Lecture 7 — Tuesday, Jan 28.

Chapter 2: Analytic Sets

Aim: Study the geometry of sets $\{f=0\}$ where f is holomorphic.

$\rightsquigarrow$ manifolds with $\mathbb{C}$-structure + singularities.

geometry of algebraic varieties over $\mathbb{C}$

real manifolds.

---

§2.1: $\mathbb{C}$-manifolds

"differentiable manifolds with holomorphic functions".

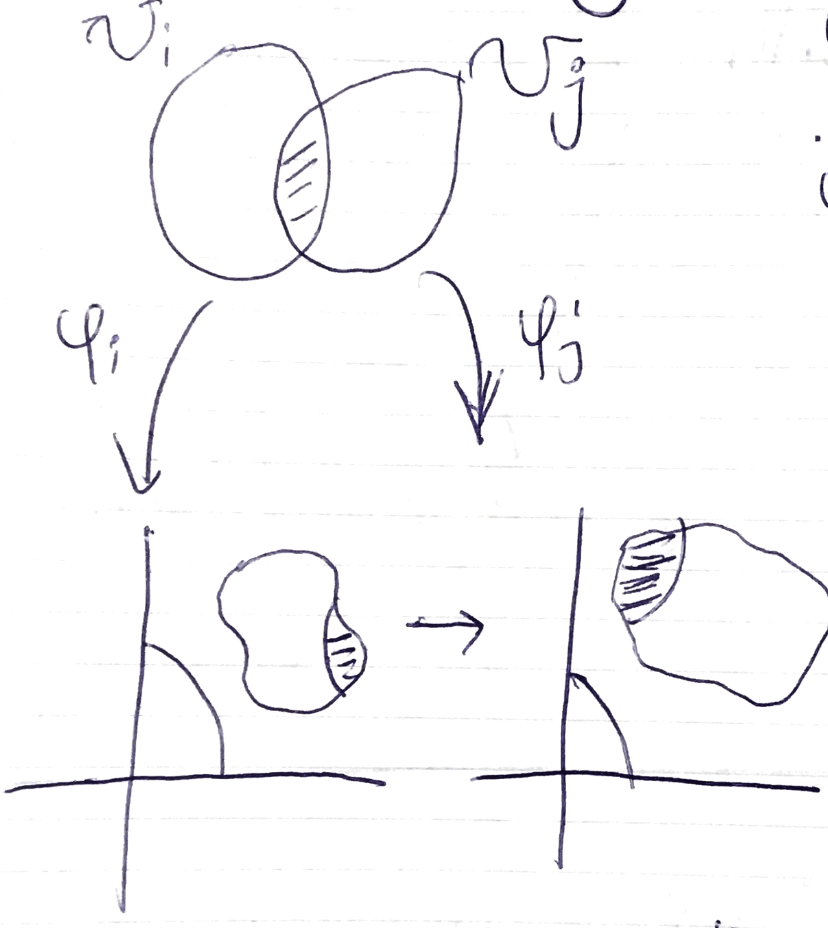
X = topological space, $n \in \mathbb{N}$

A holomorphic atlas $\mathcal{A}$ on X:

$\mathcal{A} = \{(U_i, \varphi_i)\}$ with the following conditions.

"fancy A"

- U_i is an open cover of X.
- $\varphi_i = U_i \to \mathbb{C}^n$ homeomorphism onto its image.
- The patching maps are holomorphic.



$\varphi_{ij} = \varphi_j \circ \varphi_i^{-1}$

$\varphi_i(U_i \cap U_j) \to \varphi_j(U_i \cap U_j)$ is holomorphic

Two holomorphic atlases

$\mathcal{A} = \{(U_i, \varphi_i)\}$

$\mathcal{B} = \{(U_j, \psi_j)\}$

on X are equivalent if $\psi_j \circ \varphi_i^{-1}$ are holomorphic whenever/wherever defined.

<u>Def</u>: A complex manifold of dimension $n \geq 1$ is the data of

- X = Hausdorff, second countable, topological space.

- An equivalence of holomorphic atlases (with values in $\mathbb{C}^n$).

__Remark:__ Second-countable (under the assumptions above)

$\Leftrightarrow$ X is metrizable

$\Leftrightarrow$ X is σ-compact, meaning that $X = \bigcup_{n \in \mathbb{N}} K_n$ where K_n = compact

$\Leftrightarrow$ X is paracompact (existence of partitions of unity)

__Examples:__
- $\Omega \subseteq \mathbb{C}^n$ open set (one element in the atlas, namely, Ω and one map $\varphi_1 : \mathcal{U} \xrightarrow[\text{identity}]{} \mathcal{U}$).
- Riemann Surface = $\mathbb{C}$ manifold of dim 1.

__Observation:__ The condition of being "second-countable" is actually automatic in the case $n = 1$. (This is known as Rado's theorem).

If X_1 = manifold of dim n_1

X_2 = manifold of dim n_2.

$\Rightarrow X_1 \times X_2$ is a manifold of dimension $n_1 + n_2$.

Remark: Any complex manifold is a smooth manifold.
(the underlying smooth manifold is oriented)

Terminology: A holomorphic chart on a $\mathbb{C}$-manifold X with a holomorphic atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$ is a pair (V, ψ) such that

- V is open
- $\psi : V \longrightarrow \mathbb{C}^n$ homeomorphism onto its image!
- $\psi \circ \varphi_i^{-1}$ is holomorphic $\forall i$ (on $\varphi_i(U_i \cap V)$).

Observation: $X = \mathbb{C}$-manifold with holomorphic atlas $\mathcal{A} = \{(U_i, \varphi_i)\}$.
The set of all holomorphic charts forms a holomorphic atlas $\mathcal{A}'$ compatible with $\mathcal{A}$ and maximal.

Now we are going to define a notion of a holomorphic map between two complex manifolds.

Definition: X and Y $\mathbb{C}$-manifolds of dimensions n and m.
A continuous map $f: X \to Y$ is holomorphic if $\psi_j \circ f \circ \varphi_i^{-1}$ is holom.
(Here, $\{(U_i, \varphi_i)\}$ is atlas for X, and $\{(V_j, \psi_j)\}$ is atlas for Y).

Remark: • This definition does not depend on the choice of holom. atlases on X & Y. (because composition of hol. is hol.)
• This definition is compatible with the definition of hol. maps $\mathbb{C}^n \to \mathbb{C}^m$.

• $f: X \to Y$ is called biholomorphism if f is holomorphic, bijective and f^{-1} is also holomophic.

Remark: If $X = \dim m$, $Y = \dim m$.
If $f: X \to Y$ is biholomorphism, then $n = m$.

<u>Observation</u>: f holomorphic + bijective
$\Longrightarrow f^{-1}$ is holomorphic

<u>Only True</u> for $X, Y = \mathbb{C}$-manifolds
(later on, we will learn about analytic subsets, for which the statement above will be false).

<u>Def</u>: $f: \Omega \to \mathbb{C}^m$ holomorphic
$$\Omega \subseteq \mathbb{C}^n = (z_1, z_2, \dots, z_n)$$
$$f = (f_1, f_2, \dots, f_m).$$

- f is a <u>submersion</u> if

$$df(p) = \begin{bmatrix} \frac{\partial f_1}{\partial z_1} & \cdots & \frac{\partial f_1}{\partial z_n} \\ \vdots & & \\ \frac{\partial f_m}{\partial z_1} & \cdots & \frac{\partial f_m}{\partial z_n} \end{bmatrix} \quad m \text{ rows}$$

n columns

is surjective. (it is rank $m \le n$ for all $p \in \Omega$)

Holomorphic <u>immersion</u> $\Leftrightarrow$ $df(p)$ is injective for all $p \in \Omega$.

Transport these definitions to any holomorphic map

$$f: X \longrightarrow Y$$

where X, Y are $\mathbb{C}$-manifolds.

Impose $\psi \circ f \circ \varphi^{-1}$ to be submersion/immersion to all holomorphic charts (U, φ) on X and (V, ψ) on Y.

<u>Definition</u>: $X = \mathbb{C}$-manifold of dimension n.

$Y \subseteq X$ is a $\mathbb{C}$-submanifold if for all $p \in X$, $\exists$ hol. chart (U, φ) such that
- $p \in U$, $\varphi(p) = 0$
- $\varphi(U \cap Y) = \{z_1 = z_2 = \dots = z_\ell = 0\} \cap \varphi(U)$

<u>Observation</u>: a submanifold is always a closed subset of X.

<u>Fact</u>: Each connected component of complex submanifold $Y \subseteq X$ carries structure of $\mathbb{C}$-manifold of dimension $\leq \dim(X)$ such that the injection map $Y \hookrightarrow X$ is holomorphic.

Proof: Build a holomorphic atlas $\mathcal{A}$ on Y:

$$\mathcal{A} = \{(U \cap Y, \varphi|_{U \cap Y})\}$$

$\varphi: U \longrightarrow \mathbb{C}^n$ φ has n component

$$\varphi = (\varphi_1, \dots, \varphi_n)$$

$$\varphi|_{U \cap Y} \xrightarrow{\text{restrict}} \varphi_1|_{U \cap Y} = \varphi_2|_{U \cap Y} = \dots = \varphi_n|_{U \cap Y} = 0.$$

Observation: we can interpret $\varphi|_{U \cap Y}$ maps onto open subsets of $\mathbb{C}^{n-\ell}$ that are homeomorphisms onto their images.

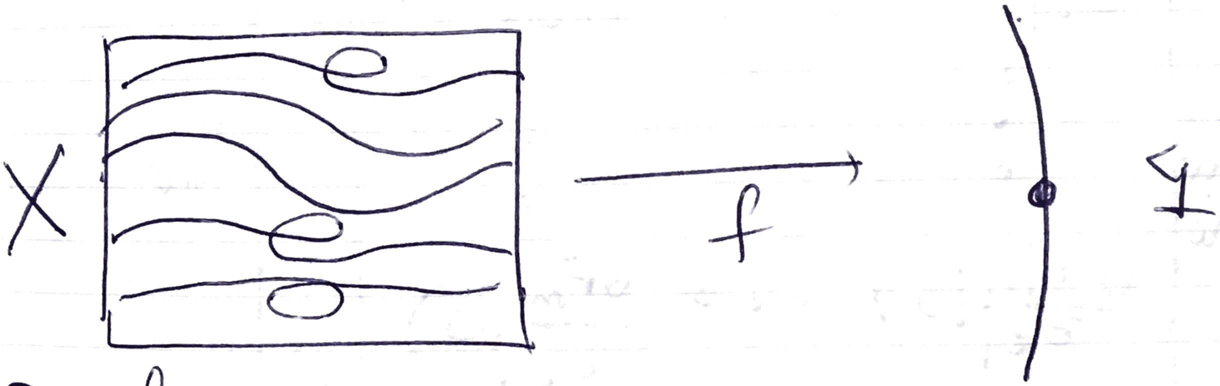
ℓ does not depend on $p \in Y$:

$$\delta : Y \longrightarrow \{0, 1, \dots, n\}$$

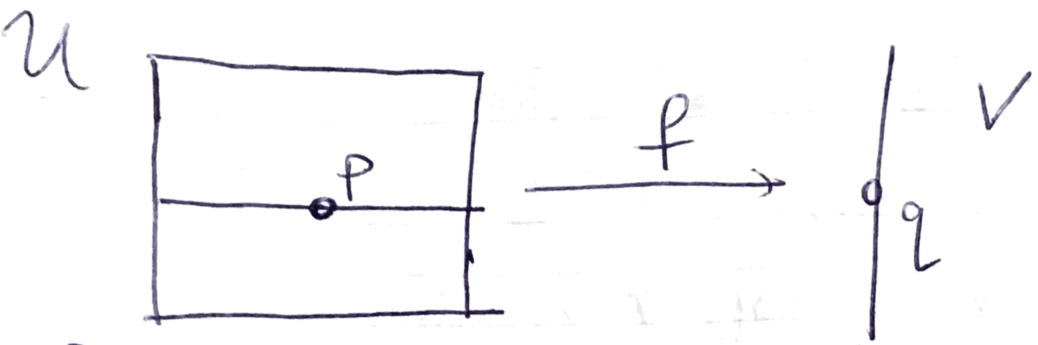
$\delta(p) = n - \ell$. δ is well-defined and locally constant.

Since Y is connected, δ is constant. (of course, $n-\ell$ is simply the dimension of Y in this case).

Thm 1: $f: X \to Y$ is a holomorphic map, where X (resp. Y) is a $\mathbb{C}$-manifold of dimension n (resp. m). Suppose that f is a submersion in a neighborhood of $f^{-1}(q)$ for some $q \in Y$. Then:
$f^{-1}(q)$ is a $\mathbb{C}$-submanifold of X with dimension $n-m$.



Proof: • If $p \notin f^{-1}(q)$, take any holomorphic chart (U, ψ) s.t. $U \cap Y = \emptyset$.
• If $p \in f^{-1}(q)$, then $f(p) = q$.
Take holomorphic ~~atlas~~ chart (U, φ) at p, and (V, ψ) at $f(p)$, such that $\psi \circ f \circ \varphi^{-1}$ is well-defined hol. submersion from an open subset of $\mathbb{C}^n$ onto $\mathbb{C}^m$.



$F = \psi \circ f \circ \varphi^{-1} : \Omega \subseteq \mathbb{C}^n \to \mathbb{C}^m$

Submersion. Let's assume $\varphi(p) = 0$.

$F = (F_1, F_2, \ldots, F_m)$, $z = (z_1, z_2, \ldots, z_n)$

Assume that the first m×m submatrix is invertible.

$$dF(0) = \begin{array}{cc} \begin{bmatrix} \frac{\partial F_1}{\partial z_1}(0) & \cdots & \frac{\partial F_1}{\partial z_n}(0) \\ \vdots & & \vdots \\ \frac{\partial F_m}{\partial z_1}(0) & \cdots & \frac{\partial F_m}{\partial z_n}(0) \end{bmatrix} & m \text{ variables} \\ n \text{ variables} & \end{array}$$

Define $\tilde{F}(z_1, z_2, \ldots, z_n) = (F_1, F_2, \ldots, F_m, x_{m+1}, \ldots, x_n)$

$d\tilde{F}(0)$ is invertible.

$\xRightarrow{\text{analytic IFT}}$ $\tilde{F}$ is a local hol. diffeom.

$\tilde{F}$ defines a hol. chart at p

in the chart $f^{-1}(q)$ is given by $\{z_1 = z_2 = \cdots = z_m = 0\}$.

Construction of $\mathbb{C}$-manifold

→ as $\mathbb{C}$-submanifold.

→ as quotient of $\mathbb{C}$-manifold by group actions.

Ex:

Take $\lambda \in \mathbb{C}$ with $0 < |\lambda| < 1$.

$M_\lambda(z) = (\lambda z_1, \lambda z_2, \dots, \lambda z_n)$

$: \mathbb{C}^n \longrightarrow \mathbb{C}^n$

holomorphic map

$\mathbb{C}^n \setminus \{0\} \longrightarrow \mathbb{C}^n \setminus \{0\}$.

biholomorphism

$X = (\mathbb{C}^n \setminus \{0\}) / \langle M_\lambda \rangle$

identify $p \sim p' \Leftrightarrow p' = M_\lambda^r(p)$ for some $r \in \mathbb{Z}$.

Claim: X can be endowed with a unique structure of $\mathbb{C}$-manifold of dim. n such that $\pi : \mathbb{C}^n \setminus \{0\} \longrightarrow X$ is holomorphic.

→ Hopf manifold.

$n=1 \rightsquigarrow$ torus $S^1 \times S^1$

$n \geq 2 \rightsquigarrow$ diffeomorphic to $S^1 \times S^{2n-1}$.

Lecture 8 Thursday, January 30

§2.2. Analytic sets (and subsets)

Let M be a complex manifold of dimension n.

Definition: $Z \subseteq M$ is called an analytic subset if for all $p \in M$, $\exists U \ni p$ open neighborhood and $f_1, f_2, \dots, f_r \in \mathcal{O}(U)$ such that $Z \cap U = \{f_1 = f_2 = \dots = f_r = 0\}$

Observation: • Any analytic subset of M is closed (in M).

• $Z_1, Z_2, \dots, Z_m$ analytic subsets of M. Then $\bigcap_{i=1}^{m} Z_i$ and $\bigcup_{i=1}^{m} Z_i$ are both analytic.

Proof: Locally, $Z_i = \{f_i^{(j)} = 0\}$

$$\Rightarrow \bigcap_{i=1}^{m} Z_i = \bigcap_{i,j} \{f_i^{(j)} = 0\} \text{ is analytic.}$$

For the union, we just consider the case $m=2$. (It follows by induction that it holds for general m).

$$Z_1 = \bigcap \{f_i = 0\}, \quad Z_2 = \bigcap \{g_j = 0\}$$

Then $Z_1 \cup Z_2 = \bigcap_{i,j} \{ f_i g_j = 0 \}$.

- $h : M \to N$ holomorphic map between complex manifolds.

Z = analytic $\Longrightarrow$ $h^{-1}(Z)$ is also analytic.

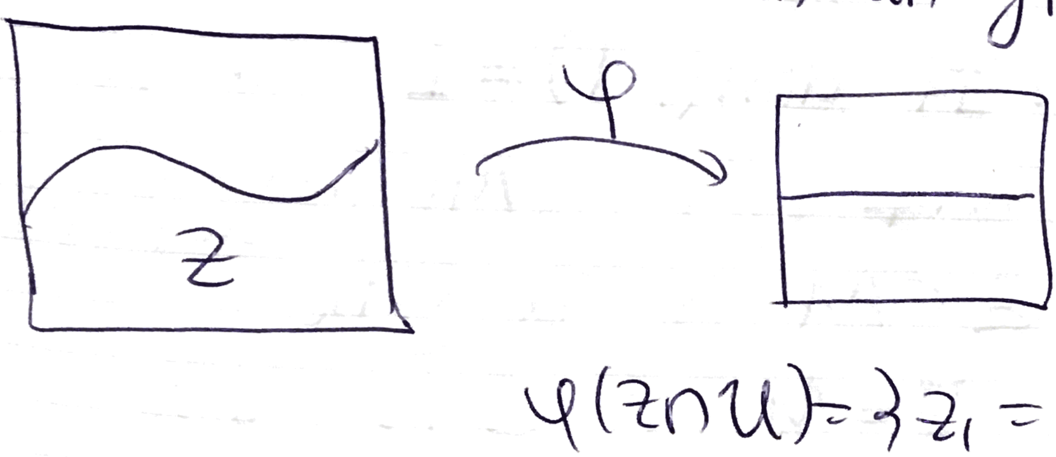
Proof: locally, $Z = \bigcap \{ f_i = 0 \}$ in N where $f_i \in \mathcal{O}(\mathcal{U})$. Then

$$h^{-1}(Z) = \bigcap \{ f_i \circ h = 0 \}.$$

- M, N are 2 $\mathbb{C}$-manifolds.

$Z \subseteq M$, $W \subseteq N$ analytic subsets $\Rightarrow$ $Z \times W \subseteq M \times N$ is analytic.

- $\mathbb{C}$-submanifold is an analytic subset.



$$\varphi(Z \cap \mathcal{U}) = \{ z_1 = z_2 = \dots = z_r = 0 \}$$

Definition: $Z \subseteq M$ is a $\mathbb{C}$-submanifold.
Z = analytic subset.
The set of regular points of Z ($\text{Reg}(Z)$) is the set of points $p \in Z$ such that $Z \cap \mathcal{U}$ is a $\mathbb{C}$-submanifold for some open neighborhood $\mathcal{U}$ of p.
$\text{Sing}(Z) = Z \setminus \text{Reg}(Z)$ singular locus of Z.

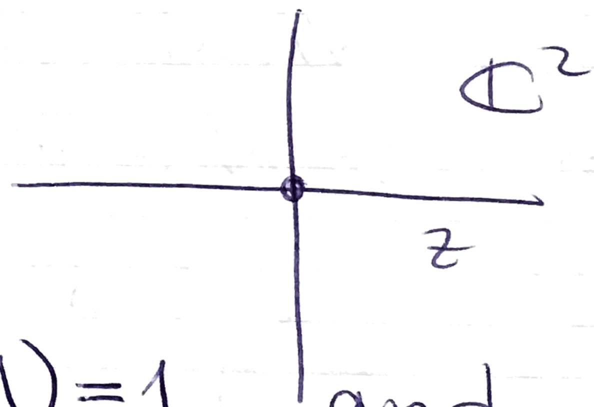
$\text{Reg}(Z)$ is open, $\text{Sing}(Z)$ is closed.

Ex: In $\mathbb{C}^2 \ni (x, y)$

$Z = \{xy = 0\}$

$\text{Reg}(Z) = Z \setminus \{(0,0)\}$

$\text{Sing}(Z) = \{(0,0)\}$.



Observation: If $\dim(M) = 1$, and M is connected (so M = Riemann surface) a subset $Z \subseteq M$ is analytic iff either $Z = M$ or Z is a discrete set.

Proof: ($\Leftarrow$) is easy.

($\Rightarrow$) Locally (reduce to $M = \mathbb{D}(0,1)$)

$Z = \cap \{f_i = 0\}$

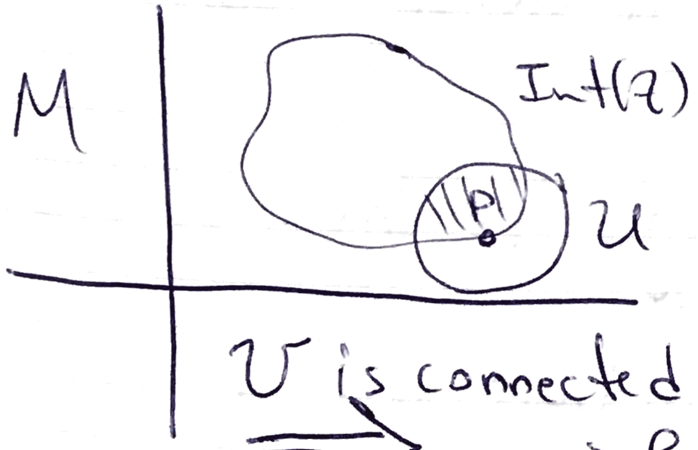
By the Principle of Analytic Continuation, we get that Z = discrete or all of $\mathbb{D}(0,1)$.

<u>Proposition</u>: Z analytic subset of M (M is a connected $\mathbb{C}$-manifold).

If $Z \subsetneq M$, then $M \setminus Z$ is an open dense and connected subset of M

(slogan: "Z is always small")

<u>Proof</u>: To show that $M \setminus Z$ is dense, need to show $\mathrm{Int}(Z) = \emptyset$. Assume, to the contrary, that $\mathrm{Int}(Z) \neq \emptyset$.



Take $p \in \partial\, \mathrm{Int}(Z)$

On some open U

$Z = \cap \{f_i = 0\}$

with $f_i \in \mathcal{O}(U)$

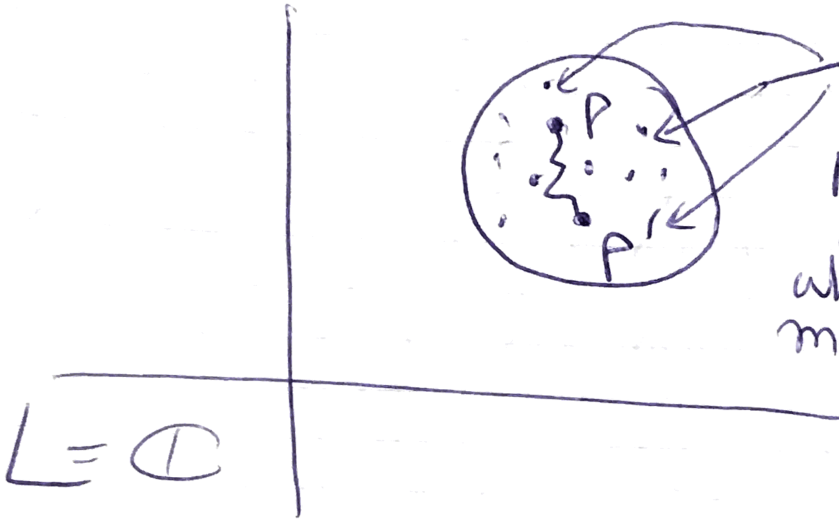
U is connected $\Longrightarrow$ $f_i|_U \equiv 0$

Principle of analytic continuation, we get that $U \subseteq Z$.

- $M \setminus Z$ is connected (we prove this locally)

We may assume $M = B^n(0,1)$.

Take $p, p' \in M\setminus Z$. Take the complex line L passing through p and p'.



$Z \cap L$ = discrete

Now we can find a path from p to p' which avoids finitely many points of $Z \cap L$.

$L = \mathbb{C}$

Theorem 2: $Z \subseteq M$ analytic subset.

- $\mathrm{Reg}(Z)$ is open and dense subset of Z
- $\mathrm{Sing}(Z)$ is closed and nowhere dense subset of Z.

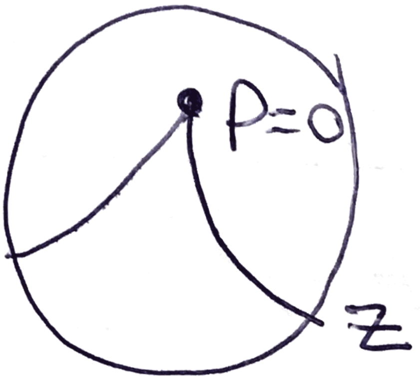
Proof: Proceed by induction on $\dim(M) = n$.

$n=1$: $Z = M$ or Z is discrete

$\Rightarrow \mathrm{Sing}(Z) = \emptyset$. There is nothing to prove.

Inductive hypothesis: $n-1 \Rightarrow n$

$M = B^n(0,1) \subseteq \mathbb{C}^n$



Have to prove that for any analytic subset of the ball $B^n(0,1)$ contains at least one regular point.

$$Z = \bigcap_{i=1}^{m} \{f_i = 0\} \qquad f_i \in \mathcal{O}(B(0,1)).$$

Assume that $f_1 \neq 0$.

Expand f_1 in power series:

$$f_1(z) = \sum a_I z^I$$

<u>Claim</u>: $\exists I$ such that:

$$\left.\frac{\partial^{|I|} f_1}{\partial z^I}\right|_Z \equiv 0 \quad \text{but} \quad \left.\frac{\partial}{\partial z_1}\left(\frac{\partial^{|I|} f_1}{\partial z^I}\right)\right|_Z \neq 0.$$

$I = (i_1, i_2, \dots, i_n)$. It may appear that $I = (0)$ works; this happens when $df_1(0) \neq 0$. In this case,

$\{f_1 = 0\} \supseteq Z$ $\quad$ $\mathbb{C}$-submanifold of $B^n(0,1)$ of dimension $\leq n-1$.

After a suitable change of coordinates,
$\{f_1=0\}=\{z_1=0\}$, and
$Z \subseteq \{0\} \times B^{n-1}(0,1)$.
By the induction assumption,
$Reg(Z) \cap (\{0\} \times B^{n-1}(0,1)) \neq \emptyset$.

<u>Definition</u>: $Z \subseteq M$ analytic subset, $p \in Z$, we define
$\dim_p(Z) =$ local dimension of Z at p.
$$= \limsup_{\substack{q \to p \\ q \in Reg(Z)}} \dim_q(Reg(Z)) \in \{0,1,\dots,n\}$$

<u>Observation</u>: If $\dim_p(Z) = \dim(M)$
$\Rightarrow Z$ is open at p
If $\dim_p(Z)=0 \Rightarrow Z \cap U = \{p\}$
for some open neighborhood $U \ni p$.

<u>Thm (*)</u>: For any analytic subset $Z \subseteq M$, $Sing(Z)$ is an analytic subset of M with dimension $\dim_p(Sing Z) < \dim_p(Z)$ $\forall p \in Z$.

$\Omega \subseteq \mathbb{C}^n$ open set.

$f : \Omega \to \mathbb{C}$ holomorphic

$Z = \{f = 0\}$.

Theorem (†): Sing(Z) is analytic, and is equal to:

$$0 = f = \frac{\partial f}{\partial z_1} = \dots = \frac{\partial f}{\partial z_n}$$

Assume $\frac{\partial f}{\partial z_1}$ is not vanishing on any connected component of Z

Note Theorem (†) $\Rightarrow$ Theorem (*) in the case of hypersurfaces. But...

$\triangle$ S $f(x,y) = x^2$, $Z = \{x = 0\}$

$\mathrm{Sing}(Z) = \emptyset$. Need this!

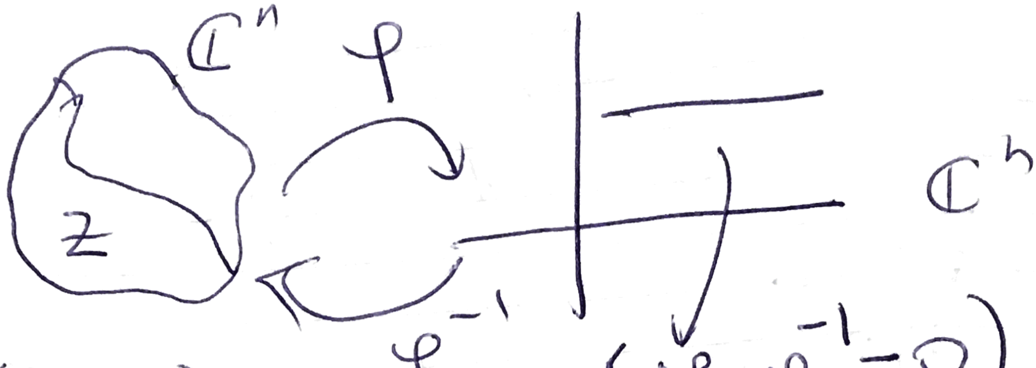
Proof: • Pick $p \in Z$ if $\frac{\partial f}{\partial z_i}(p) \neq 0$.

$\underset{\text{IFT}}{\Longrightarrow} \{f=0\}$ is a $\mathbb{C}$-submanifold near the neighborhood of p.

• Suppose $p \in \mathrm{Reg}(Z)$. Need to show that $\frac{\partial f}{\partial z_i}(p) \neq 0$ for some index i.

Indeed, we can do a change of coordinates so that $Z = \{z_1 = \dots = z_\ell = 0\}$.

Claim: $\ell = 1$



$Z=(f=0)$

$\tilde{f}=f\circ\varphi^{-1}$

$\{z_1=0\}=\{\tilde{f}=0\}$

Expand $\tilde{f}$ into power series:

$\tilde{f}(z_1,z_2)=z_1^k\cdot h(z_1,z_2)$

Such that $h(0,z_2)\not\equiv 0$.

$(h=0)=\{0\}$. $\Longrightarrow$ $h(0)\neq 0$

Since $(h=0)=(0)$, we may consider the holomorphic function $\frac{1}{h}$ on $B\setminus\{0\}$.

Hartogs $\Rightarrow$ $\frac{1}{h}$ is holomophic on B

$\Rightarrow$ $h(0)\neq 0$. Contradiction

<u>Conclusion</u>: $\tilde{f}=z^k\cdot h$ where $h(0)\neq 0$

But then $k=1$, because otherwise

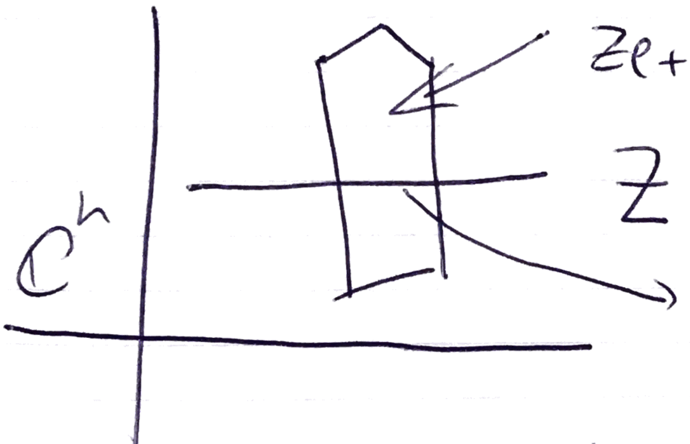
$d\tilde{f}|_{\{\tilde{f}=0\}}=0$ contradicting our assumption.

$\Rightarrow \frac{\partial \tilde{f}}{\partial z_1}(0)\neq 0$.

The claim (that $l = 1$) follows from Hartogs Theorem.

Suppose $n = 2$. If $l \geq 2$, then $Z = \{z_1 = z_2 = 0\}$ is the origin, which we have seen is not possible.

If $n \geq 3$, if $l \geq 2$, for all $(\hat{z}_{l+1}, \ldots, \hat{z}_n) \in \mathbb{C}^{n-l}$,



So the idea is to slice Z by the dimension l plane $\{z_i = \hat{z}_i \mid i = l+1, \ldots, n\}$ and apply Hartogs.