Lecture 7

Tuesday, Jan 28.

Chapter 2: Analytic Sets

Aim: Study the geometry of sets \( f = 0 \) where \( f \) is holomorphic.

manifolds with \( \mathbb{C} \)-structure

+ singularities.

\[ \Rightarrow \]

real manifolds.

Geometry of algebraic varieties over \( \mathbb{C} \)

\[ \Rightarrow \]

g 2.1: \( \mathbb{C} \)-manifolds

"differentiable manifolds with holomorphic functions".

\( X \) = topological space, new

A holomorphic atlas \( A \) on \( X \):

\( A = \{ (U_i, \phi_i) \} \) with the following conditions

"fancy A"
• $U_i$ is an open cover of $X$.
• $\varphi_i : U_i \to \mathbb{C}^n$ homeomorphism onto its image.
• The patching maps are holomorphic. $\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$

$$\varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$$

is holomorphic

Two holomorphic atlases

$\mathcal{A} = \{(U_i, \varphi_i)^3 \}$

$\mathcal{B} = \{(U_j, \varphi_j)^3 \}$
on $X$ are equivalent if

$\varphi_j \circ \varphi_i^{-1}$ are holomorphic whenever/wherever defined.

**Def:** A complex manifold of dimension $n \geq 1$ is the data of

• $X$=Hausdorff, second countable, topological space.
holomorphic.

• An equivalence of atlases
  (with values in $\mathbb{C}^n$).

Remarks: second-countable (under the assumptions above)

$\iff$ $X$ is metrizable

$\iff$ $X$ is $\sigma$-compact, meaning that $X = \bigcup_{\mathfrak{K}_n}$, where $\mathfrak{K}_n$ is compact

$\iff$ $X$ is paracompact
  (existence of partitions of unity)

Examples:
  • $\Omega \subseteq \mathbb{C}^n$ open set
    (one element in the atlas, namely, $\Omega$ and one map $\mathbf{1} : \Omega \to \Omega$).

• Riemann Surface = $\mathbb{C}$ manifold of dim 1.

Observation: The condition of being "second-countable" is actually automatic in the case $n = 1$.
(This is known as Radó's theorem).

If $X_1 = \text{manifold of dim } n_1$
  $X_2 = \text{manifold of dim } n_2$.

$\implies$ $X_1 \times X_2$ is a manifold of dimension $n_1 + n_2$. 
Remark: Any complex manifold is a smooth manifold. (The underlying smooth manifold is oriented.)

Terminology: A holomorphic chart on a $\mathbb{C}$-manifold $X$ with a holomorphic atlas $\mathcal{A} = \{ (U_i, \psi_i) \}$ is a pair $(V, \psi)$ such that

- $V$ is open
- $\psi : V \to \mathbb{C}^n$ homeomorphism onto its image!
- $\psi \circ \psi_i^{-1}$ is holomorphic $\forall i$ (on $\psi_i(U_i \cap V)$).

Observation: $X = \mathbb{C}$-manifold, with holomorphic atlas $\mathcal{A} = \{ (U_i, \psi_i) \}$. The set of holomorphic charts forms a holomorphic atlas $\mathcal{A}'$ compatible with $\mathcal{A}$ and maximal.

Now we are going to define a notion of a holomorphic map between two complex manifolds.
**Definition:** Let $X$ and $Y$ be $C^r$-manifolds of dimensions $n$ and $m$.

A continuous map $f : X \to Y$ is holomorphic if $\psi_j \circ f \circ \varphi_i^{-1}$ is holomorphic.

(Here, $\{(\varphi_i, \psi_i)\}$ is atlas for $X$, and $\{(\psi_j, \varphi_j)\}$ is atlas for $Y$).

**Remark:** This definition does not depend on the choice of holomorphic atlases on $X$ & $Y$ (because composition of holomorphic is holomorphic).

- This definition is compatible with the definition of holomorphic maps $C^n \to C^m$.

- $f : X \to Y$ is called **biholomorphism** if $f$ is holomorphic, bijective, and $f^{-1}$ is also holomorphic.

**Remark:** If $X = \dim m$, $Y = \dim m$.

If $f : X \to Y$ is biholomorphism, then $n = m$. 
Observation: $f$ holomorphic $+$ bijective $\Rightarrow f^{-1}$ is holomorphic

(Only true for $X,Y = \mathbb{C}$-manifolds
(take on, we will learn about analytic subsets, for which the statement above will be false).

Def: $f: \mathbb{S} \to \mathbb{C}^m$ holomorphic

$\mathbb{C}^n = (z_1, z_2, \ldots, z_n)$

$f = (f_1, f_2, \ldots, f_m)$.

- $f$ is a **submersion** if

$$d f(p) = \left[ \begin{array}{cccc} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial z_1} & \cdots & \frac{\partial f_m}{\partial z_n} \end{array} \right]$$

is surjective. (it is rank $m \leq n$ for all $p \in \mathbb{S}$)

**Holomorphic immersion** $\iff d f(p)$ is injective for all $p \in \mathbb{S}$.
Transport these definitions to any holomorphic map \( f: X \to Y \) where \( X, Y \) are C-manifolds.

Impose \( \psi \circ f \circ \psi^{-1} \) to be submersion/immersion to all holomorphic charts \((U, \psi)\) on \( X \) and \((V, \varphi)\) on \( Y \).

**Definition:** \( X = \text{C-manifold of dimension } n \).

\( \Sigma \subseteq X \) is a C-submanifold of \( X \) if for all \( p \in \Sigma \), \( \exists \) hol. chart \((U, \psi)\) such that \( \forall p \in U \), \( \psi(p) = 0 \)

\( \psi(U \cap \Sigma) = \{ z_1 = z_2 = \ldots = z_n = 0 \} \cap \psi(U) \)

**Observation:** a submanifold is always a closed subset of \( X \).

**Fact:** Each connected component of complex submanifold \( \Sigma \subseteq X \) carries structure of C-manifold of dimension \( \leq \dim(X) \) such that the injection map \( \Sigma \hookrightarrow X \) is holomorphic.
Proof: Build a holomorphic atlas \( A \) on \( X \):
\[
A = \{ (U_\alpha, \phi_\alpha |_{U_\alpha}) \}
\]
\( \phi: U \rightarrow \mathbb{C}^n \), \( \phi \) has \( n \) component
\( \phi = (\phi_1, \ldots, \phi_n) \)
\( \phi |_{U_{\alpha}} \quad \Rightarrow \quad \phi_1 |_{U_{\alpha}} = \phi_2 |_{U_{\alpha}} = \cdots = \phi_n |_{U_{\alpha}} = 0. \)

Observation: we can interpret \( \phi |_{U_{\alpha}} \) maps onto open subsets of \( \mathbb{C}^{n-l} \) that are homeomorphisms onto their images.

\( l \) does not depend on \( p \in X \):
\( \delta: X \rightarrow \{ 0, 1, \ldots, n \} \)
\( \delta(p) = n - l. \) \( \delta \) is well-defined and locally constant.

Since \( V \) is connected, \( \delta \) is constant, -(of course, \( n - l \) is simply the dimension of \( X \) in this case).
Thm 1: \( f: X \to Y \) is a holomorphic map, where \( X \) (resp. \( Y \)) is a \( C \)-manifold of dimension \( n \) (resp. \( m \)). Suppose that \( f \) is a submersion in a neighborhood of \( f^{-1}(q) \) for some \( q \in Y \). Then \( f^{-1}(q) \) is a \( C \)-submanifold of \( X \) with dimension \( n - m \).

\[
\begin{array}{c}
X \\
\end{array} \xrightarrow{f} \begin{array}{c}
Y \\
\end{array}
\]

Proof: If \( p \notin f^{-1}(q) \), take any holomorphic chart \( (U, \varphi) \) s.t. \( \varphi(U) = \mathbb{C}^n \).

If \( p \in f^{-1}(q) \), then \( f(p) = q \).

Take holomorphic chart \( (U_1, \psi) \) at \( p \), and chart \( (V, \varphi) \) at \( f(p) \), such that \( \psi \circ f \circ \varphi^{-1} \) is well-defined holomorphic from an open subset of \( \mathbb{C}^n \) onto \( \mathbb{C}^m \).
\( F \) is a local holomorphic diffeomorphism.

Define \( \tilde{F}(z_1, z_2, \ldots, z_m) = (F_1, F_2, \ldots, F_m, x_{m+1}, \ldots, x_n) \)

\( \tilde{F}(0) \) is invertible.

\( F \) defines a holomorphic chart at \( p \)

in the chart \( \tilde{F}^{-1}(q) \) is given by
\[ z_1 = z_2 = \ldots = z_m = 0. \]
Construction of $C$-manifold

→ as $C$-submanifold,
→ as quotient of $C$-manifold by group actions.

Example:
Take $\lambda \in C$ with $0 < |\lambda| < 1$.

$M_\lambda(z) = (\lambda z_1, \lambda z_2, \ldots, \lambda z_n)$

\[ C^n \to C^n \]

holomorphic map

$C^n \setminus \Theta \to C^n \setminus \Theta$.

biholomorphism

$X = (C^n \setminus \Theta) / \langle M_\lambda \rangle$

identity $p \sim p' \iff p = M_\lambda^p(p')$ for some $p \in \mathbb{Z}$.

Claim: $X$ can be endowed with a unique structure of $C$-manifold of dim. $n$ such that $\pi: C^n \setminus \Theta \to X$ is holomorphic.

Hopf manifold.

$n=1 \to$ torus $S^1 \times S^1$

$n \geq 2 \to$ diffeomorphic to $S^1 \times S^{2n-1}$.
§2.2. Analytic sets (and subsets)

Let $M$ be a complex manifold of dimension $n$.

**Definition:** $Z \subseteq M$ is called an analytic subset if for all $p \in M$, $\exists U \ni p$ open neighborhood and $f_1, f_2, \ldots, f_r \in \mathcal{O}(U)$ such that $Z \cap U = \{ f_1 = f_2 = \ldots = f_r = 0 \}$.

**Observation:** Any analytic subset of $M$ is closed (in $M$).

- $Z_1, Z_2, \ldots, Z_m$ analytic subsets of $M$.

Then $\bigcap_{i=1}^{m} Z_i$ and $\bigcup_{i=1}^{m} Z_i$ are both analytic.

**Proof:** Locally, $Z_i = \{ f_i^{(3)} = 0 \}$

$\Rightarrow \bigcap_{i=1}^{m} Z_i = \bigcap_{i=1}^{m} \{ f_i^{(3)} = 0 \}$ is analytic.

For the union, we just consider the case $m=2$. (It follows by induction that it holds for general $m$).

$Z_1 = \bigcap \{ f_i = 0 \}$, $Z_2 = \bigcap \{ g_j = 0 \}$. 
\[ Z_1 \cup Z_2 = \bigcup_{i,j} f_i \cap g_j = \emptyset \]

- \( h : M \to N \) holomorphic map between complex manifolds.
- \( Z \) analytic \( \implies h^{-1}(Z) \) is also analytic.

**Proof:** locally, \( Z = \bigcap f_i = \emptyset \) in \( N \)
where \( f_i \in \mathcal{O}(Z) \). Then
\[ h^{-1}(Z) = \bigcap f_i \circ h = \emptyset \]

- \( M, N \) are 2 C-manifolds.
- \( Z \subseteq M, W \subseteq N \) analytic subsets \( \implies Z \times W \subseteq M \times N \) is analytic.

- C-submanifold is an analytic subset.

\[ \forall (Z \cap W) \implies Z_1 = Z_2 = \ldots = Z = \emptyset \]
Definition: \( Z \subset \mathcal{M} \) is a \( C \)-submanifold if \( Z \) is an analytic subset.

The set of regular points of \( Z \) (\( \text{Reg}(Z) \)) is the set of points \( p \in Z \) such that \( Z \cap U \) is a \( C \)-submanifold for some open neighborhood \( U \) of \( p \).

\( \text{Sing}(Z) = Z \setminus \text{Reg}(Z) \) is the singular locus of \( Z \).

\( \text{Reg}(Z) \) is open, \( \text{Sing}(Z) \) is closed.

Example: In \( \mathbb{C}^2 \), \( Z(x, y) = 3xy = 0 \)

\( \text{Reg}(Z) = Z \setminus \{ (0,0) \} \)

\( \text{Sing}(Z) = \{ (0,0) \} \).

Observation: If \( \dim(M) = 1 \), and \( M \) is connected (so \( M = \) Riemann surface), a subset \( Z \subset M \) is analytic iff either \( Z = M \) or \( Z \) is a discrete set.

Proof: \((\Leftarrow)\) is easy.
(\Rightarrow) Locally (reduce to \(M = D(0,1)\))
\[ Z = \bigcap f_i = 0 \]
By the Principle of Analytic Continuation, we get that \(Z\) is discrete or all of \(D(0,1)\).

**Proposition:** \(Z\) analytic subset of \(M\)
(M is a connected \(C\)-manifold). dense

If \(Z \not\subset M\), then \(M \setminus Z\) is an open and connected subset of \(M\)
(slogan: "\(Z\) is always small")

**Proof:** To show that \(M \setminus Z\) is dense, need to show \(\text{Int}(Z) = \emptyset\). Assume, to the contrary, that \(\text{Int}(Z) \neq \emptyset\).

\[
\begin{array}{c|c|c|c}
M & \text{Int}(Z) & U & \text{Take } p \in \partial \text{Int}(Z) \\
\hline
& & & \text{On some open } U^- \\
\hline
U \text{ is connected} & Z = \bigcap \{ f_i = 0 \} & \text{with } f_i \in \mathcal{O}(U) \\
\hline
\end{array}
\]

Principle of analytic continuation, we get that \(U \subset \subset Z\).
\* $M \setminus Z$ is connected (we prove this locally. 

We may assume $M = B^n(0, 1)$. 

Take $p, p' \in M \setminus Z$. Take the complex line $L$ passing through $p$ and $p'$. 

$\mathbb{Z} \cap L = \text{discrete}$ 

Now we can find a path from $p$ to $p'$, which avoids finitely many points of $\mathbb{Z} \cap L$.

$L = \mathbb{C}$

\[ \overline{\text{Theorem} \, 2} : \, \mathbb{Z} \subseteq M \text{ analytic subset.} \]

\* $\operatorname{Reg}(Z)$ is open and dense subset of $\mathbb{Z}$

\* $\operatorname{Sing}(Z)$ is closed and nowhere dense subset of $\mathbb{Z}$. 

\[ \underline{\text{Proof}} : \text{Proceed by induction on } \dim(M) = n. \]

\[ n = 1 : \, Z = M \text{ or } Z \text{ is discrete} \]

\[ \Rightarrow \operatorname{Sing}(Z) = \emptyset. \text{ There is nothing to prove.} \]
Inductive hypothesis: \( n-1 \Rightarrow n \)

\[ M = B^n(0,1) \subseteq C^n \]

Have to prove that for any analytic subset of the ball \( B^n(0,1) \), it contains at least one regular point.

\[ Z = \bigcap_{i=1}^{m} \{ f_i = 0 \} \quad f_i \in C^\infty(B(0,1)) \]

Assume that \( f_1 \not\equiv 0 \).

Expand \( f_1 \) in power series:

\[ f_1(z) = \sum_{I} a_I z^I \]

Claim: \( \exists I \) such that:

\[ \frac{\partial^{\left| I \right|} f_1}{\partial z^I} \bigg|_{Z} = 0 \text{ but } \frac{\partial}{\partial z} \frac{\partial^{\left| I \right|} f_1}{\partial z^I} \bigg|_{Z} \neq 0. \]

\( I = (i_1, i_2, \ldots, i_n) \). It may appear that \( I = (0) \) works; this happens when \( df_1(0) \neq 0 \). In this case, \( \rho f_1 = 0 \) is a \( C \)-submanifold of \( B^n(0,1) \) of dimension \( \leq n-1 \).
After a suitable change of coordinates, $y_1 = y_3 = y_2 = 0$, and $Z \subseteq \mathbb{R}^3 \times B^r(0, t)$. By the induction assumption, $\text{Reg}(Z) \cap (B^3 \times B^r(0, t)) \neq \emptyset$.

**Definition**: $Z \subseteq M$ analytic subset. For $p \in Z$, we define $\dim_p(Z) = \text{local dimension of } Z \text{ at } p$.

$$\dim_p(Z) = \limsup_{q \to p} \dim_q(\text{Reg}(Z))$$

$q \in \text{Reg}(Z) \subseteq \mathbb{R}^3, t, 1, 000, n^3$

**Observation**: If $\dim_p(Z) = \dim(M)$

$\Rightarrow Z$ is open at $p$

If $\dim_p(Z) = 0$ $\Rightarrow Z \cap U = \emptyset$ $p$

for some open neighborhood $U \ni p$.

**Thm** (*): For any analytic subset $Z \subseteq M$, $\text{Sing}(Z)$ is an analytic subset of $M$ with dimension $\dim_p(\text{Sing}(Z)) < \dim_p(Z)$ $\forall p \in Z$. 
\( \Omega \subset \mathbb{C}^n \) open set.
\[ f : \Omega \to \mathbb{C} \text{ holomorphic} \]
\[ \exists f = 0^k. \]

**Theorem:** \( \text{Sing}(\Omega) \) is analytic, and is equal to:

\[ 0 = f = \frac{\partial f}{\partial z_1} = \ldots = \frac{\partial f}{\partial z_n}. \]

Note: **Theorem (+) \Rightarrow Theorem (** in the case of hypersurfaces. But...

\[ f(x, y) = x^2, \quad \Omega = \{ x = 0^k \}, \quad \text{Sing}(\Omega) = \emptyset. \text{ Need this!} \]

**Proof:**
- Pick \( p \in \Omega \) if \( \frac{\partial f}{\partial z^i}(p) \neq 0. \)

\[ \implies \{ f = 0^k \} \text{ is a C-submanifold near the neighborhood of } p. \]

- Suppose \( p \in \text{Reg}(\Omega). \) Need to show that \( \frac{\partial f}{\partial z^i}(p) \neq 0 \) for some index \( i. \)

Indeed, we can do a change of coordinates so that \( \Omega = \{ z_1 = \ldots = z_p = 0^k \}. \)

**Claim:** \( l = 1, \)
\( z = (f=0) \)

\( \Phi = f \circ \varphi^{-1} \)

\[ \varphi(z_1, z_2) = z_1^k \cdot h(z_1, z_2) \]

such that \( h(0, z_2) \neq 0 \).

\( (h=0) \Rightarrow h(0) \neq 0 \)

Since \( (h=0) = 0 \), we may consider the holomorphic function \( \frac{1}{h} \) on \( B \setminus \text{supp} \).

Hartogs \( \Rightarrow \frac{1}{h} \) is holomorphic on \( B \)

\( \Rightarrow h(0) \neq 0 \). \text{ Contradiction} \)

\text{Conclusion:} \quad \Phi = z^k \cdot h \text{ where } h(0) \neq 0

But then \( k = 1 \), because otherwise

\[ \frac{\partial f}{\partial z} \Big|_{z=0} = 0 \quad \Rightarrow \quad \frac{\partial f}{\partial z} (0) \neq 0. \]
The claim (that \( l = 1 \)) follows from Hartogs' Theorem.

Suppose \( n = 2 \). If \( l \geq 2 \), then \( \mathbb{Z} = \{ z_1 = z_2 = 0 \} \) is the origin, which we have seen is not possible.

If \( n \geq 3 \), if \( l \geq 2 \), for all \((\hat{z}_{l+1}, \ldots, \hat{z}_n) \in \mathbb{C}^{n-l}\),

\[
\hat{z}_{l+1} = \hat{z}_{l+1}, \ldots, \hat{z}_n = \hat{z}_n
\]

Now \( \mathbb{Z} = \mathbb{R}^2 \) in this hyperplane.

So the idea is to slice \( \mathbb{Z} \) by the dimension \( l \) plane and apply Hartogs:

\[
\forall z_i = \hat{z}_i \quad i = l+1, \ldots, n
\]