§ 2.3. Weierstrass Theory

Germ of a holomorphic function at $0 \in \mathbb{C}^n$

Given $f: U \rightarrow \mathbb{C}$ holomorphic map $U \ni 0$ open, and $g: V \rightarrow \mathbb{C}$ holomorphic map $V \ni 0$ open. We say that $f \sim g$ if $f = g$ in a common neighborhood of $0$.

Definition: A germ of holomorphic function at $0 \in \mathbb{C}^n$ is an equivalence class of equivalence relation $\sim$.

Observation: $f \sim g \iff$ their power series expansions at $0$ coincide.

$\iff \frac{\partial^{1+i}\ f}{\partial z^i}(0) = \frac{\partial^{1+i}\ g}{\partial z^i}(0) \ \forall i.$

Now we can put together all the germs in one space.
Notation:
\[ \mathcal{O}(\mathbb{C}^n, 0) = \{ \text{holomorphic germs at 0 in } \mathbb{C}^n \} \]

= \{ \sum_{I} a_I z^I : \mathbf{z} = (z_1, \ldots, z_n) \in \mathbb{C}^n, \forall I \in \mathbb{C} \}

\exists \rho > 0 \text{ s.t. } \sum_{I} |a_I| \rho^{|I|} < \infty.

Other notations: \( \mathcal{O}(\mathbb{C}^n, 0) = \mathcal{O}_n = \mathcal{O}_n \)

Observation: If \( M \) is a \( \mathbb{C} \)-manifold, define analogously:

\[ \mathcal{O}(M, p) = \{ \text{hol. germs at } M \text{ at } p \} \]

Fact: \( \mathcal{O}(\mathbb{C}^n, 0) \) is a ring

\[ f + g \]

* has a unit \( (f \equiv 1) \)

* it is an integral domain

\( fg = 0 \Rightarrow f = 0 \text{ or } g = 0 \)

* it is a \( \mathbb{C} \)-algebra

\[ \lambda \in \mathbb{C} \rightarrow \lambda f \in \mathcal{O}(\mathbb{C}^n, 0) \]

\[ f \in \mathcal{O}(\mathbb{C}^n, 0) \]
There is a canonical evaluation morphism:
\[ \text{ev}_0: \mathcal{O}(\mathbb{C}^n, 0) \longrightarrow \mathbb{C} \]
\[ f \longmapsto f(0) \]
This is a morphism of \( \mathbb{C} \)-algebras.
Let
\[ m_{(\mathbb{C}^n, 0)} = \{ f \in \mathcal{O}(\mathbb{C}^n, 0) \mid f(0) = 0 \} \]
is a maximal ideal.

\[ \mathcal{O}(\mathbb{C}^n, 0) / m_{(\mathbb{C}^n, 0)} = \{ f \in \mathcal{O}(\mathbb{C}^n, 0) \text{ s.t. } f(0) \neq 0 \} \]
\[ = \mathcal{O}(\mathbb{C}^n, 0) \times \{ \text{units in } \mathcal{O}(\mathbb{C}^n, 0) \} \]
\[ = \mathcal{O}(\mathbb{C}^n, 0) \]
\[ \Rightarrow m_{(\mathbb{C}^n, 0)} \text{ is the unique maximal ideal of } \mathcal{O}(\mathbb{C}^n, 0). \]
Thus, \( \mathcal{O}(\mathbb{C}^n, 0) \) is a local ring.

\[ \mathcal{O}(\mathbb{C}^n, 0) \twoheadrightarrow \mathcal{O}(\mathbb{C}^n, 0) / m_{(\mathbb{C}^n, 0)} \cong \mathbb{C} \]
\[ \text{ev}_0 \]

Observation: \((n = 1)\)

\[\mathcal{O}(c, 0) = \{ \sum a_n z^n \mid \sum |a_n| r^n < \infty \\text{ for some } p > 0 \} \]

If \(\alpha = (z^k)\) for some \(k > 0\)

\[\alpha = \mathcal{M}(c, 0) \]

\[\Rightarrow \mathcal{O}(c, 0) \text{ is a PID (principal ideal domain)} \]

In particular, it is Noetherian.

"Weierstrass theory allows one to make induction on the dimension and to view holomorphic functions as polynomials in one variable \(z\) with coefficients that are holomorphic in the other variables \(z_1, z_2, \ldots, z_{n-1}\)."

Notation: \(z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n\)

\[= (z', w)\]

where \(z' = (z_1, \ldots, z_{n-1})\) and \(w = z_n\).
Definition: $f \in \mathcal{O}(\mathbb{C}^n, 0)$ is distinguished in $\mathbb{Z}_n$ with degree $p \geq 0$, if

$$f(0, \mathbb{Z}_n) = \mathbb{Z}^p \times \text{(unit)} \quad \text{"big Oh notation"}$$

$$= a_0 \mathbb{Z}_n^p + O(\mathbb{Z}_n^{p+1})$$

- A Wermer-strap polynomial of degree $p \geq 1$ is a holomorphic function of the form

$$f(z) = w^p + a_1(z)w^{p-1} + \ldots + a_p(z)$$

where $a_i \in \mathcal{O}(\mathbb{C}^{n-1}, 0)$ and $a_i(0) = 0$ (i.e. $a_i \in \mathcal{M}(\mathbb{C}^{n-1}, 0)$).

Remark: In Hörmander's book, distinguished is called normalized.

In Gunning's book, distinguished is called regular.
Theorem: (Weierstrass division theorem)
Given \( f \in \mathcal{O}(\mathbb{C}^n, 0) \) distinguished in \( \mathbb{C}^n \) of degree \( p \). Then:

\( \exists \Delta \) open neighborhood, \( \exists C > 0 \),

for all \( g \in \mathcal{O}(\Delta) \), \( \sup_{\Delta} |g| < +\infty \),

there exists a Weierstrass polynomial \( p \) of degree \( \leq p-1 \), and \( q \in \mathcal{O}(\Delta) \)
such that

\[ g = qf + r \quad (\star) \]

\[ \max\left(\sup_{\Delta} |q|, \sup_{\Delta} |r|\right) \leq C \cdot \sup_{\Delta} |g| \]

Moreover, this decomposition is unique.

Corollary (Weierstrass Preparation Theorem)
\( f \in \mathcal{O}(\mathbb{C}^n, 0) \) is distinguished in \( \mathbb{C}^n \) of degree \( p \).

The \( \mathcal{O}_x^\times (\mathbb{C}^n, 0) \) and \( \mathcal{O}_x (\mathbb{C}^n, 0) \) Weierstrass polynomial of degree \( p \)
such that \( f = h \cdot W \).

Furthermore, this decomposition is unique.
Proof of Corollary: Apply Weierstrass division theorem to \( q = Z_n^p \)

\[ z^p = q \cdot f + r \quad z = (z', w) \]

\[ r = \sum_{j=0}^{p-1} a_j(z')w^j \]

Look at \((0, w)\) (points of this form)

\[ w^p - r(0, w) = q(0, w) \cdot f(0, w) \]

monic poly. of degree \( p \)

\[ (aw^p + \alpha(w^{p+1})) \]

\[ \Rightarrow q(0) \neq 0 \text{ and } r(0, w) = 0. \]

\[ \Rightarrow q \in \mathcal{O}^\times(C^n, 0) \text{ and so } \]

\[ \Phi = q^{-1}(Z_n^p - r) \text{ Weierstrass poly. of degree } p. \]

Uniqueness is left as an exercise (it is a consequence of the uniqueness of the Weierstrass division theorem)
Theorem (Weierstrass Division Theorem)

f \in O(c^n, 0) distinguished in \mathbb{C}^n at degree p. 
\exists \Delta \in \mathbb{C}^n \text{ open neighborhood } 
\exists C > 0, \text{ for all } g \in O(\Delta), \sup_{\Delta} |g| < +\infty
\exists g \in O(c^{n-1}, 0) \{ \mathbb{C} \} \text{ of degree } \leq p - 1
\exists \varphi \in O(\Delta) \text{ such that }
\varphi = pf + r
\max \{ \sup_{\Delta} |f|, \sup_{\Delta} |r| \} \leq C \cdot \sup_{\Delta} |g|

\text{Obs: } g(z', w) = \sum_{j=0}^{\infty} a_j(z') w^j

where \ a_j \in O(c^{n-1}, 0). \text{ We split the sum:}

\| g(z', w) = \sum_{j=0}^{p-1} a_j(z') w^j + \left( \sum_{j=p}^{\infty} \varphi_j(z') w^j \right)^p \| R_g \| \varphi_g \|

If \Delta \in \mathbb{C}^n \text{ is a polydisk,}
\text{ } \varphi \in O(\Delta), \sup_{\Delta} |\varphi| < +\infty
\quad \Rightarrow \ R_g, \varphi \in O(\Delta)
\( \Delta = 3 \{ z', \ell \} < \rho, \mid w \mid < R^2, \rho, R > 0. \)

Cauchy estimate applied to
\( w \rightarrow g(z', w) \) for a fixed \( \mid z' \mid < \rho. \)

\[ \mid a_j(z') \mid \leq \sup_{\Delta} \mid g \mid \cdot \frac{1}{R^j} \]

**Lemma:**
\[ \sup_{\Delta} \mid Rg \mid \leq \rho \cdot \sup_{\Delta} \mid g \mid \]

\[ \sup_{\Delta} \mid a_j \mid \leq \frac{(p+1)}{R^p} \sup_{\Delta} \mid g \mid \]

**Proof:**
\[ \mid a_j(z') \cdot w^j_1 \mid \leq \sup_{\Delta} \mid g \mid \cdot \frac{1}{R^j} \]

\[ R^2 = \sup_{\Delta} \mid g \mid \]

\[ \Rightarrow \sup_{\Delta} \mid Rg \mid \leq \rho \sup_{\Delta} \mid g \mid. \]

\[ \mid w \mid^p \cdot \mid a_j(z', w) \mid \leq \mid g \mid + \mid Rg(z', w) \mid \leq (p+1) \sup_{\Delta} \mid g \mid \]

\[ \mid a_j \mid \leq \frac{p+1}{R^p} \sup_{\Delta} \mid g \mid \text{ true on } \mid w \mid = R \]

hence everywhere by the maximum principle.
\[ f = f_1 + \omega \rho f_2 \]

If distinguished \( \Rightarrow \) \( f_2(0) = 1 \).

\[ g = q f + r \iff g = q (f_1 + \omega \rho f_2) + r \]

Define \( h(z', w) = f_1 f_2' (z', w) = \text{unitpoly in w with degree} \leq p-1 \text{ and self in } \mathcal{O}(e^{-1}, 0) \).

Note \( f_1(0, w) = 0 \) so \( h(0, w) = 0 \).

\[ \text{Aim: solve } g = (\omega \rho + h), \tilde{z} + \tilde{r} \]

Idea: successive approximation.

\[ g = w \cdot q_1 + r_1 \text{ with } q_1 = \partial g, r_1 = Rg \]

\[ g - h(z', w) q_1 = \omega \rho q_2 + r_2 \]

with \( q_2 = \partial g - h q_1 \) and \( r_2 = R \cdot g - h q_1 \)

\( \text{(continue inductively)} \)

\[ g - h q_n = w \rho, q_{n+1} + r_{n+1} \]
where \( q_{n+1} \in Q_{g-hq_n} \) and \( r_{n+1} \in R_{g-hq_n} \).

After subtracting two levels:

\[
-h(q_n - q_{n-1}) = \underbrace{w'p(q_{n+1} - q_n) + (r_{n+1} - r_n)}_{\Phi''} \\
\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \text{by linearity of} \quad \Phi\quad (\text{-h(q_n-q_{n-1})}) \quad \text{of the operator} \quad \Phi.
\]

\[
\sup_{\Delta} |q_{n+1} - q_n| \leq \frac{\sup_{\Delta} |h|}{RP} \frac{1}{(p+1) \sup_{\Delta} |q_n - q_{n-1}|}
\]

Since \( h(0,n) \equiv 0 \), by choosing \( p \ll 1 \), we can ensure that

\[
\sup_{ |z'| < R} \left| \frac{1}{h} \right| \leq \frac{1}{2} \frac{RP}{p+1}
\]

\[
\implies \quad \sup_{\Delta} |q_{n+1} - q_n| \leq \frac{1}{2} \sup_{\Delta} |q_n - q_{n-1}|
\]

\[
q_n \xrightarrow{\sup} q
\]

\[
\sup |\tilde{g}| \leq C \sup |q_{n+1}| \leq C' \sup_{\Delta} |g|
\]
\[ r_n \xrightarrow{\sup} r \]

\[ g - h \hat{q} = w^p q + r \implies \sup_\Delta |\hat{r}| \leq C \sup_\Delta |g| \]

so in the limit, we set
\[ g - h \hat{q} = w^p q + \tilde{r} \]
\[ g = (w^p + h) \hat{q} + \tilde{r} \]

**Uniqueness:** \((w^p + h) q + r = 0\)

\(r \in \mathcal{O}(\mathbb{C}^{n-1} \setminus \{0\}) \) with \(\deg \leq p - 1\).

\[-hq = w^p q + \tilde{r}\]

where \(q = \Phi(-hq)\)

\[ \implies \sup_\Delta |q| \leq \left( \frac{p + 1}{R^p} \sup_\Delta |h| \right) \sup_\Delta |q| \]

\[ \leq \frac{1}{2} \]

\[ \implies \sup_\Delta |q| = 0 \implies q = 0. \]

and so we also get \(r = 0\).
Lecture 10

\( \Theta_n = \{ \text{hol. germs at } 0 \text{ in } \mathbb{C}^n \text{ for } n \geq 1 \} \)

\[ \{ \sum a_I z^I \mid \sum |a_I| \rho^{|I|} < \infty \text{ for some } \rho > 0 \} \]

Recall that for \( R \) an integral domain:

R is \underline{Noetherian} if one of the following equivalent conditions is satisfied:

\( \bullet \) \( \text{In} \subseteq \text{Int} + \text{I} \) ideals \( \Rightarrow \exists \text{ no. s.t. } \text{In} = \text{In}_0 \) \( \forall n \geq 0 \).

\( \bullet \) any ideal is finitely generated.

\underline{Theorem }\( \circ \) (Hilbert Basis Theorem)

\( R \) Noetherian \( \Rightarrow \) \( R[T] \) is Noetherian.

\underline{Observation }\( \circ \) \( R[T_1, T_2, \ldots, T_N]/I \) is Noetherian for any ideal \( I \).

\( \bullet \) divisibility in \( R \): \( f \mid g \) if \( g = fxh \) for some \( h \in R \).

\( \bullet \) \( g \) is irreducible if \( f \mid g \) \( \Rightarrow \) \( f \in R^x \) or \( g \in R^x \) (Here \( R^x \) = units in \( R \)).

\( R \) is \underline{factorial} if any \( f \in R \) admits a unique decomposition as a product of irreducible elements.
In other words, given any two different factorizations, 
\[ f = f_1 f_2 \cdots f_r \quad (f_i \text{ is irreducible}) \]
\[ \tilde{f_1} \tilde{f_2} \cdots \tilde{f}_s \quad (\tilde{f}_i \text{ is irreducible}) \]
\[ \Rightarrow r = s \text{ and a permutation } \Pi: 1, 2, \ldots, r \rightarrow S \text{ such that} \]
\[ f_i / f_{\Pi(i)} \in R^* \text{ for all } i. \]

Examples of factorial rings: Euclidean domains, PID, etc.

**Thm 4:** \( \mathbb{Q}_n \) is both Noetherian and factorial.

**Lemma:** \( R \) Noetherian, \( n \geq 0 \)

Any submodule of \( R^n \) is finitely generated.

**Proof:** \( n=1 \), \( I \subseteq R \), \( I = \text{ideal} \)

\[ \Rightarrow I \text{ is finitely generated (as } R \text{ is Noetherian).} \]

**Induction on } n:** Given \( M \subseteq R^n \) a submodule, consider projection \( \Pi: R^{n+1} \rightarrow R \)

\[ (a_1, a_2, \ldots, a_{n+1}) \mapsto a_1 \]

\( \Pi(M) \) is an ideal of \( R \), so we get that \( \Pi(M) \) is finitely generated.
\[ \pi(M) = \langle \pi(m_1), \ldots, \pi(m_r) \rangle \text{ for some elements } m_1, m_2, \ldots, m_r \in M. \]

Now, given \( m \in M \), note that
\[ m - \sum_{i=1}^{r} m_i \in \ker(\pi) \]
so by induction, \( \ker(\pi) \leq \mathbb{R}^n \) is finitely generated, and we can combine the generators of \( \ker(\pi) \) together with \( m, \ldots, m_r \) to get a generating set for \( M \).

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**Proof that \( O_n \) is Noetherian**

\[ z = (z_1, z_2, \ldots, z_n) \in \mathbb{C}^n \]
\[ z = (z', w) \text{ where } z' = (z_1, \ldots, z_{n-1}) \in \mathbb{C}^{n-1} \text{ and } w = z_n \]

\( f \in O_n \) is distinguished in \( z_n \) of order \( p \geq 0 \)
\[ f(0, w) = aw^p + O(w^{p+1}) \text{ with } a \neq 0. \]
Lemma: \( \exists f \in \mathcal{O}_{n} \). There exists an open and dense set \( U \subseteq \text{GL}_{n}(\mathbb{C}) \) such that: for every \( A \in U \), we have:

\[ f(A \cdot z) \text{ is distinguished in } \mathbb{Z}_{n}. \]

Proof: Expand \( f \) into power series:

\[ f(z) = \sum a_{i} z^{i} = f_{k} + f_{k+1} + \ldots \]

\[ f_{j} = \sum_{|I|=j} a_{I} z^{i} \quad \text{homogeneous poly. of degree } j. \]

Suppose \( f_{k} \neq 0 \).

\[ V = \{ v \in \mathbb{C}^{n} \mid f_{k}(v) = 0 \} \quad \text{closed nowhere else.} \]

\[ U = \{ A \in \text{GL}_{n}(\mathbb{C}) : A \cdot e_{n} \notin V \} \]

\( U \) is open and dense.

\( A \in U \), \( z \rightarrow f(A \cdot z) = f_{A}(z) \)

\[ t \mapsto f(A \cdot t e_{n}) = f_{A}(0, t) = f_{k}(tA \cdot e_{n}) + f_{k+1}(tA \cdot e_{n}) + \ldots \]

\[ = t^{k} f_{k}(A \cdot e_{n}) + o(t^{k+1}) \]

\( \neq 0 \) \( \square \)
Proof that $O_n$ is Noetherian

We argue by induction on $n$.

$n=1$: $O_1$ is PID (so in particular Noetherian).

Inductive step: $n-1 \rightarrow n$

Pick an ideal $I \subseteq O_n$.

Take $f \neq 0 \in I$. May assume that $f$ is distinguished in $O_n$ of degree $p \geq 0$.

Weierstrass division theorem:

$\forall g \in I$, $g = qf + r$, $r \in O_{n-1}$, $\deg f(I) \leq p-1$

$O_{n-1} = \bigoplus_{i=1}^{\infty} O_{n-1}(z_i)$

$r = \sum_{j=0}^{p-1} r_j(z_j)$, $w_j$

$M = 3 \cap (g); g \in I$ is a submodule of $O_{n-1} \oplus p$

so it is finitely-generated by induction + Lemma.

Finally, $O = <f> + M$ is also finitely-generated.
Proof of factoriality of $O_n$

We want to prove:

1. Suppose $(f_i) \subset O_n$ is a sequence such that $f_{i+1} | f_i$. Then $f_i / f_{i+1} \subset O_n^\times$ for all $i \gg 0$.

2. If $f$ is irreducible and $f|\,(g,h)$ then $f/g$ or $f/h$.

We proceed by induction on $n$.

**Base case ($n=1$):** clear $\mathcal{O} = \mathbb{Z} 

**Inductive step:** Gauss Lemma.

$O_{n-1}$ is factorial $\implies O_{n-1}[w]$ is factorial.

**Observation:** $O_n^\times = \{ f \in O_n \mid f(0) \neq 0 \}$.

$O_{n-1}[w]^\times = O_{n-1}^\times \neq O_n^\times \cap O_{n-1}[w]$.  

(for $n \geq 2$)

We prove (2): $f$ irreducible in $O_n$ and $f|\,(g,h)$. We may assume that $f$ is distinguished in $\mathbb{Z}_n$ of order $p \geq 1$ and that $f \not\in O_n^\times$.  

Weierstrass preparation \Rightarrow f = \text{unit in } \mathcal{O}_n. \text{ Weierstrass polynomial.}

We may assume that
\[ f(z', w) \text{ is a Weierstrass polynomial} \]
\[ \Rightarrow f(z', w) = w^p + \sum_{i=0}^{p-1} a_i(z') w^i, \quad a_i(0) = 0. \]

Weierstrass division \Rightarrow g = q \cdot f + g_0
\[ h = q \cdot f + h_0 \]
where \( g_0, h_0 \in \mathcal{O}_{n-1}[w]. \]
\[ \Rightarrow f | g_0 h_0 \text{ in } \mathcal{O}_n \]
(because \( f | gh \)).

We need two lemmas:

\underline{Lemma A} \text{ : } f \text{ Weierstrass poly, } g \in \mathcal{O}_{n-1}[w].
\[ f \mid f \text{ in } \mathcal{O}_n \iff f \mid g \text{ in } \mathcal{O}_{n-1}[w] \]

\underline{Lemma B} \text{ : } f \text{ Weierstrass poly.}
\[ f \text{ is irreducible in } \mathcal{O}_n \text{ iff it is irreducible in } \mathcal{O}_{n-1}[w]. \]
Let's use lemmas A and B to finish the proof of factoriality.

\[ f / g_0 h_0 \text{ in } \mathbb{O}_n \overset{\text{A}}{\longrightarrow} f / g_0 h_0 \text{ in } \mathbb{O}_{n-1}[w] \]

\[ \overset{\text{B}}{\longrightarrow} f \text{ irreducible in } \mathbb{O}_{n-1}[w] \]

\[ \overset{\text{induction}}{\longrightarrow} f / g_0 \text{ or } f / h_0 \text{ in } \mathbb{O}_{n-1}[w] \]

\[ f / g_0 \text{ or } f / h_0 \text{ in } \mathbb{O}_n. \quad \square \]

Lemma A (trivial direction)

Now we can prove lemmas A + B.

Note, before we start, that

\[ \mathbb{O}_n^x = \{ f \neq 1 \mid f(0) \neq 0 \} \]

\[ (\mathbb{O}_{n-1}[w])^x = \mathbb{O}_n^x \]

Proof of (A) : \((\leq)\) clear. \((\mathbb{O}_{n-1}[w], \mathbb{O}_n)\).

\((\Rightarrow)\) \(f, p \) in \( \mathbb{O}_n \). Both \( f, p \) are

Weierstrass polynomials in variable \( w \),

and \( f \) is monic.

We do Euclidean division inside \( \mathbb{O}_{n-1}[w] \).
\[ y = q f + r \text{ where } q \in \mathcal{O}_{n-1}[w] \]
\[ \deg(r) \leq p - 1 \]
But we also have \[ y = h \cdot f \ (h \in \mathcal{O}_n) \]
By uniqueness of Weierstrass division theorem, we get \[ r = 0. \]

**Proof of Lemma B**

\[ \implies \text{exercise} \]
\[ \implies f \text{ is reducible in } \mathcal{O}_{n-1}[w] \]
\[ \varphi_1, \varphi_2 \text{ non-invertible in } \mathcal{O}_{n-1}[w] \]
\[ f = \varphi_1 \varphi_2 \]
\[ \implies f(0, w) = \varphi_1(0, w) \cdot \varphi(0, w) \]
\[ \implies f(0, w) = a \cdot w^k \]
\[ \implies \varphi_1(0, w) = b w^{p-k} \]
\[ \implies \varphi_1 \text{ & } \varphi_2 \text{ are both Weierstrass polynomials of degree } k \text{ & } p-k. \]

If \[ k = 0, \varphi_1 \in \mathcal{O}_{n-1}^* \] which contradicts the hypothesis that \( \varphi_1 \) was non-invertible.
thus, $k \geq 1$, and similarly
\[ p - k \geq 1 \implies \psi \text{ is reducible in } \mathfrak{o}_n \]
as $\psi_1, \psi_2 \in \mathfrak{m}_{0,n}$
(maximal ideal)

Now we prove ①:
\[ f_{i+1} | f_i \text{ for all } i \implies f_i \text{ is a Weierstrass polynomial.} \]
deg$_W(f_i)$ is decreasing by Lemma A.
for $i > 0$, deg$_W(f_i) = 5$ (stabilizes)
\[ = \deg_w(f_{i+1}) \]
\[ f_i = h \cdot f_{i+1}, \quad f_i (0, w) = h(0, w) f_{i+1}(0, w) \]
\[ w_0 \in \mathfrak{o}_n^* \]

§2.4. Applications to the geometry
of analytic sets

- A germ of analytic set in a $C$-manifold $M$ at $p$ is an analytic subset $Z$ of
an open set $U$ at $p$ modulo the relation $Z \sim_p Z'$ if $\psi$
\[ ZnV = Z' \cup V \text{ for some open set } V \subset U \cup U' \text{ which contains } p, \]

\[ Z \leq M \text{ analytic subset} \]

Any point \( p \) determines a germ \( (Z, p) \) at \( p \).

**Observation:** \( p \in Z \Rightarrow (Z, p) = (\emptyset, p) \)

\( (Z, p) \) = germ of analytic subset at \( p \in M \).

\[ \mapsto I (Z, p) = \exists t \in \theta_{n, p} \mid t|_Z = 0 \]

is an ideal of \( \theta_{M, p} \) (isom. \( \cong \mathbb{C}_n \))

- If \( I \leq \theta_{M, p} \) is ideal, since \( \theta_n \) is Noetherian, we get \( X = \langle f_1, f_2, \ldots, f_r \rangle \). Consider the set \( V(X) := \cap_{i=1}^r \{ f_i = 0 \} \) is analytic.

\( V(X, p) \) is a germ of analytic set at \( p \) which does not depend on choice of generators.
Observations

\[ J \subseteq I(V(J), p) \]

\[ V(I((z, x), x)) = (z, x) \]

This is exactly going to be the analogue of the usual Hilbert's Nullstellensatz.