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The Nonaccretive Radiative Transfer Equations: Existence of Solutions and Rosseland Approximation

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We present an existence theory and an asymptotic analysis for the radiative transfer equations

$$\frac{\partial u_\varepsilon}{\partial t}+\frac{\Omega\cdot\nabla_x u_\varepsilon}{\varepsilon}+\frac{\sigma(\tilde{u}_\varepsilon)}{\varepsilon^2}(u_\varepsilon-\tilde{u}_\varepsilon)=0 \qquad \text{in } X,$$
$$u_\varepsilon|_{(\partial X\times S^N)}=k, \qquad u_\varepsilon|_{t=0}=u_0, \tag{1}$$

where $u_\varepsilon \equiv u_\varepsilon(t,x,\Omega)$, $t\in\mathbb{R}_+$, $x\in X\subset\mathbb{R}^{N+1}$, $\Omega\in S^N$, and $\tilde{u}_\varepsilon(t,x)=1/|S^N|\int u_\varepsilon(t,x,\Omega)\,d\Omega$. We prove that, even if σ has a singularity ($\sigma(0)=+\infty$), (1) has a solution $u_\varepsilon\in L^\infty(R_+\times X\times S^N)$. As $\varepsilon\to 0$, we show that u_ε converges pointwise to a function $u\in L^\infty(R_+\times X)$, solution of the degenerate parabolic equation

$$\frac{\partial u}{\partial t}-\Delta F(u)=0 \qquad \text{in } X,$$
$$u|_{\partial X}=k, \qquad u|_{t=0}=u_0.$$

This is achieved without any monotonicity assumption on σ and therefore one cannot use the theory of nonlinear contraction semigroups. © 1988 Academic Press, Inc.

We consider the problem of the existence of solutions and the diffusion approximation for the Radiative Transfer (R.T.) equations. These equations describe the transport of photons in a starlike medium and are, mathematically, a nonlinear version of the transport of neutrons. The results that we prove here are wellknown in the linear case: the existence follows from the study of the spectrum of the transport operator, which has been carried out by many authors [1, 3, 8, 12, 19, 26, 28], and the dif-

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fusion approximation has also been studied extensively [3, 7, 12, 20, 21, 26, 27]. The R.T. equation has been studied more recently and these results (existence of solutions and Rosseland approximation) are known only under some monotonicity assumptions on the opacity (the term σ in the equation below), which ensure that the R.T. operator is accretive [2, 17, 22]. Our purpose here is to prove these results without any accretiveness assumptions since they have no physical meaning (see [10, 14, 23, 25]).

This is achieved for the "grey problem," which reads, for a fixed number $\varepsilon > 0$,

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\Omega \cdot \nabla_x u_\varepsilon}{\varepsilon} + \frac{\sigma(\tilde{u}_\varepsilon)}{\varepsilon^2}(u_\varepsilon - \tilde{u}_\varepsilon) = 0, \qquad u_\varepsilon|_{(\partial X \times S^N)_-} = k, \qquad u_\varepsilon|_{t=0} = u_0, \tag{1}$$

or, in a stationary case,

$$\lambda u_\varepsilon + \frac{\Omega \cdot \nabla_x u_\varepsilon}{\varepsilon} + \frac{\sigma(\tilde{u}_\varepsilon)}{\varepsilon^2}(u_\varepsilon - \tilde{u}_\varepsilon) = 0, \qquad u_\varepsilon|_{(\partial X \times S^N)_-} = k, \tag{2}$$

where $u_\varepsilon \equiv u_\varepsilon(t, x, \Omega)$ (resp. $u_\varepsilon \equiv u_\varepsilon(x, \Omega)$) with $x \in X$ (some smooth bounded convex open subset of R^{N+1}), $\Omega \in S^N$ (the unit sphere of R^{N+1}), $t \geqslant 0$. In (1), (2), $\tilde{v}$ denotes the integral

$$\tilde{v} = \int_{S^N} v(\Omega)\, d\Omega,$$

where $d\Omega$ is the normalized Lebesgue measure on S^N ($\int_{S^N} d\Omega = 1$) and

$$(\partial X \times S^N)_- = \{(x, \Omega) \in \partial X \times S^N, n(x) \cdot \Omega < 0\},$$

where $n(x)$ is the outward normal to X at the point x of ∂X. Finally, let us point out that the positive function $\sigma(T)$ may have a singularity at $T = 0$: $\sigma(0) = +\infty$ which is one of the main difficulties of the model.

It is wellknown, from physical arguments, that, as ε goes to 0, u_ε converges to a function $v(x)$ which is the solution of a nonlinear degenerate parabolic (resp. elliptic) equation

$$\frac{\partial v}{\partial t} - \Delta F(v) = 0, \qquad v|_{\partial X} = k, \qquad v|_{t=0} = u_0, \tag{3}$$

or

$$\lambda v - \Delta F(v) = 0, \quad v|_{\partial X} = k, \tag{4}$$

at least when k is constant and $u_0 = u_0(x)$. This is called the "Rosseland approximation." The function F is given by

$$F'(T) = \frac{1}{N+1} \frac{1}{\sigma(T)}.$$

We will study here Eqs. (1), (2) and their approximations (3), (4). First let us notice that when k, u_0, f are nonnegative, any solution u_ε is also nonnegative and that the maximum principle holds so that u_ε is bounded a priori. However, this is not enough to get the existence of solutions of (1) or (2) because of the nonlinear term and because the expression $\sigma(\tilde{u}_\varepsilon)(u_\varepsilon - \tilde{u}_\varepsilon)$ has no clear meaning when $\tilde{u}_\varepsilon = 0$. Thus we will need a stronger regularity property. It is given by a compactness result introduced in [16, 18] that we will use in an essential way, first to prove the existence of solutions of (1), (2) by Schauder's fixed point theorem and then to get the strong convergence of u_ε as ε goes to 0.

This paper is organized as follows. In Section I, we study the existence of solutions of (1), (2). In Section II, we prove the Rosseland approximation. In Section III, we study the existence of solutions for a more complete model than (1), (2) (model with frequency),

$$\lambda u + \Omega \cdot \nabla_x u + \sigma_\nu(T)(u - B_\nu(T)) = g,$$

$$\lambda T + \int_{\mathbb{R}^+} \sigma_\nu(T)(B_\nu(T) - \tilde{u})\, d\nu = f,$$

$$u|_{(\partial X \times S^N)_-} = h,$$

where the unknown functions are now $T(x)$ and $u(x, \Omega, \nu)$, where $\nu \in \mathbb{R}_+$ denotes the frequency of the photons. The treatment of this system uses a generalization of the methods developed in Sections I and II. However, it requires more technical arguments and, for the sake of simplicity, we give only partial results (we do not study the evolution equation or the case $\lambda = 0$). Finally, let us recall that the Rosseland approximation for the model with frequency is still an open question when σ and B do not satisfy monotonicity assumptions as in [2].

I. Existence of Solutions

I.1. *Assumptions and Main Results*

Before stating our main existence results, let us say more precisely what is a solution of (1) or (2) and introduce some assumptions. First, the function $\sigma \in C(\mathbb{R}^{+*}, \mathbb{R}^{+*})$ is supposed to satisfy

$$\exists p > 1, \quad \exists C, \quad \exists \sigma_m, \quad 0 < \sigma_m \leqslant \sigma(u) \leqslant C/u^{1/p}, \quad \forall u \leqslant 1, \tag{A1}$$

and (to treat the case $\lambda = 0$)

> $\exists u_\infty$, such that for $u \geqslant u_\infty$, σ is nonincreasing, $\sigma(u)\,u$ is nondecreasing, and $\sigma(u) \to_{u \to \infty} 0$. (A2)

In particular, from (A1) we deduce that $\sigma(u)\,u \to 0$ as $u \to 0$, and thus the term $\sigma(\tilde{u})\,u$ always has meaning if $\tilde{u} \in L^\infty(X)$, $u \geqslant 0$, since we may take as a convention $\sigma(0)\,0 = 0$.

Finally, since we treat the existence of solutions of (1), (2), we choose $\varepsilon = 1$ in this section to simplify notations.

THEOREM 1 (Existence for the Stationary Problem). *Under the assumption* (A1), *let* $f \geqslant 0$, $k \geqslant 0$ *belong to* $L^\infty(X \times S^N)$; *then, if* $\lambda > 0$, *or if* $\lambda = 0$ *and* (A2) *holds, the equation*

$$\begin{aligned} \lambda u + \Omega \cdot \nabla_x u + \sigma(\tilde{u})(u - \tilde{u}) &= f, \\ u|_{(\partial X \times S^N)_-} &= k \end{aligned} \tag{5}$$

has at least one nonnegative solution $u \in L^\infty(X \times S^N)$ *such that* $\Omega \cdot \nabla_x u \in L^p(X \times S^N)$, $\sigma(\tilde{u})\,u \in L^p(X \times S^N)$, *and* $\sigma(\tilde{u})\,\tilde{u} \in L^\infty(X)$.

THEOREM 2 (Existence for the Evolution Problem). *Under the assumption* (A1), *let* $u_0 \geqslant 0$, $k \geqslant 0$ *belong to* $L^\infty(\partial X \times S^N)$, *then the equation*

$$\begin{aligned} &\frac{\partial u}{\partial t} + \Omega \cdot \nabla_x u + \sigma(\tilde{u})(u - \tilde{u}) = 0, \\ &u|_{(\partial X \times S^N)_-} = k, \qquad u|_{t=0} = u_0 \end{aligned} \tag{6}$$

has at least one generalized nonnegative bounded solution $u \in C([0, T]; L^p(X \times S^N)) \cap L^\infty(\mathbb{R}^+ \times X \times S^N)$. *Moreover* $\partial u/\partial t + \Omega \cdot \nabla_x u$ *and* $\sigma(\tilde{u})\,u$ *belong to* $L^\infty(\mathbb{R}^+; L^p(X \times S^N))$.

Remarks. (1) With the assumption (A1), (1) is not a Lipschitz perturbation of the transport operator in L^1; this motivates Theorem 2.

(2) The monotonicity assumption which implies the accretiveness of the R.T. operator is that σ is nonincreasing and $\sigma(T)\,T$ is nondecreasing. This set of assumptions has been introduced by Mercier [22]. Assumption (A2) is weaker.

(3) In (5) or (6) the boundary condition holds in the sense of traces for the transport space. See [9] for weaker assumptions on k.

Before proving these theorems, let us recall some compactness results which are the cornerstone of the proof.

PROPOSITION 1 [16]. (1) *Let* $u \in L^q(X \times S^N)$, $q > 1$, *and* $\Omega \cdot \nabla_x u \in L^q(X \times S^N)$, $u|_{(\partial X \times S^N)_-} = k \in L^\infty((\partial X \times S^N)_-)$. *Then* $\tilde{u} \in W^{s,q}(X)$ *for any* s, $0 < s < \inf(1/q,\ 1 - 1/q)$, *and*

$$\|\tilde{u}\|_{W^{s,q}} \leqslant C(k)(\|u\|_{L^q} + \|\Omega \cdot \nabla_x u\|_{L^q}).$$

(2) *Let* $u \in L^q([0, T] \times X \times S^N)$, $q > 1$, *and* $\partial u/\partial t + \Omega \cdot \nabla_x u \in L^q([0, T] \times X \times S^N)$, *and* $u|_{t=0} = u_0 \in L^\infty(X \times S^N)$, *and* $u|_{(\partial X \times S^N)_-} = k \in L^\infty([0, T] \times (\partial X \times S^N)_-)$. *Then* $\tilde{u} \in W^{s,q}([0, T] \times X)$ *for any* s, $0 < s < \inf(1/q,\ 1 - 1/q)$, *and*

$$\|\tilde{u}\|_{W^{s,q}} \leqslant C(k, u_0)(\|u\|_{L^q} + \|\partial u/\partial t + \Omega \cdot \nabla_x u\|_{L^q}).$$

Here we will use only the easy consequence that in both cases $\tilde{u}$ is compact in the corresponding L^q spaces.

2. *Proof of Theorems* 1 *and* 2.

We prove Theorem 1 in three steps. First we study the case where $\lambda > 0$ and σ remains bounded (i.e., we make a regularization of σ and we consider $\sigma_M(T) = \sigma(T + 1/M)$ in place of σ), then we let λ tend to 0, and finally we let M tend to infinity.

(i) *Case of* $\lambda > 0$, σ *Bounded.* Here we study the existence of solutions of (5) with σ replaced by $\sigma_M(T) = \sigma(T + 1/M)$ with $M > 0$. Thus for $T \in L^\infty(X)$, let us introduce the equation

$$\lambda v + \Omega \cdot \nabla_x v + \sigma_M(T)(v - \tilde{v}) = f, \qquad v|_{(\partial X \times S^N)_-} = k. \tag{7}$$

It is well known that this linear transport equation has a unique solution $v \in L^\infty(X \times S^N)$, such that $\Omega \cdot \nabla_x v \in L^\infty(X \times S^N)$. Moreover, since λ is positive we have

$$0 \leqslant v \leqslant \operatorname{Max}\left(\|k\|_{L^\infty}, \frac{\|f\|_{L^\infty}}{\lambda}\right). \tag{8}$$

Let us define the set

$$S=\left\{T\in L^2(X), 0\leqslant T\leqslant \mathrm{Max}\left(\|k\|_{L^\infty}, \frac{\|f\|_{L^\infty}}{\lambda}\right)\right\},$$

and $\mathscr{C}$, the operator defined by $\mathscr{C}(T)=\tilde{v}$ for $T\in S$, and v being the solution of (7). $\mathscr{C}$ is clearly continuous for the L^2 topology from the convex set S into itself. Moreover, since we have for any $v=\mathscr{C}(T)$, $T\in S$,

$$|\Omega\cdot\nabla_x v|\leqslant \|f\|_{L^\infty}+(\lambda+2M)\left(\|v\|_{L^\infty}+\frac{\|f\|_{L^\infty}}{\lambda}\right),$$

we may apply Lemma 1 and obtain that the range of $\mathscr{C}$ is bounded in $H^{1/2}(X)$ and thus is compact in $L^2(X)$. This shows, using Schauder's fixed point theorem, that $\mathscr{C}$ has a fixed point and Theorem 1 is proved for $\lambda>0$ and σ bounded.

(ii) *Case of $\lambda=0$, σ Bounded.* We use the same argument to let λ go to 0. First let us prove an L^∞ a priori estimate for a solution u_λ of

$$\begin{aligned}\lambda u_\lambda+\Omega\cdot\nabla_x u_\lambda+\sigma_M(\tilde{u}_\lambda)(u_\lambda-\tilde{u}_\lambda)&=f,\\ u_\lambda|_{(\partial X\times S^N)_-}&=k.\end{aligned}\tag{5'}$$

We set $C_0=1+\|f\|_\infty$ and we choose $C_1\geqslant u_\infty$ large enough so that the function $v=C_1+C_0\Omega\cdot x$ satisfies, for any $\lambda\geqslant 0$, $M\geqslant 1$,

$$\begin{aligned}\lambda v+\Omega\cdot\nabla_x u_\lambda+\sigma_M(\tilde{v})(v-\tilde{v})&\geqslant f,\\ v|_{(\partial X\times S^N)_-}&\geqslant k.\end{aligned}\tag{9}$$

Then, subtracting (5') from (9), multiplying by $\mathbb{1}_{\{u_\lambda\geqslant v\}}$, and integrating, we get

$$\int_{X\times S^N}\lambda(u_\lambda-v)^+ +\int_{X\in S^N}\left\{\sigma_M(\tilde{u}_\lambda)(u_\lambda-\tilde{u}_\lambda)-\sigma_M(\tilde{v})(v-\tilde{v})\right\}\mathbb{1}_{\{u_\lambda\geqslant v\}}\leqslant 0. \tag{10}$$

The second term of the left-hand side of (10) may be written as

$$\begin{aligned}&\int_{X\times S^N}[\sigma_M(\tilde{u}_\lambda)(\tilde{u}_\lambda+1/M)-\sigma_M(\tilde{v})(\tilde{v}+1/M)][\mathbb{1}_{\{\tilde{u}_\lambda\geqslant\tilde{v}\}}-\mathbb{1}_{\{u_\lambda\geqslant v\}}]\\ &\quad+\int_{X\times S^N}\sigma_M(\tilde{u}_\lambda)(u_\lambda-v)[\mathbb{1}_{\{u_\lambda\geqslant v\}}-\mathbb{1}_{\{\tilde{u}_\lambda\geqslant\tilde{v}\}}]\\ &\quad+\int_{X\times S^N}(1/M+v)[\sigma_M(\tilde{u}_\lambda)-\sigma_M(\tilde{v})][\mathbb{1}_{\{u_\lambda\geqslant v\}}-\mathbb{1}_{\{\tilde{u}_\lambda\geqslant\tilde{v}\}}].\end{aligned}$$

But on the set $\{\tilde{u}_\lambda \geqslant \tilde{v}\}$ we have $\tilde{u}_\lambda \geqslant u_\infty$, since $\tilde{v} \geqslant C_1 \geqslant u_\infty$, and thus (A2) shows that each of these three integrals is nonnegative and, reporting this in (10), we obtain that $\int_{X \times S^N} (u_\lambda - v)^+ \leqslant 0$, and therefore $u_\lambda \leqslant v$ a.e. This proves:

LEMMA 1. *Under the assumptions of Theorem* 1 *and for* $M \geqslant 1$, *the solution* u_λ *of* (5′) *satisfies*

$$0 \leqslant u_\lambda \leqslant K,$$

where K depends only on $\| f \|_\infty$, σ, *and* $\| k \|_\infty$.

Remark. In particular, notice that the above estimate is independent of M and holds for any solution of (5) with $\Omega \cdot \nabla_x u \in L^1(X \times S^N)$.

To complete the proof of Theorem 1 in the case of σ bounded, we argue as in (i). Since u_λ is uniformly bounded, $\Omega \cdot \nabla_x u_\lambda$ remains also uniformly bounded in $L^\infty(X \times S^N)$ and thus, by Proposition 1, $\tilde{u}_\lambda$ is compact in $L^2(X)$. Extracting a subsequence, u_λ converges weakly and $\tilde{u}_\lambda$ converges strongly in L^2, and we may pass to the limit in (5′). We do not present the details since they will be worked out below when M tends to infinity.

(iii) σ *Unbounded.* We denote by u^M the solution, for a $\lambda \geqslant 0$, of

$$\begin{aligned} \lambda u^M + \Omega \cdot \nabla_x u^M + \sigma_M(\tilde{u}^M)(u^M - \tilde{u}^M) &= f, \\ u^M|_{(\partial X \times S^N)_-} &= k. \end{aligned} \tag{5''}$$

We already know by Lemma 1 that, for any $M \geqslant 1$,

$$\| u^M \|_{L^\infty(X \times S^N)} \leqslant K.$$

Thus, we have

$$\int (\sigma_M(\tilde{u}^M)\, u^M)^p \, dx \, d\Omega \leqslant K^{p-1} \int_X [\sigma_M(\tilde{u}^M)]^p \, \tilde{u}^M \, dx \leqslant C. \tag{11}$$

Thus $\Omega \cdot \nabla_x u^M$ is bounded in $L^p(X \times S^N)$ and by Proposition 1, $\tilde{u}^M$ is compact in $L^p(X)$ and also in any $L^q(X)$, $1 \leqslant q < \infty$, since it is bounded.

With this compactness, we may pass to the limit in every term of (5″). Indeed, there exists a $u \in L^\infty(X \times S^N)$ such that, for any q, $1 \leqslant q < \infty$,

$$u^M \xrightarrow[M \to \infty]{} u \qquad \text{weakly in } L^q(X \times S^N),$$

$$\Omega \cdot \nabla_x u^M \xrightarrow[M \to \infty]{} \Omega \cdot \nabla_x u \qquad \text{weakly in } L^q(X \times S^N),$$

$$\tilde{u}^M \xrightarrow[M \to \infty]{} \tilde{u} \qquad \text{strongly in } L^q(X).$$

Therefore, for any $M' \geqslant 1$, $\inf(M', \sigma_M(\tilde{u}^M)) \mathbb{1}_{\{\tilde{u}>0\}}$ converges strongly in $L^2(X)$ to $\inf(M', \sigma(\tilde{u})) \mathbb{1}_{\{\tilde{u}>0\}}$ and thus we have

$$\sigma_M(\tilde{u}^M)\, u^M \geqslant \inf(M', \sigma_M(\tilde{u}^M))\, u^M\, \mathbb{1}_{\{\tilde{u}>0\}} \xrightarrow{M \to \infty} \inf(M', \sigma(\tilde{u}))\, u \quad \text{weakly in } L^1(X \times S^N). \tag{12}$$

By (11) we also know that $\sigma_M(\tilde{u}^M)\, u^M$ converge weakly to some function $q(x, \Omega) \in L^p(X \times S^N)$ and we have

$$q(x, \Omega) \geqslant \inf(M', \sigma(\tilde{u}))\, u \qquad \forall M' \geqslant 1.$$

Letting M' go to infinity gives

$$q(x, \Omega) \geqslant \sigma(\tilde{u})\, u.$$

But clearly

$$\tilde{q}(x) = \lim_{M \to \infty} \sigma_M(\tilde{u}^M)\, \tilde{u}^M = \sigma(\tilde{u})\, \tilde{u},$$

and thus

$$q(x, \Omega) = \sigma(\tilde{u})\, u \qquad \text{a.e.}$$

Therefore u satisfies the first equation of (5). The boundary condition also holds, since the traces converge weakly in $L^p((\partial X \times S^N)_-)$. This concludes the proof of Theorem 1. The proof of Theorem 2 is exactly the same, using Proposition 1(ii) in place of (i), and thus we do not repeat it.

II. Rosseland Approximation

II.1. *Setting the Problem*

We consider now the rescaled model (1) or (2) and are interested in the limit of u_ε as ε goes to 0. Let us recall that the behavior of u_ε is known in various cases. First, in the linear case ($\sigma = C^{st}$), it has been shown that u_ε converges strongly to the solution of a linear diffusion equation (see [3, 7, 12, 20, 26]). In the nonlinear accretive case (A2) holds for $u_\infty = 0$) the same result has been proved [2] (see [21] too) and the limit equation (4) is now a nonlinear diffusion equation. Moreover, since in (4) we have $F'(T) = 1/[(N+1)\, \sigma(T)]$, this diffusion equation may become degenerate ($F'(0) = 0$).

For $\sigma(T) = T^{-\alpha}$, (4) is very classical: it is a porous media equation. The kind of problem has been studied by many authors, and we refer the interested reader to the papers [4–6, 11, 13, 24] and their references. Let us point out that the limit u of $u_\varepsilon(t, x, \Omega)$ depends only on (t, x). Thus, in general, a boundary layer appears where the dependence of u_ε on Q vanishes exponentially. This problem has been studied, even in the nonlinear case (see [2, 15, 27]). Here we will avoid this difficulty by setting $k = C^{st}$ (k is the entering flux) and, for Eq. (1), $u_0 \equiv u_0(x)$.

Thus we consider the equations

$$\frac{\partial u_\varepsilon}{\partial t} + \frac{\Omega \cdot \nabla_x u_\varepsilon}{\varepsilon} + \frac{\sigma(\tilde{u}_e)}{\varepsilon^2}(u_\varepsilon - \tilde{u}_\varepsilon) = 0,$$
$$u_\in |_{(\partial X \times S^N)_-} = k_0, \qquad u_\varepsilon|_{t=0} = u_0(x), \tag{13}$$

or

$$\lambda u_\varepsilon + \frac{\Omega \cdot \nabla_x u_\varepsilon}{\varepsilon} + \frac{\sigma(\tilde{u}_\varepsilon)}{\varepsilon^2}(u_\varepsilon - \tilde{u}_\varepsilon) = f,$$
$$u_\in |_{(\partial X \times S^N)_-} = k_0, \tag{14}$$

where k_0 is some nonnegative constant. We introduce also the limit equations

$$\frac{\partial u}{\partial t} - \Delta F(u) = 0,$$
$$u|_{\partial X} = k_0, \qquad u|_{t=0} = u_0(x), \tag{15}$$

or

$$\lambda u - \Delta F(u) = f,$$
$$u|_{\partial X} = k_0, \tag{16}$$

where

$$F(T) = \frac{1}{N+1}\int_0^T \frac{ds}{\sigma(s)}. \tag{17}$$

Again we state separately our results for the stationary and the evolution problem.

THEOREM 3 (Rosseland Approximation for the Stationary Problem). *Under assumption* (A1), *and* (A2) *if* $\lambda = 0$, *let* $f \geqslant 0$ *belong to* $L^\infty(X \times S^N)$ *and let* $k_0 \geqslant 0$ *be a nonnegative constant. Then any family* $(u_\varepsilon)_{\varepsilon > 0}$ *of solutions of* (14) *in the sense of Theorem* 1 *is uniformly bounded in* $L^\infty(X \times S^N)$. *Moreover, we may extract a subsequence* $(u_{\varepsilon_n})_{n \geqslant 0}$ *which converges pointwise to a function* $u \in L^\infty(X)$ *which is a solution of* (16).

THEOREM 4 (Rosseland Approximation for the Evolution Problem). *Under assumption* (A1), *let* k_0 *be a nonnegative constant and let* $u_0 \geqslant 0$ *belong to* $L^\infty(X)$. *Then any family* $(u_\varepsilon)_{\varepsilon>0}$ *of solutions of* (15) *in the sense of Theorem* 2 *is uniformly bounded in* $L^\infty(X \times S^N \times \mathbb{R}^+)$. *Moreover, we may extract a subsequence* $(u_{\varepsilon_n})_{n \geqslant 0}$ *which converges pointwise to the unique solution* $u \in C(\mathbb{R}^+; L^1(X)) \cap L^\infty(X \times \mathbb{R}^+)$ *of* (15).

II.2. *Proof of Theorem* 3

Again, we divide our proof in several steps. In the first one, we prove the uniform L^∞ bounds, then we give some a priori estimates on $\Omega \cdot \nabla_x u_\varepsilon$. In the third step, we show that u_ε is compact in $L^2(X)$ and, finally, we pass to the limit.

(i) L^∞ *Bounds for* $\lambda = 0$. First, let us prove an L^∞ estimate for solutions of (14). Using the same argument as in the proof of Theorem 1, it is enough to find a supersolution g_ε with $\tilde{g}_\varepsilon \geqslant u_\infty$ (the case $\lambda > 0$ is clear enough). Thus let us introduce the constant $\gamma = \| f \|_\infty (N+1)$ and, since F is increasing, we may define a function V_0 by $V_0 = F^{-1}(C_0 - \gamma x^2)$, where C_0 is chosen large enough such that

$$\begin{aligned} &V_0 \geqslant u_\infty, \qquad \forall x \in X, \\ &v_0|_{\partial X} \geqslant k_0 + 1. \end{aligned} \tag{18}$$

Then, let us define g_ε for ε small enough,

$$g_\varepsilon = V_0 - \varepsilon \Omega \cdot \nabla_x F(V_0)/(N+1).$$

We have

$$\tilde{g}_\varepsilon = V_0,$$

and

$$\begin{aligned} &\frac{\Omega \cdot \nabla_x g_\varepsilon}{\varepsilon} + \frac{\sigma(\tilde{g}_\varepsilon)}{\varepsilon^2}(g_\varepsilon - \tilde{g}_\varepsilon) \\ &\quad = \frac{\Omega \cdot \nabla_x V_0}{\varepsilon} - \left\{ D^2F(V_0)(\Omega, \Omega) + \frac{\sigma(V_0)}{\varepsilon} \Omega \cdot \nabla_x F(V_0) \right\} \Big/ (N+1) = 2\gamma \\ &\quad \geqslant \| f \|_\infty. \end{aligned}$$

This proves, as in Section I.2, that $u_\varepsilon \leqslant g_0$ and we have proved the L^∞ estimates for $\lambda = 0$.

(ii) *Some a Priori Estimates.* We now prove some a priori estimates which will be used later to obtain the compactness we need to send ε to 0.

Here, to simplify notations, we set $\lambda = 0$. The general case holds with the same argument.

First we multiply (14) by $(u_\varepsilon - k_0)$ and integrate over $X \times S^N$. This gives

$$\frac{1}{2\varepsilon}\int_{(\partial X \times S^N)_-} (u_\varepsilon - k_0)^2\, \Omega \cdot n d\sigma(x)\, d\Omega + \int_{X \times S^N} \frac{\sigma(\tilde{u}_\varepsilon)}{\varepsilon^2}(u_\varepsilon - \tilde{u}_\varepsilon)^2$$

$$= \int_{X \times S^N} f(u_\varepsilon - k_0)$$

and thus

$$\int_{X \times S^N} \sigma(\tilde{u}_\varepsilon)(u_\varepsilon - \tilde{u}_\varepsilon)^2 \leqslant C\varepsilon^2. \tag{19}$$

Since $\sigma(\tilde{u}_\varepsilon) \geqslant \sigma_m$, we have the estimates

$$\begin{gathered} \| u_\varepsilon - \tilde{u}_\varepsilon \|_{L^2(X \times S^N)} \leqslant C\varepsilon, \\ \| \mathbb{1}_{\{\tilde{u}_\varepsilon > 0\}}\, \sigma^{-1/2}(\tilde{u}_\varepsilon)\, \Omega \cdot \nabla_x u_\varepsilon \|_{L^2(X \times S^N)} \leqslant C. \end{gathered} \tag{20}$$

(iii) *Strong Convergence of* u_ε. We use these estimates as follows.

LEMMA 2. *Let $u_\varepsilon \geqslant 0$ be bounded in $L^\infty(X \times S^N)$ independently of ε and $\Omega \cdot \nabla_x u_\varepsilon \in L^1(X \times S^N)$. Let σ satisfy* (A1) *and assume* (20). *Then there exist a function $u \in L^\infty(X \times S^N)$ and a subsequence $u_n \equiv u_{\varepsilon_n}$ such that $u_n \to_{n \to \infty} u$ pointwise.*

Proof. We deduce from (20) that

$$\begin{aligned} \Omega \cdot \nabla_x u_\varepsilon^2 &= 2u_\varepsilon \Omega \cdot \nabla_x u_\varepsilon \mathbb{1}_{\{\tilde{u}_\varepsilon > 0\}} \\ &= 2(\sigma^{1/2}(\tilde{u}_\varepsilon)\, u_\varepsilon)(\sigma^{-1/2}(\tilde{u}_\varepsilon)\, \Omega \cdot \nabla_x u_\varepsilon \mathbb{1}_{\{\tilde{u}_\varepsilon > 0\}}) \qquad \text{a.e.} \end{aligned}$$

(Since $\Omega \cdot \nabla_x u_\varepsilon \in L^1$ and $u_\varepsilon = 0$ a.e. on the set $\{\tilde{u}_\varepsilon = 0\}$). But $\sigma^{1/2}(\tilde{u}_\varepsilon)\, u_\varepsilon$ is bounded in $L^{2p}(X \times S^N)$ (with the same p as in (A1)) since

$$\int_{X \times S^N} (\sigma^{1/2}(\tilde{u}_\varepsilon)\, u_\varepsilon)^{2p}\, dx\, d\Omega \leqslant \int_{X \times S_N} \sigma^p(\tilde{u}_\varepsilon)\, u_\varepsilon u_\varepsilon^{2p-1}\, dx\, d\Omega$$

$$\leqslant \| u_\varepsilon \|_{L^\infty}^{2p-1} \int_X \sigma^p(\tilde{u}_\varepsilon)\, \tilde{u}_\varepsilon\, dx \leqslant C.$$

Thus (20) shows that $\Omega \cdot \nabla_x u_\varepsilon^2$ is bounded in $L^{2p/(1+p)}(X \times S^N)$. Therefore, we may apply Proposition 1 and there exists a subsequence u_{ε_n} (that we write u_n) and a nonnegative function u such that

$$\widetilde{(u_n^2)} \xrightarrow[n \to \infty]{} u^2 \qquad \text{in any } L^p(X),\ 1 \leqslant p < \infty.$$

Since we have

$$u_n^2 - (\tilde{u}_n)^2 = (u_n + \tilde{u}_n)(u_n - \tilde{u}_n) \to 0 \qquad \text{in } L^2(X \times S^N),$$

we obtain

$$(\tilde{u}_n)^2 \xrightarrow[n \to \infty]{} u^2 \qquad \text{a.e.}$$

and, finally, this is enough to assert that $u_n \to u$ in any $L^r(X \times S^N)$, $1 \leqslant r < +\infty$.

(iv) *Passage to the Limit.* Now, we use this convergence to pass to the limit in (14). Let us integrate (14) over S^N. We obtain

$$\lambda \tilde{u}_\varepsilon \mathbb{1}_{\{\tilde{u}_\varepsilon > 0\}} + (1/\varepsilon) \nabla_x \cdot \widetilde{\Omega u_\varepsilon} \mathbb{1}_{\{\tilde{u}_\varepsilon > 0\}} = \tilde{f}. \tag{21}$$

Then, multiplying (14) by Ω and integrating, we have

$$\lambda \widetilde{\Omega u_\varepsilon} \mathbb{1}_{\{\tilde{u}_\varepsilon > 0\}} + \mathbb{1}_{\{\tilde{u}_\varepsilon > 0\}} (1/\varepsilon) \nabla_x \widetilde{\Omega \otimes \Omega u_\varepsilon}$$
$$+ (1/\varepsilon^2) \sigma(\tilde{u}_\varepsilon) \widetilde{\Omega u_\varepsilon} \mathbb{1}_{\{\tilde{u}_\varepsilon > 0\}} = \widetilde{f\Omega} \mathbb{1}_{\{\tilde{u}_\varepsilon > 0\}}. \tag{21'}$$

Combining (21) and (21′), we obtain

$$\lambda \tilde{u}_\varepsilon \mathbb{1}_{\{\tilde{u}_\varepsilon > 0\}} + \nabla_x \cdot \left(\varepsilon \frac{\widetilde{f\Omega}}{\sigma(\tilde{u}_\varepsilon)} \mathbb{1}_{\{\tilde{u}_\varepsilon > 0\}} - \frac{\mathbb{1}_{\{\tilde{u}_\varepsilon > 0\}}}{\sigma(\tilde{u}_\varepsilon)} \nabla_x \cdot \widetilde{\Omega \otimes \Omega u_\varepsilon} \right.$$
$$\left. - \mathbb{1}_{\{\tilde{u}_\varepsilon > 0\}} \frac{\varepsilon \lambda \widetilde{\Omega u_\varepsilon}}{\sigma(\tilde{u}_\varepsilon)} \right) = \tilde{f} \qquad \text{in } \mathscr{D}'(S \times S^N). \tag{22}$$

In (22), we have the convergences (with the notations of (iii))

$$u_n \xrightarrow[n \to \infty]{} u \qquad \text{in } L^2(X \times V),$$

$$\varepsilon_n \frac{\widetilde{f\Omega}}{\sigma(\tilde{u}_n)} \mathbb{1}_{\{\tilde{u}_n > 0\}} \xrightarrow[n \to \infty]{} 0 \qquad \text{in } L^2(X),$$

$$\varepsilon_n \frac{\lambda \widetilde{\Omega u_n}}{\sigma(\tilde{u}_n)} \mathbb{1}_{\{\tilde{u}_n > 0\}} \xrightarrow[n \to \infty]{} 0 \qquad \text{in } L^2(X).$$

To pass to the limit in the last term of (22) we use (20), which shows that there exists $q \in L^2(X \times S^N)$ such that (extracting again a subsequence)

$$\mathbb{1}_{\{\tilde{u}_n > 0\}} \sigma^{-1/2}(\tilde{u}_n) \nabla_x (\Omega \otimes \Omega u_n) \xrightarrow[n \to \infty]{} q \qquad \text{weakly in } L^2(X \times S^N). \tag{23}$$

Multiplying by $u_n \sigma^{1/2}(\tilde{u}_n)$ we get

$$u_n \nabla_x(\Omega \otimes \Omega u_n) \xrightarrow[n\to\infty]{} q \cdot (u\sigma^{1/2}(u)) \qquad \text{weakly in } L^1(X \times S^N).$$

Indeed $u_n \sigma^{1/2}(\tilde{u}_n)$ converges in $L^2(X \times S^N)$ to $u\sigma^{1/2}(u)$ since

$$|u_n \sigma^{1/2}(\tilde{u}_n) - u\sigma^{1/2}(u)| \leqslant |u_n \sigma^{1/2}(\tilde{u}_n) - \tilde{u}_n \sigma^{1/2}(\tilde{u}_n)| + |\tilde{u}_n \sigma^{1/2}(\tilde{u}_n) - u\sigma^{1/2}(u)|$$
$$\xrightarrow[L^2(X \times S_N)]{} 0 \qquad \text{(by (A1) and (19))}.$$

Thus, we have proved that

$$\tfrac{1}{2}\nabla_x(\Omega \otimes \Omega u_n^2) \rightharpoonup qx(u\sigma^{1/2}(u)) \qquad \text{weakly in } L^1,$$
$$\rightharpoonup \tfrac{1}{2}\nabla_x(\Omega \otimes \Omega u) \qquad \text{in } D'(X \times S^N),$$

therefore

$$q(u\sigma^{1/2}(u)) = \tfrac{1}{2}\nabla_x(\Omega \otimes \Omega u^2). \tag{24}$$

We now prove that this implies that

$$\mathbb{1}_{\{u>0\}} \frac{\tilde{q}}{\sigma^{1/2}(u)} = \nabla_x F(u). \tag{25}$$

To do so, we work with $w = u^2$ and (24) gives that

$$\mathbb{1}_{\{w>0\}} \frac{\tilde{q}}{\sigma^{1/2}(w^{1/2})} = \frac{1}{2(N+1)} G(w) \nabla_x w,$$

where

$$G(z) = \frac{1}{z^{1/2}\sigma(z^{1/2})}$$

and we know that G is continuous on $\mathbb{R}^{+*}$, $G(z) \to_{z\to 0^+} +\infty$, $0 \leqslant G(z) \leqslant C/z^{1/2}$ (by (A1)), and thus we may define

$$H(t) = \int_0^t G(s)\, ds,$$

and we want to show that $\mathbb{1}_{\{w>0\}} G(w) \nabla_x w = \nabla_x H(w)$ in $\mathscr{D}'$. But we have proved that $w \in L^p(X)$, $\nabla_x w \in L^p(X)$ for some $p > 1$ (see (24)), Thus denoting $G_M = \inf(M, G)$ and $H_M(t) = \int_0^t G_M(s)\, ds$, H_M is C^1, and we have $\nabla_x H_M(w) = \mathbb{1}_{\{w>0\}} G_M(w) \nabla_x w$. As M tends to infinity, $H_M(w)$ converges pointwise (and thus in any $L^q(X)$) to $H(w)$. Thus $\nabla_x H_M(w)$ converges in $\mathscr{D}'(X)$ to $\nabla_x H(w)$, and $\mathbb{1}_{\{w>0\}} G_M(w) \nabla_x w$ converges pointwise to $\mathbb{1}_{\{w>0\}} G(w) \nabla_x w$. But $\mathbb{1}_{\{w>0\}} |G_M(w) \nabla_x w| \leqslant |\nabla_x w|\, |G(w) \mathbb{1}_{\{w>0\}}| \in L^2(X)$. Hence, by dominated convergence, the two limits are identical and we have proved (25).

Therefore, we may pass to the limit in the last term of (22):

$$\frac{\mathbb{1}_{\{\tilde{u}_n>0\}}}{\sigma(\tilde{u}_n)} \nabla_x \cdot \widetilde{\Omega \otimes \Omega u_n} = 1_{\{\tilde{u}_n>0\}} \frac{1}{\sigma^{1/2}(\tilde{u}_n)} \frac{1}{\sigma^{1/2}(\tilde{u}_n)} \nabla_x \cdot \widetilde{\Omega \otimes \Omega u_n}$$

$$\xrightarrow[n\to\infty]{} \mathbb{1}_{\{u>0\}} \frac{\tilde{q}}{\sigma^{1/2}(u)} = \nabla_x F(u)$$

(since $\mathbb{1}_{\{\tilde{u}_n>0\}}/\sigma^{1/2}(\tilde{u}_n) \to \mathbb{1}_{\{u>0\}}/\sigma^{1/2}(u)$ in $L^2(X)$). This proves that u satisfies

$$u - \Delta F(u) = \bar{f}. \tag{16'}$$

Finally, u satisfies the boundary condition of (16) since $\Omega \cdot \nabla_x u_\varepsilon^2$ is bounded in some $L^p(X \times S^N)$, $p>1$. Thus u_ε^2 has a trace which passes to the limit. Equation (16′) shows that $F(u)$ also has a trace and they coincide by a density argument. This concludes the proof of Theorem 3.

II.3. *Proof of Theorem* 4

The main steps of the proof of this theorem are the same as in the previous one. First the L^∞ bounds are clear since the maximum principle asserts that

$$\| u_\varepsilon \|_{L^\infty(X\times S^N)} \leqslant \sup(k_0, \| u_0 \|_{L^\infty(X\times S^N)}).$$

As before, we obtain the estimate

$$\int_{X\times S^N\times R^+} \sigma(\tilde{u}_\varepsilon)(u_\varepsilon - \tilde{u}_\varepsilon)^2 \leqslant C\varepsilon^2, \tag{19'}$$

i.e.,

$$\begin{aligned} &\| u_\varepsilon - \tilde{u}_\varepsilon \|_{L^2(X\times S^N\times R^+)} \leqslant C\varepsilon, \\ &\left\| \sigma^{-1/2}(\tilde{u}_\varepsilon)\, \mathbb{1}_{\{\tilde{u}_\varepsilon>0\}} \cdot \left(\varepsilon \frac{\partial u_\varepsilon}{\partial t} + \Omega \cdot \nabla_x u_\varepsilon \right) \right\|_{L^2(X\times S^N\times R^+)} \leqslant C. \end{aligned} \tag{20'}$$

We have the

LEMMA 3. *With the above notations and assumptions, for any* $T \geqslant 0$, $(\widetilde{u_\varepsilon^\alpha})^{1/\alpha}$ *is bounded in* $L^q([0, T], W^{\beta,q}(X))$ *where* $q = 2p/(p+1)$, $\alpha = 1 + 1/2p$, *for any* $\beta < (p-1)/(2p+1)$.

Proof of Lemma 3. First proceed as if $\sigma^{1/2}(\tilde{u}_\varepsilon)\ u_\varepsilon^{1/2p}$ was bounded in $L^\infty(X\times S^N)$. Thus

$$\left\| \varepsilon \frac{\partial u_\varepsilon^\alpha}{\partial t} + \Omega \cdot \nabla_x u_\varepsilon^\alpha \right\|_{L^2(X\times S^N\times R^+)} \leqslant C.$$

Following [16, 18], we make a Fourier transform in (t, x) (after an extension of the functions outside $R^+\times X$). Denoting (τ, ξ) the Fourier variable and $\hat{u}$ the Fourier transform of u, we have

$$\int_{\tau,\xi} |\xi|\,|\widetilde{(\widehat{u_\varepsilon^\alpha})}|^2 \leqslant \int_{\tau,\xi} |\xi| \left[\int_{S^N} |\widehat{u_\varepsilon^\alpha}|^2 \frac{\lambda^2 + (\xi\cdot\Omega/|\xi| + \varepsilon\tau/|\xi|)^2}{\lambda}\, d\Omega \right.$$
$$\left. \times \int_{S^N} \frac{\lambda}{\lambda^2 + (\xi\cdot\Omega/|\xi| + \varepsilon\tau/|\xi|)^2}\, d\Omega\right] d\xi\, d\tau,$$

and the last term of the right member of this inequality is bounded independently of ε, ξ, τ, λ so that we may choose, for each (τ, ξ),

$$\lambda = \left[\frac{\int |\widehat{u_\varepsilon^\alpha}|^2\,((\xi\cdot\Omega/|\xi| + \varepsilon(\tau/|\xi|))^2}{\int |(\widehat{u_\varepsilon^\alpha})|^2\, d\Omega}\, d\Omega\right]^{1/2},$$

and we obtain

$$\int_{\tau,\xi} |\xi|\,|\widetilde{(\widehat{u_\varepsilon^\alpha})}|^2$$
$$\leqslant \int_{\tau,\xi} \left[\int |(\widehat{u_\varepsilon^\alpha})|^2\ (\xi\cdot\Omega + \varepsilon\tau)^2\, d\Omega\right]^{1/2} \left[\int |(\widehat{u_\varepsilon^\alpha})|^2\, d\Omega\right]^{12} d\xi\, d\tau$$
$$\leqslant \left[\int_\Omega d\Omega \int_{\tau,\xi} |(\widehat{u_\varepsilon^\alpha})|^2\, d\tau\, d\xi \cdot \int_{\Omega,\xi,\tau} |(\widehat{u_\varepsilon^\alpha})|^2\,(\xi\cdot\Omega + \varepsilon\tau)^2\, d\Omega\, d\xi\, d\tau\right]^{1/2}$$
$$\leqslant C^{st}.$$

This would prove that $\widetilde{u_\varepsilon^\alpha}$ is bounded in $L^2([0, T]; H^{1/2}(X))$. But, in fact, $\sigma^{1/2}(u_\varepsilon)\, u_\varepsilon^{1/2p}$ is bounded in $L^{2p}([0, T]\times X\times S^N)$, so that one only knows that

$$\varepsilon \frac{\partial u_\varepsilon^\alpha}{\partial t} + \Omega\cdot\nabla_x u_\varepsilon^\alpha$$

is bounded in $L^{2p/(p+1)}([0, T] \times X \times S^N)$. Consider now the map $\mathscr{T}: f \to \widetilde{u_\varepsilon^\alpha}$ where

$$u_\varepsilon^\alpha + \varepsilon \frac{\partial u_\varepsilon^\alpha}{\partial t} + \Omega \cdot \nabla_x u_\varepsilon^\alpha = f_\varepsilon$$

$$u^\alpha_{\varepsilon|(\partial X \times S^N)_-} = 0$$

$$u^\alpha_{\varepsilon|_{t=0}} = 0.$$

From the above computation, we know that $\mathscr{T}$ maps $L^\infty(\varepsilon > 0; L^2([0, T] \times X \times S^N))$ into $L^\infty(\varepsilon > 0; L^2([0, T]; H^{1/2}(X)))$; it also maps obviously $L^\infty(\varepsilon > 0; L^1([0, T] \times X \times S^N))$ into $L^\infty(\varepsilon > 0; L^1([0, T] \times X))$. Therefore by a standard interpolation argument, we know that $\widetilde{u_\varepsilon^\alpha}$ is bounded in $L^{2p/(p+1)}([0, T]; W^{s, 2p/(p+1)}(X))$ for any $s < (p-1)/2p$. It remains to show that if $v \in W^{s,q}(\mathbb{R}^{N+1})$ and v has a compact support, then $v^{1/\alpha} \in W^{\beta,q}(\mathbb{R}^{N+1})$, for $0 < \beta < s/\alpha$. We may compute

$$\| v^{1/\alpha} \|^q_{W^{\beta,q}} = \int_{R^{N+1} \times B} \frac{| v^{1/\alpha}(x+h) - v^{1/\alpha}(x)|^q}{|h|^{N+1+q\beta}} \, dx \, dh + | v^{1/\alpha} |^q_{L^q},$$

where $B = \{h \in R^{N+1} / |h| \leqslant 1\}$. But we have, since $1/\alpha < 1$,

$$\begin{aligned} \| v^{1/\alpha} \|^q_{W^{\beta,q}} &\leqslant C \int_{R^{N+1} \times B} \frac{| v(x+h) - v(x)|^{q/\alpha}}{|h|^{N+1+q\beta}} \, dx \, dh + C(| v |^q_{L^q})^{1/\alpha} \\ &\leqslant C \left(\int_{R^{N+1} \times B} \frac{| v(x+h) - v(x)|^q}{|h|^{N+1+sq}} \right)^{1/\alpha} + C(| v |^q_{L^q})^{1/\alpha} \\ &\leqslant C \| v \|^{1/\alpha}_{W^{s,q}} + C \| v \|^{q/\alpha}_{L^q}. \end{aligned}$$

This proves that $(\widetilde{u_\varepsilon^\alpha})^{1/\alpha}$ is bounded in $L^q([0, T]; W^{\beta,q}(X))$ with $q = 2p/(p+1)$ and $\beta < (p-1)/2p\alpha$, and Lemma 3 is proved.

Remark. To write the Fourier transform we have used several times extensions of u_ε to the whole space. This is possible thanks to the results of [9] and the assumption on $u_\varepsilon|_{(\partial X \times S^N)_-}$, $u_\varepsilon|_{t=0}$. We may assume that these extensions have the same compact support.

Let us continue the proof of Theorem 4. Following the computation of the previous section, we write

$$\frac{\partial}{\partial t} \tilde{u}_\varepsilon - \nabla_x \left[\frac{\mathbb{1}_{\{\widetilde{u_\varepsilon} > 0\}}}{\sigma(\tilde{u}_\varepsilon)} \left(\varepsilon \frac{\partial \widetilde{\Omega u_\varepsilon}}{\partial t} + \nabla_x \widetilde{\Omega \otimes \Omega u_\varepsilon} \right) \right] = 0. \tag{26}$$

Thus (20′) shows that $\partial \tilde{u}_\varepsilon / \partial t$ is bounded in $L^q([0, T]; W^{-1,q}(X))$.

Setting $v_\varepsilon = (\widetilde{u_\varepsilon^\alpha})^{1/\alpha}$, we have proved that

$$v_\varepsilon \in L^q([0, T]; W^{\beta,q}(X)),$$

$$|\tilde{u}_\varepsilon - v_\varepsilon|_{L^q([0,T]\times X)} \xrightarrow[\varepsilon\to 0]{} 0 \qquad \text{(since } \tilde{u}_\varepsilon, v_\varepsilon \text{ are bounded)} \tag{27}$$

$$\frac{\partial \tilde{u}_\varepsilon}{\partial t} \in L^q([0, T]; W^{-1,q}(X))$$

with uniform bounds in ε. The next step of our proof is to deduce from (27) that $\tilde{u}_\varepsilon$ has a subsequence which converges in $L^q([0, T] \times X)$.

From (27), we deduce that $\tilde{u}_\varepsilon$ has a subsequence which converges in $L^q([0, T] \times X)$, according to the following lemma.

LEMMA. *Let $V \subset W \subset Z$ be three Banach spaces, and assume that the inclusion $V \subset W$ is compact, and the inclusion $W \subset Z$ continuous. Let f_ε and g_ε be families such that*

— *f_ε is bounded in $L^q([0, T]; V)$;*
— *$f_\varepsilon - g_\varepsilon \to 0$ in $L^q([0, T]; W)$ when $\varepsilon \to 0$;*
— *$\partial_t g_\varepsilon$ is bounded in $L^q([0, T]; Z)$.*

Then f_ε is compact in $L^q([0, T]; W)$.

Indeed, we apply this lemma to $V = W^{s,q}(X)$, $W = L^q(X)$, and $Z = W^{-1,q}(X)$.

This lemma is almost classical, but we give the proof of it for the sake of completeness.

Proof of Lemma. We can obviously assume that we are in the situation where $V \subset Y \subset W \subset Z$, where the inclusions $V \subset Y$ and $Y \subset W$ are compact. Define

$$\omega_f^\varepsilon(t, h) = f_\varepsilon(t+h) - f_\varepsilon(t); \qquad \omega_g^\varepsilon(t, h) = g_\varepsilon(t+h) - g_\varepsilon(t).$$

Since the inclusion $V \subset Y$ is compact, for any $\alpha > 0$, there exists $C_\alpha > 0$ such that

$$|\omega_f^\varepsilon|_Y \leqslant \alpha\, |\omega_f^\varepsilon|_V + C_\alpha(|\omega_f^\varepsilon - \omega_g^\varepsilon|_W + |\omega_g^\varepsilon|_Z).$$

Replacing ω_g^ε by $\int_t^{t+h} \partial_t g_\varepsilon(s)\, ds$, one has, by integration of the above inequality on any compact t-set,

$$\int |\omega_f^\varepsilon(t,h)|_Y^q\,dt$$

$$\leqslant C\alpha^p \int |\omega_f^\varepsilon(t,h)|_V^q\,dt$$

$$+ CC_\alpha^q\left(\int |\omega_f^\varepsilon(t,h)-\omega_g^\varepsilon(t,h)|_W^q\,dt + h^{q-1}\iint_t^{t+h} |\partial_t g_\varepsilon(s)|_Z^q\,ds\,dt\right)$$

$$\leqslant C\alpha + CC_\alpha^q(\rho(\varepsilon)+Ch^q).$$

The latter estimate ensures the compactness of f_ε in $L^q([0,T];W)$, by using Kolmogorov's L^q-compactness criterion derived from the Ascoli theorem, and the compactness of the inclusion $Y\subset W$.

The end of the proof of Theorem 4 is now the same as point (iv) of the proof of Theorem 3 and we do not repeat it.

III. Existence of Solutions for the Model with Frequencies

In this section, we consider the complete model of Radiative Transfer. Thus we consider the solution $T(x)$, $u(x,\Omega,v)$, $x\in X$, $\Omega\in S^N$, and $v\in\mathbb{R}^+$, of the system

$$\lambda u+\Omega\cdot\nabla_x u+\sigma_v(T)(u-B_v(T))=f,$$

$$\lambda T+\int_{\mathbb{R}^+}\sigma_v(T)(B_v(T)-\tilde{u})\,dv=g, \tag{28}$$

$$u|_{(\partial X\times S^N)_-}=k.$$

We extend the results of Section I to (28) and prove the existence of solutions even for a singular opacity ($\sigma_v(0)=+\infty$). Again we begin by studying a "regular case" where σ_v is regularized (Section III.1), then we treat the general case (Section III.2).

To simplify the proofs we only consider the case $\lambda>0$.

III.1. *Regular Opacities*

Let us introduce some assumptions on the opacities (σ_v) and the Planckian reemission (B_v). We will assume that there exists a positive number δ and two functions Σ_v, P_v for $v\geqslant 0$ such that

$$P_v\Sigma_v\in L^1(R^+), \tag{29}$$

$$0\leqslant B_v(T)\leqslant P_v,\ 0\leqslant\sigma_v(T)\leqslant\Sigma_v,\ \forall T\in[0,\delta]\ \forall v\geqslant 0, \tag{30}$$

$$B_v(\cdot),\ \sigma_v(\cdot)\text{ are continuous for a.e. } v\geqslant 0, \tag{31}$$

$$0 \leqslant f(x, \Omega, \nu),\ k(x, \Omega, \nu) \leqslant \lambda P_\nu,\ 0 \leqslant g(x) \leqslant \lambda \delta, \tag{32}$$

for a.e. ν, the functions f, k, $\sigma_\nu(T)$, $\sigma_\nu(T)\, B_\nu(T)$ are continuous with respect to frequencies at point ν uniformly for $T \in [0, \delta]$, $x \in X$, $\Omega \in S^N$. (33)

Finally, beyond these technical assumptions, we need a fundamental assumption to be able to solve the second equation of the system (28). The most general one seems to be

for each measurable function $w(\nu)$, $0 \leqslant w(\nu) \leqslant P_\nu$, there exists a unique measurable function $\Phi_w(x) \in [0, \delta]$ s.t.

$$\lambda \Phi_w + \int_0^\infty \sigma_\nu(\Phi_w)\, B_\nu(\Phi_w)\, d\nu = \int_0^\infty \sigma_\nu(\Phi_w)\, w\, d\nu + g.$$

Remarks. (1) Assumption (34) is satisfied in particular when the function $\sigma_\nu(\Phi)\, B_\nu(\Phi)$ is nondecreasing in Φ and $\sigma_\nu(\Phi)$ nonincreasing in Φ for a.e. ν. This case corresponds to accretive operators; see [22, 17]. We can even generalize this by assuming only that $\int_0^\infty \sigma_\nu(\Phi)\, B_\nu(\Phi)\, d\nu$ is nondecreasing.

(2) The assumptions (29)–(33) are compatible with Planck's values of B_ν and Kramer's values of σ_ν with uniformly positive temperature. The case of zero temperature is treated in Section III.2. Let us recall that Planck and Kramer's functions are (assuming that the local temperature and energy are proportional)

$$B_\nu(T) = \frac{\nu^3}{e^{\nu/T} - 1} \tag{B}$$

$$\sigma_\nu(T) = \frac{1 - e^{-\nu/T}}{\sqrt{T}\, \nu^3} \tag{C}$$

and the assumption (34) holds since Remark 1 applies.

(3) Here we have assumed that $\nu \in \mathbb{R}^+$ since it is the physically relevant case. For numerical applications, it is useful to assume that ν belongs to some discrete set. This subsection holds in this case with very few changes.

Let us state the main result of this subsection.

PROPOSITION 2. *Under assumptions* (29)–(34) *there exists at least one solution* (u, T) *of* (28) *and it satisfies*

$$0 \leqslant u \leqslant P_\nu,\ 0 \leqslant T \leqslant \delta,$$

and (28) *holds in the sense that* $\Omega \cdot \nabla_x u \in L^\infty(X \times S^N)$ *for a.e.* ν.

Proof. The proof of Proposition 2 uses the same arguments as the proof of Theorem 2 and thus we only prove the new points. We introduce the set

$$\mathscr{C} = \{\Phi \in L^1(X), 0 \leqslant \Phi \leqslant \delta\} \qquad \text{(with its } L^1 \text{ topology).}$$

For any $\Phi \in \mathscr{C}$, we may solve the linear transport equation with parameter v:

$$\begin{gathered} \lambda u + \Omega \cdot \nabla_x u + \sigma_v(\Phi)\, u = \sigma_v(\Phi)\, B_v(\Phi) + f, \\ u\,|_{(\partial X \times S^N)_-} = k. \end{gathered} \tag{35}$$

The maximum principle gives

$$0 \leqslant u \leqslant P_v, \tag{36}$$

since we have with a simple calculation that

$$\begin{aligned} &\lambda(u - P_v) + \Omega \cdot \nabla_x (u - P_v)^+ + \sigma_v(\Phi)(u - P_v)^+ \\ &\leqslant \sigma_v(\Phi)(B_v(\Phi) - P_v)\, \mathbb{1}_{\{u \geqslant P_v\}} + \lambda(f - P_v)\, \mathbb{1}_{\{u \geqslant P_v\}} \\ &\leqslant 0 \end{aligned}$$

and

$$(u - P_v)^+\,|_{(\partial X \times S^N)_-} \leqslant 0.$$

Following Section I, we set $\tilde{u} = T_1 \Phi$, and, using (34), we may define an operator T_2 which associates to $w(x, v)$, $0 \leqslant w(x, v) \leqslant P_v$, the solution $\Psi(x)$, $0 \leqslant \Psi(x) \leqslant \delta$, of

$$\lambda \Psi + \int \sigma_v(\Psi)\, B_v(\Psi)\, dv = \int \sigma_v(\Psi)\, w\, dv + g.$$

To solve (28) is now equivalent to finding a fixed point for $T = T_2 \cdot T_1$ and thus it is enough to prove that T is compact from $\mathscr{C}$ to $\mathscr{D}$.

To this end, we introduce the set

$$\mathscr{D} = \{w \in L^1_\Sigma\,(X \times \mathbb{R}^+),\ 0 \leqslant w \leqslant B_v\},$$

with the topology of $L^1_\Sigma(X \times \mathbb{R}^+)$, i.e., L^1 with weight Σ,

$$\|w\|_{L^1_\Sigma} = \int_{X \times \mathbb{R}^+} |w(x, v)|\ \Sigma_v\, dx\, dv,$$

and our theorem may be reduced to the following lemmas.

LEMMA 4. *T_1 is continuous compact from $\mathscr{C}$ into $\mathscr{D}$.*

LEMMA 5. *T_2 is continuous from $\mathscr{D}$ into $\mathscr{C}$.*

Proof of Lemma 4. The continuity of T_1 is deduced from the continuity of the solution of (35) with respect to Φ for every v and from the dominated convergence (using (29) and (30)); we leave it to the reader and we prove the compactness. We divide this proof in three steps.

First Step. Continuity of $T_1 \Phi$ in v.

We will need this continuity, and more precisely that

$$\sup_{\Phi \in \mathscr{C}} \| T_1 \Phi(x, v) - T_1 \Phi(x, v') \|_{L^1(X)} \xrightarrow[v' \to v]{} 0 \qquad \text{for a.e. } v. \tag{37}$$

To prove (37), take v such that (33) holds and set $w(x, \Omega) = u(x, \Omega, v) - u(x, \Omega, v')$ for a solution u of (35). Then

$$w + \Omega \cdot \nabla_x w + \sigma_v(\Phi)\, w = f_v - f_{v'} + \sigma_v(\Phi)\, B_v(\Phi) - \sigma'_v(\Phi)\, B_{v'}(\Phi)$$
$$+ [\sigma_{v'}(\Phi) - \sigma_v(\Phi)]\, u(x, \Omega, v'),$$
$$w|_{(\partial X \times S^N)_-} = k_v - k_{v'},$$

i.e.,

$$|w| + \Omega \cdot \nabla_x |w| \leqslant \sup_{x, \Omega} |f_v - f_{v'}| + \sup_{\Phi \leqslant \sigma} |\sigma_v B_v(\Phi) - \sigma_{v'} B_{v'}(\Phi)|$$
$$+ \sup_{\Phi \leqslant \delta} |\sigma_v B_v(\Phi) - \sigma_{v'} B_{v'}(\Phi)| + \sup_{\Phi \leqslant \delta} |\sigma_v(\Phi) - \sigma_{v'}(\Phi)|\, P_{v'}.$$

After integration, we obtain

$$\int_{X \times S^N} |w|\, dx\, d\Omega \leqslant C \sup_{x, \Omega} \{ |f_v - f_{v'}| + |k_v - k_{v'}|$$
$$+ \sup_{\Phi \leqslant \delta} |\sigma_v B_v(\Phi) - \sigma_{v'} B_{v'}(\Phi)|$$
$$+ \sup_{\Phi \leqslant \delta} |\sigma_v(\Phi) - \sigma_{v'}(\Phi)|\, P_{v'} \}.$$

By (33), the second member of this inequality converges to 0 and (37) is proved.

Second Step. Compactness of $T_1 \Phi$ for a.e. v.

Now, we prove that for a.e. v, the family $(T_1 \Phi)_{\Phi \in \mathscr{C}}$ is relatively compact. For a solution u of (35), we have

$$\Omega \cdot \nabla_x u = -\lambda u - \sigma_v(\Phi)\, u + f + \sigma_v B_v(\Phi).$$

Introducing the solution $v \in L^{\infty}(X \times S^N)$ of (ν is a parameter of v)

$$\Omega \cdot \nabla_x v = 0,$$
$$v|_{(\partial X \times S^N)_-} = k,$$

we obtain

$$|\Omega \cdot \nabla_x (u - v)| \leqslant 2\lambda P_\nu + 2\Sigma_\nu P_\nu,$$
$$(u - v)|_{(\partial X \times S^N)_-} = 0.$$

Hence, for a.e. ν, $\Omega \cdot \nabla_x (u - v)$ remains bounded in $L^{\infty}(X \times S^N)$ and, by Proposition 1, it shows that the family $(\tilde{u}(x))_{\Phi \in \mathscr{C}}$ is relatively compact in $L^2(X)$ (and thus in $L^1(X)$).

Third Step. Compactness of T_1 in $\mathscr{D}$.

Now, let us choose a dense family $(\nu_h)_{h \in \mathbb{N}}$ for which (33) holds. For any sequence $\Phi^n \in \mathscr{C}$ and $w^n = T_1 \Phi^n$, we may extract by a diagonal procedure (and using step 2) a subsequence, still denoted w^n, such that for some $w(x, \nu_h)$,

$$w^n(x, \nu_h) \xrightarrow[n \to \infty]{} w(x, \nu_h) \qquad \text{in } L^1(X) \quad \forall \nu_h.$$

Using the first step, we deduce that $w^n(x, \nu)$ converges in $L^1(X)$ to some $w(x, \nu)$ for a.e. ν, and thus, by dominated convergence, we obtain that

$$\int_{\mathbb{R}^+} \Sigma_\nu \int_X |w^n - w| \, dx \, d\nu \xrightarrow[n \to \infty]{} 0.$$

This proves the compactness of T_1 and it remains to prove Lemma 5.

Proof of Lemma 5. Let a sequence $w^n \in \mathscr{D}$ converge to w in L^1_Σ, then it converges a.e. (extracting subsequence). By compactness, for a.e. $x \in X$, we may extract from $\Psi^n = T_2(w^n)$ a subsequence $\Psi^n(x)$ which converges to some $\Psi(x) \in [0, \delta]$. Passing to the limit in (34), we obtain that $\Psi(x)$ is the unique solution of

$$\Psi(x) + \int_0^{\infty} \sigma_\nu(\Psi(x)) \, B_\nu(\Psi(x)) \, d\nu = \int_0^{\infty} \sigma_\nu(\Psi(x)) \, w(\nu, x) \, d\nu + g(x).$$

This shows that for a.e. x, the full sequence $\Psi^n(x)$ converges to $\Psi(x) = T_2 w$ in $L^1(X)$ and the proof of Lemma 2 is complete, thus proving Proposition 2.

III.2. *Singular Opacities*

In this section, we treat the case of a more general opacity than in Section III.1. Mainly we extend Proposition 2 to the case where $\sigma_\nu(T)$

blows up as $T \to 0$ ($\sigma_\nu(0) = +\infty$) in order to admit the Planck's function B_ν and Kramer's opacity σ_ν given in Remark (2).

We could give a set of general assumptions for which the following theorem holds, but these assumptions are rather technical and numerous. Thus we prefer to state our results for the particular case of Remark (2).

First, we introduce a "regularized" version of (28) by setting

$$\begin{aligned} &\lambda u_\alpha + \Omega \cdot \nabla_x u_\alpha + \sigma_\nu(T_\alpha)\, u_\alpha = \sigma_\nu(T_\alpha)\, B_\nu(T_\alpha) + f, \\ &\lambda T_\alpha + \int_{\mathbb{R}^+} \sigma_\nu(T_\alpha)\, B_\nu(T_\alpha)\, d\nu = \int_{\mathbb{R}^+} \sigma_\nu(T_\alpha)\, \tilde{u}_\alpha\, d\nu + g + \alpha. \end{aligned} \tag{28_α}$$

Since, in the case of Remark (2), $\int \sigma_\nu(T)\, B_\nu(T)\, d\nu$ is increasing and $\sigma_\nu(T)$ is decreasing, we obtain an a priori lower bound on T_α, $T_\alpha \geqslant T_{\min}(\alpha)$, where $T_{\min}(\alpha)$ is the solution of

$$\lambda T_{\min}(\alpha) + \int_0^\infty \sigma_\nu B_\nu(T_{\min}(\alpha))\, d\nu = \alpha.$$

This allows us to apply Proposition 2 and thus $(28)_\alpha$ has a solution (u_α, T_α) which satisfies

$$0 \leqslant u_\alpha \leqslant B_\nu(\delta + \alpha) = P_\nu, \qquad 0 \leqslant T_\alpha \leqslant \delta + \alpha, \tag{38}$$

when f, g, k satisfy (32).

Then, our goal is to prove the

THEOREM 5. *We assume that B_ν, σ_ν are given by* (B)–(C), *and that f, g, k satisfy* (32), *and that f and k are continuous in ν, for* a.e. ν, *uniformly for $x \in X$, $\Omega \in S^N$. Denote (u_α, T_α) the solution of $(28)_\alpha$ obtained through Proposition 2. Then we may extract from (u_α, T_α) a subsequence (u_n, T_n)* s.t. *$u_n \rightharpoonup u$ in $L^\infty(X \times S^N \times R^+)$ weak-$*$ and $T_n \to T$ in $L^1(X)$ and (u, T) is a solution of* (28). *(u, T) satisfy* (38) *and* (28) *holds in the sense that $\Omega \cdot \nabla_x u \in L^\infty(X \times S^N)$ for* a.e. ν *and $\sigma_\nu(T)\, u \in L^q(X \times S^N; L^1(R_\nu^+))$ for some $q > 1$.*

Remark. Again we use the convention $\sigma_\nu(0) \cdot 0 = 0$.

Proof. As before, we first prove some compactness and we pass to the limit in $(28)_\alpha$ using this compactness.

First Step. Compactness of $\int_\nu \tilde{u}_\alpha\, d\nu$.

Denoting $w_\alpha(x, \Omega) = \int u_\alpha\, d\nu$, our goal is to prove that $\tilde{w}_\alpha$ is compact in $L^1(X)$. w_α satisfies the equation

$$\Omega \cdot \nabla_x w_\alpha + \int_{\mathbb{R}^+} \sigma_\nu(T_\alpha)\, u_\alpha\, d\nu = \int_{\mathbb{R}^+} (\sigma_\nu B_\nu(T_\alpha) + f - \lambda u_\alpha)\, d\nu, \tag{39}$$

and, by Proposition 1, it is enough to prove that there exists $q>1$ such that

$$\Omega \cdot \nabla_x w_\alpha \in L^q(X \times S^N). \tag{40}$$

The right side of (39) is clearly in this set and we compute, for any $q>1$, $s>1$,

$$\begin{aligned}
&\int_{X \times S^N} \left(\int_{\mathbb{R}^+} \sigma_v(T_\alpha)\, u_\alpha dv \right)^q dx\, d\Omega \\
&\quad \leqslant \int_{X \times S^N} d\Omega\, dx \left[\int_{\mathbb{R}^+} \sigma_v^q(T_\alpha)\, u_\alpha dv \left(\int_{\mathbb{R}^+} u_\alpha^{q-1}\, dv \right)^{q-1} \right] \\
&\quad \leqslant C \int_{X \times \mathbb{R}^+} \sigma_v^q(T_\alpha)\, \tilde{u}_\alpha dv\, dx \\
&\quad \leqslant C \left[\int_{\mathbb{R}^+ \times X} \sigma_v(T_\alpha)\, \tilde{u}_\alpha \right]^{1/s} \left[\int_{\mathbb{R}^+ \times X} \sigma_v(T_\alpha)^{(qs-1)/(s-1)}\, B_v(\delta) \right]^{(s-1)/s} \\
&\quad \leqslant C \sup_{T \leqslant \delta} \left[\lambda T + \int_{\mathbb{R}^+ \times X} \sigma_v B_v(T)\, dv \right]^{1/s} \\
&\qquad \times \left[\int_{\mathbb{R}^+ \times X} \sigma_v(T)^{(qs-1)/(s-1)}\, B_v(\delta) \right]^{(s-1)/s}.
\end{aligned}$$

One can check using assumptions (B)–(C) that this remains bounded by choosing q, $s>1$ properly. In the following we call q such a choice. Then, we have obtained that $\tilde{w}_\alpha$ is compact in $L^1(X)$ and in any $L^p(X)$, $1 \leqslant p < +\infty$.

Second Step. Compactness of $\tilde{u}_\alpha(x, v)$.

We now prove that $\tilde{u}_\alpha$ is compact in $L^1(X \times \mathbb{R}^+)$.

Using step 1 and (38) (with $P_v \in L^1(\mathbb{R}^+)$) and step 1 of the proof of Lemma 4, it remains to estimate

$$\begin{aligned}
&\int_{X \times S^N} \int_{v \geqslant h} |u_\alpha(x, \Omega, v+h) - u_\alpha(x, \Omega, v)|\, dv\, dx\, d\Omega \\
&\quad \leqslant C \int_{R^+} \left\{ \sup_{x, \Omega} \left[|f_v - f_{v+k}| + |k_v - k_{v+h}| \right] \right. \\
&\qquad \left. + \sup_{T \leqslant \delta} |\sigma_v B_v(T) - \sigma_{v+h} B_{v+h}(T)| \right\} dv \\
&\qquad + \frac{2}{\lambda} \int_{v \geqslant h} \int_X |\sigma_{v+h}(T_\alpha) - \sigma_v(T_\alpha)|\, \tilde{u}_\alpha(x, v)\, dv\, dx.
\end{aligned}$$

Thus there exists a modulus of continuity ρ such that the right-hand side of this inequality is bounded by

$$\rho(h)+C\int_{v}\int_{\{x,\,T_\alpha(x)\geqslant\varepsilon\}}|\sigma_{v+h}(T_\alpha)-\sigma_v(T_\alpha)|\,P_v\,dv\,dx$$

$$+C\int_{v\geqslant h}\int_{\{x,\,T_\alpha(x)\leqslant\varepsilon\}}\sup_{T\leqslant\varepsilon}\left|\frac{\sigma_{v+h}-\sigma_v}{\sigma_v}\right|(T)\,\sigma_x(T_\alpha)\,\tilde{u}_\alpha\,dv\,dx$$

$$\leqslant\rho(h)+C\int_{\mathbb{R}^+}\sup_{\varepsilon\leqslant T\leqslant\delta}|\sigma_{v+h}(T)-\sigma_v(T)|\,P_v\,dv$$

$$+C\sup_{\substack{T\leqslant\varepsilon\\ v\geqslant h}}\left|\frac{\sigma_{v+h}-\sigma_v}{\sigma_v}\right|(T)\left(\lambda\varepsilon+\int_{\mathbb{R}^+}\sigma_v(\varepsilon)\,B_v(\varepsilon)\,dv\right)$$

$$\leqslant\rho(h)+\rho'(\varepsilon,h)+\rho''(\varepsilon),$$

where $\rho''(\varepsilon)\to_{\varepsilon\to 0}0$, $\forall\varepsilon$, $\rho'(\varepsilon,h)\to_{h\to 0}0$ (use the special form (B)–(C)). Thus we have obtained

$$\int_X\int_{v\in\mathbb{R}^+}|\tilde{u}_\alpha(x,v+h)-\tilde{u}_\alpha(x,v)|\,dv\ \,dx$$

$$\leqslant 2\int_{v\leqslant h}P_v\,dv+\rho(h)+\inf_\varepsilon\{\rho''(\varepsilon)+\rho'(\varepsilon,h)\}\xrightarrow[h\to 0]{}0$$

and have proved the compactness of $\tilde{u}_\alpha$ in $L^1(X\times R^+)$.

Third Step. Compactness of T_α.

Therefore, we may extract from $\tilde{u}_\alpha$ a subsequence denoted $\tilde{u}_n$ which converges a.e. to $v(x,v)$. Since we have

$$\lambda T_n+\int_{\mathbb{R}^+}\sigma_v B_v(T_n)\,dv=\int_{\mathbb{R}^+}\sigma_v(T_n)\,\tilde{u}_n(x,v)\ dv+g(x),$$

we check, as in the proof of Lemma 5, that $T_n(x)$ converges to $T(x)$ (for a.e. x) the unique solution of

$$\lambda T+\int_{\mathbb{R}^+}\sigma_v B_v(T)\,dv=\int_{\mathbb{R}^+}\sigma_v(T)\,v(x,v)\,dv+g(x).\tag{41}$$

Fourth Step. Passage to the limit.

Following Section I, we may extract from u_n a sequence (still denoted u_n) such that

$$
\begin{aligned}
u_n(x, \Omega, \nu) &\to u(x, \Omega, \nu) && \text{in } L^\infty(X \times S^N \times \mathbb{R}^+) \text{ weak-}*,\\
\tilde{u}_n(x, \nu) &\to \tilde{u}(x, \nu) = v(x, \nu) && \text{in } L^1(X \times \mathbb{R}^+),\\
\Omega \cdot \nabla_x u_n &\to \Omega \cdot \nabla_x u && \text{in } \mathscr{D}'(X \times S^N \times \mathbb{R}^+),\\
\sigma_\nu B_\nu(T_n) &\to \sigma_\nu B_\nu(T) && \text{in } L^1(X \times \mathbb{R}^+).
\end{aligned}
$$

Moreover, we know that (41) holds and it remains to prove that $\sigma_\nu(T_n)\, u_n$ converges to $\sigma_\nu(T)\, u$.

Since $\sigma_\nu(T_n)\, u_n$ is nonnegative and bounded in $L^1(X \times S^N \times \mathbb{R}^+)$, it converges weakly to a nonnegative measure m on $X \times S^N \times \mathbb{R}^+$. On the other hand, for any $M > 1$, $\inf(M, \sigma_\nu(T_n))\, \mathbb{1}_{\{T>0\}}$ converges, as n tends to infinity, to $\inf(M, \sigma_\nu(T))\, \mathbb{1}_{\{T>0\}}$ for a.e. x, ν. Thus, for any $M' > 1$, we have

$$
\sigma_\nu(T_n)\, u_n \geqslant \inf(M, \sigma_\nu(T_n))\, u_n\, \mathbb{1}_{\{T>0\}} \xrightarrow{n \to \infty} \inf(M, \sigma_\nu(T))\, u
$$
$$
\text{weakly in } L^p(X \times S^N \times [0, M']),\ 1 < p < \infty.
$$

Therefore, we have obtained (sending M to infinity)

$$
m \geqslant \sigma_\nu(T)\, u\, \mathbb{1}_{T>0} = \sigma_\nu(T)\, u,
$$

but, since

$$
\int_{S^N} m\, d\Omega \leqslant \lim_{n \to \infty} \sigma_\nu(T_n)\, \tilde{u}_n = \sigma_\nu(T)\, \tilde{u},
$$

we have $m = \sigma_\nu(T)\, u$ and Theorem 5 is proved.

References

1. G. Albertoni and B. Montagnini, On the spectrum of neutron transport equation in finite bodies, *J. Math. Anal. Appl.* **13** (1966), 19–48.
2. C. Bardos, F. Golse, and B. Perthame, The Rosseland approximation for the radiative transfer equations, *Comm. Pure Appl. Math.* **40** (1987), 691–721.
3. C. Bardos, R. Santos, and R. Sentis, Diffusion approximation and computation of the critical size, *Trans. Amer. Math. Soc.* **284** (2) (1984), 617–649.
4. P. Benilan, "Equations d'évolution dans un espace de Banach quelconque et applications," Thesis, Orsay, 1972.
5. P. Benilan, H. Brézis, and M. G. Crandall, A semilinear equation in $L^1(R)$, *Ann. Scuola Norm. Sup. Pisa* **2** (1975), 523–555.
6. M. Bertsch, Thesis, Leyden, 1984.
7. A. Bensoussan, J. L. Lions, and G. C. Papanicolaou, Boundary layers and homogenization of transport processes, *J. Publ. Res. Inst. Math. Sci. Kyoto* **15** (1979), 53–157.
8. K. M. Case and P. F. Zweifel, "Linear Transport Theory," Addison–Wesley, Reading, M.A.

9. M. CESSENAT, Théorème de trace pour des espaces de fonctions de la neutronique, *C. R. Acad. Sci. Paris* **300** (1)(1985), 89–92.
10. S. CHANDRASEKHAR, "Radiative Transfer," Dover, New York, 1960.
11. M. G. CRANDALL AND T. S. LIGETT, Generation of semigroups of nonlinear transformations on general Banach spaces, *Amer. J. Math.* **93** (1971), 265–297.
12. R. DAUTRAY AND J. L. LIONS, "Analyse mathématique et calcul numérique pour les sciences et les techniques," Vol. III, Masson, Paris, 1985.
13. L. C. EVANS, Applications of nonlinear semigroup theory to certain partial differential equations, *in* "Nonlinear Evolution Equations" (M. G. Crandall, Ed.), Academic Press, Orlando, FL (1978).
14. H. FRISCH AND M. FAUROBERT, Boundary layer conditions for the transport of radiation in stars, *Astron. Astrophys.* **140** (1984).
15. F. GOLSE, The Milne problem for the radiative transfer equations (with frequency dependence), *Trans. Amer. Math. Soc.* **303** (1987), 125–143.
16. F. GOLSE, P. L. LIONS, B. PERTHAME, AND R. SENTIS, Regularity of the moments of the solution of a transport equation, *J. Funct. Anal.* **16** (1988).
17. F. GOLSE AND B. PERTHAME, Generalized solutions of the radiative transfer equations in a singular case, *Comm. Math. Phys.* **106** (1986), 211–239.
18. F. GOLSE, B. PERTHAME, AND R. SENTIS, Un résultat de compacité pour les équations de transport et application au calcul de la limite de la valeur propre principale d'un opérateur de transport, *C. R. Acad. Sci. Paris, Sér. I Math.* **301** (1985), 341–344.
19. M. MOKHTAR KHARROUBI, "Propriétés spectrales de l'opérateur de transport dans le cas anisotrope," Thèse, Université Paris VI, Paris, 1982.
20. E. W. LARSEN AND J. B. KELLER, Asymptotic solutions of neutrons transport problem, *J. Math. Phys.* **15** (1974), 75–81.
21. E. W. LARSEN, G. C. POMRANING, AND V. C. BADHAM, Asymptotic analysis of radiative transfer problems, *J. Quant. Spectros. Radiat. Transfer* **29** (1983), 285–310.
22. B. MERCIER, Applications of accretive operators theory to the Radiative Transfer equations, *SIAM J. Math. Anal.* **18** (1987), 393–408.
23. B. W. MIHALAS AND D. MIHALAS, "Foundations of Radiative Hydrodynamics," Pergamon, New York, (1984).
24. L. A. PELETIER, The porous media equation, *in* "Applications of Nonlinear Analysis in the Physical Sciences" (H. Amann, N. Bazley, and K. Kirchgässner, Eds.), pp. 229–241, Pitman, Boston, 1981.
25. G. C. POMRANING, "Radiation Hydrodynamics," Pergamon, Elmsford, NY, (1983).
26. R. SENTIS, Thèse, Université Paris IX, Paris, 1981.
27. R. SENTIS, Half-space problems for frequency dependent transport equation. Application to the Rosseland approximation, *J. Transport Theory Stat. Phys.* **16** (1987), 653–700.
28. I. VIDAV, Spectrum of perturbed semigroups with application to transport theory, *J. Math. Anal. Appl.* **30** (1970), 244–279.