Lecture 1 | Rennes June 2015 | (1)

Aim of this series of talks: introduce the notion of b-divisors and explain one application of it to the study of the asymptotic properties of linear series and more precisely to the volume function on the Néron-Severi space of an algebraic variety

TALK 1 : b-divisors = definition and examples.

(1) Basics.

$X^d / \mathbb{C}$ ~~proj~~ projective normal variety (reduced separated irreducible)

Weil divisor $Z = \sum a_i [Z_i]$

Z_i codimension 1 subspace

$a_i \in \mathbb{Z}$

free abelian group $:= W(X)$

$W_{\mathbb{R}}(X) = W(X) \otimes_{\mathbb{Z}} \mathbb{R}$ $\mathbb{R}$-vector space infinite dimension

Cartier divisor

$Z \in C(X) \subseteq W(X)$ is Cartier if locally in any affine chart U one has $Z|_U = \mathrm{div}(\varphi)$ $\varphi \in \mathbb{C}(U)$ rational function.

$C_{\mathbb{R}}(X) = C(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

observation . X smooth then $C(X) = W(X)$

. X has mild singularities (quotient ones, $\mathbb{Q}$-factorial)

$C_{\mathbb{Q}}(X) = W_{\mathbb{Q}}(X)$

in general $C_{\mathbb{R}}(X) \subsetneq W_{\mathbb{R}}(X)$.

Natural transformations

X', X projective normal variety same dimension

$f : X' \to X$ regular dominant map.

- $f_* : W(X') \to W(X)$ group morphism / linear map $W_{\mathbb{R}}(X') \to W_{\mathbb{R}}(X)$

$$f_*(\sum a_i [Z_i]) = \sum a_i f_*[Z_i].$$

if $\operatorname{codim} f(Z_i) < 1$ $\quad f_*[Z_i] = 0$.

if $\operatorname{codim} f(Z_i) = 1$ $\quad f_*[Z_i] = e_i [f(Z_i)]$

where $e_i = [\mathbb{C}(Z_i) : \mathbb{C}(f(Z_i))]$

$=$ topo. degree of $f : Z_i \to f(Z_i)$. ≥ 1

- $f^* : C(X) \to C(X')$. U affine chart $Z|_U = \operatorname{div}(\varphi)$.

$$f^* Z|_{f^{-1}U} = \operatorname{div}(\varphi \circ f).$$

$\boxed{2}$ b-divisors.

Heuristic: a b-divisor is a divisor that arises in some birational model of X. They arise in two ways depending on whether one works with Cartier or Weil divisors. To define these objects properly I need to introduce some more terminology.

Set of models

$$\mathcal{B}(X) = \left\{ \begin{array}{c} \text{birational maps } \pi : X_\pi \to X \\ X_\pi \text{ projective smooth} \end{array} \right\} \Big/ \text{up to isomorphism}$$

an element in $\mathcal{B}(X)$ is a " smooth birational model of X ".

remark: this is a set since it is a quotient of the set of all coherent sheaves of ideals of X $\qquad \mathfrak{a} \rightsquigarrow \pi_{\mathfrak{a}} =$ blow-up of $\mathfrak{a}$.

• <u>domination relation</u> on $\mathcal{B}(X)$

$\pi' \geq \pi$ if

$$\begin{tikzcd} X_{\pi'} \arrow[rr, dashed, "\mu"] \arrow[dr] & & X_\pi \arrow[dl] \\ & X & \end{tikzcd}$$

$\mu = \pi^{-1} \circ (\pi')$ is a <u>regular</u> map.

<u>Fact</u> ~~given~~ $\mathcal{B}(X)$ is an directed poset

given any $\pi, \pi' \in \mathcal{B}(X)$ there exists $\pi'' \geq \pi$ and $\pi'' \geq \pi'$.

* Look at the graph $\Gamma \subseteq X_{\pi'} \times X_\pi$ of μ and pick any desingularization of Γ. (not unique).
* In arbitrary characteristic $\pi = \pi_{\mathfrak{a}}$. $\pi' = \pi_{\mathfrak{b}}$ may take $\pi'' = \pi_{\mathfrak{a}\mathfrak{b}}$. $\square$.

<u>Weil b-divisor</u>

it is a function $\pi \in \mathcal{B}(X) \longmapsto Z_\pi \in W(X)$

with compatibility $\mu_* Z_{\pi'} = Z_\pi$ if $X_{\pi'} \xrightarrow{\mu} X_\pi$

$\rightarrow$ $b\text{-}W(X)$ = set of all b-divisors $b\text{-}W_{\mathbb{R}}(X) = b\text{-}W(X) \otimes_{\mathbb{Z}} \mathbb{R}$!

~~alternative way~~

~~$Z \in b\text{-}W(X)$~~

~~a) take a Weil divisor $Z_X \in W(X)$~~

~~b) for any $\pi \in \mathcal{B}(X)$ write $Z_\pi = \pi^{-1} Z_X + E_\pi$~~

~~strict transform~~ ~~supported on the exceptional locus of π~~

<u>alternative way</u> = a divisorial valuation is a function

$v : \mathbb{C}(X) \rightarrow \mathbb{Z}$ such that $v(f) = \mathrm{ord}_E(f)$ for some irreducible hypersurface E in some X_π.

$b\text{-}W(X) \Leftrightarrow \mathbb{Z}$-valued function g_Z on the set of all divisorial valuations on X

with a finiteness condition: for all π $\{E$ hyp. of $X_\pi \mid g_Z(\mathrm{ord}_E) \neq 0\}$ is fini

⚠ This is an interesting point of view especially when one can extend g_Z to more general valuations.

Prop | $\mu : X' \to X$ birational map.
$\mathcal{B}(X') \subseteq \mathcal{B}(X)$.
$\mu_* : W(X') \to W(X)$ is an isomorphism

<u>Main example of Weil b-divisor</u> = Cartier b-divisor.

$Z \in b\text{-}W(X)$ is Cartier if there exists $\pi_0 \in \mathcal{B}(X)$
s.t. for all $\pi \geq \pi_0$ $\quad Z_\pi = \mu^* Z_{\pi_0}$.

where $\quad X_\pi \xrightarrow{\mu} X_{\pi_0}$.

$b\text{-}C(X) \subseteq b\text{-}W(X)$
$= \{$set of all Cartier b-divisors$\}$.

Prop One has a natural map
$C(X_\pi) \xrightarrow{i_\pi} b\text{-}C(X)$ that is an injection

Prop $Z \in C(X_\pi) \qquad Z' \in C(X_{\pi'})$
$i_\pi(Z) = i_{\pi'}(Z')$ in $b\text{-}C(X)$
iff there exists $\pi'' \geq \pi, \pi'$

$$\begin{tikzcd} & X_{\pi''} \arrow[dl, "\mu'"'] \arrow[dr, "\mu"] & \\ X_{\pi'} \arrow[dr] & & X_\pi \arrow[dl] \\ & X & \end{tikzcd} \qquad \mu^* Z_\pi = (\mu')^* Z_{\pi'}.$$

<u>Obs.</u> $\mu : X' \to X$ birational map proper $C(X') \simeq C(X)$

⚠ Not all Weil b-divisors are Cartier

Fix a canonical divisor $K_X = \mathrm{div}(\omega)$ on X, ω meromorphic d-form.
For any π, set $K_{X_\pi} = \mathrm{div}(\pi^*\omega)$ on X_π.

Then $\pi_* K_{X_\pi} = K_X$

$\{K_{X_\pi}\}$ is a Weil b-divisor that is <u>not</u> Cartier.

$$X_{\pi'} \xrightarrow[\mu]{\text{blow-up of } p} X_\pi \qquad K_{X_{\pi'}} = \mu^* K_{X_\pi} - E_p$$

$$E_p = \mu^{-1}\{p\}.$$

③ <u>Historical comments</u> not me who introduced first the notion of b-divisor but appeared in the literature of the russian school

∘ <u>Manin</u> Cubic forms 70's.
he used the notion to study birational transformations of smooth cubic surfaces over perfect fields

∘ <u>Shokurov</u> preliminary flips 3 fold log models (see Ambro)
he introduced the notion in his work on the MMP. Not crucial but useful to define and study discrepancy (compare K_{X_π} and $\pi^* K_X$).
used to further define classes of singularities.

Main applications ⇝ <u>dynamical</u> systems [Cantat, BFJ]
used to study the growth of degrees of a rational map, the point of working with all birational models and divisors on them gives a way to eliminate all pts of indeterminacy of a rational map.

• geometry of linear series ⤳ explain this in more detail.

b-divisor is the natural language to speak of Zariski decomposition in dimension ≥ 3.

[BFJ – Kuronya Maclean]

$\boxed{4}$ b-classes

In most applications we shall need to work with a smaller set than $b\text{-}W(X)$ or $b\text{-}C^1(X)$.

$\boxed{\text{def}}$ $NS(X) = C^1(X) / _{\text{numerical equivalence}}$ $\qquad Z \equiv Z'$ iff $Z \cdot C = Z' \cdot C$ for all curve $C \subset X$

Abelian group that is finitely generated

$NS_{\mathbb{R}}(X) = NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$.

obs: X is smooth. $Z \longmapsto$ fundamental class $\in H^{1,1}(X) \cap H^2(X, \mathbb{Z})$.

Lefschetz 1,1 thm $NS_{\mathbb{R}}(X) = H^{1,1}(X) \cap H^2(X, \mathbb{R})$.

def] Weil b-class

$\alpha = \{\alpha_\pi\}_{\pi \in \mathcal{B}(X)}$ $\qquad \mu_* \alpha_{\pi'} = \alpha_\pi$ $\qquad X_{\pi'} \xrightarrow{\mu} X_\pi$

Cartier b-class

$\alpha \in b\text{-}NS^1(X)$ s.t. $\exists \pi_0$ $\qquad \alpha_\pi = \mu^* \alpha_{\pi_0}$ for all $\pi \geq \pi_0$ $\qquad X_\pi \xrightarrow{\mu} X_{\pi_0}$.

$\boxed{\text{Exp}}$ $NS^1_{\mathbb{R}}(X_\pi) \xrightarrow{i_\pi} CNS_{\mathbb{R}}(X)$ is an injection.

$CNS_{\mathbb{R}}(X) = \bigcup_{\pi \in \mathcal{B}(X)} i_\pi(NS_{\mathbb{R}}(X_\pi))$.

~~$b\text{-}NS_{\mathbb{R}} \xrightarrow{\lambda_\pi} NS_{\mathbb{R}}(X_\pi)$ surjective~~.

One can translate the definition by saying.

$$b\text{-}NS_{\mathbb{R}}(X) = \varprojlim_{\pi \in \mathcal{B}(X)} NS(X_\pi) \qquad \text{pull-back maps}$$

$$C\text{-}NS(X) = \varinjlim_{\pi \in \mathcal{B}(X)} NS(X_\pi) \qquad \text{push-forward maps.}$$

Lecture 2 The Riemann-Roch problem (14:30)

① volume of line bdls
② volume on the NS
③ convex cone in NS
④ example

In this talk we shall forget about b-divisors and discuss an old and basic problem in algebraic geometry.

$X^d/\mathbb{C}$ projective smooth $L \to X$ line bundle.

<u>question</u> = compute $h^0(nL) = \dim_{\mathbb{C}} H^0(X, L^{\otimes n})$.

- $\to$ give a formula for all n
- $\to$ describe the asymptotics when $n \to \infty$.

in 1D $\deg(L) = \#$ zeros $- \#$ poles of a meromorphic section.

- $\deg(L) < 0$ $h^0(nL) = 0$.
- $\deg(L) = 0$ $h^0(nL) \in \{0,1\}$. is a periodic function
- $\deg(L) > 0$ $\cdot h^0(nL) = 1 - g(X) + n \deg(L)$ if $n \deg(L) > 2g(X) - 2$.

in 2D problem was solved by Zariski (contribution by Cutkosky-Srinivas) in a celebrated paper where he introduced what is called now the Zariski decomposition ~~...~~

$\boxed{\text{thm}}$ $h^0(nL) = P(n) + \lambda(n)$ for all $n \gg 0$.

- $P =$ quadratic polynomial with rational coefficients
- $\lambda =$ periodic function

in 3D we shall see that the situation is much more complicated.
We shall thus focus our attention to the growth rate of $h^0(nL)$ which is usually measured by the <u>volume</u> of L. This will define a real-valued function on $NS(X)$ and our aim is to study this function.

[I] Volume of a line bundle

recall

$NS(X) = \mathrm{Pic}(X)/$ num equiv. Abel. gp. f-g
$NS_{\mathbb{R}}(X) = NS(X) \otimes_{\mathbb{Z}} \mathbb{R}$ f-dim $\subset H^{1,1} \cap H^2(X, \mathbb{R})$

Recall $L \to X$ line bundle

$c_1(L) =$ class in $NS_{\mathbb{R}}(X)$ of the divisor of poles and zeros of a ~~[illegible]~~ rational section of L.

equivalently = put a smooth Hermit- metric on L. Take its curvature = get a closed $(1,1)$ smooth form $\Theta = c_1(L)$ = cohomological class of Θ in $H^2(X)$.

Intersection products

- using cup-products: $c_1(L)^d := \int_X \Theta^d \in \mathbb{R}$.
- algebraically more involved.

$$\chi(X, L) = \sum_0^d (-1)^k \dim_{\mathbb{C}} H^i(X, L)$$

for $n \gg 0$ $\quad \chi(X, L^{\otimes n}) = \frac{c_1(L)^d}{d!} n^d + O(n^{d-1})$.

Ample line bundle "analog of divisor of positive degree in arbitrary dimension".

$\hookrightarrow$ cohomological definition $H^i(X, \mathcal{F} \otimes L^{\otimes n}) = 0 \quad \forall \mathcal{F}\ \forall i > 0\ \forall n \gg 0$

Cor $h^0(nL) = \frac{c_1(L)^d}{d!} n^d + O(n^{d-1})$ when L is ample

Volume $L \to X$ arbitrary.

$$\mathrm{vol}(L) = \limsup_{n \to \infty} \frac{h^0(nL)}{n^d} \times d!.$$

Fact
- $L \geq L'$ (in the sense that $L - L'$ admits a regular section) $\Rightarrow \mathrm{vol}(L) \geq \mathrm{vol}(L')$
- $\mathrm{vol}(aL) = a^d \, \mathrm{vol}(L)$.

[2] The volume on the Néron-Severi space

I would like to spend some time now explaining.

Thm | • If $c_1(L) = c_1(L')$ ie $L \equiv L'$ then $\mathrm{Vol}(L) = \mathrm{vol}(L')$.

In particular vol induces a function on $NS_{\mathbb{Q}}(X) \subseteq NS_{\mathbb{R}}(X)$.

Thm | Vol extends to a unique ~~function~~ continuous and homogeneous function on $NS_{\mathbb{R}}(X)$

<u>idea of proof</u>.

* Fujita vanishing theorem (extension of vanishing of cohomology to a larger class of line bundles than ample ones)

$\Rightarrow \exists N$ s.t. for all $\mathcal{L} \equiv 0$ $\quad H^0(N+\mathcal{L}) \neq 0$.

* $L' = L + \mathcal{L} \qquad \mathcal{L} \equiv 0$.

$aL + N \geq aL + a\mathcal{L}$ for all a.

$\mathrm{Vol}(aL+N) \geq a^d \, \mathrm{Vol}(L+\mathcal{L})$.

* for all $\varepsilon > 0$ $\quad |\mathrm{vol}(aL+N) - \mathrm{vol}(aL)| \leq \varepsilon a^d$.

write $N = N' - N''$ $\quad N', N''$ ample (very ample)

may assume $-N$ is very ample

<u>key exact sequence</u> work in terms of sheaves $L = \mathcal{O}_X(D)$ $-N = \mathcal{O}_X(E)$.

may assume E smooth having no component in common with D

$$0 \to \mathcal{O}_X(D-E) \xrightarrow{\times E} \mathcal{O}_X(D) \xrightarrow{\text{rest.}} \mathcal{O}_E(D) \to 0$$

$\ker(\text{rest.})(U) = \{ f \in \mathcal{E}(U) \quad \mathrm{div}(f) \geq -D \quad f|_E = 0 \}$.

$\Rightarrow \exists g \in \mathcal{E}(U)$ $\mathrm{div}(g) \geq -D+E$ $\quad f = g\,\varphi_E$

$\mathrm{div}(\varphi_E) = E$.

More generally $E_1 \dots E_p \in |N|$.

$$0 \to \mathcal{O}_X(apD - pE) \to \mathcal{O}_X(apD) \to \bigoplus_{i=1}^{p} \mathcal{O}_{E_i}(apD) \to 0$$

$$h^0(apD - pE) - h^0(apD) + p\, h^0(E, apL) \geq 0$$

$$0 \geq h^0(p(aL-N)) - h^0(p\,aL) \geq -p\, c^{te}\, (ap)^{d-1}$$

$$0 \geq vol(aL-N) - vol(aL) \geq -c^{te}\, a^{d-1}$$

remark = continuity statement ~~s~~ of the same flavor.

[3] Convex cones in $NS(X)$

* ample class $\alpha \in NS_{\mathbb{R}}(X)$ iff L_i ample $a_i > 0$ $\alpha = \sum_{i=1}^{k} a_i c_1(L_i)$

$vol(\alpha) = \alpha^d$. α ample

* nef class α nef iff $\exists\, \alpha_n$ ample $\alpha_n \to \alpha$.

$vol(\alpha) = \alpha^d$ α nef

discussion : ample and nef classes are closely intertwined

+ thm: α ample iff for Z dim k $\alpha^k \cdot [Z] = \int_Z \alpha^k > 0$.
[Nakai-Moishezon] induction on dimension + cohomological characterization of ampleness.

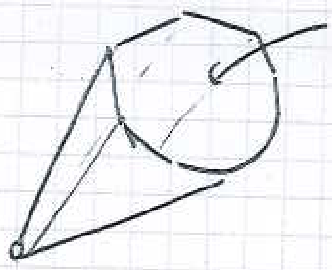
+ thm. α nef iff for all curve $\alpha \cdot [C] \geq 0$.
[Kleiman]

$\rightsquigarrow$ $\alpha_n \cdot C > 0 \Rightarrow \alpha \cdot C \geq 0$

$\rightsquigarrow$ $\alpha \cdot [C] \geq 0\ \forall C \Rightarrow \alpha^k \cdot [Z] \geq 0$ for Z irred. dim k (induction on dim).

$\alpha_n = \alpha + \frac{1}{n}\beta$ β fixed ample class □.

Picture



ample / nef cone are strictly convex.

~~[illegible]~~.

closure (ample) = nef cone

interior (nef) = ample con

$vol(\alpha) = \alpha^d$ polynomial.

* the big cone

α is a big class iff $vol(\alpha) > 0$.

To understand more geometrically the meaning of this condition I need to recall a few facts ~~from~~ about Iitaka fibration

$L \to X$ arbitrary line bundle ~~set~~ $N(L) = \{n \in \mathbb{N},\ h^0(L^{\otimes n}) \neq 0\}$.

fix $n \in N(L)$ for any $\sigma \in H^0(nL)$ look at $div(\sigma)$ = zero divisor

~~[illegible]~~ Fixed part $F_n := \max\{$ effective integral divisor $\leq div(\sigma)$ for all $\sigma\}$.

Not difficult to see that

$div(\sigma) = M(\sigma) + F_n$ and outside a proper alg. subset of $H^0(nL)$

$M(\sigma)$ has no component in common with F_n, $\bigcup M(\sigma) = X$.

if $\dim H^0(nL) \geq 2$

$$X \overset{\Phi_{n,L}}{\dashrightarrow} \mathbb{P}(H^0(X, L^{\otimes n})).$$ well defined on $X \setminus |F_n|$.

$$x \longmapsto [\{\sigma,\ \sigma(x) = 0\}].$$

Thm / i) for all $n \in N(L)$ large enough

$\dim \Phi_{n,L}(X) \equiv cst =: \kappa(L)$ Iitaka dimension

ii) $\exists\, a, b > 0$ st.

$$a\, n^{\kappa(L)} < h^0(nL) < b\, n^{\kappa(L)}$$

Cor : L is big $\Leftrightarrow$ $\kappa(L) = d$.

Thm α is big $\iff$ $\exists$ a ample class and e effective st.

$$\alpha = a + e.$$

only $\Rightarrow$] ~~~~ $\alpha = c_1(L)$ take A ample line bundle.

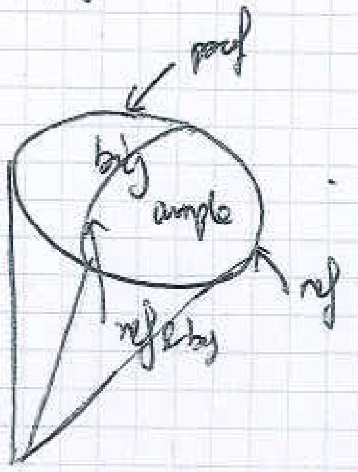
claim $h^0(nL - A) \neq 0$ for $n \gg 0$.

$$0 \to \mathcal{O}_X(nL-A) \to \mathcal{O}_X(nL) \to \mathcal{O}_A(nL) \to 0.$$

$$h^0(nL-A) \geq h^0(nL) - h^0_A(nL) \xrightarrow{\text{asympt. RR}} \leq$$

$$\geq \text{cte}\, n^d - \text{cte}\, n^{d-1}$$

def psef cone: α psef $\iff$ $\exists\, \alpha_n$ big $\to \alpha$.



interior (psef) = big.
closure (big) = psef
obviously convex cone.
(complex argument $\alpha = \{T\}$ $T \geq 0$ closed $(1,1)$
$\pm$ psef $\Rightarrow \pm\|T\| \geq 0$).

$\boxed{\text{Thm}}$ The volume function is $\mathcal{C}^1$ in the big cone.

4 Examples

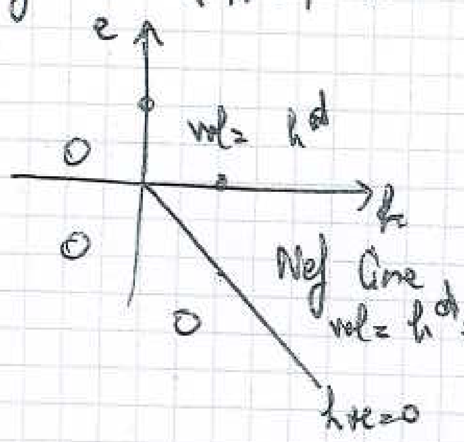
For interesting example need to assume dim NS ≥ 2 and not all classes are nef. Rapidly you end up with delicate computations.

a) Blow-up of $\mathbb{P}^d$ at a single point. $\pi : X \to \mathbb{P}^d$

$H = \pi^* \mathcal{O}(1)$ $\quad E$ = exceptional divisor.

$NS = \mathbb{R} H \oplus \mathbb{R} E.$ $\quad \alpha = hH + eE$

$\mathrm{Psef} = \langle H - E, E \rangle$



continuous vol: $d\mathrm{vol} = (d h^{d-1}, \pm d e^{d-1}) \neq 0$ on $\{h+e\} = 0$.

~~b) nice behaviour of the volume: Mori dream spaces~~

b) line bundle with f.g. ring of sections

$L \to X$ $\quad R[L] = \bigoplus_{n \in \mathbb{N}} H^0(X, L^{\otimes n})$ $\quad \mathbb{C}$-algebra

$L = \mathcal{O}_X(D)$ $\quad \begin{matrix} \mathrm{div}(f_n) \geq -nD \\ \mathrm{div}(f_m) \geq -mD \end{matrix} \Rightarrow \mathrm{div}(f_n f_m) \geq -(n+m) D.$

Assume $R(L)$ is finitely generated as an algebra: Hilbert's thm $\Rightarrow \mathrm{vol}(L) \in \mathbb{N}$.

$\varphi_L : X \dashrightarrow \mathrm{Proj}\, R(L)$

Concretely suppose $\sigma_0 - \sigma_k$ generate $R(L)$ (for simplicity $\in H^0(L)$).

$\mathcal{I}_L$ = ideal generated by homogeneous polynomials of degree ℓ

P s.t. $P(\sigma_0 - \sigma_k) = 0 \in H^0(L^{\otimes \ell})$

$\mathrm{Proj}\, R(L) = V(J_L) \subseteq \mathbb{P}^{\ell}$.

$\varphi_L(x) = [\sigma_0(x) : \cdots : \sigma_\ell(x)] \in V(J_L)$.

$$\begin{tikzcd} X' \arrow[d, "\pi"'] \arrow[dr, "\mu"] & \\ X \arrow[r, dashed] & \mathrm{Proj}\, R(L) \end{tikzcd} \qquad \pi^* L = \mu^*(A) + E, \qquad h^0(kA) = h^0(kL)$$

$\Rightarrow \mathrm{vol}(L) \in \mathbb{Q}_+$.   rem: big nef + $R(L)$ f.g. $\Rightarrow$ semi-ample by Zariski

c) Mori dream spaces (Hu–Keel)

space where all rings of sections are f.g. ($\mathbb{Q}$-factorial)

(i) $\mathrm{Pic}_{\mathbb{Q}}(X) \cong NS_{\mathbb{Q}}(X) = \bigoplus_{i=1}^{k} L_i \mathbb{Q}$

(ii) $\mathrm{Cox}(X) = \bigoplus_{v \in \mathbb{Z}^k} H^0(X, L^v)$ is f.g.

[proof by ELMNP asymptotic invariants of base loci]

Thm X Mori dream space

~~there exists $NS(X)$ is rational polyhedral cone in~~

- Then $\mathrm{Psef}(X) \subseteq NS(X)$ is a rational polyhedral convex cone in $NS(X)$
- there exists a finite subdivision of $\mathrm{Psef}(X)$ into rational polyhedral convex cones $Q_1 \ldots Q_N$ such that for each i ~~there exists a birational map $f_i : X \dashrightarrow X_i$ (or in codim 1)~~ $\mathrm{vol}|_{Q_i}$ is polynomial of degree d.

c') ex) Toric varieties are Mori dream spaces: in this case

$L \rightsquigarrow h_L \in PL(\Delta) \rightsquigarrow g_L = \sup\{g \text{ convex} \leq h_L\} \in PL(\Delta') \cap \text{Convex}$

$\Delta(g_L) = \text{polytope} \qquad \mathrm{vol}(L) = \frac{1}{d!}\mathrm{vol}(\Delta(g_L))$

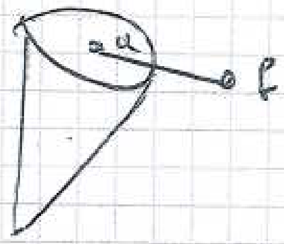
a) bad examples = Cutkosky Zariski

~~Küronya~~

dim 3 $\qquad \mathrm{vol}(L) \notin \mathbb{Q}$

E elliptic curve general $\qquad V = E \times E$

$$\overline{\mathrm{Eff}}(V) = \mathrm{Nef}(V) = \overline{\{\alpha^2 > 0\}} \in NS(V) = \mathbb{R}^3.$$



$$X = \mathbb{P}\big(\mathcal{O}(A) \oplus \mathcal{O}(B)\big)$$

$$L = \mathcal{O}(1)$$

$\boxed{\text{Lecture 3}}$ (1h30)

① Zariski decomposition: surface case
② C^1-diff. in surface case
③ Zariski decomp. in higher dimension

Recall $X^d / \mathbb{C}$ smooth projective $L \to X$ line bundle $\mathrm{vol}(L) = \limsup \dfrac{h^0(nL)}{n^d/d!}$

vol extends to $NS_{\mathbb{R}}(X)$ as a continuous function.

$\mathrm{Big}(X) = \{\mathrm{vol} > 0\} = \{$ a.i.e., a ample + effective $\}$

remark: ~~[illegible]~~ $=3$ ~~[illegible]~~ (D): vol is not C^2.

Thm: $\mathrm{vol} : \mathrm{Big}(X) \to \mathbb{R}_+^{\times}$ is C^1-differentiable.

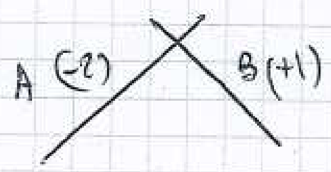
I The surface case: the Zariski decomposition

Statement = key statement to answer the Riemann-Roch problem for surfaces

$L \in \mathrm{Pic}(X)$ $L \to X$ effective $L = \mathcal{O}_X(D)$ $D = \sum_{i=1}^{k} a_i D_i$ $a_i \geq 0$.
(D_i irreducible.)

Thm $D = P + N$ unique decomposition $P = \sum p_i D_i$ $p_i \in \mathbb{Q}$ $N = \sum n_i D_i$ $n_i \in \mathbb{Q}_+$

(i) P is nef, N effective.
(ii) the support of N is contractible i.e. the intersection form $(D_i \cdot D_j)_{n_i, n_j \neq 0}$ is negative definite.
(iii) $P \cdot N = 0$

ex: 

$D = 2A + B$

$N = \frac{3}{2} A$ $\quad P = \frac{1}{2} A + B$

obs. $P > 0$.

idea of proof [Bauer - Küronya - Kennedy]

- Look at the set of nef divisors $P \leq D$.

$P = \sum p_i D_i \qquad P' = \sum p'_i D_i$

$\max\{P, P'\} = \sum \max\{p_i, p'_i\} D_i$.

lemma $\max\{P, P'\}$ nef and $\leq D$.

- $P := \max\{P' \text{ nef } \leq D\}$

(i) $\rightsquigarrow$ if the ~~form~~ form on $D-P$ is not negative definite then there is a nef and effective divisor on it [harder than the previous lemma]

(ii) $\rightsquigarrow$ $D_i \in |N|$. $P+\varepsilon D_i$ not nef. $\Rightarrow$ $P+\varepsilon D_i \cdot D_i < 0$

- uniqueness = exercise

$\square$.

Consequences

for all m sufficiently divisible $h^0(mP) = h^0(mD)$

sketch, $D' \in |D|$. $D' = \underset{\substack{\uparrow \\ \text{supported in } |N|}}{N'} + M'$ not supported of $|N|$

$D' - N \equiv^{num} P \qquad \Rightarrow (N'-N) \cdot D_i \leq 0$ for each D_i.

since $(\,,\,)|_{|N|} < 0 \qquad N'-N \geq 0$ $\square$

$vol(D) = vol(P) \in \mathbb{Q}_+$.

even though in general $R(X,L)$ is not finitely generated.

$\exists$ example in 3d s.t. $vol(D) \notin \mathbb{Q}_+$

Further consequence Bauer - Küronya - Szemberg.

there exists a locally finite polyhedral rational subdivision of $Big(X)$ such that $vol|_{\text{face of subd.}}$ is polynomial

In fact on each face the support of $N(D)$ is fixed.

[2] Proof of the main thm in 2d.

$\alpha \in NS_{\mathbb{R}}(X)$ big $\quad \gamma$ arbitrary.

$\alpha = P(\alpha) + N(\alpha)$ Zariski decomposition

$\rightsquigarrow P(\alpha) \cdot N(\alpha) = 0 \quad \mathrm{vol}(P(\alpha)) = \mathrm{vol}(\alpha) \quad N(\alpha) \geq 0$.

Want to prove $\quad \mathrm{vol}(\alpha + t\gamma) = \mathrm{vol}(\alpha) + 2t\, P(\alpha)\cdot\gamma + O(t^2)$

in other words $\quad d(\mathrm{vol})(\alpha) = 2\, P(\alpha)$.

Key estimate (Siu, numerical criterion for bigness)
(Demailly, asymptotic Morse inequalities)

$\mathrm{vol}(\alpha - \beta) \geq \alpha^2 - 2\,\alpha\cdot\beta$.

α, β nef.

Proof ~~α nef and γ nef $P(\alpha) = \alpha$ $t>0$ $\mathrm{vol}(\alpha + t\gamma) = \alpha^2 + 2t\,\alpha\cdot\gamma + O(t^2)$. $t \leq 0$ key estimate $\mathrm{vol}(\alpha + t\gamma) \geq \alpha^2 + 2t\,\alpha\cdot\gamma$~~ reduce to γ nef.

$$\mathrm{vol}(\alpha + t\gamma) \geq \mathrm{vol}(P(\alpha) + t\gamma) \geq \underset{\mathrm{vol}(\alpha)}{\mathrm{vol}(P(\alpha))} + 2t\, P(\alpha)\cdot\gamma + O(t^2)$$

| if $t \leq 0$ key estimate
| if $t \geq 0$ easy

"the check" $\quad \mathrm{vol}(\alpha) \geq \mathrm{vol}(\alpha + t\gamma) - 2t\, P(\alpha + t\gamma)\cdot\gamma + O(t^2)$

Conclude using

$\square$

Continuity of the Zariski decomposition in Big (X)

$\alpha \mapsto P(\alpha)$ is continuous in Big (X)

proof. if α is big then $\{\beta,\ (1-\varepsilon)\alpha \leq \beta \leq (1+\varepsilon)\alpha\}$ is a neighborhood of α. $\quad (1-\varepsilon)P(\alpha) \leq P(\beta) \leq (1+\varepsilon)P(\alpha)$

$\square$

Remark :

X rational surface $C_i^2 = -1$ infinite.

$\alpha_k = [C_k] \in$ psef $\quad \delta(\alpha_k) = 0$.

extract in $NS_{\mathbb{R}}(X)$ $\quad \frac{\alpha_k}{\|\alpha_k\|} \to \alpha$ then α is nef.

3 Zariski decomposition in higher dimension.

- in arbitrary dimension $\mathrm{vol}(\alpha-\beta) \geq \alpha^d - d\, \alpha^{d-1}\cdot\beta$ holds true

  [proof is analogous to the one given in the ~~first~~ second lecture and based on a restriction argument]

- Key problem = understanding the Zariski decomposition.

Long history of the problem = several nice references.

+ Prokhorov

+ Nakayama-Paclean

idea = no nice decomposition in a fixed model

But there exists a weak form of a Zariski decomposition on b-divisors.

⚠ There is a constant issue with definitions with general psef divisors. we shall always work with big classes. For our applications to the volume function this is sufficient.

⚠ • $\alpha \in NS(X_\pi)$. denote by $b(\alpha) \in \mathbb{Q}-NS(\mathcal{X})$

terminology: unique Cartier b-class such that $\begin{cases} b(\alpha)_\pi = \alpha \\ b(\alpha)_{\pi'} = \mu^* b(\alpha)_\pi \end{cases}$

$X_{\pi'} \xrightarrow{\mu} X_\pi$.

• $b(\alpha) \in \mathbb{Q}-NS(\mathcal{X})$ is nef $\Leftrightarrow$ α is nef.
big $\Leftrightarrow$ α is big.

<u>Definition using envelope</u> $\alpha \in NS(X)$ big class

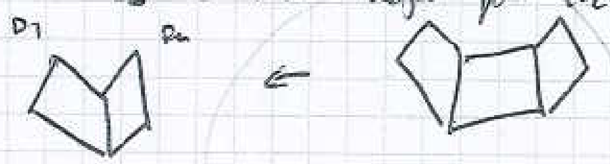
- $\mathcal{S}(\alpha) = \{ \beta \in \mathbb{Q}\text{-}NS(X)_{\mathrm{nef}}$
  $b(\alpha) - \beta = b(\gamma)$
  $\gamma \in NS_{\mathbb{Q}}(X_\pi)$ effective. $\}$.

<u>Lemma</u>: $\beta_1, \beta_2 \in \mathcal{S}(\alpha)$ then there exists $\beta \in \mathcal{S}(\alpha)$
$\beta \geq \beta_1$ and $\beta \geq \beta_2$.

- $\mathcal{S}(\alpha)$ is non empty.

<u>proof</u>. α big $\alpha = a + e$ a ample $\Rightarrow \mathcal{S}(\alpha) \neq \emptyset$.

- in X_π D_1 D_2 divisors ≥ 0. $\alpha \in NS_{\mathbb{Z}}(X) = c_1(\mathcal{O}_X(D))$
  $\pi^* D - D_1$ & $\pi^* D - D_2$ are nef
  $\to$ by blowing-up you may assume that any irred. component of D_1 is either disjoint from $|D_2|$ or included in it



  $\to$ exercise to show that $\pi^* D - \min\{D_1, D_2\}$ is nef.

  3 cases
  - $\to$ $C' \subseteq |D_1| \cap |D_2|$ $(\pi^* D - \min\{D_1, D_2\}) \cdot C' = (\pi^* D - D_i) \cdot C'$
  - $\to$ $C' \cap |D_1| = \emptyset$ $\pi^* D - \min\{D_1, D_2\} \cdot C' = (\pi^* D - D_2) \cdot C'$ $\square$.
  - $\to$ $C' \not\subseteq |D_1| \cup |D_2|$ $\pi^* D - \min\{D_1, D_2\} \cdot C' \geq 0$.

$\mathcal{P}(\alpha) := \text{"}\sup\text{"}_{\beta \in \mathcal{S}(\alpha)} \beta$. Needs to define this properly

Concretely in $NS(X_\pi)$
$\mathcal{B}(\alpha)_\pi = \{ \beta_\pi, \beta \in \mathcal{S}(\alpha) \}$
- set of effective classes
- satisfies Lemma $\Rightarrow$ compact filtered set.
- $\beta_\pi \leq \alpha_\pi$

$$\bigcap_{\gamma \in \Theta(\alpha)_\pi} \overline{\{\beta \in \Theta(\alpha)_\pi, \ \beta \geq \gamma\}}$$ is a singleton ~~[illegible]~~

non empty since $\Theta(\alpha)_\pi$ is filtered

compact subset in $NS(X_\pi)$ | $\beta_k \to \beta$, $\beta'_k \to \beta'$. ~~[illegible]~~ $\beta \geq \beta'_k \Rightarrow \beta \geq \beta'$

$P(\alpha)_\pi :=$ the singleton.

<u>Alternative definition = geometric</u>.

$\alpha = c_1(L) \in NS(X)$. $L \to X$ big.

Mobile / Fixed decomposition: for each $n \gg 0$.

- $nL = M_n + F_n$ $\quad \sigma \in H^0(L^n)$ $\operatorname{div}(\sigma) = M_n(\sigma) + F_n$.

  $\operatorname{codim} \bigcap_\sigma M_n(\sigma) \geq 2$.

- $bs(M_n)$ = coherent sheaf of ideals locally generated by f_σ where $\operatorname{div}(f_\sigma) = M_n(\sigma)$.

$$X_n \xrightarrow{\mu_n} X \qquad \mu_n \text{ dominates blow-up of } bs(M_n)$$

$$\mu_n^* M_n = \tilde{M}_n + E_n \quad \text{exceptional}$$

$\tilde{M}_n$: no base point

$$b(c_1(L)) = \frac{1}{n} b(\tilde{M}_n) + \text{Effective}.$$

<u>Thm</u> $P(\alpha) := \lim_{n\to\infty} \frac{1}{n} b(c_1(\tilde{M}_n))$.

<u>proof</u>. $\tilde{M}_n$ base pt free $\Rightarrow c_1(\tilde{M}_n)$ is nef $\quad \frac{1}{n} b(c_1(\tilde{M}_n)) \leq P(\alpha)$.

- decompose $\pi^* \alpha = A + E$ $\quad A$ ample $\quad E \geq 0$ in X_π.

  kA very ample in a higher model $\quad kA + kE = k\tilde{M}_k +$ Effective

  $kA \leq k\tilde{M}_k \Leftarrow H^0(kA + kE) \cong H^0(k\tilde{M}_k)$. $\square$

Properties of the Zariski decomposition and difficulties.

- $\alpha \in \mathrm{Big}(X) \longmapsto P(\alpha) \in b\text{-}NS(X)$

  def] $\gamma \in b\text{-}NS(X)$ is nef if for all π $\exists\, \beta_k \in \mathrm{Nef} \cap C\text{-}NS$

  $(\beta_k)_\pi \to \gamma_\pi$

  [a priori β_k lives in higher and higher model].

  $P(\alpha)$ is nef in a strong sense: $\exists\, \beta_k \in$ Nef Cartier b divisors such that for all π $(\beta_k)_\pi \to P(\alpha)_\pi$.

  Conjecture γ nef b-divisor. There exists a sequence of Cartier nef b divisors. $\beta_k \downarrow \gamma$..

- Continuity in the big cone.

  $\alpha_n \mapsto \alpha \Rightarrow P(\alpha_n) \to P(\alpha)$

  proof $(1-\varepsilon)\alpha \leq \alpha_n \leq (1+\varepsilon)\alpha$ and monotonicity $\square$.

Nakayama's example.

- There exists a divisor D on a 3-fold X such that.

  $P(D)$ is not a Cartier b-divisor.

- same example as mentioned above

$X = E \times E$ E elliptic curve generic $\mathrm{Psef}(X) = \mathrm{Nef}(X) =$ round cone $\subset NS = \mathbb{R}^3$

$\mathbb{P}(L_1 \oplus L_2 \oplus \mathcal{O}_X) \xrightarrow{p} X$

Grothendieck sense

$D = p^* \Delta + \mathcal{O}(1)$.

Lecture 4 (1h00).

- End of proof of $\mathcal{C}^1$-diff. of vol in arbitrary dimension
- Further works.

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[1] End of proof.

- $\alpha \in \mathrm{Big}(X)$ $\quad \gamma \in NS(X)$

$$\mathrm{vol}(\alpha + t\gamma) = \mathrm{vol}(\alpha) + t L_\alpha(\gamma) + \mathcal{O}(t^2).$$

"linear form on $NS(X)$

- $\mathcal{D}(\alpha) = \{ \beta$ Cartier b-divisor, nef $\beta \leq \alpha \}$

$$\mathrm{vol}(\alpha + t\gamma) \geq \mathrm{vol}(\beta + t\gamma) \geqslant \beta^d + dt\, \beta^{d-1} \cdot \gamma + \mathcal{O}(t^2).$$

key estimate

⚠ not careful about $\mathcal{O}(t^2)$ ~ check BFJ.

"take the sup over all $\beta \in \mathcal{D}(\alpha)$"

Lemma ~~[illegible]~~ for each γ $\quad \lim_{\beta \in \mathcal{D}(\alpha)} \beta^{d-1} \cdot \gamma$ exists

proof. γ effective

$\beta' \geq \beta$ both nef $\quad (\beta')^{d-1} \cdot \gamma \geq \beta^{d-1} \cdot \gamma$.

$\beta \in \mathcal{D}(\alpha) \leq \alpha \leq a$ = ample so $\beta^{d-1} \cdot \gamma \leq a^{d-1} \cdot \gamma < +\infty$

□

notation $\quad P(\alpha)^{d-1} \cdot \gamma := \lim_{\beta \in \mathcal{D}(\alpha)} \beta^{d-1} \cdot \gamma$.

linear form ≥ 0 on effective classes.

Deep fact similarly $\lim_{p \in \mathcal{O}(L)} p^{d} \cdot$ exists $=: P(L)^{d}$

thm | Fujita $P(L)^{d} = \mathrm{vol}(L)$

remark : conclude the proof of $\mathcal{C}^1$-diff .

proof of Fujita (discussion) ~

× one approach = use Okounkov bodies . Works over any field .

× other approaches = use vanishing theorems .

→ if $|pL| = |M_p| + F_p$ and $|M_p|$ has no base points then $|M_p| + A$ ample $L \geq A$.

→ otherwise $|M_p|$ has base locus $\mathcal{J}_p$. and the point is somehow to control these base loci .

$\Downarrow$

use multiplier ideals + subadditivity to get a hand on it .

Applications.

* Orthogonality. $\frac{d}{dt}\Big|_{t=0} \mathrm{vol}(\alpha + t\alpha) = d\,\mathrm{vol}(\alpha) = d\, P(\alpha)^{d}$.

$$= d\, P(\alpha)^{d-1}\cdot \alpha$$

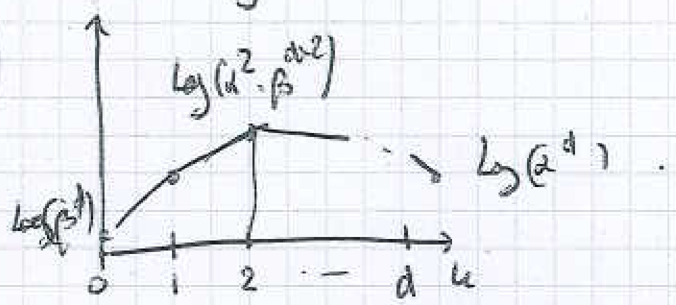
Hence $P(\alpha)^{d-1}\cdot(\alpha - P(\alpha)) = 0$.

Obs: this is the key to prove the duality of the psef cone in NS and the cone of movable curves [BDPP]

* Khovanskii – Teissier inequalities

$\alpha, \beta \in NS(X)$ nef.

$k \mapsto \alpha^k \cdot \beta^{d-k}$ is log concave



Csq of Hodge-Riemann relation (reduce to Hodge index thm)

Thm α, β big and nef

(i) $k \to \mathrm{Log}(\alpha^k\cdot\beta^{d-k})$ is affine

$\Updownarrow$

(ii) α & β are proportional.

Further results:

| Kähler case = Boucksom, Xiao, Popovici
| Higher codimension = Lehmann, Fulger

→ Kähler case basics are due to Boucksom.

X Kähler compact $\quad \alpha \in H^{1,1}(X) \cap H^2(X,\mathbb{R})$.

| psef $\quad \alpha = \{T\} \quad T \geq 0 \ (1,1)$ closed.
| big $\quad \alpha \geq \varepsilon\{\omega\} \quad \omega$ Kähler.
| nef $\quad$ for all $\varepsilon > 0 \quad \{\alpha + \varepsilon\omega\}$ Kähler class $\quad \omega$ Kähler fixed
| Kähler $\quad \alpha = \{\omega\} \quad \omega$ Kähler form

$$\mathrm{vol}(\alpha) = \sup_{\substack{\{T\} = \alpha \\ T \geq 0}} T_{ac}^d .$$

(Fact that this is finite follows from regularization thm of Demailly)

Boucksom

thm $\mathrm{vol}(\alpha) = 0 \iff \alpha$ is big.

thm vol is $\mathcal{C}^0$.

thm α big $\iff$ for all $\varepsilon > 0 \ \exists\, \omega$ Kähler and e
$\alpha = \{\omega\} + e \quad e \geq 0$ and $|\mathrm{vol}(\alpha) - \omega^d| \leq \varepsilon$.

recently Xiao + Popovici : $\mathrm{vol}(\alpha - \beta) \geq \alpha^d - d \cdot \alpha^{d-1} \cdot \beta \quad \alpha, \beta$ nef.

$\overset{?}{\Rightarrow}$ vol is $\mathcal{C}^1$-differentiable

Comments – applications of b-divisors to dynamics

$f : X \to X$ rational dominant map. induces linear maps

$$f_* : b\text{-}NS(X) \to b\text{-}NS(X).$$

$$f^* : b\text{-}CNS(X) \to b\text{-}CNS(X).$$