\big|_{t=0} = 0\,. \end{cases}$$

This is a linear ODE with variable amplification or damping rate, and one checks easily that implies that

$$g(t,X(t,0,y)) = 0\,, \quad 0 < t < T \text{ and } y \in \mathbf{R}^N\,.$$

Since $X(t,0,\cdot)$ is a C^1-diffeomorphism for all $t \in [0,T]$, we conclude that

$$g(t,x) = 0\,, \quad \text{for all } x \in \mathbf{R}^N \text{ and all } t \in [0,T]\,.$$

This proves the uniqueness of the solution.

Step 2: Conversely, split f^{in} as follows:

$$f^{in} = f_1^{in} - f_2^{in}\,, \qquad \text{with } f_1^{in} := \sqrt{1+(f^{in})^2} \text{ and } f_2^{in} := f_1^{in} - f^{in}\,.$$

By Theorem 2.3.5, the transported measures

$$\mu_1(t) := X(t,0,\cdot)\#(f_1^{in}\mathscr{L}^N) \quad \text{ and } \mu_2(t) := X(t,0,\cdot)\#(f_2^{in}\mathscr{L}^N)$$

are both weak solutions of the transport equation in conservation form

$$\partial_t \mu + \operatorname{div}_x(\mu V) = 0\,,$$

with the initial data prescribed above. By linearity of this equation, the signed measure-valued function $t \mapsto \mu_1(t) - \mu_2(t)$ that is defined for all $t \in [0,T]$ satisfies

$$\begin{cases} \partial_t(\mu_1 - \mu_2) + \operatorname{div}_x((\mu_1 - \mu_2)V) = 0\,, \\ (\mu_1(0) - \mu_2(0)) = f^{in}\,. \end{cases}$$

On the other hand,

$$\begin{aligned} \mu_1(t) &:= X(t,0,\cdot)\#(f_1^{in}\mathscr{L}^N) = f_1(t,\cdot)\mathscr{L}^N \\ &\quad \text{with } f_1(t,x) = f_1^{in}(X(0,t,x))J(0,t,x)\,, \\ \mu_2(t) &:= X(t,0,\cdot)\#(f_2^{in}\mathscr{L}^N = f_2(t,\cdot)\mathscr{L}^N \\ &\quad \text{with } f_2(t,x) = f_2^{in}(X(0,t,x))J(0,t,x)\,, \end{aligned}$$

and this gives the formula for $f = f_1 - f_2$.

Together with assumption (H3) on V with $k=1$, these formulas show that $f \in C^1([0,T]\times \mathbf{R}^N)$.

So far, we only know that $f(t,\cdot)\mathscr{L}^N$ is a weak solution of the conservative transport equation and that its is given by the formula above. Since we already know that $f \in C^1([0,T]\times \mathbf{R}^N)$, we conclude from Proposition 2.3.4 that f is a classical solution of the conservative transport equation. ■

Chapter 3

From Particle Systems to Mean Field PDEs

3.1 A general formalism for mean field limits in classical mechanics

We first introduce a formalism for mean field limits in classical mechanics that encompasses all the examples discussed above.

Consider a system of N particles, whose state at time t is defined by phase space coordinates $\hat{z}_1(t), \ldots, \hat{z}_N(t) \in \mathbf{R}^d$. For instance, z_j is the position x_j of the jth vortex center in the case of the two dimensional Euler equations for incompressible fluids, and the phase space dimension is $d = 2$. In the case of the Vlasov-Poisson system, the phase space is $\mathbf{R}^3 \times \mathbf{R}^3 \simeq \mathbf{R}^6$, so that $d = 6$, and $z_j = (x_j, v_j)$, where x_j and v_j are respectively the position and the velocity of the jth particle.

The interaction between the ith and the jth particle is given by $K(\hat{z}_i, \hat{z}_j)$, where

$$K : \mathbf{R}^d \times \mathbf{R}^d \to \mathbf{R}^d$$

is a map whose properties will be discussed below.

The evolution of $\hat{z}_1(t), \ldots, \hat{z}_N(t) \in \mathbf{R}^d$ is governed by the system of ODEs

$$\frac{d\hat{z}_i}{dt}(t) = \sum_{\substack{j=1\\ j\neq i}}^{N} K(\hat{z}_i(t), \hat{z}_j(t)) \,, \quad i, j = 1, \ldots, N \,.$$

Problem: to describe the behavior of $\hat{z}_1(t), \ldots, \hat{z}_N(t) \in \mathbf{R}^d$ in the large N limit and in some appropriate time scale.

First we need to rescale the time variable, and introduce a new time variable $\hat{t}$ so that, in new time scale, the action on any one of the N particles due to the

$N-1$ other particles is of order 1 as $N \to +\infty$. In other words, the new time variable $\hat{t}$ is chosen so that

$$\frac{d\hat{z}_i}{d\hat{t}} = O(1) \text{ for each } i = 1, \dots, N \text{ as } N \to \infty \, .$$

The action on the ith particle of the $N-1$ other particles is

$$\sum_{\substack{j=1 \\ j \neq i}}^{N} K(\hat{z}_i, \hat{z}_j) \, ,$$

and it obviously contains $N-1$ terms of order 1 (assuming each term $K(\hat{z}_i, \hat{z}_j)$ to be of order 1, for instance). Set $\hat{t} = t/N$, then

$$\frac{d\hat{z}_i}{d\hat{t}} = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} K(\hat{z}_i, \hat{z}_j) \, .$$

From now on, we drop hats on all variables and consider as our starting point the rescaled problem

$$\dot{z}_i(t) = \frac{1}{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} K(z_i(t), z_j(t)) \, , \qquad i = 1, \dots, N \, .$$

At this point, we introduce an important assumption on the interaction kernel: the action of the jth particle on the ith particle must exactly balance the action of the ith particle on the jth particle. When the interaction is a force, this is precisely Newton's third law of mechanics. Thus we assume that the interaction kernel satisfies

$$K(z, z') = -K(z', z) \, , \qquad z, z' \in \mathbf{R}^d \, .$$

We have assumed here that the interaction kernel K is defined on the whole $\mathbf{R}^d \times \mathbf{R}^d$ space; in particular, the condition above implies that K vanishes identically on the diagonal, i.e.

$$K(z, z) = 0 \, , \qquad z \in \mathbf{R}^d \, .$$

Hence the restriction $j \neq i$ can be removed in the summation that appears on the right hand side of the ODEs governing the N-particle dynamics: since $K(z_i(t), z_i(t)) = 0$ for all $i = 1, \dots, N$, one has

$$\dot{z}_i(t) = \frac{1}{N} \sum_{j=1}^{N} K(z_i(t), z_j(t)) \qquad i = 1, \dots, N \, .$$

At this point, we can explain the key idea in the mean field limit: if the points $z_j(t)$ for $j = 1, \ldots, N$ are "distributed at time t under the probability measure $f(t, dz)$" in the large N limit, then, one expects that

$$\frac{1}{N}\sum_{j=1}^{N} K(z_i(t), z_j(t)) \to \int_{\mathbf{R}^d} K(z_i(t), z')f(t, dz') \qquad \text{as } N \to +\infty\,.$$

This suggests replacing the N-particle system of differential equations with the single differential equation

$$\dot{z}(t) = \int_{\mathbf{R}^d} K(z(t), z')f(t, dz')\,.$$

Here $f(t, dz)$ is unknown, as is $z(t)$, so that it seems that this single differential equation is insufficient to determine both these unknowns.

But one recognizes in the equality above the equation of characteristics for the mean field PDE

$$\partial_t f + \operatorname{div}_z(f\mathcal{K}f) = 0\,,$$

where the notation $\mathcal{K}$ designates the integral operator defined by the formula

$$\mathcal{K}f(t, z) := \int_{\mathbf{R}^d} K(z, z')f(t, dz')\,.$$

Now, this is a single PDE (in fact an integro-differential equation) for the single unknown f.

A priori f is a time dependent Borel probability measure on $\mathbf{R}^d$, so that the mean field PDE is to be understood in the sense of distributions on $\mathbf{R}^d$. In other words,

$$\frac{d}{dt}\int_{\mathbf{R}^d} \phi(z)f(t, dz) = \int_{\mathbf{R}^d} \mathcal{K}f(t, z) \cdot \nabla\phi(z)f(t, dz)$$

for each test function[1] $\phi \in C_b^1(\mathbf{R}^d)$.

A very important mathematical object in the mathematical theory of the mean field limit is the empirical measure, which is defined below.

Definition 3.1.1 *To each N-tuple $Z_N = (z_1, \ldots, z_N) \in (\mathbf{R}^d)^N \simeq \mathbf{R}^{dN}$, one associates its empirical measure*

$$\mu_{Z_N} := \frac{1}{N}\sum_{j=1}^{N} \delta_{z_j}\,.$$

[1] For each topological space X and each finite dimensional vector space E on $\mathbf{R}$, we denote by $C_b(X, E)$ the set of continuous functions defined on X with values in E that are bounded on X. For each $n, k \geq 1$, we denote by $C_b^k(\mathbf{R}^n, E)$ the set of functions of class C^k defined on $\mathbf{R}^n$ with values in E all of whose partial derivatives are bounded on $\mathbf{R}^n$: for each norm $|\cdot|_E$ on E, one has

$$C_b^k(\mathbf{R}^n, E) := \{f \in C^k(\mathbf{R}^n, E) \text{ s.t. } \sup_{x\in\mathbf{R}^n} |\partial^\alpha f(x)|_E < \infty \text{ for each } \alpha \in \mathbf{N}^n\}\,.$$

We also denote $C_b(X) := C_b(X, \mathbf{R})$ and $C_b^k(\mathbf{R}^n) := C_b^k(\mathbf{R}^n, \mathbf{R})$.

The empirical measure of a N-tuple $Z_N \in (\mathbf{R}^d)^N$ is a Borel probability measure on $\mathbf{R}^d$. As we shall see in the next section, the N-tuple

$$t \mapsto Z_N(t) = (z_1(t), \ldots, z_N(t))$$

is a solution of the N-particle ODE system

$$\dot{z}_i(t) = \frac{1}{N} \sum_{j=1}^{N} K(z_i(t), z_j(t)), \quad i = 1, \ldots, N$$

if and only if the empirical measure $\mu_{Z_N(t)}$ is a solution of the mean field PDE

$$\partial_t \mu_{Z_N(t)} + \operatorname{div}_z(\mu_{Z_N(t)} \mathcal{K} \mu_{Z_N(t)}) = 0 .$$

We conclude this section with a few exercises where the reader can verify that the formalism introduced here encompasses the two main examples of mean-field theories presented above, i.e. the two dimensional Euler equation and the Vlasov-Poisson system.

Exercise:
1) Compute $\Delta \ln |x|$ in the sense of distributions on $\mathbf{R}^2$ (answer: $2\pi\delta_0$).
2) Define

$$K(x, x') := -\tfrac{1}{2\pi} \frac{J(x - x')}{|x - x'|^2}, \quad x \neq x' \in \mathbf{R}^2 ,$$

where J designates the rotation of an angle $-\frac{\pi}{2}$:

$$J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

For each $\omega \equiv \omega(t, x)$ belonging to $C^1_b(\mathbf{R}_+ \times \mathbf{R}^2)$ such that $\operatorname{supp}(\omega(t, \cdot))$ is compact for each $t \geq 0$, prove that the vector field u defined by

$$u(t, x) := \int_{\mathbf{R}^2} K(x, x')\omega(x')dx'$$

is of class C^1_b on $\mathbf{R}_+ \times \mathbf{R}^2$ and satisfies

$$\operatorname{div}_x u(t, x) = 0 , \quad \operatorname{div}_x(Ju)(t, x) = \omega(t, x) .$$

3) Conclude that the two dimensional Euler equation for incompressible fluids can be put in the formalism described in the present section, except for the fact that the interaction kernel K is singular on the diagonal of $\mathbf{R}^2 \times \mathbf{R}^2$.

Exercise: Let (f, ϕ) be a solution of the Vlasov-Poisson system such that $f \in C^\infty(\mathbf{R}_+ \times \mathbf{R}^3 \times \mathbf{R}^3)$ and $\phi \in C^\infty(\mathbf{R}_+ \times \mathbf{R}^3)$, while $(x, v) \mapsto f(t, x, v)$ and $x \mapsto \phi(t, x)$ belong to $\mathcal{S}(\mathbf{R}^3 \times \mathbf{R}^3)$ and $\mathcal{S}(\mathbf{R}^3)$ respectively. Assume further that

$$\iint_{\mathbf{R}^3 \times \mathbf{R}^3} f(0, x, v)dxdv = 1 , \quad \text{and} \quad \iint_{\mathbf{R}^3 \times \mathbf{R}^3} vf(0, x, v)dxdv = 0 .$$

1) Prove that

$$\iint_{\mathbf{R}^3\times\mathbf{R}^3} f(t,x,v)dxdv=1 \quad \text{and} \quad \iint_{\mathbf{R}^3\times\mathbf{R}^3} vf(t,x,v)dxdv=0 \quad \text{for all } t\geq 0\,.$$

2) Set $z=(x,v)$ and

$$K(z,z')=K(x,v,x',v'):=\left(v-v',\tfrac{q^2}{4\pi\epsilon_0 m}\frac{x-x'}{|x-x'|^3}\right)\,.$$

Prove that

$$\iint_{\mathbf{R}^3\times\mathbf{R}^3} K(x,v,x',v')f(t,x',v')dx'dv'=(v,-\tfrac{q}{m}\nabla_x\phi(t,x))\,,$$

where

$$-\Delta_x\phi(t,x)=\frac{q}{\epsilon_0}\int_{\mathbf{R}^3} f(t,x,v)dv\,.$$

3) Conclude that the Vlasov-Poisson system can be put in the formalism described in the present section, except for the fact that the interaction kernel K is singular on the set $\{(x,v,x',v')\in(\mathbf{R}^3)^4 \text{ s.t. } x=x'\}$.

3.2 The mean field characteristic flow

Henceforth we assume that the interaction kernel $K:\mathbf{R}^d\times\mathbf{R}^d\to\mathbf{R}^d$ satisfies the following assumptions.

First K is skew-symmetric:

$$(HK1) \qquad K(z,z')=-K(z',z) \quad \text{for all } z,z'\in\mathbf{R}^d\,.$$

Besides, $K\in C^1(\mathbf{R}^d\times\mathbf{R}^d;\mathbf{R}^d)$, with bounded partial derivatives of order 1. In other words, there exists a constant $L\geq 0$ such that

$$(HK2) \qquad \sup_{z'\in\mathbf{R}^d}|\nabla_z K(z,z')|\leq L\,, \quad \text{and} \quad \sup_{z\in\mathbf{R}^d}|\nabla_{z'}K(z,z')|\leq L\,.$$

Applying the mean value theorem shows that assumption (HK2) implies that K is Lipschitz continuous in z uniformly in z' (and conversely):

$$\left\{\begin{aligned}&\sup_{z'\in\mathbf{R}^d}|K(z_1,z')-K(z_2,z')|\leq L|z_1-z_2|\,,\\&\sup_{z\in\mathbf{R}^d}|K(z,z_1)-K(z,z_2)|\leq L|z_1-z_2|\,.\end{aligned}\right.$$

Assumption (HK2) also implies that K grows at most linearly at infinity:

$$|K(z,z')|\leq L(|z|+|z'|)\,,\quad z,z'\in\mathbf{R}^d\,.$$

Notice also that the integral operator $\mathcal{K}$ can be extended to the set of Borel probability measures[2] on $\mathbf{R}^d$ with finite moment of order 1, i.e.

$$\mathcal{P}_1(\mathbf{R}^d) := \left\{ p \in \mathcal{P}(\mathbf{R}^d) \text{ s.t. } \int_{\mathbf{R}^d} |z| p(dz) < \infty \right\} ,$$

in the obvious manner, i.e.

$$\mathcal{K}p(z) := \int_{\mathbf{R}^d} K(z, z') p(dz') .$$

The extended operator $\mathcal{K}$ so defined maps $\mathcal{P}_1(\mathbf{R}^d)$ into the class $\mathrm{Lip}(\mathbf{R}^d; \mathbf{R}^d)$ of Lipschitz continuous vector fields on $\mathbf{R}^d$.

With the assumptions above, one easily arrives at the existence and uniqueness theory for the N-body ODE system.

Theorem 3.2.1 *Assume that the interaction kernel $K \in C^1(\mathbf{R}^d \times \mathbf{R}^d, \mathbf{R}^d)$ satisfies assumptions (HK1-HK2). Then*
a) for each $N \geq 1$ and each N-tuple $Z_N^{in} = (z_1^{in}, \dots, z_N^{in})$, the Cauchy problem for the N-particle ODE system

$$\begin{cases} \dot{z}_i(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} K(z_i(t), z_j(t)) , \qquad i = 1, \dots, N , \\ z_i(0) = z_i^{in} , \end{cases}$$

has a unique solution of class C^1 on $\mathbf{R}$

$$t \mapsto Z_N(t) = (z_1(t), \dots, z_N(t)) =: T_t Z_N^{in} ;$$

b) the empirical measure $f(t, dz) := \mu_{T_t Z_N^{in}}$ is a weak solution of the Cauchy problem for the mean field PDE

$$\begin{cases} \partial_t f + \mathrm{div}_z (f \mathcal{K} f) = 0 , \\ f\big|_{t=0} = f^{in} . \end{cases}$$

Statement a) follows from the Cauchy-Lipschitz theorem. Statement b) follows from the method of characteristics for the transport equation. For the sake of being complete, we sketch the main steps in the proof of statement b), and leave the details as an exercise to be treated by the reader.

Exercise: Let $b \equiv b(t, y) \in C([0, \tau]; \mathbf{R}^d)$ be such that $D_y b \in C([0, \tau]; \mathbf{R}^d)$ and

$$(H) \qquad |b(t, y)| \leq \kappa (1 + |y|)$$

for all $t \in [0, \tau]$ and $y \in \mathbf{R}^d$, where κ is a positive constant.

[2] Henceforth, the set of Borel probability measures on $\mathbf{R}^d$ will be denoted by $\mathcal{P}(\mathbf{R}^d)$.

1) Prove that, for each $t \in [0, \tau]$, the Cauchy problem for the ODE

$$\begin{cases} \dot{Y}(s) = b(s, Y(s)) \,, \\ Y(t) = y \,, \end{cases}$$

has a unique solution $s \mapsto Y(s, t, y)$. What is the maximal domain of definition of this solution? What is the regularity of the map Y viewed as a function of the 3 variables s, t, y?
2) What is the role of assumption (H)?
3) Prove that, for each $t_1, t_2, t_3 \in [0, \tau]$ and $y \in \mathbf{R}^d$, one has

$$Y(t_3, t_2, Y(t_2, t_1, y)) = Y(t_3, t_1, y) \,.$$

4) Compute

$$\partial_t Y(s, t, y) + b(t, y) \cdot \nabla_y Y(s, t, y) \,.$$

5) Let $f^{in} \in C^1(\mathbf{R}^d)$. Prove that the Cauchy problem for the transport equation

$$\begin{cases} \partial_t f(t, y) + b(t, y) \cdot \nabla_y f(t, y) = 0 \,, \\ f\big|_{t=0} = f^{in} \,, \end{cases}$$

has a unique solution $f \in C^1([0, \tau] \times \mathbf{R}^d)$, and that this solution is given by the formula

$$f(t, y) = f^{in}(Y(0, t, y)) \,.$$

6) Let μ^{in} be a Borel probability measure on $\mathbf{R}^d$. Prove that the push-forward measure[3]

$$\mu(t) := Y(t, 0, \cdot)\#\mu^{in}$$

is a weak solution of

$$\begin{cases} \partial_t \mu + \operatorname{div}_y(\mu b) = 0 \,, \\ \mu\big|_{t=0} = \mu^{in} \,. \end{cases}$$

Hint: for $\phi \in C^1_c(\mathbf{R}^d)$, compute

$$\frac{d}{dt} \int_{\mathbf{R}^d} \phi(Y(t, 0, y)) \mu^{in}(dy) \,.$$

7) Prove that the unique weak solution[4] $\mu \in C([0, \tau], w - \mathcal{P}(\mathbf{R}^d))$ of the Cauchy problem considered in 6) is the push-forward measure defined by the formula

$$\mu(t) := Y(t, 0, \cdot)\#\mu^{in}$$

[3]Given two measurable spaces $(X, \mathcal{A})$ and $(Y, \mathcal{B})$, a measurable map $\Phi : (X, \mathcal{A}) \to (Y, \mathcal{B})$ and a measure m on $(X, \mathcal{A})$, the push-forward of m under Φ is the measure on $(Y, \mathcal{B})$ defined by the formula

$$\Phi\#m(B) = m(\Phi^{-1}(B)) \,, \quad \text{for all } B \in \mathcal{B} \,.$$

[4]We designate by $w - \mathcal{P}(\mathbf{R}^d)$ the set $\mathcal{P}(\mathbf{R}^d)$ equipped with the weak topology of probability measures, i.e. the topology defined by the family of semi-distances

$$d_\phi(\mu, \nu) := \left| \int_{\mathbf{R}^d} \phi(z) \mu(dz) - \int_{\mathbf{R}^d} \phi(z) \nu(dz) \right|$$

as ϕ runs through $C_b(\mathbf{R}^d)$.

for each $t \in [0,\tau]$. (Hint: for $\phi \in C_c^1(\mathbf{R}^d)$, compute

$$\frac{d}{dt}\langle Y(0,t,\cdot)\#\mu(t),\phi\rangle$$

in the sense of distributions on $(0,\tau)$.)

For a solution of this exercise, see chapter 1, section 1 of [1].

Our next step is to formulate and solve a new problem that will contain both the N-particle ODE system in the mean-field scaling and the mean-field PDE.

Theorem 3.2.2 *Assume that the interaction kernel $K \in C^1(\mathbf{R}^d \times \mathbf{R}^d, \mathbf{R}^d)$ satisfies assumptions (HK1-HK2). For each $\zeta^{in} \in \mathbf{R}^d$ and each Borel probability measure $\mu^{in} \in \mathcal{P}_1(\mathbf{R}^d)$, there exists a unique solution denoted by*

$$\mathbf{R} \ni t \mapsto Z(t,\zeta^{in},\mu^{in}) \in \mathbf{R}^d$$

of class C^1 of the problem

$$\begin{cases} \partial_t Z(t,\zeta^{in},\mu^{in}) = (\mathcal{K}\mu(t))(Z(t,\zeta^{in},\mu^{in}))\,,\\ \mu(t) = Z(t,\cdot,\mu^{in})\#\mu^{in}\,,\\ Z(0,\zeta^{in},\mu^{in}) = \zeta^{in}\,. \end{cases}$$

Notice that the ODE governing the evolution of $t \mapsto Z(t,\zeta^{in},\mu^{in})$ is set in the single-particle phase space $\mathbf{R}^d$, and not in the N-particle phase space, as is the case of the ODE system studied in Theorem 3.2.1.

Obviously, the ODE appearing in Theorem 3.2.2 is precisely the equation of characteristics for the mean field PDE. Henceforth, we refer to this ODE as the equations of "mean field characteristics", and to its solution Z as the "mean field characteristic flow".

How the mean field characteristic flow Z and the flow T_t associated to the N-particle ODE system are related is explained in the next proposition.

Proposition 3.2.3 *Assume that the interaction kernel $K \in C^1(\mathbf{R}^d \times \mathbf{R}^d, \mathbf{R}^d)$ satisfies assumptions (HK1-HK2). For each $Z_N^{in} = (z_1^{in},\ldots,z_N^{in})$, the solution*

$$T_t Z_N^{in} = (z_1(t),\ldots,z_N(t))$$

of the N-body problem and the mean field characteristic flow $Z(t,\zeta^{in},\mu^{in})$ satisfy

$$z_i(t) = Z(t,z_i^{in},\mu_{Z_N^{in}})\,,\quad i=1,\ldots,N\,,$$

for all $t \in \mathbf{R}$.

Proof of Proposition 3.2.3. Define

$$\zeta_i(t) := Z(t,z_i^{in},\mu_{Z_N^{in}})\,,\quad i=1,\ldots,N\,.$$

Then[5]

$$\mu(t) = Z(t, \cdot, \mu_{Z_N^{in}})\#\mu_{Z_N^{in}} = \frac{1}{N}\sum_{j=1}^{N} \delta_{\zeta_j(t)}$$

for all $t \in \mathbf{R}$. Therefore, ζ_i satisfies

$$\dot{\zeta}_i(t) = (\mathcal{K}\mu(t))(\zeta_i(t)) = \frac{1}{N}\sum_{j=1}^{N} K(\zeta_i(t), \zeta_j(t))\,, \quad i = 1, \ldots, N\,,$$

for all $t \in \mathbf{R}$. Moreover

$$\zeta_i(0) = Z(0, z_i^{in}, \mu^{in}) = z_i^{in}\,, \quad i = 1, \ldots, N\,.$$

Therefore, by uniqueness of the solution of the N-particle equation (Theorem 3.2.1), one has

$$\zeta_i(t) = z_i(t)\,,$$

for all $i = 1, \ldots, N$ and all $t \in \mathbf{R}$. ∎

The proof of Theorem 3.2.2 is a simple variant of the proof of the Cauchy-Lipschitz theorem.

Proof of Theorem 3.2.2. Let $\mu^{in} \in \mathcal{P}_1(\mathbf{R}^d)$, and denote

$$C_1 := \int_{\mathbf{R}^d} |z|\mu^{in}(dz)\,.$$

Let

$$X := \left\{ v \in C(\mathbf{R}^d; \mathbf{R}^d) \text{ s.t. } \sup_{z \in \mathbf{R}^d} \frac{|v(z)|}{1 + |z|} < \infty \right\}\,,$$

which is a Banach space for the norm

$$\|v\|_X := \sup_{z \in \mathbf{R}^d} \frac{|v(z)|}{1 + |z|}\,.$$

[5] The reader should be aware of the following subtle point. In classical references on distribution theory, such as [8], the Dirac mass is viewed as a distribution, therefore as an object that generalizes the notion of function. There is a notion of pull-back of a distribution under a C^∞ diffeomorphism such that the pull-back of the Dirac mass at y_0 with a C^∞ diffeomorphism $\chi : \mathbf{R}^N \to \mathbf{R}^N$ satisfying $\chi(x_0) = y_0$ is

$$\delta_{y_0} \circ \chi = |\det(D\chi(x_0))|^{-1}\delta_{x_0}\,.$$

This notion is not to be confused with the push-forward under χ of the Dirac mass at δ_{x_0} viewed as a probability measure, which, according to the definition in the previous footnote is

$$\chi\#\delta_{x_0} = \delta_{y_0}\,.$$

In particular

$$\chi\#\delta_{x_0} \neq \delta_{x_0} \circ \chi^{-1}$$

unless χ has Jacobian determinant 1 at x_0.

By assumption (HK2) on the interaction kernel K, for each $v, w \in X$, one has

$$\begin{aligned}\left|\int_{\mathbf{R}^d} K(v(z), v(z'))\mu^{in}(dz') - \int_{\mathbf{R}^d} K(w(z), w(z'))\mu^{in}(dz')\right| \\ \leq L\int_{\mathbf{R}^d} (|v(z) - w(z)| + |v(z') - w(z')|)\mu^{in}(dz') \\ \leq L\|v - w\|_X(1 + |z|) + L\|v - w\|_X \int_{\mathbf{R}^d} (1 + |z'|)\mu^{in}(dz') \\ = L\|v - w\|_X(1 + |z| + 1 + C_1) \\ \leq L\|v - w\|_X(2 + C_1)(1 + |z|)\,.\end{aligned}$$

Define a sequence $(Z_n)_{n\geq 0}$ by induction, as follows:

$$\begin{cases} Z_{n+1}(t,\zeta) = \zeta + \displaystyle\int_0^t \int_{\mathbf{R}^d} K(Z_n(t,\zeta), Z_n(t,\zeta'))\mu^{in}(d\zeta')ds\,, \quad n \geq 0\,, \\ Z_0(t,\zeta) = \zeta\,. \end{cases}$$

One checks by induction with the inequality above that, for each $n \in \mathbf{N}$,

$$\|Z_{n+1}(t,\cdot) - Z_n(t,\cdot)\|_X \leq \frac{((2 + C_1)L|t|)^n}{n!}\|Z_1(t,\cdot) - Z_0(t,\cdot)\|_X\,.$$

Since

$$\begin{aligned}|Z_1(t,\zeta) - \zeta| &= \left|\int_0^t \int_{\mathbf{R}^d} K(\zeta,\zeta')\mu^{in}(d\zeta')ds\right| \\ &\leq \int_0^{|t|} \int_{\mathbf{R}^d} L(|\zeta| + |\zeta'|)\mu^{in}(d\zeta')ds \\ &= \int_0^{|t|} L(|\zeta| + C_1)ds \leq L(1 + C_1)(1 + |\zeta|)|t|\,,\end{aligned}$$

one has

$$\|Z_{n+1}(t,\cdot) - Z_n(t,\cdot)\|_X \leq \frac{((2 + C_1)L|t|)^{n+1}}{n!}\,.$$

Thus, for each $\tau > 0$,

$$Z_n(t,\cdot) \to Z(t,\cdot) \quad \text{in } X \text{ uniformly on } [-\tau,\tau]\,,$$

where $Z \in C(\mathbf{R}; X)$ satisfies

$$Z(t,\zeta) = \zeta + \int_0^t \int_{\mathbf{R}^d} K(Z(s,\zeta), Z(s,\zeta'))\mu^{in}(d\zeta')ds$$

for all $t \in \mathbf{R}$ and all $\zeta \in \mathbf{R}^d$.

If Z and $\tilde{Z} \in C(\mathbf{R}; X)$ satisfy the integral equation above, then

$$Z(t,\zeta) - \tilde{Z}(t,\zeta) = \int_{\mathbf{R}^d} (K(Z(s,\zeta), Z(s,\zeta')) - K(\tilde{Z}(s,\zeta), \tilde{Z}(s,\zeta')))\mu^{in}(d\zeta')\,,$$

so that, for all $t \in \mathbf{R}$, one has

$$\|Z(t,\cdot) - \tilde{Z}(t,\cdot)\|_X \leq L(2 + C_1) \left| \int_0^t \|Z(s,\cdot) - \tilde{Z}(s,\cdot)\|_X ds \right| .$$

This implies that

$$\|Z(t,\cdot) - \tilde{Z}(t,\cdot)\|_X = 0$$

by Gronwall's inequality, so that $Z = \tilde{Z}$. Hence the integral equation has only one solution $Z \in C(\mathbf{R}; X)$.

Since $Z \in C(\mathbf{R}_+; X)$, $K \in C^1(\mathbf{R}^d \times \mathbf{R}^d, \mathbf{R}^d)$ satisfies (HK2) and $\mu^{in} \in \mathcal{P}_1(\mathbf{R}^d)$, the function

$$s \mapsto \int_{\mathbf{R}^d} K(Z(s,\zeta), Z(s,\zeta'))\mu^{in}(d\zeta')$$

is continuous on $\mathbf{R}$.

Using the integral equation shows that the function $t \mapsto Z(t,\zeta)$ is of class C^1 on $\mathbf{R}$ and satisfies

$$\begin{cases} \partial_t Z(t,\zeta) = \displaystyle\int_{\mathbf{R}^d} K(Z(t,\zeta), Z(t,\zeta'))\mu^{in}(d\zeta') , \\ Z(0,\zeta) = \zeta . \end{cases}$$

Substituting $z' = Z(t,\zeta')$ in the integral above, one has

$$\int_{\mathbf{R}^d} K(Z(t,\zeta), Z(t,\zeta'))\mu^{in}(d\zeta') = \int_{\mathbf{R}^d} K(Z(t,\zeta), z')Z(t,\cdot)\#\mu^{in}(dz')$$

so that the element Z of $C(\mathbf{R}; X)$ so constructed is the unique solution of the mean field characteristic equation. ■

References for this and the previous section are [2, 16].

3.3 Dobrushin's stability estimate and the mean field limit

3.3.1 The Monge-Kantorovich distance

For each $r > 1$, we denote by $\mathcal{P}_r(\mathbf{R}^d)$ the set of Borel probability measures on $\mathbf{R}^d$ with a finite moment of order r, i.e. satisfying

$$\int_{\mathbf{R}^d} |z|^r \mu(dz) < \infty .$$

Given $\mu, \nu \in \mathcal{P}_r(\mathbf{R}^d)$, we define $\Pi(\mu,\nu)$ to be the set of Borel probability measures π on $\mathbf{R}^d \times \mathbf{R}^d$ with first and second marginals μ and ν respectively. Equivalently, for each $\pi \in \mathcal{P}(\mathbf{R}^d \times \mathbf{R}^d)$,

$$\pi \in \Pi(\mu,\nu) \Leftrightarrow \iint_{\mathbf{R}^d \times \mathbf{R}^d} (\phi(x)+\psi(y))\pi(dxdy) = \int_{\mathbf{R}^d} \phi(x)\mu(dx) + \int_{\mathbf{R}^d} \psi(y)\nu(dy)$$

for each $\phi, \psi \in C(\mathbf{R}^d)$ such that $\phi(z) = O(|z|^r)$ and $\psi(z) = O(|z|^r)$ as $|z| \to \infty$.

Probability measures belonging to $\Pi(\mu, \nu)$ are sometimes referred to as "couplings of μ and ν".

Exercise: Check that, if μ and $\nu \in \mathcal{P}_r(\mathbf{R}^d)$ for some $r > 0$, then one has $\Pi(\mu, \nu) \subset \mathcal{P}_r(\mathbf{R}^d \times \mathbf{R}^d)$.

With these elements of notation, we now introduce the notion of Monge-Kantorovich distance.

Definition 3.3.1 *For each $r \geq 1$ and each $\mu, \nu \in \mathcal{P}_r(\mathbf{R}^d)$, the Monge-Kantorovich distance* $\operatorname{dist}_{MK,r}(\mu, \nu)$ *between μ and ν is defined by the formula*

$$\operatorname{dist}_{MK,r}(\mu, \nu) = \inf_{\pi \in \Pi(\mu,\nu)} \left(\iint_{\mathbf{R}^d \times \mathbf{R}^d} |x - y|^r \pi(dxdy) \right)^{1/r} .$$

These distances also go by the name of "Kantorovich-Rubinstein distances" or "Wasserstein distances" — although the minimization problem in the right hand side of the formula defining $\operatorname{dist}_{MK,r}$ had been considered for the first time by Monge[6] and systematically studied by Kantorovich.

We shall use the Monge-Kantorovich distances only as a convenient tool for studying the stability of the mean field characteristic flow. Therefore, we shall not attempt to present the mathematical theory of these distances and refer instead to the C. Villani's books [21, 22] for a very detailed discussion of this topic.

However, it is useful to know the following property that is special to the case $r = 1$.

Proposition 3.3.2 *The Monge-Kantorovich distance with exponent* 1 *is also given by the formula*

$$\operatorname{dist}_{MK,1}(\mu, \nu) = \sup_{\substack{\phi \in \operatorname{Lip}(\mathbf{R}^d) \\ \operatorname{Lip}(\phi) \leq 1}} \left| \int_{\mathbf{R}^d} \phi(z)\mu(dz) - \int_{\mathbf{R}^d} \phi(z)\nu(dz) \right| ,$$

with the notation

$$\operatorname{Lip}(\phi) := \sup_{x \neq y \in \mathbf{R}^d} \frac{|\phi(x) - \phi(y)|}{|x - y|} .$$

for the Lipschitz constant of ϕ.

The proof of this proposition is based on a duality argument in optimization: see for instance Theorems 1.14 and 7.3 (i) in [21].

[6] Monge's original problem was to minimize over the class of all Borel measurable transportation maps $T : \mathbf{R}^d \to \mathbf{R}^d$ such that $T\#\mu = \nu$ the transportation cost

$$\int_{\mathbf{R}^d} |x - T(x)|\mu(dx) .$$

3.3.2 Dobrushin's estimate

As explained in Proposition 3.2.3, the mean field characteristic flow contains all the relevant information about both the mean field PDE and the N-particle ODE system.

Dobrushin's approach to the mean field limit is based on the idea of proving the stability of the mean field characteristic flow $Z(t,\zeta^{in},\mu^{in})$ in *both* the initial position in phase space ζ^{in} and the initial distribution μ^{in}. As we shall see, the Monge-Kantorovich distance is the best adapted mathematical tool to measure this stability.

Dobrushin's idea ultimately rests on the following key computation. Let $\zeta_1^{in},\zeta_2^{in}\in\mathbf{R}^d$, and let $\mu_1^{in},\mu_2^{in}\in\mathcal{P}_1(\mathbf{R}^d)$. Then

$$\begin{aligned}Z(t,\zeta_1,\mu_1^{in})-Z(t,\zeta_2,\mu_2^{in})&=\zeta_1-\zeta_2\\&+\int_0^t\int_{\mathbf{R}^d}K(Z(s,\zeta_1,\mu_1^{in}),z')\mu_1(s,dz')ds\\&-\int_0^t\int_{\mathbf{R}^d}K(Z(s,\zeta_2,\mu_2^{in}),z')\mu_2(s,dz')ds\,.\end{aligned}$$

Since $\mu_j(t)=Z(t,\cdot,\mu_j^{in})\#\mu_j^{in}$ for $j=1,2$, each inner integral on the right hand side of the equality above can be expressed as follows:

$$\begin{aligned}&\int_{\mathbf{R}^d}K(Z(s,\zeta_j,\mu_j^{in}),z')\mu_j(s,dz')\\&\qquad=\int_{\mathbf{R}^d}K(Z(s,\zeta_j,\mu_j^{in}),Z(s,\zeta_j',\mu_j^{in}))\mu_j^{in}(d\zeta_j')\end{aligned}$$

for $j=1,2$. Therefore, for each coupling $\pi^{in}\in\mathcal{P}_1(\mu_1^{in},\mu_2^{in})$, one has

$$\begin{aligned}&\int_{\mathbf{R}^d}K(Z(s,\zeta_1,\mu_1^{in}),Z(s,\zeta_1',\mu_1^{in}))\mu_1^{in}(d\zeta_1')\\&-\int_{\mathbf{R}^d}K(Z(s,\zeta_2,\mu_2^{in}),Z(s,\zeta_2',\mu_2^{in}))\mu_2^{in}(d\zeta_2')\\&=\iint_{\mathbf{R}^d\times\mathbf{R}^d}(K(Z(s,\zeta_1,\mu_1^{in}),Z(s,\zeta_1',\mu_1^{in}))\\&\qquad-K(Z(s,\zeta_2,\mu_2^{in}),Z(s,\zeta_2',\mu_2^{in})))\pi^{in}(d\zeta_1',d\zeta_2')\,,\end{aligned}$$

so that

$$\begin{aligned}Z(t,\zeta_1,\mu_1^{in})-Z(t,\zeta_2,\mu_2^{in})&=\zeta_1-\zeta_2\\&+\int_0^t\iint_{\mathbf{R}^d\times\mathbf{R}^d}(K(Z(s,\zeta_1,\mu_1^{in}),Z(s,\zeta_1',\mu_1^{in}))\\&\qquad-K(Z(s,\zeta_2,\mu_2^{in}),Z(s,\zeta_2',\mu_2^{in})))\pi^{in}(d\zeta_1',d\zeta_2')ds\,.\end{aligned}$$

This last equality is the key observation in Dobrushin's argument, which explains the role of couplings of μ_1^{in} and μ_2^{in} in this problem, and therefore why it is natural to use the Monge-Kantorovich distance.

After this, the end of the argument is plain sailing. By assumption (HK2) on the interaction kernel K, for all $a, a', b, b' \in \mathbf{R}^d$, one has

$$\begin{aligned}|K(a,a') - K(b,b')| &\leq |K(a,a') - K(b,a')| + |K(b,a') - K(b,b')| \\ &\leq L|a-b| + L|a'-b'|\,.\end{aligned}$$

Therefore

$$\begin{aligned}&|Z(t,\zeta_1,\mu_1^{in}) - Z(t,\zeta_2,\mu_2^{in})| \\ &\qquad \leq |\zeta_1 - \zeta_2| + L\int_0^t |Z(s,\zeta_1,\mu_1^{in}) - Z(s,\zeta_2,\mu_2^{in})|ds \\ &\qquad + L\int_0^t \iint_{\mathbf{R}^d\times\mathbf{R}^d} |Z(s,\zeta_1',\mu_1^{in}) - Z(s,\zeta_2',\mu_2^{in})|\pi^{in}(d\zeta_1' d\zeta_2')ds\,.\end{aligned}$$

It is convenient at this point to introduce the notation

$$D[\pi](s) := \iint_{\mathbf{R}^d\times\mathbf{R}^d} |Z(s,\zeta_1',\mu_1^{in}) - Z(s,\zeta_2',\mu_2^{in})|\pi(d\zeta_1' d\zeta_2')$$

for each $\pi \in \mathcal{P}_1(\mathbf{R}^d \times \mathbf{R}^d)$. Thus, the previous inequality becomes

$$\begin{aligned}&|Z(t,\zeta_1,\mu_1^{in}) - Z(t,\zeta_2,\mu_2^{in})| \leq |\zeta_1 - \zeta_2| \\ &\qquad + L\int_0^t |Z(s,\zeta_1,\mu_1^{in}) - Z(s,\zeta_2,\mu_2^{in})|ds + L\int_0^t D[\pi^{in}](s)ds\,.\end{aligned}$$

Integrating both sides of the inequality above with respect to $\pi^{in}(d\zeta_1 d\zeta_2)$ leads to

$$\begin{aligned}D[\pi^{in}](t) &\leq \iint_{\mathbf{R}^d\times\mathbf{R}^d} |\zeta_1 - \zeta_2|\pi^{in}(d\zeta_1 d\zeta_2) + 2L\int_0^t D[\pi^{in}](s)ds \\ &= D[\pi^{in}](0) + 2L\int_0^t D[\pi^{in}](s)ds\,.\end{aligned}$$

By Gronwall's inequality, we conclude that, for all $t \in \mathbf{R}$, one has

$$D[\pi^{in}](t) \leq D[\pi^{in}](0)e^{2L|t|}\,.$$

Now we can state Dobrushin's stability theorem.

Theorem 3.3.3 (Dobrushin) *Assume that $K \in C^1(\mathbf{R}^d \times \mathbf{R}^d, \mathbf{R}^d)$ satisfies (HK1-HK2). Let $\mu_1^{in}, \mu_2^{in} \in \mathcal{P}_1(\mathbf{R}^d)$. For all $t \in \mathbf{R}$, let*

$$\begin{cases}\mu_1(t) = Z(t,\cdot,\mu_1^{in})\#\mu_1^{in}\,, \\ \mu_2(t) = Z(t,\cdot,\mu_2^{in})\#\mu_2^{in}\,,\end{cases}$$

where Z is the mean field characteristic flow defined in Theorem 3.2.2.

Then, for all $t \in \mathbf{R}$, one has

$$\operatorname{dist}_{MK,1}(\mu_1(t),\mu_2(t)) \leq e^{2L|t|}\operatorname{dist}_{MK,1}(\mu_1^{in},\mu_2^{in})\,.$$

Proof. We have seen that, for all $\mu_1^{in}, \mu_2^{in} \in \mathcal{P}_1(\mathbf{R}^d)$ and all $\pi^{in} \in \Pi(\mu_1^{in}, \mu_2^{in})$, one has

$$D[\pi^{in}](t) \le D[\pi^{in}](0)e^{2L|t|}$$

for all $t \in \mathbf{R}$.

Since $Z(t, \cdot, \mu_j^{in})\#\mu_j^{in} = \mu_j(t)$ for $j = 1, 2$, the map

$$\Phi_t : (\zeta_1, \zeta_2) \mapsto (Z(t, \zeta_1, \mu_1^{in}), Z(t, \zeta_2, \mu_2^{in}))$$

satisfies

$$\Phi_t\#\pi^{in} = \pi(t) \in \Pi(\mu_1(t), \mu_2(t))$$

for all $t \in \mathbf{R}$, since $\pi^{in} \in \Pi(\mu_1^{in}, \mu_2^{in})$.

Thus

$$\begin{aligned}
\operatorname{dist}_{MK,1}(\mu_1(t), \mu_2(t)) &= \inf_{\pi \in \Pi(\mu_1(t), \mu_2(t))} \iint_{\mathbf{R}^d \times \mathbf{R}^d} |\zeta_1 - \zeta_2| \pi(d\zeta_1 d\zeta_2) \\
&\le \inf_{\pi^{in} \in \Pi(\mu_1^{in}, \mu_2^{in})} \iint_{\mathbf{R}^d \times \mathbf{R}^d} |Z(t, \zeta_1, \mu_1^{in}) - Z(t, \zeta_2, \mu_2^{in})| \pi^{in}(d\zeta_1 d\zeta_2) \\
&= \inf_{\pi^{in} \in \Pi(\mu_1^{in}, \mu_2^{in})} D[\pi^{in}](t) \le e^{2L|t|} \inf_{\pi^{in} \in \Pi(\mu_1^{in}, \mu_2^{in})} D[\pi^{in}](0) \\
&= e^{2L|t|} \operatorname{dist}_{MK,1}(\mu_1^{in}, \mu_2^{in})
\end{aligned}$$

which concludes the proof. ■

The discussion in this section is inspired from [4]; see also [14]. The interested reader is also referred to the very interesting paper [12] where Monge-Kantorovich distances with exponents different from 1 are used in the same context — see also [7].

3.3.3 The mean field limit

The mean field limit of the N-particle system is a consequence of Dobrushin's stability theorem, as explained below.

Theorem 3.3.4 *Assume that the interaction kernel $K \in C^1(\mathbf{R}^d \times \mathbf{R}^d)$ and satisfies assumptions (HK1-HK2). Let f^{in} be a probability density on $\mathbf{R}^d$ such that*

$$\int_{\mathbf{R}^d} |z| f^{in}(z) dz < \infty \,.$$

Then the Cauchy problem for the mean field PDE

$$\begin{cases} \partial_t f(t,z) + \operatorname{div}_z(f(t,z)\mathcal{K}f(t,z)) = 0\,, \quad z \in \mathbf{R}^d\,,\ t \in \mathbf{R}\,, \\ f\big|_{t=0} = f^{in} \end{cases}$$

has a unique weak solution $f \in C(\mathbf{R}; L^1(\mathbf{R}^d))$.

For each $N \geq 1$, let $\mathfrak{Z}(N) = (z^{in}_{1,N}, \ldots, z^{in}_{N,N}) \in (\mathbf{R}^d)^N$ be such that

$$\mu_{\mathfrak{Z}(N)} = \frac{1}{N} \sum_{=1}^{N} \delta_{z^{in}_{j,N}}$$

satisfies

$$\mathrm{dist}_{MK,1}(\mu_{\mathfrak{Z}(N)}, f^{in}) \to 0 \quad \text{as } N \to \infty\,.$$

Let $t \mapsto T_t\mathfrak{Z}(N) = (z_{1,N}(t), \ldots, z_{N,N}(t)) \in (\mathbf{R}^d)^N$ be the solution of the N-particle ODE system with initial data $\mathfrak{Z}(N)$, i.e.

$$\begin{cases} \dot{z}_i(t) = \dfrac{1}{N} \displaystyle\sum_{j=1}^{N} K(z_i(t), z_j(t))\,, \quad i = 1, \ldots, N, \\ z_i(0) = z_i^{in}\,. \end{cases}$$

Then[7]

$$\mu_{T_t\mathfrak{Z}(N)} \rightharpoonup f(t,\cdot)\mathscr{L}^d \text{ as } N \to \infty$$

in the weak topology of probability measures, with convergence rate

$$\mathrm{dist}_{MK,1}(\mu_{T_t\mathfrak{Z}(N)}, f(t,\cdot)\mathscr{L}^d) \leq e^{2L|t|}\,\mathrm{dist}_{MK,1}(\mu_{\mathfrak{Z}(N)}, f^{in}) \to 0$$

as $N \to \infty$ for each $t \in \mathbf{R}$.

Proof. By Theorem 3.2.2 and questions 6 and 7 in the exercise on the method of characteristics before Theorem 3.2.2, one has

$$f(t,\cdot)\mathscr{L}^d = Z(t,\cdot, f^{in}\mathscr{L}^d)\# f^{in}\mathscr{L}^d$$

for all $t \in \mathbf{R}$. This implies in particular the uniqueness of the solution of the Cauchy problem in $C(\mathbf{R}; L^1(\mathbf{R}^d))$ for the mean field PDE.

By Proposition 3.2.3,

$$\mu_{T_t\mathfrak{Z}(N)} = Z(t,\cdot,\mu_{\mathfrak{Z}(N)})\#\mu_{\mathfrak{Z}(N)}$$

for all $t \in \mathbf{R}$.

By Dobrushin's stability estimate,

$$\mathrm{dist}_{MK,1}(\mu_{T_t\mathfrak{Z}(N)}, f(t,\cdot)\mathscr{L}^d) \leq e^{2L|t|}\,\mathrm{dist}_{MK,1}(\mu_{\mathfrak{Z}(N)}, f^{in})$$

for all $t \in \mathbf{R}$, and since we have chosen $\mathfrak{Z}(N)$ so that

$$\mathrm{dist}_{MK,1}(\mu_{\mathfrak{Z}(N)}, f^{in}) \to 0$$

as $N \to \infty$, we conclude that

$$\mathrm{dist}_{MK,1}(\mu_{T_t\mathfrak{Z}(N)}, f(t,\cdot)\mathscr{L}^d) \to 0$$

[7]The notation $\mathscr{L}^d$ designates the Lebesgue measure on $\mathbf{R}^d$.

as $N \to \infty$ for each $t \in \mathbf{R}$.

As for weak convergence, pick $\phi \in \operatorname{Lip}(\mathbf{R}^d)$; then

$$\begin{aligned}\left|\int_{\mathbf{R}^d} \phi(z)\mu_{T_t\mathfrak{Z}(N)}(dz) - \int_{\mathbf{R}^d} \phi(z)f(t,z)dz\right| \\ = \left|\iint_{\mathbf{R}^d\times\mathbf{R}^d} (\phi(x)-\phi(y))\pi(dxdy)\right| \\ \le \iint_{\mathbf{R}^d\times\mathbf{R}^d} |\phi(x)-\phi(y)|\pi(dxdy) \\ \le \operatorname{Lip}(\phi)\iint_{\mathbf{R}^d\times\mathbf{R}^d} |x-y|\pi(dxdy)\end{aligned}$$

for each $\pi \in \Pi(\mu_{T_t\mathfrak{Z}(N)}, f(t,\cdot)\mathscr{L}^d)$. Thus

$$\begin{aligned}\left|\int_{\mathbf{R}^d} \phi(z)\mu_{T_t\mathfrak{Z}(N)}(dz) - \int_{\mathbf{R}^d} \phi(z)f(t,z)dz\right| \\ \le \operatorname{Lip}(\phi) \inf_{\pi\in\Pi(\mu_{T_t\mathfrak{Z}(N)},f(t,\cdot)\mathscr{L}^d)} \iint_{\mathbf{R}^d\times\mathbf{R}^d} |x-y|\pi(dxdy) \\ = \operatorname{Lip}(\phi)\operatorname{dist}_{MK,1}(\mu_{T_t\mathfrak{Z}(N)}, f(t,\cdot)\mathscr{L}^d) \to 0\end{aligned}$$

for each $t \in \mathbf{R}$ as $N \to \infty$. (Notice that the inequality above is an obvious consequence of the definition of $\operatorname{dist}_{MK,1}$, so that the equality in Proposition 3.3.2 is not needed here.)

This is true in particular for each $\phi \in C_c^1(\mathbf{R}^d)$, and since $C_c^1(\mathbf{R}^d)$ is dense in $C_c(\mathbf{R}^d)$, we conclude that

$$\int_{\mathbf{R}^d} \phi(z)\mu_{T_t\mathfrak{Z}(N)}(dz) \to \int_{\mathbf{R}^d} \phi(z)f(t,z)dz$$

as $N \to \infty$ for each $\phi \in C_c(\mathbf{R}^d)$. Since

$$\int_{\mathbf{R}^d} \mu_{T_t\mathfrak{Z}(N)}(dz) = \int_{\mathbf{R}^d} f(t,z)dz = 1$$

for all $t \in \mathbf{R}$, we conclude that the convergence above holds for each $\phi \in C_b(\mathbf{R}^d)$, which means that

$$\mu_{T_t\mathfrak{Z}(N)} \to f(t,\cdot)\mathscr{L}^d$$

as $N \to \infty$ in the weak topology of probability measures, by applying Theorem 6.8 in chapter II of [13], sometimes referred to as the "portmanteau theorem". ■

The theorem above is the main result on the mean field limit in [16, 2, 4].

3.3.4 On the choice of the initial data

In practice, using Theorem 3.3.4 as a rigorous justification of the mean field limit requires being able to generate N-tuples of the form $\mathfrak{Z}(N) = (z_{1,N}^{in}, \dots, z_{N,N}^{in}) \in$

$(\mathbf{R}^d)^N$ such that

$$\mu_{\mathfrak{z}(N)} = \frac{1}{N}\sum_{=1}^{N} \delta_{z_{j,N}^{in}}$$

satisfies

$$\mathrm{dist}_{MK,1}(\mu_{\mathfrak{z}(N)}, f^{in}) \to 0 \quad \text{as } N \to \infty\,.$$

Assume that f^{in} is a probability density on $\mathbf{R}^d$ such that

$$\int_{\mathbf{R}^d} |z|^2 f(z)dz < \infty\,.$$

Let $\Omega := (\mathbf{R}^d)^{\mathbf{N}^*}$, the set of sequences of points in $\mathbf{R}^d$ indexed by $\mathbf{N}^*$. Let $\mathcal{F}$ be the σ-algebra on Ω generated by cylinders, i.e. by sets of the form

$$\prod_{n\ge 1} B_n \quad \text{with } B_n \text{ Borel set in } \mathbf{R}^d$$
$$\text{and } B_n = \mathbf{R}^d \text{ for all but finitely many } n\,.$$

Finally, we endow the measurable space $(\Omega, \mathcal{F})$ with the probability measure $\mathbf{P} := (f^{in})^{\otimes\infty}$, defined on the set of cylinders of Ω by the formula

$$\mathbf{P}\left(\prod_{n\ge 1} B_n\right) = \prod_{n\ge 1} f^{in}(B_n)\,.$$

(Notice that $f^{in}(B_n) = 1$ for all but finitely many n, since $B_n = \mathbf{R}^d$ except for finitely many n.)

Theorem 3.3.5 *For each $\mathbf{z}^{in} = (z_k^{in})_{k\ge 1} \in \Omega$, let $Z_N^{in} = (z_1^{in}, \ldots, z_N^{in})$. Then*

$$\mathrm{dist}_{MK,1}(\mu_{Z_N^{in}}, f^{in}\mathscr{L}^d) \to 0$$

as $N \to \infty$ for $\mathbf{P}$-a.e. $\mathbf{z}^{in} \in \Omega$.

Proof. For $\phi \in C_c(\mathbf{R}^d)$ or $\phi(z) = |z|$, consider the sequence of random variables on $(\Omega, \mathcal{F})$ defined by

$$Y_n(\mathbf{z}) = \phi(z_n)\,,$$

where

$$\mathbf{z} := (z_1, \ldots, z_n, \ldots) \in \Omega\,.$$

The random variables Y_n are identically distributed, since

$$\mathbf{P}(Y_n \ge a) = \int_{\mathbf{R}^d} \mathbf{1}_{\phi(z)\ge a} f^{in}(z)dz$$

is independent of n.

The random variables Y_n are also independent, since for all $N \geq 1$ and all $g_1, \ldots, g_N \in C_b(\mathbf{R})$, one has

$$\mathbf{E}^{\mathbf{P}}(g_1(Y_1)\ldots g_N(Y_N)) = \prod_{k=1}^{N} \int_{\mathbf{R}^d} g_k(\phi(z)) f^{in}(z)dz = \prod_{k=1}^{N} \mathbf{E}^{\mathbf{P}}(g_k(Y_k)) .$$

Finally, the random variables Y_n have finite variance since

$$\mathbf{E}^{\mathbf{P}}(|Y_n|^2) = \int_{\mathbf{R}^d} |z|^2 f^{in}(z)dz < \infty .$$

By the strong law of large numbers (see Theorem 3.27 in [3]), one has

$$\left\langle \frac{1}{N}\sum_{k=1}^{N} \delta_{z_k}, \phi \right\rangle = \frac{1}{N}\sum_{k=1}^{N} Y_k \to \mathbf{E}^{\mathbf{P}}(Y_1) = \int_{\mathbf{R}^d} \phi(z) f^{in}(z)dz$$

for $\mathbf{P}$-a.e. $\mathbf{z}$.

Since $C_c(\mathbf{R}^d)$ is separable, one can assume that the $\mathbf{P}$-negligible set is the same for all $\phi \in C_c(\mathbf{R}^d)$, and take its union with the one corresponding to $\phi(z) = |z|$. This means precisely that

$$\frac{1}{N}\sum_{k=1}^{N} \delta_{z_k} \to f^{in}\mathscr{L}^d$$

weakly in $\mathcal{P}_1(\mathbf{R}^d)$ for $\mathbf{P}$-a.e. $\mathbf{z} \in \Omega$. One concludes the proof with the lemma below. ■

Lemma 3.3.6 *The Monge-Kantorovich distance* $\mathrm{dist}_{MK,1}$ *metricizes the topology of weak convergence on* $\mathcal{P}_1(\mathbf{R}^d)$. *In other words, given a sequence* $(\mu_n)_{n\geq 1}$ *of elements of* $\mathcal{P}_1(\mathbf{R}^d)$ *and* $\mu \in \mathcal{P}_1(\mathbf{R}^d)$, *the two following statements are equivalent:*

(1) $\mathrm{dist}_{MK,1}(\mu_n, \mu) \to 0$ *as* $n \to \infty$;

(2) $\mu_n \to \mu$ *weakly in* $\mathcal{P}(\mathbf{R}^d)$ *as* $n \to \infty$ *and*

$$\sup_n \int_{\mathbf{R}^d} |z|\mathbf{1}_{|z|\geq R}\mu_n(dz) \to 0 \qquad \textit{as } R \to \infty .$$

For a proof of Lemma 3.3.6, see [21].

Exercise: The reader is invited to verify the fact that one can choose the $\mathbf{P}$-negligible set that appears in the proof of Theorem 3.3.5 to be the same for all $\phi \in C_c(\mathbf{R}^d)$ and for $\phi(z) = |z|$. Here is an outline of the argument.

a) Let $R > 0$; let E_R be the set of real-valued continuous functions defined on $[-R, R]^d$ that vanish identically on $\partial[-R, R]^d$, equipped with the sup-norm

$$\|\phi\| := \sup_{x\in[-R,R]^d} |\phi(x)| .$$

Prove that E_R is a separable Banach space.

Denote by $\mathcal{N}_\phi$ be the set of $\mathbf{z} \in \Omega$ such that

$$\left\langle \frac{1}{N} \sum_{k=1}^{N} \delta_{z_k}, \phi \right\rangle$$

does not converge to

$$\int_{\mathbf{R}^d} \phi(z) f^{in}(z) dz$$

as $N \to \infty$. Let $R > 0$ and let $(\phi_n)_{n \geq 1}$ be a dense sequence of elements of E_R, extended by 0 to $\mathbf{R}^d$. Define

$$\mathcal{N}_R := \bigcup_{n \geq 1} \mathcal{N}_{\phi_n} .$$

b) Prove that

$$\left\langle \frac{1}{N} \sum_{k=1}^{N} \delta_{z_k}, \phi \right\rangle \to \int_{\mathbf{R}^d} \phi(z) f^{in}(z) dz$$

as $N \to \infty$ for all $\phi \in E_R$ and all $\mathbf{z} \notin \mathcal{N}_R$. (Hint: pick $\phi \in E_R$ and $\epsilon > 0$, and choose $m := m(\phi, \epsilon)$ such that $\|\phi - \phi_m\| < \epsilon$. With the decomposition

$$\begin{aligned}
\left\langle \frac{1}{N} \sum_{k=1}^{N} \delta_{z_k}, \phi \right\rangle - \int_{\mathbf{R}^d} \phi(z) f^{in}(z) dz &= \left\langle \frac{1}{N} \sum_{k=1}^{N} \delta_{z_k}, \phi - \phi_m \right\rangle \\
&+ \left\langle \frac{1}{N} \sum_{k=1}^{N} \delta_{z_k}, \phi_m \right\rangle - \int_{\mathbf{R}^d} \phi_m(z) f^{in}(z) dz \\
&+ \int_{\mathbf{R}^d} (\phi_m(z) - \phi(z)) f^{in}(z) dz \,,
\end{aligned}$$

prove that

$$\left| \left\langle \frac{1}{N} \sum_{k=1}^{N} \delta_{z_k}, \phi \right\rangle - \int_{\mathbf{R}^d} \phi(z) f^{in}(z) dz \right| < 3\epsilon$$

for all $\mathbf{z} \notin \mathcal{N}_R$ provided that $N \geq N_0 = N_0(\epsilon, \phi)$.)

c) Complete the proof of Theorem 3.3.5.

Thus, using Theorem 3.3.4 to prove the mean field limit requires choosing

$$\mathfrak{z}(N) = (z_{1,N}^{in}, \ldots, z_{N,N}^{in}) \in (\mathbf{R}^d)^N$$

for each $N \geq 1$ so that

$$\operatorname{dist}_{MK,1}(\mu_{\mathfrak{z}(N)}, f^{in} \mathscr{L}^d) \to 0 \quad \text{ as } N \to \infty \,.$$

Theorem 3.3.5 provides us with a strategy for making this choice, which is to draw an infinite sequence z_j^{in} at random and independently with distribution

$f^{in}\mathscr{L}^d$, and to set $z_{j,N}^{in} := z_j^{in}$. This strategy avoids the unpleasant task of having to change the first terms in $\mathfrak{Z}(N)$ as $N \to \infty$.

Since Dobrushin's estimate bounds $\mathrm{dist}_{MK,1}(f(t,\cdot)\mathscr{L}^d, \mu_{T_t\mathfrak{Z}(N)})$ in terms of $\mathrm{dist}_{MK,1}(f^{in}\mathscr{L}^d, \mu_{\mathfrak{Z}(N)})$, having an explicit bound on $\mathrm{dist}_{MK,1}(f^{in}\mathscr{L}^d, \mu_{\mathfrak{Z}(N)})$ would provide us with a quantitative error estimate for the mean field limit. Bounds of this type have been studied in detail by several authors: see for instance Theorem 1.1 in [9] and Lemma 4.2 in [15] for a quick overview with further references.

More details on the topics discussed in the present section are to be found in [2], as well as a precise statement concerning the behavior of fluctuations around the mean field limit — in some sense, the asymptotic behavior at next order after the mean field limit (see Theorem 3.5 in [2]).

Chapter 4

The Cauchy Problem for Vlasov-Poisson

This chapter reviews the existence, uniqueness and regularity theory for solutions of the Cauchy problem for the Vlasov-Poisson system

$$\begin{cases} \partial_t f(t,x,v) + v\cdot\nabla_x f(t,x,v) - \nabla_x\Phi(t,x)\cdot\nabla_v f(t,x,v) = 0\,, \quad x,v,\in\mathbf{R}^d\,, \\ -\Delta_x\Phi(t,x) = \int_{\mathbf{R}^d} f(t,x,v)dv\,, \\ f(0,x,v) = f^{in}(x,v)\,. \end{cases}$$

4.1 Elementary Inequalities

We recall from the previous chapter the following a priori estimates satisfied by classical solutions (f,E) of the Vlasov-Poisson system with appropriate decay as $|x|+|v|\to\infty$:
(a) conservation of mass (or particle number):

$$\mathcal{M}(t) := \iint_{\mathbf{R}^d\times\mathbf{R}^d} f(t,x,v)dxdv = \iint_{\mathbf{R}^d\times\mathbf{R}^d} f^{in}(x,v)dxdv =: \mathcal{M}^{in}\,;$$

(b) energy conservation: for each $t\geq 0$, one has

$$\begin{aligned}\mathcal{E}(t) : &= \iint_{\mathbf{R}^d\times\mathbf{R}^d} \tfrac12|v|^2 f(t,x,v)dxdv + \int_{\mathbf{R}^d}\tfrac12|\nabla_x\Phi(t,x)|^2dx \\ &= \iint_{\mathbf{R}^d\times\mathbf{R}^d} \tfrac12|v|^2 f^{in}(x,v)dxdv + \int_{\mathbf{R}^d}\tfrac12|\nabla_x\Phi^{in}(x)|^2dx =: \mathcal{E}^{in}\,,\end{aligned}$$

where $\nabla_x\Phi^{in}$ is obtained by solving the Poisson equation

$$-\Delta_x\Phi^{in}(x) = \int_{\mathbf{R}^d} f^{in}(x,v)dv\,, \quad \nabla_x\Phi^{in}(x)\to 0 \text{ as } |x|\to\infty\,.$$

Other elementary inequalities are needed in the theory of the Cauchy problem for the Vlasov-Poisson system. We have gathered them together in the present section.

Positivity and maximum principle:

$$
\begin{aligned}
0 \le f^{in}(x,v) \le M & \text{ for a.e. } (x,v) \in \mathbf{R}^d \times \mathbf{R}^d \\
&\Rightarrow 0 \le f(t,x,v) \le M \text{ for a.e. } (x,v) \in \mathbf{R}^d \times \mathbf{R}^d\,, \quad \text{for all } t \ge 0\,.
\end{aligned}
$$

Interpolation inequality: for each $m \ge r > 0$, there exists a positive constant $C(d,m,r)$ such that, for each measurable function $f \equiv f(x,v)$ defined a.e. on $\mathbf{R}^d_x \times \mathbf{R}^d_v$, one has

$$
\left\| \int_{\mathbf{R}^d} |v|^r f(\cdot,v)dv \right\|_{L^{\frac{m+d}{r+d}}(\mathbf{R}^d)} \le C(d,m,r) \|f\|^{\frac{m-r}{m+d}}_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)} \||v|^m f\|^{\frac{r+d}{m+d}}_{L^1(\mathbf{R}^d\times\mathbf{R}^d)}\,.
$$

Basic facts on the Coulomb potential: let $G_d \equiv G_d(x)$ be the function defined on $\mathbf{R}^d \setminus \{0\}$ by the formula

$$
G_d(x) := \begin{cases} -\frac{1}{2\pi}\ln|x| & \text{if } d=2\,, \\ \frac{1}{c(d)}\dfrac{1}{|x|^{d-2}} & \text{if } d \ge 3\,, \end{cases}
$$

where

$$
c(d) := (d-2)|\mathbf{S}^{d-1}|\,.
$$

Then

$$
-\Delta G_d = \delta_0 \quad \text{in } \mathcal{D}'(\mathbf{R}^d)\,.
$$

A straightforward computation shows that

$$
-\nabla G_d = \frac{1}{|\mathbf{S}^{d-1}|}\frac{x}{|x|^d} \quad \text{in } \mathcal{D}'(\mathbf{R}^d)\,,
$$

so that[1]

$$
\nabla G_d \in L^{\frac{d}{d-1},\infty}(\mathbf{R}^d)\,.
$$

[1]If $(X,\mathcal{A},\mu)$ is a σ-finite measured space, for all $p \in [1,\infty)$, the space $L^{p,\infty}(X,\mu)$ is defined as the set of equivalence classes of measurable functions defined μ-a.e. on X and satisfying

$$
\mu(\{x \in X \text{ s.t. } |f(x)|^p \ge t\}) \le C/t\,, \quad t > 0\,.
$$

By the Bienaymé-Chebyshev inequality, $f \in L^p(X) \Rightarrow f \in L^{p,\infty}(X)$ since

$$
\mu(\{x \in X \text{ s.t. } |f(x)|^p \ge t\}) \le \frac{1}{t}\int_X |f(x)|^p dx = \frac{\|f\|^p_{L^p}}{t}\,.
$$

The converse is obviously wrong: for instance the function $f : (1,\infty) \ni x \mapsto 1/x \in \mathbf{R}$ belongs to $L^{1,\infty}(1,\infty)$ but not to $L^1(1,\infty)$. The space $L^{p,\infty}(X,\mu)$ is sometimes referred to as the weak L^p space, or the Marcinkiewicz L^p space. It belongs to the more general class of Lorentz spaces defined in terms of L^1 and L^∞ by the Lions-Peetre real interpolation method. The reader interested in further details on this subject is advised to read [19].

Moreover[2]

$$\nabla^2 G_d = \tfrac{1}{|\S^{d-1}|}\,\mathrm{vp}\,\frac{|x|^2 I - dx^{\otimes 2}}{|x|^{d+2}} - \tfrac{1}{d}\delta_0 I \quad \text{ in } \mathcal{D}'(\mathbf{R}^d)\,.$$

Estimates on the force field: the force field is given by

$$E(t,\cdot) = -\nabla G_d \star \rho(t,\cdot)\,,$$

so that

$$\|E(t,\cdot)\|_{L^q} \le C\|\rho(t,\cdot)\|_{L^p}\,,$$

for all $t \ge 0$, with

$$1 + \frac{1}{q} = \frac{1}{p} + \frac{d-1}{d} \Leftrightarrow \frac{1}{q} = \frac{1}{p} - \frac{1}{d} \text{ if } 1 < p < d\,.$$

This follows from the weak Young inequality for the convolution product (see section 4.3 on p. 107 in [11]).

Estimates on the force field: in addition, for all $t \ge 0$, one has

$$\nabla_x E(t,\cdot) := -\nabla^2 G_d \star \rho(t,\cdot)$$

so that

$$\|\nabla_x E(t,\cdot)\|_{L^p(\mathbf{R}^d)} \le C\|\rho(t,\cdot)\|_{L^p(\mathbf{R}^d)}$$

for all $1 < p < \infty$ by the Calderón-Zygmund continuity theorem for singular integrals (see Theorem 4.12 in [5]).

Likewise

$$\partial_t E(t,\cdot) = -\nabla G \star \partial_t \rho(t,\cdot) = \nabla G \star \mathrm{div}_x\, j(t,\cdot)$$

where we recall that the current density is defined by the formula

$$j(t,x) := \int_{\mathbf{R}^d} v(f(t,x,v)dv\,.$$

Therefore

$$\|\partial_t E(t,\cdot)\|_{L^p(\mathbf{R}^d)} \le C\|j(t,\cdot)\|_{L^p(\mathbf{R}^d)}$$

[2]Let $\Omega \in L^q(\mathbf{S}^{d-1})$ for some $q > 1$ satisfy the condition

$$\int_{\mathbf{S}^{d-1}} \Omega(\omega)ds(\omega) = 0\,,$$

where ds is the $d-1$-dimensional surface element on $\mathbf{S}^{d-1}$. The tempered distribution $\mathrm{vp}\,\frac{\Omega(x/|x|)}{|x|^d}$ is defined by the formula

$$\begin{aligned}\left\langle \mathrm{vp}\,\frac{\Omega(x/|x|)}{|x|^d}, \phi\right\rangle &= \lim_{\epsilon\to 0^+}\int_{\mathbf{R}^d}\frac{\Omega(x/|x|)}{|x|^d}\phi(x)\mathbf{1}_{|x|>\epsilon}dx\\ &= \int_{\mathbf{R}^d}\frac{\Omega(x/|x|)}{|x|^d}(\phi(x)-\phi(0))\mathbf{1}_{|x|\le R}dx + \int_{\mathbf{R}^d}\frac{\Omega(x/|x|)}{|x|^d}\phi(x)\mathbf{1}_{|x|>R}dx\end{aligned}$$

for each $R > 0$ and each $\phi \in \mathcal{S}(\mathbf{R}^d)$.

for all $1 < p < \infty$, again by the Calderón-Zygmund continuity theorem for singular integrals.

A priori bounds: assume that the initial condition f^{in} of the Cauchy problem for the Vlasov-Poisson equation satisfies

$$f^{in} \geq 0 \text{ a.e. on } \mathbf{R}^d \times \mathbf{R}^d\,, \quad f^{in} \in L^1 \cap L^\infty(\mathbf{R}^d_x \times \mathbf{R}^d_v)\,,$$

and

$$\mathcal{E}^{in} := \iint_{\mathbf{R}^d \times \mathbf{R}^d} \tfrac{1}{2}|v|^2 f^{in}(x,v)dxdv + \int_{\mathbf{R}^d} \tfrac{1}{2}|\nabla_x \Phi^{in}(x)|^2 dx < \infty\,.$$

By the energy conservation

$$\iint_{\mathbf{R}^d \times \mathbf{R}^d} \tfrac{1}{2}|v|^2 f(t,x,v)dxdv \leq \mathcal{E}^{in} \quad \text{and} \quad \int_{\mathbf{R}^d} \tfrac{1}{2}|E(t,x)|^2 dx \leq \mathcal{E}^{in}$$

so that, by the interpolation inequality

$$\|\rho(t,\cdot)\|_{L^{\frac{d+2}{d}}(\mathbf{R}^d)} + \|j(t,\cdot)\|_{L^{\frac{d+2}{d+1}}(\mathbf{R}^d)} \leq C$$

for some positive constante C. On the other hand, by the global conservation of mass

$$\begin{aligned}\|\rho(t,\cdot)\|_{L^1(\mathbf{R}^d)} &= \iint_{\mathbf{R}^d \times \mathbf{R}^d} f(t,x,v)dxdv \\ &= \iint_{\mathbf{R}^d \times \mathbf{R}^d} f^{in}(x,v)dxdv =: \mathcal{M}^{in} < \infty\,,\end{aligned}$$

while

$$\begin{aligned}\|j(t,\cdot)\|_{L^1(\mathbf{R}^d)} &\leq \iint_{\mathbf{R}^d \times \mathbf{R}^d} |v| f(t,x,v)dxdv \\ &\leq \iint_{\mathbf{R}^d \times \mathbf{R}^d} \tfrac{1}{2}(1+|v|^2) f(t,x,v)dxdv \leq \tfrac{1}{2}\mathcal{M}^{in} + \mathcal{E}^{in} < \infty\,.\end{aligned}$$

By Hölder's inequality, for all $t \geq 0$, one has

$$\|\rho(t,\cdot)\|_{L^p(\mathbf{R}^d)} \leq C < \infty \quad \text{for } 1 \leq p \leq \frac{d+2}{d}\,,$$

while

$$\|j(t,\cdot)\|_{L^p(\mathbf{R}^d)} \leq C < \infty \quad \text{for } 1 \leq p \leq \frac{d+2}{d+1}\,.$$

These a priori bounds imply that the force field satisfies

$$\|E(t,\cdot)\|_{L^q(\mathbf{R}^d)} \leq C < \infty \quad \text{for } \frac{d}{d-1} < q \leq \frac{d(d+2)}{(d-2)(d+1)}\,,$$

together with

$$\begin{aligned}&\|\nabla_x E(t,\cdot)\|_{L^q(\mathbf{R}^d)} \leq C < \infty && \text{for } 1 < p \leq \frac{d+2}{d}\,,\\ &\|\partial_t E(t,\cdot)\|_{L^q(\mathbf{R}^d)} \leq C < \infty && \text{for } 1 < p \leq \frac{d+2}{d+1}\end{aligned}$$

for all $t \geq 0$.

4.2 Global Existence of Weak Solutions

The global existence of weak solutions of the Cauchy problem for the Vlasov-Poisson system was obtained in 1975 by Arsenev.

Theorem 4.2.1 *Assume that $d \geq 2$, and let $f^{in} \in L^1 \cap L^\infty(\mathbf{R}^d_x \times \mathbf{R}^d_v)$ satisfy*

$$f^{in} \geq 0 \text{ a.e. and } \iint_{\mathbf{R}^d \times \mathbf{R}^d} \tfrac{1}{2}|v|^2 f(t,x,v)dxdv + \int_{\mathbf{R}^d} \tfrac{1}{2}|E(t,x)|^2 dx = \mathcal{E}^{in} < \infty\,.$$

Then there exists a global weak solution $f \in L^\infty(\mathbf{R}_+; L^1(\mathbf{R}^d_x \times \mathbf{R}^d_v))$ of the Cauchy problem for the Vlasov-Poisson system with initial data f^{in}. This solution satisfies

$$0 \leq f(t,x,v) \leq \|f^{in}\|_{L^\infty} \text{ for a.e. } (x,v) \in \mathbf{R}^d \times \mathbf{R}^d \text{ for all } t \geq 0\,,$$

together with the mass bound

$$\iint_{\mathbf{R}^d \times \mathbf{R}^d} f(t,x,v)dxdv \leq \mathcal{M}^{in} < \infty$$

for all $t \geq 0$ and the energy bound

$$\iint_{\mathbf{R}^d \times \mathbf{R}^d} \tfrac{1}{2}|v|^2 f(t,x,v)dxdv + \int_{\mathbf{R}^d} \tfrac{1}{2}|\nabla_x \Phi(t,x)|^2 dx \leq \mathcal{E}^{in} < \infty$$

for a.e. $t \geq 0$. The initial condition is verified in the sense of distributions, i.e. for all $\phi \in C^\infty_c(\mathbf{R}^d \times \mathbf{R}^d)$, the function

$$t \mapsto \iint_{\mathbf{R}^d \times \mathbf{R}^d} f(t,x,v)\phi(x,v)dxdv$$

is continuous on $\mathbf{R}_+$ and satisfies

$$\iint_{\mathbf{R}^d \times \mathbf{R}^d} f(0,x,v)\phi(x,v)dxdv = \iint_{\mathbf{R}^d \times \mathbf{R}^d} f^{in}(x,v)\phi(x,v)dxdv\,.$$

4.2.1 The approximate Vlasov-Poisson system

Let $\zeta \in C^\infty(\mathbf{R}^d)$ satisfy

$$\zeta(x) = \zeta(-x) \geq 0 \text{ for all } x \in \mathbf{R}^d\,, \quad \operatorname{supp}(\zeta) \subset B(0,1) \text{ and } \int_{\mathbf{R}^d} \zeta(x)dx = 1\,,$$

and set $\zeta_\epsilon(x) = \epsilon^{-d}\zeta(x/\epsilon)$. Set $\xi_\epsilon(x,v) := \zeta_\epsilon(x)\zeta_\epsilon(v)$.

The reason for using an even mollifier ζ_ϵ is explained by the following lemma.

Lemma 4.2.2 *Let $\chi \in C^\infty_c(\mathbf{R}^N)$ satisfy*

$$\chi(x) = \chi(-x)\,, \quad \text{for all } x \in \mathbf{R}^N\,.$$

Then the convolution operator $C_\chi : \phi \mapsto \chi \star \phi$ is self-adjoint on $L^2(\mathbf{R}^N)$.

The (elementary) proof of this lemma is left as an exercise.

Exercise: Prove that C_χ is a bounded operator on $L^2(\mathbf{R}^N)$. (Hint: apply Young's inequality). Compute the operator norm of C_χ. (Answer: one finds $\|C_\chi\| = \|\hat{\chi}\|_{L^\infty(\mathbf{R}^N)}$, not $\|C_\chi\|_{L^1(\mathbf{R}^N)}$.)

The approximate Vlasov-Poisson system (VP$_\epsilon$) is defined as follows

$$\left\{\begin{aligned}&\partial_t f_\epsilon(t,x,v) + v\cdot\nabla_x f_\epsilon(t,x,v) - \nabla_x\Phi_\epsilon(t,x)\cdot\nabla_v f_\epsilon(t,x,v) = 0\,,\\ &-\Delta_x\Phi_\epsilon(t,\cdot) = \zeta_\epsilon\star\zeta_\epsilon\star\rho_\epsilon(t,\cdot)\,, \qquad \nabla_x\Phi_\epsilon\to 0 \text{ as } |x|\to\infty\,,\\ &\rho_\epsilon(t,x) = \int_{\mathbf{R}^d} f_\epsilon(t,\cdot,v)dv\,,\\ &f_\epsilon\big|_{t=0} = \xi_\epsilon\star(\mathbf{1}_{\epsilon|x|<1}\mathbf{1}_{\epsilon|v|<1}f^{in}) =: f_\epsilon^{in}\,.\end{aligned}\right. \tag{VP_ϵ}$$

Thus, for all $t\geq 0$,

$$-\nabla_x\Phi_\epsilon(t,\cdot) = (\zeta_\epsilon\star\zeta_\epsilon\star\nabla G_d)\star\rho_\epsilon(t,\cdot)\,,$$

and, for each $\epsilon>0$, one has

$$\zeta_\epsilon\star\zeta_\epsilon\star\nabla G_d\in C^\infty(\mathbf{R}^d)\cap L^{d/(d-1),\infty}(\mathbf{R}^d)\cap L^\infty(\mathbf{R}^d)\,.$$

Proposition 4.2.3 *For each $f^{in}\in L^1\cap L^\infty(\mathbf{R}^d_x\times\mathbf{R}^d_v)$ satisfying*

$$f^{in}\geq 0 \text{ a.e.}$$

and

$$\iint_{\mathbf{R}^d\times\mathbf{R}^d}\tfrac12|v|^2 f_\epsilon^{in}(x,v)dxdv + \int_{\mathbf{R}^d}\tfrac12|\tilde{E}_\epsilon^{in}(x)|^2dx = \mathcal{E}^{in}<\infty\,,$$

with

$$\tilde{E}_\epsilon^{in} = -\zeta_\epsilon\star\nabla G_d\star\rho_\epsilon^{in}\,,$$

where

$$\rho_\epsilon^{in}(x) := \int_{\mathbf{R}^d} f_\epsilon^{in}(x,v)dv\,,$$

there exists a unique weak solution $f_\epsilon\in C(\mathbf{R}_+;L^1(\mathbf{R}^d_x\times\mathbf{R}^d_v))$ of (VP_ϵ). This solution satisfies

$$0\leq f_\epsilon(t,x,v)\leq\|f^{in}\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)} \text{ for all } (x,v)\in\mathbf{R}^d\times\mathbf{R}^d \text{ and } t\geq 0\,,$$

together with the mass conservation

$$\iint_{\mathbf{R}^d\times\mathbf{R}^d} f_\epsilon(t,x,v)dxdv = \iint_{\mathbf{R}^d\times\mathbf{R}^d} f_\epsilon^{in}(x,v)dxdv$$

and the approximate energy conservation bound

$$\iint_{\mathbf{R}^d\times\mathbf{R}^d}\tfrac12|v|^2 f_\epsilon(t,x,v)dxdv + \int_{\mathbf{R}^d}\tfrac12|\tilde{E}_\epsilon(t,x)|^2dx\leq\mathcal{E}^{in}$$

for all $t\geq 0$, where

$$\tilde{E}_\epsilon(t,\cdot) := (\zeta_\epsilon\star\nabla G_d)\star\rho_\epsilon(t,\cdot)\,.$$

Proof. WLOG assume that

$$\iint_{\mathbf{R}^d\times\mathbf{R}^d} f_\epsilon^{in}(x,v)dxdv = 1 \quad \text{ and } \iint_{\mathbf{R}^d\times\mathbf{R}^d} v f_\epsilon^{in}(x,v)dxdv = 0\,.$$

Since the dynamics of the approximate Vlasov-Poisson system preserves the total mass and total momentum, one has

$$\iint_{\mathbf{R}^d\times\mathbf{R}^d} f_\epsilon(t,x,v)dxdv = 1 \text{ for all } t \geq 0\,,$$

and

$$\iint_{\mathbf{R}^d\times\mathbf{R}^d} v f_\epsilon(t,x,v)dxdv = 0 \text{ for all } t \geq 0\,.$$

Thus, as observed before (see the last exercise in section 3.1), the system (VP$_\epsilon$) can be put in the form

$$\partial_t f_\epsilon(t,z) + \operatorname{div}_z \left(f_\epsilon(t,z)\int_{\mathbf{R}^{2d}} K_\epsilon(z,z')f_\epsilon(t,z')dz'\right) = 0\,,$$

with $z=(x,v)$ and

$$K_\epsilon(x,v,x',v') = (v-v', \zeta_\epsilon \star \zeta_\epsilon \star \nabla G_d)\,.$$

The existence and uniqueness of the solution of (VP$_\epsilon$) has already been obtained as a consequence of the construction of the mean field flow. Indeed, by Theorem 3.2.2, there exists a unique map

$$\mathbf{R}_+ \times \mathbf{R}^{2d} \times \mathcal{P}_1(\mathbf{R}^d) \ni (t, z^{in}, \mu^{in}) \mapsto Z_\epsilon(t,z^{in},\mu^{in}) \in \mathbf{R}^{2d}$$

such that $t \mapsto Z_\epsilon(t,z^{in},\mu^{in})$ is the integral curve of the vector field

$$z \mapsto \int_{\mathbf{R}^{2d}} K_\epsilon(z,z')\mu_\epsilon(t,dz') =: (\mathcal{K}\mu_\epsilon(t))(z)$$

passing through z^{in} at time $t=0$, where $\mu_\epsilon(t) := Z_\epsilon(t,\cdot,\mu^{in})\#\mu^{in}$.

Set

$$Z_\epsilon(t,z^{in}) := Z_\epsilon(t,z^{in}, f_\epsilon^{in}\mathscr{L}^{2d})$$

and

$$V_\epsilon(t,z) := (\mathcal{K}\mu_\epsilon(t))(z) \quad \text{ where } \mu_\epsilon(t) := Z_\epsilon(t,\cdot)\#(f_\epsilon^{in}\mathscr{L}^{2d})\,.$$

Observe that $V_\epsilon \in C(\mathbf{R}_+\times\mathbf{R}^{2d};\mathbf{R}^{2d})$, that $V_\epsilon(t,\cdot) \in C^\infty(\mathbf{R}^{2d};\mathbf{R}^{2d})$ and that

$$|V_\epsilon(t,z)| \leq \|\nabla G_d \star \zeta_\epsilon\|_{L^\infty(\mathbf{R}^d)} + |z|\,.$$

Therefore $Z_\epsilon \in C^1(\mathbf{R}_+\times\mathbf{R}^{2d};\mathbf{R}^{2d})$ and $Z_\epsilon(t,\cdot)$ is a C^∞-diffeomorphism of $\mathbf{R}^{2d}$ on itself for all $t \geq 0$. Moreover, one verifies that

$$\operatorname{div}_z V_\epsilon(t,z) = \operatorname{div}_x v - \operatorname{div}_v((\nabla G_d \star \zeta_\epsilon \star \zeta_\epsilon \star \rho_\epsilon(t,\cdot))(x)) = 0\,,$$

so that, for all $t \geq 0$,

$$Z_\epsilon(t,\cdot)\#\mathscr{L}^{2d} = \mathscr{L}^{2d}\,.$$

In particular, the solution of (VP$_\epsilon$) is

$$\mu_\epsilon(t) := Z_\epsilon(t,\cdot)\#(f_\epsilon^{in}\mathscr{L}^{2d}) = f_\epsilon(t,\cdot)\mathscr{L}^{2d}\,,$$

(see questions 6 and 7 in the exercise on the method of characteristics before Theorem 3.2.2), with

$$f_\epsilon(t,z) := f_\epsilon^{in}(Z_\epsilon(t,\cdot)^{-1}(z))\,,$$

for all $t \geq 0$. With this formula, all the properties of f_ϵ are obvious, except the energy conservation.

Observe that, for each $\epsilon > 0$, the initial data $f_\epsilon^{in} \in C_c^\infty(\mathbf{R}^d \times \mathbf{R}^d)$. Thus, for each $\epsilon > 0$, one has

$$f_\epsilon \in C^1(\mathbf{R}_+ \times \mathbf{R}^d \times \mathbf{R}^d) \text{ and } \operatorname{supp}(f_\epsilon(t,\cdot,\cdot)) \text{ is compact for all } t \geq 0\,.$$

Proceeding as the in proof of the energy conservation in the Vlasov-Poisson system, we compute

$$\begin{aligned}
\frac{d}{dt}\iint_{\mathbf{R}^d\times\mathbf{R}^d} \tfrac{1}{2}|v|^2 f_\epsilon(t,x,v)dxdv &= -\iint_{\mathbf{R}^d\times\mathbf{R}^d} v\cdot\nabla_x\Phi_\epsilon(t,x)f_\epsilon(t,x,v)dxdv \\
&= -\int_{\mathbf{R}^d} j_\epsilon(t,x)\cdot\nabla_x\Phi_\epsilon(t,x)dx \\
&= \int_{\mathbf{R}^d} j_\epsilon(t,x)\cdot(\zeta_\epsilon\star\tilde{E}_\epsilon(t,\cdot)(x)dx \\
&= \int_{\mathbf{R}^d} (\zeta_\epsilon\star j_\epsilon(t,\cdot))(x)\cdot\tilde{E}_\epsilon(t,x)dx
\end{aligned}$$

for all $t \geq 0$ by Lemma 4.2.2. Then

$$\begin{aligned}
\int_{\mathbf{R}^d} (\zeta_\epsilon\star j_\epsilon(t,\cdot))(x)\cdot\tilde{E}_\epsilon(t,x)dx &= \int_{\mathbf{R}^d} (\zeta_\epsilon\star\operatorname{div}_x j_\epsilon(t,\cdot)(x))\tilde{\Phi}_\epsilon(t,x)dx \\
&= -\int_{\mathbf{R}^d} (\zeta_\epsilon\star\partial_t\rho_\epsilon(t,\cdot)(x))\tilde{\Phi}_\epsilon(t,x)dx \\
&= \int_{\mathbf{R}^d} (\partial_t\Delta_x\tilde{\Phi}_\epsilon(t,x))\tilde{\Phi}_\epsilon(t,x)dx \\
&= -\int_{\mathbf{R}^d} \partial_t\nabla_x\tilde{\Phi}_\epsilon(t,x)\cdot\nabla_x\tilde{\Phi}_\epsilon(t,x)dx \\
&= -\frac{d}{dt}\int_{\mathbf{R}^d} \tfrac{1}{2}|\nabla_x\tilde{\Phi}_\epsilon(t,x)|^2dx\,,
\end{aligned}$$

for all $t \geq 0$, and the energy conservation follows. ■

4.2.2 Convergence to the Vlasov-Poisson system

Now we let $\epsilon \to 0$, and pass to the limit in (VP$_\epsilon$), using the a priori estimates on f_ϵ and Φ_ϵ that are uniform in ϵ.
Step 1: Uniform estimates. First, one has

$$0 \leq f_\epsilon(t,x,v) \leq \|f_\epsilon^{in}\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)} \text{ for a.e. } (x,v) \in \mathbf{R}^d \times \mathbf{R}^d\,,$$

together with

$$\begin{aligned}\iint_{\mathbf{R}^d\times\mathbf{R}^d} f_\epsilon(t,x,v)dxdv &= \iint_{\mathbf{R}^d\times\mathbf{R}^d} f_\epsilon^{in}(x,v)dxdv \\ &\leq \iint_{\mathbf{R}^d\times\mathbf{R}^d} f_\epsilon^{in}(x,v)dxdv =: \mathcal{M}^{in}\,,\end{aligned}$$

for all $t \geq 0$.

Next

$$\begin{aligned}&\iint_{\mathbf{R}^d\times\mathbf{R}^d} \tfrac{1}{2}|v|^2 f_\epsilon(t,x,v)dxdv + \int_{\mathbf{R}^d} \tfrac{1}{2}|\tilde{E}_\epsilon(t,x)|^2 dx \\ &\qquad = \iint_{\mathbf{R}^d\times\mathbf{R}^d} \tfrac{1}{2}|v|^2 f_\epsilon^{in}(x,v)dxdv + \int_{\mathbf{R}^d} \tfrac{1}{2}|\tilde{E}_\epsilon^{in}(x)|^2 dx = \mathcal{E}_\epsilon^{in}\,,\end{aligned}$$

where

$$E_\epsilon^{in} = -\zeta_\epsilon \star \nabla G_d \star \rho_\epsilon^{in}\,,$$

with

$$\rho_\epsilon^{in}(x) := \int_{\mathbf{R}^d} f_\epsilon^{in}(x,v)dv\,.$$

By the interpolation inequality and the field estimate

$$\begin{aligned}\|E_\epsilon^{in}\|_{L^q(\mathbf{R}^d)} &\leq \|\nabla G_d \star \rho_\epsilon^{in}\|_{L^q(\mathbf{R}^d)} \\ &\leq C_d \|\rho_\epsilon^{in}\|_{L^{\frac{d+2}{d}}(\mathbf{R}^d)} \\ &\leq C_d C(p,d) \|f^{in}\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)}^{\frac{2}{d+2}} (\mathcal{E}^{in})^{\frac{d}{d+2}}\end{aligned}$$

with

$$\frac{1}{q} = \frac{d}{d+2} - \frac{1}{d}\,, \qquad \text{or equivalently } q = \frac{d(d+2)}{(d-2)(d+1)}\,.$$

On the other hand

$$\begin{aligned}\|E_\epsilon^{in}\|_{L^{\frac{d}{d-1}}(\mathbf{R}^d)} &\leq \|\nabla G_d \star \rho_\epsilon^{in}\|_{L^{\frac{d}{d-1}}(\mathbf{R}^d)} \\ &\leq C_d \|\rho_\epsilon^{in}\|_{L^1(\mathbf{R}^d)} \leq C_d \|\rho^{in}\|_{L^1(\mathbf{R}^d)} = C_d \mathcal{M}^{in}\,.\end{aligned}$$

By Hölder's inequality

$$\sup_{\epsilon>0} \|\tilde{E}_\epsilon^{in}\|_{L^2(\mathbf{R}^d)} < \infty$$

provided that

$$\frac{d}{d-1} \leq 2 \leq \frac{d(d+2)}{(d-2)(d+1)}\,, \qquad \text{or equivalently } 2 \leq d \leq 5\,.$$

Hence

$$\begin{aligned}\sup_{t,\epsilon>0} &\iint_{\mathbf{R}^d\times\mathbf{R}^d} \tfrac{1}{2}|v|^2 f_\epsilon(t,x,v)dxdv + \int_{\mathbf{R}^d} \tfrac{1}{2}\tilde{E}_\epsilon(t,x)dx \\ &= \sup_{\epsilon>0} \iint_{\mathbf{R}^d\times\mathbf{R}^d} \tfrac{1}{2}|v|^2 f_\epsilon^{in}(x,v)dxdv + \int_{\mathbf{R}^d} \tfrac{1}{2}\tilde{E}_\epsilon^{in}(x)dx =: \overline{\mathcal{E}}^{in} < \infty\,.\end{aligned}$$

Step 2: Weak compactness. By the Banach-Alaoglu theorem, there exist subsequences of (f_ϵ, E_ϵ) (still denoted (f_ϵ, E_ϵ) for the sake of simplicity) such that

$$f_\epsilon \to f \text{ in } L^\infty(\mathbf{R}_+ \times \mathbf{R}^d \times \mathbf{R}^d) \text{ weak-}*$$

and

$$\rho_\epsilon \to \rho \text{ in } L^\infty(\mathbf{R}_+; L^{(d+2)/d}(\mathbf{R}^d)) \text{ weak-}*\,,$$

while

$$\tilde{E}_\epsilon \to E \text{ in } L^\infty(\mathbf{R}_+; L^2(\mathbf{R}^d)) \text{ weak-}*\ .$$

In particular

$$0 = \partial_{x_i}\partial_{x_j}\tilde{\Phi}_\epsilon - \partial_{x_j}\partial_{x_i}\tilde{\Phi}_\epsilon = \partial_{x_j}(\tilde{E}_\epsilon)_i - \partial_{x_i}(\tilde{E}_\epsilon)_j \to \partial_{x_j}E_i - \partial_{x_i}E_j$$

in $\mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^d)$ as $\epsilon \to 0$, so that E is a gradient field (see Theorem VI in chapter II of [18]). In other words, there exists $\Phi \in \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^d)$ such that

$$E = -\nabla_x \Phi\,.$$

Since

$$\iint_{\mathbf{R}^d\times\mathbf{R}^d} \tfrac{1}{2}|v|^2 f_\epsilon(t,x,v)dxdv \leq \overline{\mathcal{E}}^{in} < \infty\,,$$

we conclude that

$$\rho \in L^\infty(\mathbf{R}_+; L^1(\mathbf{R}^d))\,,$$

and that

$$\int_{\mathbf{R}^d} f(t,x,v)dv = \rho(t,x)\,, \qquad \text{for a.e. } x \in \mathbf{R}^d \text{ and } t > 0\,,$$

where the equality above follows from the tightness[3] in the variable v of the sequence f_ϵ.

[3] A sequence μ_n of bounded, signed Radon measures on $\mathbf{R}^N$ is said to be *tight* if

$$\mu_n(\mathbf{R}^N \setminus B(0,R)) \to 0 \quad \text{if } R \to \infty \text{ uniformly in } n\,.$$

By the "portmanteau theorem" (Theorem 6.8 in chapter of [13]), if a sequence μ_n of bounded, signed Radon measures on $\mathbf{R}^N$ is tight, then the convergence

$$\int_{\mathbf{R}^N} \phi(z)\mu_n(dz) \to 0 \quad \text{as } n \to \infty$$

holds for all $\phi \in C_b(\mathbf{R}^N)$ if (and only if) it holds for all $\phi \in C_c(\mathbf{R}^N)$.

Integrating further in the variable x and applying Fatou's lemma shows that

$$\iint_{\mathbf{R}^d\times\mathbf{R}^d} f(t,x,v)dxdv \le \iint_{\mathbf{R}^d\times\mathbf{R}^d} f^{in}(x,v)dxdv = \mathcal{M}^{in}$$

for a.e. $t\ge 0$. (The inequality comes from the lack of tightness in the x-variable and the resulting potential loss of mass as $|x|\to\infty$.)

Observe that

$$\tilde{E}^{in}_\epsilon = -\zeta_\epsilon \star \nabla G_d \star \rho^{in}_\epsilon \to -\nabla G_d \star \rho^{in} =: E^{in}$$

in $\mathcal{D}'(\mathbf{R}^d)$, since $\zeta_\epsilon \to \delta_0$ in $\mathcal{D}'(\mathbf{R}^d)$ and $\mathrm{supp}(\zeta_\epsilon)\subset B(0,1)$ for all $\epsilon\in(0,1)$ (see Theorem V in chapter VI of [18]), while $\rho^{in}_\epsilon \to \rho^{in}$ in $L^1(\mathbf{R}^d)$ by dominated convergence. Since

$$\begin{aligned}\iint_{\mathbf{R}^d\times\mathbf{R}^d} \tfrac12|v|^2 f_\epsilon(t,x,v)dxdv + \int_{\mathbf{R}^d} \tfrac12|\tilde{E}_\epsilon(t,x)|^2dx\\ = \iint_{\mathbf{R}^d\times\mathbf{R}^d} \tfrac12|v|^2 f^{in}_\epsilon(x,v)dxdv + \int_{\mathbf{R}^d} \tfrac12|\tilde{E}^{in}_\epsilon(x)|^2dx \le \overline{\mathcal{E}}^{in}\end{aligned}$$

for all $t\ge 0$, one has

$$\begin{aligned}\iint_{\mathbf{R}^d\times\mathbf{R}^d} \tfrac12|v|^2 f(t,x,v)dxdv + \int_{\mathbf{R}^d} \tfrac12|E(t,x)|^2dx\\ \le \iint_{\mathbf{R}^d\times\mathbf{R}^d} \tfrac12|v|^2 f^{in}(x,v)dxdv + \int_{\mathbf{R}^d} \tfrac12|E^{in}(x)|^2dx = \mathcal{E}^{in}\end{aligned}$$

for a.e. $t\ge 0$ by the usual argument[4] involving Fatou's lemma (as above for the bound on the total mass), convexity and weak limit in the energy integral.

[4]Since $\tilde{E}_\epsilon \to E$ in $L^\infty(\mathbf{R}_+;L^2(\mathbf{R}^d))$ weak-* as $\epsilon\to 0$, one has

$$\int_a^b\int_{\mathbf{R}^d} E(s,x)\cdot(\tilde{E}_\epsilon(s,x)-E(s,x))dxds \to 0$$

and therefore

$$\varliminf_{\epsilon\to 0}\int_a^b\int_{\mathbf{R}^d} |\tilde{E}_\epsilon(t,x)|^2dxds \ge \int_a^b\int_{\mathbf{R}^d} |\tilde{E}(t,x)|^2dxdt$$

for each $a<b\in\mathbf{R}_+$. On the other hand, since $f_\epsilon \to f$ in $L^\infty(\mathbf{R}_+\times\mathbf{R}^d\times\mathbf{R}^d)$ weak-*, one has

$$\int_a^b\iint_{\mathbf{R}^d\times\mathbf{R}^d} \mathbf{1}_{|x|+|v|\le R}|v|^2 f_\epsilon(t,x,v)dxdvdt \to \int_a^b\iint_{\mathbf{R}^d\times\mathbf{R}^d} \mathbf{1}_{|x|+|v|\le R}|v|^2 f(t,x,v)dxdvdt$$

and since $f_\epsilon\ge 0$ a.e. on $\mathbf{R}_+\times\mathbf{R}^d\times\mathbf{R}^d$

$$\varliminf_{\epsilon\to 0}\int_a^b\iint_{\mathbf{R}^d\times\mathbf{R}^d} |v|^2 f_\epsilon(t,x,v)dxdvdt \ge \int_a^b\iint_{\mathbf{R}^d\times\mathbf{R}^d} \mathbf{1}_{|x|+|v|\le R}|v|^2 f(t,x,v)dxdvdt\,.$$

Letting $R\to\infty$ and applying Fatou's lemma shows that

$$\varliminf_{\epsilon\to 0}\int_a^b\iint_{\mathbf{R}^d\times\mathbf{R}^d} |v|^2 f_\epsilon(t,x,v)dxdvdt \ge \int_a^b\iint_{\mathbf{R}^d\times\mathbf{R}^d} |v|^2 f(t,x,v)dxdvdt\,.$$

Passing to the limit in the sense of distributions in the (linear) Poisson equation, we conclude that

$$-\Delta_x \Phi = \rho = \int_{\mathbf{R}^d} f dv \quad \text{ in } \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^d) .$$

Step 3: Passing to the limit in the nonlinearities. Next we pass to the limit in the Vlasov equation, which we recast as

$$(\partial_t + v \cdot \nabla_x) f_\epsilon = \operatorname{div}_v (f_\epsilon \nabla_x \Phi_\epsilon) .$$

Since the left hand side is linear in f_ϵ, we can pass to the limit in the sense of distributions and find that

$$(\partial_t + v \cdot \nabla_x) f_\epsilon \to (\partial_t + v \cdot \nabla_x) f \text{ in } \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^d \times \mathbf{R}^d)$$

as $\epsilon \to 0$.

It remains to pass to the limit in the nonlinear term $f_\epsilon \nabla_x \Phi_\epsilon$. Since ζ_ϵ is uniformly bounded in $L^1(\mathbf{R}^d)$, and since

$$-\nabla_x^2 \Phi_\epsilon(t, \cdot) = \nabla_x^2 G_d \star (\zeta_\epsilon \star \zeta_\epsilon \star \rho_\epsilon(t, \cdot))$$

the a priori estimate on the derivatives of the force field can be applied and shows that, for all $t \geq 0$

$$\|\nabla_x^2 \Phi_\epsilon(t, \cdot)\|_{L^p(\mathbf{R}^d)} \leq C \|\rho_\epsilon(t, \cdot)\|_{L^p(\mathbf{R}^d)} \leq \text{Const.}, \quad 1 < p \leq \frac{d+2}{d} .$$

By the same token, for all $t \geq 0$, one has

$$\|\partial_t \nabla_x \Phi_\epsilon(t, \cdot)\|_{L^q(\mathbf{R}^d)} \leq C \|j_\epsilon(t, \cdot)\|_{L^p(\mathbf{R}^d)} \leq \text{Const.}, \quad 1 < q \leq \frac{d+2}{d+1} .$$

In view of the interpolation inequality, we conclude that

$$\sup_{t, \epsilon > 0} \|\partial_t \nabla_x \Phi_\epsilon(t, \cdot)\|_{L^q(\mathbf{R}^d)} + \|\nabla_x^2 \Phi_\epsilon(t, \cdot)\|_{L^p(\mathbf{R}^d)} < \infty$$

for

$$1 < p \leq \frac{d+2}{d} \text{ and } 1 < q \leq \frac{d+2}{d+1} .$$

By the Rellich compactness theorem, we conclude that

$$\chi \nabla_x \Phi_\epsilon \to \chi \nabla_x \Phi \text{ in } L^1(\mathbf{R}_+ \times \mathbf{R}^d \times \mathbf{R}^d) \text{ strong,}$$

Finally

$$\int_a^b \iint_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 f(t, x, v) dx dv dt + \int_a^b \int_{\mathbf{R}^d} |\tilde{E}(t, x)|^2 dx dt$$
$$\leq \varliminf_{\epsilon \to 0} \left(\int_a^b \iint_{\mathbf{R}^d \times \mathbf{R}^d} |v|^2 f_\epsilon(t, x, v) dx dv dt + \int_a^b \int_{\mathbf{R}^d} |\tilde{E}_\epsilon(t, x)|^2 dx ds \right) = 2 \mathcal{E}^{in} .$$

Since this inequality holds for all $a < b \in \mathbf{R}$, it also holds for a.e. $t \in \mathbf{R}_+$.

for each $\chi \in C_c(\mathbf{R}_+^* \times \mathbf{R}^d \times \mathbf{R}^d)$ as $\epsilon \to 0$. Therefore

$$\begin{aligned}&\iiint_{\mathbf{R}_+ \times \mathbf{R}^d \times \mathbf{R}^d} \chi f_\epsilon(t,x,v) \nabla_x \Phi_\epsilon(t,x) dx dv dt \\ \to &\iiint_{\mathbf{R}_+ \times \mathbf{R}^d \times \mathbf{R}^d} \chi f(t,x,v) \nabla_x \Phi(t,x) dx dv dt\end{aligned}$$

as $\epsilon \to 0$. In other words,

$$f_\epsilon \nabla_x \Phi_\epsilon \to f \nabla_x \Phi \text{ in } \mathcal{D}'(\mathbf{R}_+^* \times \mathbf{R}^d \times \mathbf{R}^d)$$

as $\epsilon \to 0$, so that

$$(\partial_t + v \cdot \nabla_x) f = \operatorname{div}_v (f \nabla_x \Phi) = -\operatorname{div}_v (fE)\,.$$

Step 4: Initial condition. It remains to check that f satisfies the initial condition.

Observe that

$$\partial_t f_\epsilon = -\operatorname{div}_x (v f_\epsilon) + \operatorname{div}_v (f_\epsilon \nabla_x \Phi_\epsilon)\,.$$

By the mass and energy bounds, and the maximum principle

$$\sup_{t,\epsilon>0} \|v f_\epsilon(t,\cdot,\cdot)\|_{L^1(\mathbf{R}^d \times \mathbf{R}^d)} + \sup_{t,\epsilon>0} \|f_\epsilon(t,\cdot,\cdot) \nabla_x \Phi_\epsilon(t,\cdot)\|_{L^\infty(\mathbf{R}_v^d; L^2(\mathbf{R}^d))} < \infty\,.$$

Therefore, for each $\chi \in C_c^\infty(\mathbf{R}^d \times \mathbf{R}^d)$

$$\sup_{t,\epsilon>0} \left| \frac{d}{dt} \iint_{\mathbf{R}^d \times \mathbf{R}^d} f_\epsilon(t,x,v) \chi(x,v) dx dv \right| < \infty\,,$$

and we conclude from the Ascoli-Arzela theorem that

$$\iint_{\mathbf{R}^d \times \mathbf{R}^d} f_\epsilon(t,x,v) \chi(x,v) dx dv \to \iint_{\mathbf{R}^d \times \mathbf{R}^d} f(t,x,v) \chi(x,v) dx dv$$

uniformly on $[0,T]$ for all $T > 0$ as $\epsilon \to 0^+$. Thus the function

$$t \mapsto \iint_{\mathbf{R}^d \times \mathbf{R}^d} f(t,x,v) \chi(x,v) dx dv$$

is continuous on $\mathbf{R}_+$ (being the uniform limit of continuous functions on $[0,T]$ for all $T > 0$).

In particular, for $t = 0$, this implies that

$$\iint_{\mathbf{R}^d \times \mathbf{R}^d} f^{in}(x,v) \chi(x,v) dx dv = \iint_{\mathbf{R}^d \times \mathbf{R}^d} f(0,x,v) \chi(x,v) dx dv\,,$$

and since this equality holds for all $\chi \in C_c^\infty(\mathbf{R}^d \times \mathbf{R}^d)$, we conclude that

$$f\big|_{t=0} = f^{in}\,.$$

The proof is complete.

4.3 Propagation of Moments in Dimension 2

In this and the next section, we seek to bound moments in the variable v of solutions of the Vlasov-Poisson system. More precisely, we seek to construct weak solutions $f \equiv f(t,x,v)$ of the Vlasov-Poisson systems for which quantities of the form

$$\iint_{\mathbf{R}^2\times\mathbf{R}^2} |v|^k f(t,x,v)dxdv$$

are bounded for all $t \geq 0$ provided that they are bounded at $t=0$. Estimates of this kind are the key step in the proof of global existence of classical — instead of weak — solutions of the Vlasov-Poisson system.

Henceforth we proceed by formal a priori estimates; while our computations are not justified for all weak solutions of the Vlasov-Poisson system, the same computations could be done on the approximate Vlasov-Poisson system (VP$_\epsilon$), and since the resulting estimates are uniform as $\epsilon \to 0^+$, one would get the desired bounds on all weak solutions constructed as in the previous section.

As a warm-up, we first consider the case $d=2$, that is by far the easiest. In this section and the next one, we designate by C various constants that may depend on quantities that are fixed (and in any case independent of ϵ, such as the space dimension d, the initial data $f^{in}\dots$)

Start from the differential inequality satisfied by moments of the distribution function:

$$\begin{aligned}\frac{d}{dt}\iint_{\mathbf{R}^2\times\mathbf{R}^2} &|v|^k f(t,x,v)dxdv\\ &= k\iint_{\mathbf{R}^2\times\mathbf{R}^2}|v|^{k-2}v\cdot E(t,x)f(t,x,v)dxdv\\ &\leq k\int_{\mathbf{R}^2}|E(t,x)|\left(\int_{\mathbf{R}^2}|v|^{k-1}f(t,x,v)dv\right)dx\\ &\leq k\|E(t,\cdot)\|_{L^{k+2}(\mathbf{R}^2)}\left\|\int_{\mathbf{R}^2}|v|^{k-1}f(t,\cdot,v)dv\right\|_{L^{\frac{k+2}{k+1}}(\mathbf{R}^2)}.\end{aligned}$$

By the interpolation inequality (with $r=k-1$, $d=2$ and $m=k$), one has

$$\left\|\int_{\mathbf{R}^2}|v|^{k-1}f(t,\cdot,v)dv\right\|_{L^{\frac{k+2}{k+1}}(\mathbf{R}^2)} \leq C\left(\iint_{\mathbf{R}^2\times\mathbf{R}^2}|v|^k f(t,x,v)dxdv\right)^{\frac{k+1}{k+2}}.$$

Denoting

$$\mu_k(t) := \iint_{\mathbf{R}^2\times\mathbf{R}^2}|v|^k f(t,x,v)dxdv\,,$$

one has therefore

$$\dot{\mu}_k(t) \leq C\|E(t,\cdot)\|_{L^{k+2}(\mathbf{R}^2)}\mu_k(t)^{\frac{k+1}{k+2}}.$$

Our goal is to obtain an inequality of the form

$$\|E(t,\cdot)\|_{L^{k+2}(\mathbf{R}^2)} \leq C\mu_k(t)^{\frac{1}{k+2}}.$$

(Any power larger that $\frac{1}{k+2}$ in this estimate would lead to a blow-up on the upper bound of $\mu_k(t)$ and is therefore meaningless.)

Applying the a priori bound on the force field

$$\|E(t,\cdot)\|_{L^{k+2}(\mathbf{R}^2)} \le C\|\rho(t,\cdot)\|_{L^{\frac{2k+4}{k+4}}(\mathbf{R}^2)} .$$

On the other hand, applying again the interpolation inequality shows that

$$\|\rho(t,\cdot)\|_{L^{\frac{k+2}{2}}(\mathbf{R}^2)} \le C\mu_k(t)^{\frac{2}{k+2}} .$$

Then, by Hölder's inequality

$$\|\rho(t,\cdot)\|_{L^{\frac{2k+4}{k+4}}(\mathbf{R}^2)} \le \|\rho(t,\cdot)\|_{L^1(\mathbf{R}^2)}^{1-\theta}\|\rho(t,\cdot)\|_{L^{\frac{k+2}{2}}(\mathbf{R}^2)}^{\theta}$$

with

$$1-\theta+\frac{2\theta}{k+2}=\frac{k+4}{2k+4} ,$$

i.e.

$$(1-\theta)\left(1-\frac{2}{k+2}\right)=(1-\theta)\frac{k}{k+2}=\frac{k}{2k+4} ,$$

so that

$$\theta=1-\theta=\tfrac{1}{2} .$$

Therefore

$$\|E(t,\cdot)\|_{L^{k+2}(\mathbf{R}^2)} \le C\|\rho(t,\cdot)\|_{L^{\frac{2k+4}{k+4}}(\mathbf{R}^2)} \le C\|\rho(t,\cdot)\|_{L^{\frac{k+2}{2}}(\mathbf{R}^2)}^{\frac{1}{2}} \le C\mu_k(t)^{\frac{1}{k+2}} .$$

Inserting this in the differential inequality for μ_k, one finds that

$$\dot{\mu}_k(t) \le C\|E(t,\cdot)\|_{L^{k+2}(\mathbf{R}^2)}\mu_k(t)^{\frac{k+1}{k+2}} \le C_k(\mathcal{M}^{in},\mathcal{E}^{in},\|f^{in}\|_{L^\infty(\mathbf{R}^2\times\mathbf{R}^2)}\mu_k(t) ,$$

and we conclude that

$$\mu_k(t) \le \mu_k(0)e^{C_k(\mathcal{M}^{in},\mathcal{E}^{in},\|f^{in}\|_{L^\infty(\mathbf{R}^2\times\mathbf{R}^2)})t} , \quad \text{for all } t,k \ge 0 .$$

Summarizing, we have proved the following result.

Theorem 4.3.1 *Let $f^{in} \in L^1 \cap L^\infty(\mathbf{R}^2 \times \mathbf{R}^2)$ be such that $f^{in} \ge 0$ a.e., and assume that*

$$\tfrac{1}{2}\iint_{\mathbf{R}^2\times\mathbf{R}^2}|v|^2 f^{in}(x,v)dxdv+\tfrac{1}{2}\int_{\mathbf{R}^2}|E^{in}(x)|^2dx =: \mathcal{E}^{in} < \infty ,$$

where

$$E^{in}=-\nabla G_2\star\rho^{in} , \quad \rho^{in}:=\int_{\mathbf{R}^2}f^{in}dv .$$

Assume further that

$$\iint_{\mathbf{R}^2\times\mathbf{R}^2}|v|^{k_0}f^{in}(x,v)dxdv<\infty$$

for some $k_0 > 2$.

Then, there exists a weak solution of the Vlasov-Poisson system in $\mathbf{R}^2 \times \mathbf{R}^2$ *with initial data* f^{in} *such that*

$$\begin{aligned} &\iint_{\mathbf{R}^2\times\mathbf{R}^2} |v|^k f(t,x,v)dxdv \\ &\leq e^{C_k(\mathcal{M}^{in},\mathcal{E}^{in},\|f^{in}\|_{L^\infty(\mathbf{R}^2\times\mathbf{R}^2)})t} \iint_{\mathbf{R}^2\times\mathbf{R}^2} |v|^2 f^{in}(x,v)dxdv \end{aligned}$$

for all $t \geq 0$ *and* $0 \leq k \leq k_0$.

4.4 Propagation of Moments in Space Dimension 3

In space dimension 3, the analogous result is stated below.

Theorem 4.4.1 (P.-L. Lions, B. Perthame [10]) *Let* $k_0 > 6$ *and* $T > 0$. *Let* $f^{in} \in L^1 \cap L^\infty(\mathbf{R}^3 \times \mathbf{R}^3)$ *be such that* $f^{in} \geq 0$ *a.e., and assume that*

$$\tfrac{1}{2}\iint_{\mathbf{R}^3\times\mathbf{R}^3} |v|^2 f^{in}(x,v)dxdv + \tfrac{1}{2}\int_{\mathbf{R}^3} |E^{in}(x)|^2 dx =: \mathcal{E}^{in} < \infty\,,$$

where

$$E^{in} = -\nabla G_3 \star \rho^{in}\,, \quad \rho^{in} := \int_{\mathbf{R}^3} f^{in} dv\,.$$

Assume further that

$$\iint_{\mathbf{R}^2\times\mathbf{R}^2} |v|^{k_0} f^{in}(x,v)dxdv < \infty$$

for some $k_0 > 3$.

Then, there exists $C_T > 0$ *and a weak solution of the Vlasov-Poisson system in* $\mathbf{R}^3 \times \mathbf{R}^3$ *with initial data* f^{in} *such that*

$$\iint_{\mathbf{R}^3\times\mathbf{R}^3} |v|^k f(t,x,v)dxdv \leq C_T\,, \quad 0 \leq t \leq T\,,$$

for all k *such that* $0 \leq k \leq k_0$.

The proof of this result is rather involved, and is split in 5 steps presented below.

To begin with, start again from the differential inequality satisfied by moments of the distribution function:

$$\begin{aligned}\frac{d}{dt}&\iint_{\mathbf{R}^3\times\mathbf{R}^3}|v|^k f(t,x,v)dxdv\\&=k\iint_{\mathbf{R}^3\times\mathbf{R}^3}|v|^{k-2}v\cdot E(t,x)f(t,x,v)dxdv\\&\le k\int_{\mathbf{R}^3}|E(t,x)|\left(\int_{\mathbf{R}^2}|v|^{k-1}f(t,x,v)dv\right)dx\\&\le k\|E(t,\cdot)\|_{L^{k+3}(\mathbf{R}^3)}\left\|\int_{\mathbf{R}^3}|v|^{k-1}f(t,\cdot,v)dv\right\|_{L^{\frac{k+3}{k+2}}(\mathbf{R}^3)}.\end{aligned}$$

By the interpolation inequality, this time with $r=k-1$, $d=3$ and $m=k$, one has

$$\left\|\int_{\mathbf{R}^3}|v|^{k-1}f(t,\cdot,v)dv\right\|_{L^{\frac{k+3}{k+2}}(\mathbf{R}^3)}\le C\left(\iint_{\mathbf{R}^3\times\mathbf{R}^3}|v|^k f(t,x,v)dxdv\right)^{\frac{k+2}{k+3}}.$$

Denoting as above

$$\mu_k(t):=\iint_{\mathbf{R}^3\times\mathbf{R}^3}|v|^k f(t,x,v)dxdv\,,$$

one arrives at the differential inequality

$$\dot\mu_k(t)\le C\|E(t,\cdot)\|_{L^{k+3}(\mathbf{R}^3)}\mu_k(t)^{\frac{k+2}{k+3}}\,.$$

In the 2-dimensional case, the force field E was estimated by using the Poisson equation and the a priori bounds on the solution of the Vlasov equation deduced from the conservation laws of mass and energy, and the positivity and maximum principle for the distribution function.

Controlling the propagation of moments in the 3-dimensional case requires using the Vlasov equation itself, and not only consequences thereof such as the conservation laws of mass and energy, or the positivity and maximum principle for the distribution function.

4.4.1 Step 1: a formula for the macroscopic density

A first important step in the proof is to obtain a formula for the macroscopic density base on solving the Vlasov equation along characteristics.

Lemma 4.4.2 *Under the same assumptions as in Theorem 4.4.1, one has*

$$\rho(t,x)=\rho_0(t,x)-\operatorname{div}_x\int_0^t s\left(\int_{\mathbf{R}^3}E(t-s,x-sv)f(t-s,x-sv,v)dv\right)ds\,,$$

where

$$\rho_0(t,x):=\int_{\mathbf{R}^3}f^{in}(x-tv,v)dv\,.$$

Proof. Indeed, solving the Vlasov equation for f by the method of characteristics, with $-E \cdot \nabla_v f$ treated as a source term, one finds that

$$f(t,x,v) = f^{in}(x-tv,v) - \int_0^t E(t-s,x-sv) \cdot \nabla_v f(t-s,x-sv,v)ds\,.$$

By the chain rule

$$\begin{aligned}&\operatorname{div}_v(E(t-s,x-sv)f(t-s,x-sv,v))\\&\qquad= E(t-s,x-sv)\cdot\nabla_v f(t-s,x-sv,v)\\&\qquad- s\operatorname{div}_x(E(t-s,x-sv)f(t-s,x-sv,v))\,,\end{aligned}$$

so that

$$\begin{aligned}&\int_{\mathbf{R}^3} E(t-s,x-sv)\cdot\nabla_v f(t-s,x-sv,v)dv\\&\qquad= s\operatorname{div}_x \int_{\mathbf{R}^3} E(t-s,x-sv)f(t-s,x-sv,v)dv\,.\end{aligned}$$

Therefore, integrating both sides of the formula above for f in the v variable, one sees that

$$\begin{aligned}\rho(t,x) = \int_{\mathbf{R}^3} f(t,x,v)dv &= \int_{\mathbf{R}^3} f^{in}(x-tv,v)dv\\&- \int_0^t s\operatorname{div}_x\left(\int_{\mathbf{R}^3} E(t-s,x-sv)f(t-s,x-sv,v)dv\right)ds\,,\end{aligned}$$

which is precisely the formula for ρ given above. ■

4.4.2 Step 2: estimating the force field

We recall that

$$E(t,\cdot) = -\nabla G_3 \star \rho(t,\cdot)\,.$$

With the formula above for ρ, one has

$$\begin{aligned}E(t,\cdot) =& -\nabla G_3 \star \rho_0(t,\cdot)\\&+ \nabla G_3 \operatorname{div}_x \int_0^t s\left(\int_{\mathbf{R}^3} E(t-s,x-sv)f(t-s,x-sv,v)dv\right)ds\,.\end{aligned}$$

Hence, by the Young inequality (for the first term on the right hand side) and the Calderón-Zygmund inequality (for the second term on the right hand side), one has

$$\begin{aligned}&\|E(t,\cdot)\|_{L^{k+3}(\mathbf{R}^3)} \le C\|\rho_0(t,\cdot)\|_{L^{\frac{3k+9}{k+6}}(\mathbf{R}^3)}\\&+C\left\|\int_0^t s\left(\int_{\mathbf{R}^3} E(t-s,x-sv)f(t-s,x-sv,v)dv\right)ds\right\|_{L^{k+3}(\mathbf{R}^3)},.\end{aligned}$$

Estimating ρ_0 is easy, as it involves the initial data f^{in} explicitly. Thus

$$\begin{aligned}\|\rho_0(t,\cdot)\|_{L^{\frac{3k+9}{k+6}}(\mathbf{R}^3)} &\le \left(\iint_{\mathbf{R}^3\times\mathbf{R}^3} |v|^l f^{in}(x-tv,v)dxdv\right)^{\frac{k+6}{3k+9}}\\ &= \left(\iint_{\mathbf{R}^3\times\mathbf{R}^3} |v|^l f^{in}(y,v)dydv\right)^{\frac{k+6}{3k+9}} = \mu_l(0)^{\frac{k+6}{3k+9}}\,,\end{aligned}$$

with

$$\frac{l+3}{3} = \frac{3k+9}{k+6}\,.$$

This follows from the interpolation inequality with $r=1$, $d=3$ and $m=l$. Since $k>3$, one has

$$\tfrac{1}{3}l = \frac{2k+3}{k+6} \le \frac{2k+k}{3+6} = \tfrac{1}{3}k \quad \text{or equivalently } l\le k\,.$$

Thus

$$\mu_l(0)\le \mu_0(0)+\mu_k(0)\,.$$

The second term, i.e.

$$\left\|\int_0^t s\left(\int_{\mathbf{R}^3} E(t-s,x-sv)f(t-s,x-sv,v)dv\right)ds\right\|_{L^{k+3}(\mathbf{R}^3)}$$

requires a much more involved discussion, presented below.

Specifically, the integral

$$\int_0^t s\left(\int_{\mathbf{R}^3} E(t-s,x-sv)f(t-s,x-sv,v)dv\right)ds$$

is split into

$$\begin{aligned}&\int_0^t s\left(\int_{\mathbf{R}^3} E(t-s,x-sv)f(t-s,x-sv,v)dv\right)ds\\ &\quad= \int_0^{t_0} s\left(\int_{\mathbf{R}^3} E(t-s,x-sv)f(t-s,x-sv,v)dv\right)ds\\ &\quad+ \int_{t_0}^t s\left(\int_{\mathbf{R}^3} E(t-s,x-sv)f(t-s,x-sv,v)dv\right)ds =: J+I\,.\end{aligned}$$

4.4.3 Step 3: the large t contribution

The second term I on the right hand side of the equality above is the easier one to estimate.

Lemma 4.4.3 *For each $p\in(1,\infty)$ and each measurable ϕ defined on $\mathbf{R}^n$, one has*

$$\|\phi\psi\|_{L^1(\mathbf{R}^n)} \le \|\phi\|_{L^1(\mathbf{R}^n)}^{\frac{1}{p}}\|\phi\|_{L^\infty(\mathbf{R}^n)}^{1-\frac{1}{p}}\|\psi\|_{L^{p,\infty}(\mathbf{R}^n)}\,.$$

Proof. Indeed, if $\phi \in L^1 \cap L^\infty(\mathbf{R}^n)$, the linear map

$$T : \psi \mapsto \phi\psi$$

satisfies

$$T(L^1(\mathbf{R}^n)) \subset L^1(\mathbf{R}^n) \text{ with } \|T\|_{\mathcal{L}(L^1(\mathbf{R}^n),L^1(\mathbf{R}^n))} \leq \|\phi\|_{L^\infty(\mathbf{R}^n)}\,,$$

and

$$T(L^\infty(\mathbf{R}^n)) \subset L^1(\mathbf{R}^n) \text{ with } \|T\|_{\mathcal{L}(L^\infty(\mathbf{R}^n),L^1(\mathbf{R}^n))} \leq \|\phi\|_{L^1(\mathbf{R}^n)}\,.$$

By real interpolation[5]

$$T((L^1(\mathbf{R}^n), L^\infty(\mathbf{R}^n))_{1/p',\infty}) = T(L^{p,\infty}(\mathbf{R}^n)) \subset L^1(\mathbf{R}^n)\,,$$

with

$$\|T\|_{\mathcal{L}(L^{p,\infty}(\mathbf{R}^n),L^1(\mathbf{R}^n))} \leq \|\phi\|^{1-\frac{1}{p'}}_{L^\infty(\mathbf{R}^n)}\|\phi\|^{\frac{1}{p'}}_{L^1(\mathbf{R}^n)} = \|\phi\|^{\frac{1}{p}}_{L^\infty(\mathbf{R}^n)}\|\phi\|^{1-\frac{1}{p}}_{L^1(\mathbf{R}^n)}$$

which leads to the estimate above for $\phi\psi = T\psi$. ■

Then[6]

$$|I(t,x)| = \left|\int_{t_0}^t s\left(\int_{\mathbf{R}^3} E(t-s,x-sv)f(t-s,x-sv,v)dv\right)ds\right|$$
$$\int_{t_0}^t s\|E(t-s,x-s\cdot)\|_{L^{\frac{3}{2},\infty}(\mathbf{R}^3_v)}\|f\|^{\frac{2}{3}}_{L^\infty}\|f(t-s,x-s\cdot,\cdot)\|^{\frac{1}{3}}_{L^1(\mathbf{R}^3_v)}ds\,.$$

Besides, an explicit computation shows that

$$\|E(t-s,x-s\cdot)\|_{L^{\frac{3}{2},\infty}(\mathbf{R}^3)} = s^{-2}\|E(t-s,\cdot)\|_{L^{\frac{3}{2},\infty}(\mathbf{R}^3)}\,,$$

[5] Given two Banach spaces E_0, E_1 with norms denoted by $\|\cdot\|_0$ and $\|\cdot\|_1$ respectively, for each $\psi \in E_0 + E_1$, set

$$K(t,\psi) := \inf_{\substack{\psi_0+\psi_1=\psi \\ \psi_0\in E_0,\,\psi_1\in E_1}} (\|\psi_0\|_0 + t\|\psi_1\|_1)\,.$$

The interpolation space $(E_0,E_1)_{\theta,\infty}$ is defined, for each $\theta \in (0,1)$, as the set of ψs such that $K(t,\psi) \leq Ct^\theta$. In the special case where $E_0 = L^1(\mathbf{R}^n)$ while $E_1 = L^\infty(\mathbf{R}^n)$, one has

$$K(t,\psi) = \int_0^t \psi^*(s)ds$$

where ψ^* is the decreasing rearrangement of ψ, and one finds that

$$(L^1(\mathbf{R}^n), L^\infty(\mathbf{R}^n))_{1/p',\infty} = L^{p,\infty}(\mathbf{R}^n)$$

for all $p \in (1,\infty)$. The fact that $T \in \mathcal{L}((L^1(\mathbf{R}^n), L^\infty(\mathbf{R}^n))_{1/p',\infty}, L^1(\mathbf{R}^n))$ with the desired estimate on the norm of T is precisely Lemma 22.3 in the book by L. Tartar, "An introduction to Sobolev spaces an interpolation spaces", Lecture Notes of the UMI no. 3, Springer-Verlag, Berlin, Heidelberg 2007.

[6] If $\phi \in L^{p,\infty}(\mathbf{R}^n)$, one denotes

$$\|\phi\|_{L^{p,\infty}(\mathbf{R}^n)} = \inf\{C > 0 \text{ s.t. } \mathscr{L}^n(\{x \in \mathbf{R}^n \text{ s.t. } |\phi(x)|^p \geq t\}) \leq C^p/t \text{ for all } t > 0\}\,.$$

so that

$$
\begin{aligned}
|I(t,x)| \leq \|E(t-s,\cdot)\|_{L^{\frac{3}{2},\infty}(\mathbf{R}^3)} \|f\|^{\frac{2}{3}}_{L^\infty(\mathbf{R}^3)} \\
\times \int_{t_0}^t \frac{1}{s} \left(\int_{\mathbf{R}^3} \|f(t-s,x-s\cdot,\cdot)\|^{\frac{1}{3}}_{L^1(\mathbf{R}^3)} dv \right) ds\,.
\end{aligned}
$$

Now

$$
\begin{aligned}
\|E(t-s,\cdot)\|_{L^{\frac{3}{2},\infty}(\mathbf{R}^3)} &= \|\nabla G_3 \star \rho(t-s,\cdot)\|_{L^{\frac{3}{2},\infty}(\mathbf{R}^3)} \\
&\leq \|\nabla G_3\|_{L^{\frac{3}{2},\infty}(\mathbf{R}^3)} \|\rho(t-s,\cdot)\|_{L^1(\mathbf{R}^3)} \\
&= \|\nabla G_3\|_{L^{\frac{3}{2},\infty}(\mathbf{R}^3)} \mathcal{M}^{in}\,,
\end{aligned}
$$

so that the estimate above reduces to

$$
|I(t,x)| \leq C \int_{t_0}^t \frac{1}{s} \left(\int_{\mathbf{R}^3} f(t-s,x-sv,v)dv \right)^{\frac{1}{3}} ds\,.
$$

Therefore

$$
\begin{aligned}
&\|I(t,\cdot)\|_{L^{k+3}(\mathbf{R}^3_x)} \\
&= \left\| \int_{t_0}^t s \left(\int_{\mathbf{R}^3} E(t-s,x-sv) f(t-s,x-sv,v)dv \right) ds \right\|_{L^{k+3}(\mathbf{R}^3_x)} \\
&\leq C \int_{t_0}^t \frac{1}{s} \left\| \left(\int_{\mathbf{R}^3} f(t-s,x-sv,v)dv \right)^{\frac{1}{3}} \right\|_{L^{k+3}(\mathbf{R}^3_x)} ds \\
&\leq C \ln \frac{t}{t_0} \sup_{t_0 \leq s \leq t} \left\| \left(\int_{\mathbf{R}^3} f(t-s,x-sv,v)dv \right)^{\frac{1}{3}} \right\|_{L^{k+3}(\mathbf{R}^3_x)} \\
&= C \ln \frac{t}{t_0} \sup_{t_0 \leq s \leq t} \left\| \int_{\mathbf{R}^3} f(t-s,x-sv,v)dv \right\|^{\frac{1}{3}}_{L^{\frac{k+3}{3}}(\mathbf{R}^3_x)} \\
&\leq C \ln \frac{t}{t_0} \sup_{t_0 \leq s \leq t} \left\| \left(\int_{\mathbf{R}^3} |v|^k f(t-s,x-sv,v)dv \right)^{\frac{3}{k+3}} \right\|^{\frac{1}{3}}_{L^{\frac{k+3}{3}}(\mathbf{R}^3_x)} \\
&= C \ln \frac{t}{t_0} \sup_{t_0 \leq s \leq t} \left(\iint_{\mathbf{R}^3 \times \mathbf{R}^3} |v|^k f(t-s,x-sv,v)dxdv \right)^{\frac{1}{k+3}} \\
&= C \ln \frac{t}{t_0} \sup_{t_0 \leq s \leq t} \mu_k(t-s)^{\frac{1}{k+3}}\,.
\end{aligned}
$$

In this chain of inequalities, we have used the following obvious identity:

Lebesgue norms of powers: for each nonnegative, measurable ϕ defined a.e. on $\mathbf{R}^n$, each $p \in [1,\infty]$ and each $\alpha \geq 1/p$, one has

$$
\|\phi^\alpha\|_{L^p(\mathbf{R}^n)} = \|\phi\|^\alpha_{L^{\alpha p}(\mathbf{R}^n)}\,.
$$

4.4.4 Step 4: the small t contribution

It remains to estimate the contribution

$$\|J(t,\cdot)\|_{L^{k+3}(\mathbf{R}^3_x)}$$
$$=\left\|\int_0^{t_0} s\left(\int_{\mathbf{R}^3} E(t-s,x-sv)f(t-s,x-sv,v)dv\right)ds\right\|_{L^{k+3}(\mathbf{R}^3_x)}.$$

By Hölder's inequality, for $\frac{1}{r}+\frac{1}{r'}=1$ with $r\in(1,\infty)$ left unspecified so far, one has

$$|J(t,x)|=\left|\int_{\mathbf{R}^3} E(t-s,x-sv)f(t-s,x-sv,v)dv\right|$$
$$\leq\left(\int_{\mathbf{R}^3}|E(t-s,x-sv)|^r dv\right)^{\frac{1}{r}}\left(\int_{\mathbf{R}^3} f(t-s,x-sv,v)^{r'}dv\right)^{\frac{1}{r'}}$$
$$\leq\frac{1}{s^{3/r}}\left(\int_{\mathbf{R}^3}|E(t-s,y)|^r dy\right)^{\frac{1}{r}}\|f\|_{L^\infty}^{\frac{1}{r}}\left(\int_{\mathbf{R}^3} f(t-s,x-sv,v)dv\right)^{\frac{1}{r'}}.$$

Hence

$$|J(t,x)|\leq\sup_{0<s<t_0}\|E(t-s,\cdot)\|_{L^r}\int_0^{t_0}s^{1-\frac{3}{r}}\left(\int_{\mathbf{R}^3} f(t-s,x-sv,v)dv\right)^{\frac{1}{r'}}ds\,,$$

so that

$$\|J(t,\cdot)\|_{L^{k+3}(\mathbf{R}^3)}$$
$$\leq\int_0^{t_0}s^{1-\frac{3}{r}}\|E(t-s,\cdot)\|_{L^r(\mathbf{R}^3)}\left\|\left(\int_{\mathbf{R}^3} f(t-s,x-sv,v)dv\right)^{\frac{1}{r'}}\right\|_{L^{k+3}(\mathbf{R}^3_x)}ds$$
$$\leq Ct_0^{2-\frac{3}{r}}\sup_{0<s<t_0}\|E(t-s,\cdot)\|_{L^r(\mathbf{R}^3_x)}\sup_{0<s<t_0}\left\|\int_{\mathbf{R}^3} f(t-s,x-sv,v)dv\right\|_{L^{\frac{k+3}{r'}}(\mathbf{R}^3_x)}^{\frac{1}{r'}},$$

by the Lebesgue norm of powers identity.

Let $m>0$ be such that

$$\frac{m+3}{3}=\frac{k+3}{r'}\,.$$

By the interpolation inequality

$$\int_{\mathbf{R}^3} f(t-s,x-sv,v)dv\leq\left(\int_{\mathbf{R}^3}|v|^m f(t-s,x-sv,v)dv\right)^{\frac{3}{3+m}},$$

so that

$$\left\|\int_{\mathbf{R}^3} f(t-s,x-sv,v)dv\right\|_{L^{\frac{k+3}{r'}}(\mathbf{R}^3_x)}^{\frac{1}{r'}}$$
$$\leq\left(\iint_{\mathbf{R}^3\times\mathbf{R}^3}|v|^m f(t-s,x-sv,v)dxdv\right)^{\frac{1}{k+3}}.$$

Therefore

$$\begin{aligned}
&\|J(t,\cdot)\|_{L^{k+3}(\mathbf{R}^3)} \\
&\le Ct_0^{2-\frac{3}{r}} \sup_{0<s<t_0} \|E(t-s,\cdot)\|_{L^r(\mathbf{R}^3)} \sup_{0<s<t_0} \mu_m(t-s)^{\frac{1}{k+3}} \\
&\le Ct_0^{2-\frac{3}{r}} \sup_{0<s<t_0} \mu_m(t-s)^{\frac{1}{k+3}}\,,
\end{aligned}$$

since

$$\|E(t,\cdot)\|_{L^r(\mathbf{R}^3)} = \|\nabla G_3 \star \rho(t,\cdot)\|_{L^r(\mathbf{R}^3)} \le C\|\rho(t,\cdot)\|_{L^p(\mathbf{R}^3)} \le C$$

with

$$\frac{1}{r} = \frac{1}{p} - \frac{1}{3} \text{ and } 1 < p \le \frac{5}{3} \quad \text{ or equivalently } \tfrac{3}{2} < r \le \tfrac{15}{4}\,.$$

Observe that this inequality involves the desired exponent $\frac{1}{k+3}$, unfortunately with μ_m, and $m \ge k$ since $r > \frac{3}{2}$ so that $r' < 3$.

Returning to the differential inequality for moments of the distribution function

$$\dot{\mu}_m(t) \le C\|E(t,\cdot)\|_{L^{3+m}(\mathbf{R}^3)}\mu_m(t)^{\frac{m+2}{m+3}}\,,$$

or equivalently

$$(m+3)\frac{d}{dt}\mu_m(t)^{\frac{1}{m+3}} \le C\|E(t,\cdot)\|_{L^{3+m}(\mathbf{R}^3)}\,,$$

we find that

$$\begin{aligned}
\mu_m(t) &\le \left(\mu_m(0)^{\frac{1}{m+3}} + \frac{C}{m+3}\int_0^t \|E(s,\cdot)\|_{L^{3+m}(\mathbf{R}^3)}ds\right)^{m+3} \\
&\le \left(\mu_m(0)^{\frac{1}{m+3}} + \frac{Ct}{m+3}\sup_{0<s<t} \|E(s,\cdot)\|_{L^{3+m}(\mathbf{R}^3)}\right)^{m+3}.
\end{aligned}$$

By the interpolation inequality with $r = 0$, $d = 3$ and $m = k$, one has

$$\|\rho(t,\cdot)\|_{L^{\frac{3+k}{3}}(\mathbf{R}^3)} \le C\mu_k(t)^{\frac{3}{k+3}}\,.$$

(As we shall see below, we need $k > 2$, so that $\mu_k(t)$ is not bounded a priori by the conservation of mass and energy.) With this estimate

$$\|E(t,\cdot)\|_{L^q(\mathbf{R}^3)} = \|\nabla G_3 \star \rho(t,\cdot)\|_{L^q(\mathbf{R}^3)} \le C\|\rho(t,\cdot)\|_{L^{\frac{k+3}{3}}(\mathbf{R}^3)} \le C\mu_k(t)^{\frac{3}{k+3}}$$

for

$$\frac{1}{q} = \frac{3}{3+k} - \frac{1}{3} \text{ i.e. } q = \frac{9+3k}{6-k}\,, \quad \text{ assuming } k < 6\,.$$

Assume that

$$q \ge 3+m \text{ i.e. } \frac{3k+9}{r'} \le \frac{3k+9}{6-k} \quad \text{ or equivalently } 6-k \le r'\,.$$

Recall that

$$\tfrac{3}{2} < r \leq \tfrac{15}{4} \text{ implies that } \tfrac{15}{11} \leq r' < 3 \text{ so that } k > 3\,.$$

Thus, let $3 < k < 6$ and $\frac{3}{2} < r \leq \frac{15}{4}$ so that $\frac{15}{11} \leq r' = \frac{r}{r-1} < 3$; pick m so that $\frac{m+3}{3} = \frac{k+3}{r'}$, and let $q = \frac{9+3k}{6-k}$. With this choice of parameters, one has $q \geq 3+m$, and therefore

$$\mu_m(t) \leq \left(\mu_m(0)^{\frac{1}{m+3}} + \frac{Ct}{m+3}\sup_{0<s<t}\|E(s,\cdot)\|^{\theta}_{L^q(\mathbf{R}^3)}\|E(s,\cdot)\|^{1-\theta}_{L^2(\mathbf{R}^3)}\right)^{m+3}$$

with

$$\frac{\theta}{q} + \frac{1-\theta}{2} = \frac{1}{3+m}$$

— in other words

$$\theta = \frac{m+1}{(m+3)}\frac{q}{q-2} = \frac{m+1}{m+3}\frac{3k+9}{5k-3}\,.$$

If $k \geq 6$, pick $\bar{k} \in (3,6)$; then, setting $q = \frac{9+3\bar{k}}{6-\bar{k}}$ and defining m by the relation $\frac{m+3}{3} = \frac{\bar{k}+3}{r'}$ with the same r as before, one has $q \geq 3+m$ as above, and one sets

$$\theta = \frac{m+1}{(m+3)}\frac{q}{q-2} = \frac{m+1}{m+3}\frac{3\bar{k}+9}{5\bar{k}-3}\,.$$

The only difference is that, by the same argument as above

$$\begin{aligned}\|E(t,\cdot)\|_{L^q} &\leq C\|\rho(t,\cdot)\|_{L^{\frac{\bar{k}+3}{3}}(\mathbf{R}^3)} \leq C\|\rho(t,\cdot)\|^{1-\alpha}_{L^1(\mathbf{R}^3)}\|\rho(t,\cdot)\|^{\alpha}_{L^{\frac{\bar{k}+3}{3}}(\mathbf{R}^3)}\\ &\leq C(1+\|\rho(t,\cdot)\|_{L^{\frac{k+3}{3}}(\mathbf{R}^3)}) \leq C(1+\mu_k(t)^{\frac{3}{k+3}})\,,\end{aligned}$$

since the total mass at time t is bounded by the initial total mass.

Therefore, applying Young's classical inequality[7]

$$\begin{aligned}\mu_m(t) &\leq \left(\mu_m(0)^{\frac{1}{m+3}} + \frac{Ct}{m+3}(1+\sup_{0<s<t}\mu_k(s)^{\frac{3}{3+k}})^{\theta}\sqrt{2\mathcal{E}^{in}}^{1-\theta}\right)^{m+3}\\ &\leq \left(\mu_m(0)^{\frac{1}{m+3}} + \frac{Ct}{m+3}\left(\frac{1}{\theta}(1+\sup_{0<s<t}\mu_k(s)^{\frac{3}{3+k}}) + \frac{1}{1-\theta}\sqrt{2\mathcal{E}^{in}}\right)\right)^{m+3}\\ &\leq C(1+t)^{m+3}(1+\sup_{0<s<t}\mu_k(s)^{\frac{3}{3+k}})^{m+3}\,.\end{aligned}$$

[7] For each $a,b>0$ and each $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}\,.$$

Finally

$$
\begin{aligned}
&\|J(t,\cdot)\|_{L^{k+3}(\mathbf{R}^3)}\\
&= \left\|\int_0^{t_0} s\left(\int_{\mathbf{R}^3} E(t-s,x-sv)f(t-s,x-sv,v)dv\right)ds\right\|_{L^{k+3}(\mathbf{R}^3_x)}\\
&\le Ct_0^{2-\frac{3}{r}}\sup_{0<s<t_0}\mu_m(t-s)^{\frac{1}{k+3}}\\
&\le Ct_0^{2-\frac{3}{r}}(1+t)^{\frac{m+3}{k+3}}(1+\sup_{0<s<t}\mu_k(s)^{\frac{3}{3+k}})^{\frac{m+3}{k+3}}\,.
\end{aligned}
$$

4.4.5 Step 6: the final propagation estimate

Define

$$
M_k(t) := \sup_{0\le s\le t}\mu_k(s)\,.
$$

Thus

$$
\begin{aligned}
\|E(t,\cdot)\|_{L^{k+3}(\mathbf{R}^3)} &\le C\|\rho_0(t,\cdot)\|_{L^{\frac{3k+9}{k+6}}(\mathbf{R}^3)} + \|I(t,\cdot)\|_{L^{k+3}(\mathbf{R}^3)} + \|J(t,\cdot)\|_{L^{k+3}(\mathbf{R}^3)}\\
&\le C\|\rho_0(t,\cdot)\|_{L^{\frac{3k+9}{k+6}}(\mathbf{R}^3)} + C\ln\frac{t}{t_0}M_k(t)^{\frac{1}{k+3}}\\
&+ Ct_0^{2-\frac{3}{r}}(1+t)^{\frac{m+3}{k+3}}(1+M_k(t))^{\frac{3(m+3)}{(k+3)^2}}\,.
\end{aligned}
$$

Henceforth, assume that $t\in[0,T]$ with $T>1$; thus the inequality above simplifies into

$$
\begin{aligned}
\|E(t,\cdot)\|_{L^{k+3}(\mathbf{R}^3)} &\le C_T(1+M_k(t))^{\frac{1}{k+3}} + C\ln\frac{1}{t_0}M_k(t)^{\frac{1}{k+3}}\\
&+ C_Tt_0^{2-\frac{3}{r}}(1+M_k(t))^{\frac{3(m+3)}{(k+3)^2}}\,.
\end{aligned}
$$

Pick $t_0\le 1<T$ such that

$$
t_0^{2-\frac{3}{r}}(1+M_k(t))^{\frac{3(m+3)}{(k+3)^2}} = 1\,.
$$

Then

$$
\begin{aligned}
\|E(t,\cdot)\|_{L^{k+3}(\mathbf{R}^3)} &\le 2C_T(1+M_k(t))^{\frac{1}{k+3}}\\
&+ C\frac{3(m+3)}{(2-\frac{3}{r})(k+3)^2}M_k(t)\ln(1+M_k(t))\\
&\le C_{k,T}(1+M_k(t))^{\frac{1}{k+3}}\ln(1+M_k(t))\,.
\end{aligned}
$$

Thus we arrive at the differential inequality

$$
\begin{aligned}
\dot\mu_k(t) &\le C_{k,T}(1+M_k(t))^{\frac{1}{k+3}}\ln(1+M_k(t))\mu_k(t)^{\frac{k+2}{k+3}}\\
&\le C_{k,T}(1+M_k(t))\ln(1+M_k(t))\,,
\end{aligned}
$$

so that, integrating both sides of this inequality on $[0,t]$, we conclude that

$$M_k(t) \le M_k(0) + C_{k,T} \int_0^t (1+M_k(s)) \ln(1+M_k(s))ds\,,$$

for all $t \in [0,T]$.

Setting $y(t) = 1 + M_k(t)$, one has

$$0 < y(t) \le y(0) + C_{k,T} \int_0^t y(s) \ln y(s) ds$$

so that

$$\frac{C_{k,T} y(t) \ln y(t)}{y(0) + C_{k,T} \int_0^t y(s) \ln y(s) ds} \le C_{k,T} \ln y(t)\,.$$

Integrating in time leads to the further inequality

$$\ln \frac{y(t)}{y(0)} \le \ln \frac{y(0) + C_{k,T} \int_0^t y(s) \ln y(s) ds}{y(0)} \le C_{k,T} \int_0^t \ln y(s) ds$$

and hence

$$\ln y(t) \le \ln y(0) + C_{k,T} \int_0^t \ln y(s) ds\,.$$

By the classical Gronwall inequality, one obtains

$$\ln y(t) \le e^{tC_{k,T}} \ln y(0)$$

and therefore

$$y(t) \le \exp(e^{tC_{k,T}} \ln y(0))\,, \quad t \in [0,T]\,.$$

This completes the proof of Theorem 4.4.1.

4.5 Propagation of C^1 Regularity in Dimensions 2 and 3

In this section, we explain how the propagation of moments obtained in the two previous sections can be used to establish the propagation for all positive times of the C^1 regularity of the initial data.

The global existence of classical (C^1) solutions of the Vlasov-Poisson system has been obtained by Ukai-Okabe [20] in the case of space dimension 2, and by Pfaffelmoser [17] and Lions-Perthame [10] independently in the case of space dimension 3.

The key to obtaining classical solutions of the Vlasov-Poisson system is to prove that the macroscopic (charge) density $\rho \in L^\infty([0,T] \times \mathbf{R}^d)$ for all $T > 0$.

One already knows that the weak solutions constructed above satisfy the (weak) maximum principle, so that $f \in L^\infty(\mathbf{R}_+ \times \mathbf{R}^d \times \mathbf{R}^d)$. Obtaining a L^∞ bound on

$$\rho(t,x) = \int_{\mathbf{R}^d} f(t,x,v) dv$$

is essentially equivalent to controlling the decay in v of the number density f.

In the work of Pfaffelmoser [17], this is done by choosing f^{in} with compact support in $\mathbf{R}^d \times \mathbf{R}^d$. Controlling the electric field E in L^∞ leads to a control of the growth of the support of $f(t,x,v)$ in the variable v for $t > 0$. With the maximum principle for f, this results in an L^∞ control of ρ.

Unfortunately, compactly supported number densities are not very natural in statistical mechanics — for instance, Maxwell-Boltzmann distributions have excellent decay properties as $|v| \to \infty$, but are not compactly supported in v. With a view towards physical applications, it is perhaps more realistic to control the decay of f as $|v| \to \infty$, without necessarily assuming that f is compactly supported. This is the Lions-Perthame approach, which we have adopted in this course.

The decay of the distribution function f in the v variable is formulated in terms of a convenient weighted estimate. Therefore, we first introduce the appropriate class of weight functions w to be used for that purpose.

Let $w \in C^1(\mathbf{R})$ be such that

$$w \geq 0\,, \quad w' \leq 0\,, \quad \text{ and } w(r) = O(r^{-\alpha}) \text{ with } \alpha > d\,.$$

Theorem 4.5.1 *Assume that*

$$0 \leq f^{in}(x,v) \leq w(|v|) \text{ and } \nabla G_d \star \rho^{in} \in L^2(\mathbf{R}^d)\,,$$

where we have denoted

$$\rho^{in} := \int_{\mathbf{R}^d} f^{in} dv\,.$$

Assume that, for some k_0, one has

$$\iint_{\mathbf{R}^d \times \mathbf{R}^d} (1 + |v|^{k_0}) f^{in}(x,v)\,dx\,dv < \infty\,,$$

with

$$k_0 > d(d-1)$$

i.e.

$$k_0 > 2 \text{ if } d = 2\,, \quad \text{ while } k_0 > 6 \text{ if } d = 3\,.$$

Then there exists a weak solution $(f,E) \in C^1(\mathbf{R}_+ \times \mathbf{R}^d \times \mathbf{R}^d) \times C^1(\mathbf{R}_+ \times \mathbf{R}^d)$ of the Cauchy problem for Vlasov-Poisson system satisfying the initial condition $f\big|_{t=0} = f^{in}$, together with the decay estimates

$$f(t,x,v) + |D_x f(t,x,v)| + |D_v f(t,x,v)| = O(|v|^{-\alpha}) \quad \text{ as } |v| \to \infty$$

uniformly in $(t,x) \in [0,T] \times \mathbf{R}^d$.

Notice that, by the estimate on the electric field

$$\|E(0,\cdot)\|_{L^2(\mathbf{R}^d)} \leq C \|\rho^{in}\|_{L^{6/5}(\mathbf{R}^d)} \qquad \text{if } d = 3$$

is a consequence of the other assumptions on f^{in} in the 3-dimensional case.

The uniqueness of the solution f is not stated in the theorem above. It is true and can be obtained as a consequence of some of the estimates already used for the propagation of regularity. More recently, a very elegant uniqueness estimate using Monge-Kantorovich distances has been proposed by Loeper [12].

4.5.1 Step 1: L^∞ bound on the field

By Theorems 4.3.1 and 4.4.1, one has

$$\mu_k(t) \le C_T \text{ for all } k = 0, \dots, k_0 \text{ and all } t \in [0, T]\,.$$

By the interpolation inequality, one concludes that

$$\|\rho(t,\cdot)\|_{L^{\frac{k_0+d}{d}}(\mathbf{R}^d)} \le C_T \text{ for all } t \in [0, T]\,.$$

On the other hand, by the mass inequality, one has

$$\|\rho(t,\cdot)\|_{L^1(\mathbf{R}^d)} \le \mathcal{M}^{in} \text{ for all } t \ge 0\,.$$

Recall that the field is given by the expression

$$\begin{aligned} E(t,\cdot) &= -\nabla G_d \star \rho(t,\cdot) \\ &= -(\mathbf{1}_{B(0,1)} \nabla G_d) \star \rho(t,\cdot) - (\mathbf{1}_{B(0,1)^c} \nabla G_d) \star \rho(t,\cdot)\,, \end{aligned}$$

and that

$$\mathbf{1}_{B(0,1)} \nabla G_d = O(|x|^{1-d} \mathbf{1}_{|x|\le 1}) \in L^m(\mathbf{R}^d) \text{ for all } 1 \le m < \frac{d}{d-1}\,,$$

while

$$\mathbf{1}_{B(0,1)^c} \nabla G_d = O(|x|^{1-d} \mathbf{1}_{|x|\ge 1}) \in L^\infty(\mathbf{R}^d)\,.$$

Therefore, since $\frac{k_0+d}{d} > d$, one has

$$\|(\mathbf{1}_{B(0,1)} \nabla G_d) \star \rho(t,\cdot)\|_{L^\infty(\mathbf{R}^d)} \le C_d \|\rho(t,\cdot)\|_{L^{\frac{k_0+d}{d}}(\mathbf{R}^d)}\,,$$

while

$$\|(\mathbf{1}_{B(0,1)^c} \nabla G_d) \star \rho(t,\cdot)\|_{L^\infty(\mathbf{R}^d)} \le C'_d \|\rho(t,\cdot)\|_{L^1(\mathbf{R}^d)}\,.$$

This implies that

$$\begin{aligned} \|E(t,\cdot)\|_{L^\infty(\mathbf{R}^d)} &\le C_d \|\rho(t,\cdot)\|_{L^{\frac{k_0+d}{d}}(\mathbf{R}^d)} \\ &\quad + C'_d \|\rho(t,\cdot)\|_{L^1(\mathbf{R}^d)} \\ &\le C_d C_T + C'_d \mathcal{M}^{in} =: A_T\,. \end{aligned}$$

4.5.2 Step 2: L^∞ bound on the charge density

Then

$$\frac{d}{dt}\iint_{\mathbf{R}^d\times\mathbf{R}^d}(f(t,x,v)-w(|v|-At))_+dxdv$$
$$=\iint_{\mathbf{R}^d\times\mathbf{R}^d}(A-E(t,x)\cdot\tfrac{v}{|v|})w'(|v|-At)\mathbf{1}_{(f(t,x,v)\geq w(|v|-At))}dxdv\leq 0$$

so that

$$f^{in}(x,v)\leq w(|v|)\quad\text{ for a.e. } (x,v)\in\mathbf{R}^d\times\mathbf{R}^d$$

implies that

$$f(t,x,v)\leq w(|v|-At)\quad\text{ for a.e. } (t,x,v)\in\mathbf{R}_+\times\mathbf{R}^d\times\mathbf{R}^d\,.$$

In particular, since w is nonincreasing,

$$\begin{aligned}\rho(t,x)&=\int_{\mathbf{R}^d}f(t,x,v)dv\leq\int_{\mathbf{R}^d}w(|v|-At)dv\\&\leq w(-AT)\int_{|v|\leq AT}dv+\int_{|v|>At}w(|v|-At)dv\\&\leq w(-AT)(AT)^d|B(0,1)|+|\mathbf{S}^{d-1}|\int_{At}^{\infty}w(r-At)r^{d-1}dr\\&\leq w(-AT)(AT)^d|\mathbf{B}^d|+|\mathbf{S}^{d-1}|\int_0^\infty w(r)(AT+r)^{d-1}dr=:R_T<\infty\,,\end{aligned}$$

for a.e. $(t,x,v)\in\mathbf{R}_+\times\mathbf{R}^d\times\mathbf{R}^d$.

4.5.3 Step 3: Estimating $D_{x,v}f$

First, we estimate

$$L(t):=\|D_xf(t,\cdot,\cdot)\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)}+\|D_vf(t,\cdot,\cdot)\|_{L^\infty(\mathbf{R}^d\times\mathbf{R}^d)}$$

in terms of

$$\|D_xE(t,\cdot)\|_{L^\infty(\mathbf{R}^d)}\,.$$

Differentiating the Vlasov equation in x and v, one has

$$\begin{aligned}&(\partial_t+v\cdot\nabla_x+E(t,x)\cdot\nabla_v)\begin{pmatrix}D_xf(t,x,v)\\D_vf(t,x,v)\end{pmatrix}\\&\qquad=\begin{pmatrix}0&D_xE(t,x)^T\\I&0\end{pmatrix}\begin{pmatrix}D_xf(t,x,v)\\D_vf(t,x,v)\end{pmatrix},\end{aligned}$$

so that

$$\begin{aligned}&(\partial_t+v\cdot\nabla_x+E(t,x)\cdot\nabla_v)(|D_xf(t,x,v)|+|D_vf(t,x,v)|)\\&\qquad\leq(1+|D_xE(t,x)|)(|D_xf(t,x,v)|+|D_vf(t,x,v)|)\,,\end{aligned}$$

for a.e. $(t,x,v) \in \mathbf{R}_+ \times \mathbf{R}^d \times \mathbf{R}^d$. Hence

$$L(t) \le L(0) \exp\left(\int_0^t (1 + \|D_x E(s,\cdot)\|_{L^\infty}(\mathbf{R}^d)) ds\right).$$

Setting

$$J(t) := \int_0^t (1 + \|D_x E(s,\cdot)\|_{L^\infty}(\mathbf{R}^d)) ds\,,$$

one has indeed

$$\begin{aligned}&(\partial_t + v\cdot\nabla_x + E(t,x)\cdot\nabla_v)\left((|D_x f(t,x,v)| + |D_v f(t,x,v)|)e^{-J(t)}\right)\\ &\le (|D_x f(t,x,v)| + |D_v f(t,x,v)|)e^{-J(t)}(|D_x E(t,x)| - \|D_x E(s,\cdot)\|_{L^\infty}) \le 0\end{aligned}$$

for a.e. $(t,x,v) \in \mathbf{R}_+ \times \mathbf{R}^d \times \mathbf{R}^d$, and one concludes by the maximum principle that

$$\begin{aligned}&\left((|D_x f(t,x,v)| + |D_v f(t,x,v)|)e^{-J(t)}\right)\\ &\le \sup_{x\in\mathbf{R}^d}(|D_x f(0,x,v)| + |D_v f(0,x,v)|)e^{-J(0)} = L(0)\,.\end{aligned}$$

4.5.4 Step 4: Estimating $D_x E$

Next we estimate

$$\|D_x E(t,\cdot)\|_{L^\infty(\mathbf{R}^d)} = \|\nabla^2 G_d \star \rho(t,\cdot)\|_{L^\infty(\mathbf{R}^d)}\,.$$

This cannot be estimated by $\rho(t,\cdot)\|_{L^\infty(\mathbf{R}^d)}$ because the Calderón-Zygmund inequality does not hold in L^∞. Instead, we use the following lemma.

Lemma 4.5.2 *Let Ω be a continuous function on $\mathbf{S}^{d-1}$ such that*

$$\int_{\mathbf{S}^{d-1}} \Omega(y) ds(y) = 0\,,$$

and let

$$K = \mathrm{vp}\,\frac{\Omega(\frac{x}{|x|})}{|x|^d}\,.$$

Then

$$\begin{aligned}&\|K\star\phi\|_{L^\infty(\mathbf{R}^d)}\\ &\le \|\Omega\|_{L^\infty(\mathbf{S}^d)}(|\mathbf{S}^{d-1}| + \|\phi\|_{L^1(\mathbf{R}^d)} + |\mathbf{S}^{d-1}|\|\phi\|_{L^\infty(\mathbf{R}^d)}\ln(1+\|D\phi\|_{L^\infty(\mathbf{R}^d)}))\,.\end{aligned}$$

Proof of Lemma 4.5.2. Split the integral as

$$\begin{aligned}\int_{|y|>\epsilon}\frac{\Omega(\frac{y}{|y|})}{|y|^d}\phi(x-y)dy &= \int_{|y|>1}\frac{\Omega(\frac{y}{|y|})}{|y|^d}\phi(x-y)dy\\ &+ \int_{r\le|y|\le 1}\frac{\Omega(\frac{y}{|y|})}{|y|^d}\phi(x-y)dy\\ &+ \int_{\epsilon<|y|<r}\frac{\Omega(\frac{y}{|y|})}{|y|^d}\phi(x-y)dy = I_1 + I_2 + I_3\,.\end{aligned}$$

First
$$|I_1| \leq \|\Omega\|_{L^\infty(\mathbf{S}^d)} \|\phi\|_{L^1(\mathbf{R}^d)} \,.$$
Next
$$\begin{aligned}|I_2| &\leq \|\Omega\|_{L^\infty(\mathbf{S}^d)} \|\phi\|_{L^\infty(\mathbf{R}^d)} \int_{r\leq|y|\leq 1} \frac{dy}{|y|^d} \\ &= \|\Omega\|_{L^\infty(\mathbf{S}^d)} \|\phi\|_{L^\infty(\mathbf{R}^d)} |\mathbf{S}^{d-1}| \int_{r\leq|y|\leq 1} \frac{dR}{R} \\ &= \|\Omega\|_{L^\infty(\mathbf{S}^d)} \|\phi\|_{L^\infty(\mathbf{R}^d)} |\mathbf{S}^{d-1}| \ln\frac{1}{r}\,,\end{aligned}$$
where the first equality comes from computing the integral in spherical coordinates.

Finally
$$I_3 = \int_{\epsilon<|y|<r} \frac{\Omega(\frac{y}{|y|})}{|y|^d}(\phi(x-y)-\phi(x))dy$$
since
$$\int_{\epsilon<|y|<r} \frac{\Omega(\frac{y}{|y|})}{|y|^d} dy = \int_\epsilon^r \frac{dR}{R} \int_{\mathbf{S}^{d-1}} \Omega(z)ds(z) = 0$$
in spherical coordinates $R = |y|$ and $z = y/R$. Thus
$$\begin{aligned}|I_3| &\leq \left| \int_{\epsilon<|y|<r} \frac{\Omega(\frac{y}{|y|})}{|y|^d}(\phi(x-y)-\phi(x))dy \right| \\ &\leq \int_{\epsilon<|y|<r} \frac{|\Omega(\frac{y}{|y|})|}{|y|^d}|\phi(x-y)-\phi(x)|dy \\ &\leq \|\Omega\|_{L^\infty(\mathbf{S}^d)} \|D\phi\|_{L^\infty(\mathbf{R}^d)} \int_{|y|<r} \frac{dy}{|y|^{d-1}} \\ &= \|\Omega\|_{L^\infty(\mathbf{S}^d)} \|D\phi\|_{L^\infty(\mathbf{R}^d)} |\mathbf{S}^{d-1}| r\,.\end{aligned}$$
Eventually, one arrives at the inequality
$$\begin{aligned}&\left| \int_{|y|>\epsilon} \frac{\Omega(\frac{y}{|y|})}{|y|^d}\phi(x-y)dy \right| \\ &\leq \|\Omega\|_{L^\infty(\mathbf{S}^d)} (\|\phi\|_{L^1(\mathbf{R}^d)} + \|\phi\|_{L^\infty(\mathbf{R}^d)} |\mathbf{S}^{d-1}| \ln\frac{1}{r} + \|D\phi\|_{L^\infty(\mathbf{R}^d)} |\mathbf{S}^{d-1}| r)\end{aligned}$$
which holds for all $r \in (0,1)$. Setting
$$r = \frac{1}{1 + \|D\phi\|_{L^\infty(\mathbf{R}^d)}}$$
in the inequality above leads to the desired estimate. ■

Applying this lemma to control the derivatives of the field, one finds that
$$\begin{aligned}&\|D_x E(t,\cdot)\|_{L^\infty(\mathbf{R}^d)} = \|\nabla^2 G_d \star \rho(t,\cdot)\|_{L^\infty(\mathbf{R}^d)} \\ &\leq C(1 + \|\rho(t,\cdot)\|_{L^1(\mathbf{R}^d)} + \|\rho(t,\cdot)\|_{L^\infty(\mathbf{R}^d)} \ln(1 + \|D\rho(t,\cdot)\|_{L^\infty(\mathbf{R}^d)})) \\ &\leq C_T(1 + \ln(1 + \|D\rho(t,\cdot)\|_{L^\infty(\mathbf{R}^d)}))\,, \quad t \in [0,T]\,.\end{aligned}$$

4.5.5 Step 5: Estimating $D_x\rho$

At this point, we return to the differential inequality

$$\begin{aligned}(\partial_t + v\cdot\nabla_x + E(t,x)\cdot\nabla_v)(|D_xf(t,x,v)| + |D_vf(t,x,v)|)\\ \le (1+|D_xE(t,x)|)(|D_xf(t,x,v)| + |D_vf(t,x,v)|)\,,\end{aligned}$$

and we recall that

$$(\partial_t + v\cdot\nabla_x + E(t,x)\cdot\nabla_v)\left((|D_xf(t,x,v)| + |D_vf(t,x,v)|)e^{-J(t)}\right) \le 0\,,$$

where

$$J(t) := \int_0^t (1+\|D_xE(s,\cdot)\|_{L^\infty(\mathbf{R}^d)})ds\,.$$

Setting

$$g(t,x,v) := (|D_xf(t,x,v)| + |D_vf(t,x,v)|)e^{-J(t)}\,,$$

one has

$$\begin{aligned}&\frac{d}{dt}\iint_{\mathbf{R}^d\times\mathbf{R}^d}(g(t,x,v) - w(|v|-At))_+dxdv\\ &\quad\le \iint_{\mathbf{R}^d\times\mathbf{R}^d}(A - E(t,x)\cdot\tfrac{v}{|v|})w'(|v|-At)\mathbf{1}_{g(t,x,v)\ge w(|v|-At)}dxdv \le 0\,.\end{aligned}$$

Therefore

$$|D_xf^{in}(x,v)| + |D_vf^{in}(x,v)| \le w(v)$$

which imples that

$$|D_xf(t,x,v)| + |D_vf(t,x,v)| \le e^{J(t)}w(|v|-At)\,.$$

for a.e. $(t,x,v)\in\mathbf{R}_+\times\mathbf{R}^d\times\mathbf{R}^d$. In particular

$$|D_x\rho(t,x)| \le \int_{\mathbf{R}^d}|D_xf(t,x,v)|dv \le e^{J(t)}\int_{\mathbf{R}^d}w(|v|-At)dv \le R_Te^{J(t)}\,.$$

4.5.6 Step 6: Conclusion

Putting together the estimates in the last two steps, we find that

$$\begin{aligned}J(t) &\le T + C_T\int_0^t(1+\ln(1+\|D\rho(s,\cdot)\|_{L^\infty(\mathbf{R}^d)}))ds\\ &\le T + C_T\int_0^t(1+\ln(1+R_Te^{J(s)}))ds\\ &\le T(1+C_T) + C_T\int_0^t\ln(1+R_Te^{J(s)})ds\\ &\le T(1+C_T) + C_T\int_0^t\ln((1+R_T)e^{J(s)})ds\\ &\le T(1+C_T(1+\ln(1+R_T))) + C_T\int_0^t J(s)ds\,.\end{aligned}$$

By Gronwall's inequality

$$J(t) \le T(1 + C_T(1 + \ln(1 + R_T)))e^{TC_T} \quad \text{for all } t \in [0, T]\,.$$

Returning to step 4, we see that

$$\|D_x\rho(t,\cdot)\|_{L^\infty(\mathbf{R}^d)} \le R_T \exp(T(1 + C_T(1 + \ln(1 + R_T)))e^{TC_T})$$

and inserting this estimate in the conclusion of step 3 shows that

$$\|D_x E(t,\cdot)\|_{L^\infty(\mathbf{R}^d)} \le C_T(1 + \ln(1 + R_T \exp(T(1 + C_T(1 + \ln(1 + R_T)))e^{TC_T})))$$

for all $t \in [0, T]$.

Finally, returning to step 2 shows that

$$\begin{aligned} \|D_x f(t,\cdot,\cdot)\|_{L^\infty(\mathbf{R}^d)} + \|D_v f(t,\cdot,\cdot)\|_{L^\infty(\mathbf{R}^d)} = L(t) \le L(0)e^{J(t)} \\ \le (\|D_x f^{in}\|_{L^\infty(\mathbf{R}^d)} + \|D_v f^{in}\|_{L^\infty(\mathbf{R}^d)}) \exp(T(1 + C_T(1 + \ln(1 + R_T)))e^{TC_T}) \end{aligned}$$

and this concludes the proof.

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