Use Weierstrass Theory to give a detailed geometric description of germs of analytic subsets in \( \mathbb{C}^n \).
(local parametrization lemma).

\[ \text{Analytic Nullstellensatz.} \]

\[ \mathfrak{a} \subseteq \mathcal{O}(\mathbb{C}^n, 0) \quad \text{ideal} \]

\[ \mathcal{I}(V(\mathfrak{a}), 0) = \sqrt{\mathfrak{a}} \]

\[ \text{Cauchy coherence Theorem} \quad \mathcal{I}_{A, x}^7 \]
is coherent.

\[ \text{Singular locus} \]

\[ \text{Sing}(A) = A \setminus \text{Reg}(A) \quad \text{is analytic.} \]
We fix, once and for all, a prime ideal \( \mathfrak{a} \subseteq \mathcal{O}(\mathbb{C}^n) \). We let \( A = V(\mathfrak{a}) \). Put \( \mathfrak{a} \) into "normal form" so that \( A \) can be viewed as an "analytic cover" over some polydisk.

\( \mathfrak{a} \subseteq \mathcal{O}(\mathbb{C}^n) \) is prime. (Assumption for now)

Construction of adapted coordinates 
\((z_1, z_2, \ldots, z_n)\) with

\[ \mathfrak{a}_k = \mathfrak{a} \cap \mathcal{O}(z_1, \ldots, z_k) \]

and depends only on the first \( k \) variables.

\( \exists d \geq 0 \) such that

* \( \mathfrak{a}_d = (0) \)

* For each \( k \) in the range
  \( d + 1 \leq k \leq n \),

\( \mathfrak{a}_k \in \) Weierstrass poly., in \( \mathbb{Z}_k \):

\[
P_k(z_1, \ldots, z_{k-1}, z_k) = \mathbb{Z}_k^{d_k} + \sum a_{jk}(\mathfrak{a}) z_k^{d_k-j}
\]
Proof: induction on \( n \).

\[ \alpha_n = (0) \text{, then done} \]

Otherwise pick \( P_n \in \mathcal{Q} \) \((= \alpha_n)\)

Weierstrass Prep. theorem \(\Rightarrow\) \( P_n = \text{Weierstrass poly., in } \mathbb{Z}_n \)

If \( (\alpha_{n-1}) = (0) \), then done \(\checkmark\)

otherwise apply the induction hypothesis.

\[ \boxed{\text{B: Noether Normalization Lemma}} \]

\[ \mathcal{O}_d = \mathbb{C}[z_1, \ldots, z_d] \xrightarrow{\phi} \mathcal{O}_n/\alpha \]

is a finite integral extension.

\[ \mathcal{O}_n = \mathcal{O}_n(\mathbb{C}^{(n)}) \]

\( \mathcal{O}_n/\alpha \) is a finite type \( \mathcal{O}_d \)-module.

Proof: Want to find \( h_1, \ldots, h_n \in \mathcal{O}_n/\alpha \)

such that \( f \in \mathcal{O}_n \), \( \exists g_1, \ldots, g_n \in \mathcal{O}_d \) with \( f = \sum g_i h_i \mod \alpha \).
Let $f \in \mathbb{R}_n$. Use WEIERSTRASS division to write:

$$f = q_n \cdot P_n + r_n$$

where $r_n = \text{poly of degree } \leq d_n - 1$ with coeff $\in \mathbb{Z}_1, \ldots, \mathbb{Z}_{d_n-1}$.

Divide all coeff. by $P_{n-1}$, etc.

End up with:

$$f = \text{poly. in } (\mathbb{Z}_{d+1}, \ldots, \mathbb{Z}_{d_n}) \text{ mod } \mathbb{A}$$

with coefficients in $\mathbb{Q}_d$ of degree

$$\leq \max 3d_k - 1, \quad k = d + 1, \ldots, n.$$

Take as a family of generators for $\mathbb{Q}_n/\mathbb{A}$ the image in $\mathbb{Q}_n/\mathbb{A}$ of $1, Z_{d+1}, \ldots, Z_{d_n}$.

Observation: any element $f \in \mathbb{Q}_n/\mathbb{A}$ integral over $\mathbb{Q}_d$, $\exists q \geq 1, \ f^q = \sum_{i=0}^{\infty} a_i \cdot f^i \text{ mod } \mathbb{A}$.
Let \( M = \text{Frac}(O_n / \alpha) \) is well-defined since \( O_n / \alpha \) is a domain (as \( \alpha \) is a prime ideal),

\[ M_d = \text{Frac}(O_d). \] We have \[ q = [M : M_d] \]

\[ 1 \leq q < \infty \]

\[ E(z_1, \ldots, z_n) = (z_1, \ldots, z_d, z_{d+1}, \ldots, z_n) \]

\[ (z_i) \]

1. \( M = M_d \lceil z_n \rfloor \)

2. \[ P_n(z_n) = z_n^q + \sum_{j=1}^{q-1} a_{j,n}(z') z_n^q \]

\[ a_{j,n}(0) = 0. \]

3. \[ P_k(z_k) = z_k^{d_k} + \sum_{j=1}^{d_k-1} a_{j,k}(z') z_k^{d_k-j} \]

\[ a_{j,k}(0) = 0 \]

holds for \( d+1 \leq k \leq n, \ d_k \leq q. \)

Obs. \( A = V(\alpha) \)

"\( O_n / \alpha \)"

= \( \{ \text{hol. functions on } A \} \)
Proof: \( \frac{\mathbb{O}_n}{\alpha} \rightarrow \frac{\mathbb{O}_n}{\alpha} \) its image in the quotient.

\( M = M_d[z_{d+1}, \ldots, z_n] \)

The primitive element theorem \( \Rightarrow \) for a generic (open dense)

\( c \in \mathbb{C}^{n-d}, \quad \sum_{i=d+1}^{n} c_i \bar{z}_i \) generates

\( M \) over \( M_d \). We may assume that \( M = M_d[z_n] \). This proves \( \triangle \).

Take the minimal poly. of \( \bar{z}_n \) \( / M_d \)

\( P = T^q + \sum_{j=0}^{q-1} b_j(z') T^j \) \quad \( b_j \in M_d \).

\( \mathbb{O}_n/\alpha \cong \bar{z}_n \) is integral over \( \mathbb{O}_d \).

\( \alpha(T) = T^{q'} + \sum a_j(z') T^j \) \quad \( a_j \in \mathbb{O}_d \)

\( \alpha'(\bar{z}_n) = 0 \).

\( \Rightarrow P | \alpha. \)
\[ \mathcal{Q} = P \circ R \quad \text{all monic!} \]

Since \( \mathcal{O}_d \) is factorial domain, Gauss lemma implies that \( P \) has its coefficients in \( \mathcal{O}_d \).

- **Weierstrass preparation theorem**: (applied to \( P \))
  \[ P = \text{unit} \circ \text{Weierstrass poly}, \]
  \[ (T, z') \]
  \[ z' = z_1, \ldots, z_d \]
  \[ \text{(lemma } \Rightarrow \text{ unit } \in \mathcal{O}_d[T]) \]
  \[ \implies \text{unit } \equiv \text{constant}. \]
  \[ \Rightarrow b_j \in \mathcal{O}_d, \ b_j(0) = 0. \]

This proves the statement \( \Box \).

Exactly the same argument is used to prove statement \( \Box \).
Write $\hat{p} = p_n$. Consider:

$\delta(\hat{p}) = \text{discriminant of } \hat{p}$.

$\in O_d$ that measures whether or not $\hat{p}$ has double roots.

Let $k$ be the splitting field of $\hat{p}$ over $\mathbb{Q}_d$. Then

$\hat{p}(T) = \prod_{i=1}^{2}(T - u_i)$ over $k$.

$\delta(\hat{p}) = \prod_{i \neq j} (u_i - u_j)^2$

belongs to $O_d$, because it is Galois-invariant (under $\text{Gal}(K/\mathbb{Q}_d)$).

$\delta(\hat{p})(z') \neq 0 \iff \exists p_{z'}(T) = 0 \ 	ext{has only 2 simple solutions}.$
For any $\mathbf{t} \in Q_{n}$, $\delta \cdot \mathbf{t} \in D_{d}[\mathbb{Z}/n]$

Here $\delta = \delta(\mathbf{p})$ is the discriminant.

This is existence of universal denominator.

Proof: $\mathbf{f} = \sum b_j \mathbf{z}^j$ by $\Theta$, $b_j \in M_d$.

Write $u_1 = \mathbf{z}^1$, $u_2, \ldots, u_q$ Galois conjugates $G K(M_d)$

$f_1 = \mathbf{f}_1, f_2, \ldots, f_q$

$\mathbf{f}_1 = \sum_{j=0}^{q-1} b_j u_j^1$

$\mathbf{f}_2 = \sum_{j=0}^{q-1} b_j u_j^2$

$\vdots$

$\mathbf{f}_q = \sum_{j=0}^{q-1} b_j u_j^q$

$$
\begin{pmatrix}
\mathbf{f}_1 \\
\mathbf{f}_2 \\
\vdots \\
\mathbf{f}_q
\end{pmatrix} = \begin{pmatrix}
1 & u_1 & \cdots & u_1^{q-1} \\
1 & u_2 & \cdots & u_2^{q-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & u_q & \cdots & u_q^{q-1}
\end{pmatrix}
\begin{pmatrix}
b_1 \\
b_2 \\
\vdots \\
b_q
\end{pmatrix}
$$

Multiply both sides again by $b_j \in \mathbb{Z}[u_1, u_2, \ldots, u_q]$

$\prod_{i=1}^{q} u_i$ multiply both sides again by $b_j \in \text{det}(M)$

$\Rightarrow \delta \cdot b_j$ is integral over $\Theta_d$. 

Since $\Theta_d$ is factorial, we get that $\delta \cdot b \in \Theta_d$.

$\Rightarrow \delta \cdot f \in \Theta_d[z_n]$

\[ \delta z_{d+1} = B_{d+1}(z', z_n) + \alpha \]

\[ \delta z_{n-1} = B_{n-1}(z', z_n) + \alpha \]

\[ \text{deg} (B, z_n) \leq q-1 \]

\[ \Box \quad B = < \hat{P}, \delta z_j - B_j(z', z_n) >_{j=d+1}^{n-1} \]

\[ \exists \mu, \delta^\mu \alpha c B \subseteq \alpha \]

Proof of $\Box$: Let $f \in \alpha$.

Make successive division to get:

W. -division theorem

\[ f = q \hat{P} + \delta z_{j+1} \]

\[ \delta = 0, \ldots, z_{n-1} \]

\[ \sum_{j=0}^{q-1} C_j z_n^j \]

\[ f = \sum_{j=0}^{q-1} C_j z_n^j \]
Repeat this process to get

\[ f = \sum_{d+1}^n p_i \cdot q_i^d + R^d \}

of degree \( \leq q-1 \).

\( \delta \) \( \delta' \), \( \forall \in \mathcal{B} \)

\( \delta^{(n-d)(q-1)} \quad R \in \mathcal{B} \)

\( \delta^{(n-d)(q-1)} \quad R = \sum a_i(z') \left( (\delta_i - B_j) + B_j \right)^I \)

\( \delta^{(n-d)(q-1)} \quad R = \sum a_i(z') \beta_{d+1} \cdots \beta_{i-1} \beta_{i} \quad \text{mod} \mathcal{B} \quad \epsilon \in \mathcal{B} \)

Hence, \( \Phi \) divides the right hand side.

because \( \Phi \) is the minimal polynomial.
\( P_k \in \mathfrak{a}, \quad P_k(\bar{x}_k) = 0 \)

(minimal poly. of \( \bar{x}_k / M_d \))

\[
z_k = \frac{B_k(z, \bar{z}_n)}{\delta} + \text{element of } \mathcal{B}
\]

\( \deg \leq q \).

\[
P_k(z_k) = \left( \frac{B_k}{\delta} \right)^{d_k} + \sum \delta a_{i,j,k}(z)
\]

\( B_k \) only depends

on \( z_n \).

\[
\delta \cdot P_k \quad \text{similar argument as before.}
\]

in \( \Omega_n / \mathfrak{a} \), \( \delta^q P_k(\frac{B_k}{\delta}) = 0. \)
Local Parametrization Lemma (but really, it is a theorem)

\( \mathfrak{a} \leq 0^{(c^n,0)} \) is a prime ideal.

Set \( A = V(\mathfrak{a}, 0) \) closed analytic subset of \( \Delta^n \)

In a polydisk: \( \Delta = \{ (z', z'') \mid z' = z_1, \ldots, z_d \} \)
\( z'' = z_{d+1}, \ldots, z_n \) (for suitable coordinates).

- \( \pi : A \rightarrow \Delta^d \), \( \pi(z', z'') = z' \) is ramified covering of degree \( q \geq 1 \),
  whose ramification locus is included in \( \delta = 0^q \).

\( S = 3 \delta = 0^q \leq \Delta^d \)

\[ A \setminus \pi^{-1}(S) \rightarrow \Delta^d \mid S \]
  is an unramified covering map of degree 2
  (this is the usual notion of covering in topology).

\[ a \] \( A \setminus \pi^{-1}(S), A \) are both connected
  and \( \overline{A \setminus \pi^{-1}(S)} = A \).
\[ p \in \Delta^d \Rightarrow \#\pi'(p) \leq q \] with equality if \( p \notin S' \).

\[ A \leq \frac{1}{2}z''1 \leq C1z'1^2 \] for some \( C > 0 \).

**Proof**: We produced \( \hat{P}(z', z_n) = z_n + \sum_{j} a_j(z')z_n^{q-j} \)
and \( S = \text{discriminant of } \hat{P}(z', 0) \)
\[ d+1 \leq j \leq n-1 \quad \delta_j z_j = B_j(z', z_n) + \alpha \]
\[ B = \langle \hat{P}(z', z_n), \delta(z'), z_j - B_j(z', z_n) \rangle \leq C \]
\[ \exists m \text{ s.t. } \delta^m \alpha \leq B \leq \alpha. \]
\[ V(B) \triangleright V(\emptyset) = A \]
\[ V(B) \setminus (pr')^{-1}(S') = A \setminus \pi'(S) \]

* Pick \( z' \notin S \). We consider \( \pi'(z') \leq A \)

\[ (z', z'') \in A \iff (z', z'') \in V(B) \]

\[ \Leftrightarrow \hat{P}(z', z_n) = 0, \quad \text{and} \quad \delta(z') z_j = B_j(z', z_n) \] for \( d + 1 \leq j \leq n - 1 \).

Since \( \delta(z') \neq 0 \), \( \hat{P} \) has exactly \( q \) solutions

\[ z_n^{(1)}, z_n^{(2)}, \ldots, z_n^{(q)} \]

\[ \pi'(z') = \left\{ (z', B_l(z', z_n^{(l)}), z_n^{(l)}) \middle| \frac{\delta(z')}{\delta(z')} \right\} \]

Consists of \( q \) distinct points.

[We also need * \( |a_j(z')| \leq O(1z'^1j) \)

* all solutions of \( \hat{P}(z', \cdot) = 0 \) are included in a fixed polydisk]
Claim: \( z' \rightarrow z_n^{(e)}(z') \) are holomorphic consequence of the analytic implicit function theorem applied to the polynomial \( \hat{P} \), which can be applied as \( \frac{\partial \hat{P}}{\partial z_n} \neq 0 \). This proves the statement \( \square \).

- \( |a_j(z')| = O(1|z'|^j) \)

Otherwise, we could perturb the coordinates \( z_n' = z_n + \text{linear}(z') \), such that \( \text{deg}(\hat{P}) \) drops.

- Continuity of roots \( \rightarrow \square \)

Lemma: \( P(T) = T^q + a_1 T^{q-1} + \ldots + a_q \)

\( \implies \) solution to \( |_{P(T) = 0} \leq q \max |a_j|^{1/j} \)

\( \implies \) solutions to \( \hat{P}(z', z_n) = 0 \)

\( |a_j| \leq \frac{1}{q^j} \) we would get a contradiction.

\( P(w) = 0, \quad -1 = \frac{a_1}{w^3} + \ldots + \frac{a_q}{w^q} \leq \frac{1}{q^3} \leq \frac{1}{q^j} \)
Lemma: \( P_n = T^q + a_1^{(n)} T^{q-1} + \ldots + a_i^{(n)} \to P \)

If \( P(0) = 0 \), then \( P_n(0) \) s.t. \( w_n \to w \).

Proof of connectedness in \( \mathbb{D} \):

\( A \setminus \pi^{-1}(S') \) is connected \( \iff \) we want to show this. \( A_1, A_2, \ldots, A_N \) connected components of \( A \setminus \pi^{-1}(S') \).

\[
P^{(e)}(z', T) = \prod_{(z', z'') \in A_e} (T - z_n)
\]

\( \delta(z') \neq 0 \)

for \( e = 1, \ldots, N \)

\( P^{(e)} \) polynomials in one variable, and

coefficients in \( O(\Delta_d \setminus S') \) bounded.

(lemma: If \( f \in O(\Delta_d \setminus S') \) \( \iff \) \( f = 0 \) on \( S' \) and \( f \) extends hol. to \( \Delta_d \).

So, all these \( P^{(e)} \in O(\Delta_d)[T] \)

\[
\prod_{e=1}^{N} P^{(e)}(z', T) = \hat{P}(z', T) \quad \text{when } \delta(z') \neq 0
\]

for all \( z' \) by continuity.
But \( \hat{\mathbf{P}} \) is irreducible, so \( \sqrt{\lambda} = 1 \) and so \( A \setminus \pi'(S') \) is connected.

**Claim:** \( A \setminus \pi'(S) = A \) (density).

- **Case:** \( d = n - 1 \), \( B = \langle \hat{\mathbf{P}} \rangle \)
  
  \( (z', z_n) \in A \), \( \delta(z') = 0 \).

Pick \( z'_\varepsilon \to z' \) as \( \varepsilon \to 0 \), and assume that \( \delta(z'_\varepsilon) \neq 0 \).

\[ \pi'(z'_\varepsilon) = \left\{ (z'_\varepsilon, w_n) \mid \hat{\mathbf{P}}(z'_\varepsilon, w_n) = 0 \right\} \]

\[ \hat{\mathbf{P}}(z'_\varepsilon, \ast) \]

By continuity of solutions, \( \hat{\mathbf{P}}(z', z_n) = 0 \) exists \( z_n(\varepsilon) \to z_n \) such that \( \hat{\mathbf{P}}(z'_\varepsilon, z_n(\varepsilon)) = 0 \).

So, \((z'_\varepsilon, z_n(\varepsilon)) \to (z', z_n)\), which proves the density.
Case: $d < n - 1$. Use analytic Nullstellensatz.

Let's prove the analytic Nullstellensatz.

Thm: $\mathfrak{a} \leq 0_{(\mathbb{C}^n, 0)}$, $I(V(\mathfrak{a}), 0) = \sqrt{\mathfrak{a}}$

Proof: $\sqrt{\mathfrak{a}} \subseteq I(V(\mathfrak{a}), 0)$ easy.

Suppose first that $\mathfrak{a}$ is prime. In this case $\sqrt{\mathfrak{a}} = \mathfrak{a}$, and so the claim is that $I(V(\mathfrak{a}), 0) \leq \mathfrak{a}$ ($= \sqrt{\mathfrak{a}}$).

Let $f \in I(V(\mathfrak{a}), 0)$, $f \in \mathcal{O}_n / \mathfrak{a}$.

We want to show that $f = 0$.

Since $\mathcal{O}_n / \mathfrak{a}$ is finite $\mathcal{O}_d$-module.

\[ f + a_1 f + \ldots + a_r = 0 \text{ in } \mathcal{O}_n / \mathfrak{a} \]

(Here, $a_j \in \mathcal{O}_d$ (so depends only on first $d$ variables, i.e. on $z'$).

\[ f + a_1(z') f + \ldots + a_r(z') = 0 \in \mathfrak{a} \]

\[ \Rightarrow a_r(z') \big| A = 0 \text{. By } \mathfrak{a}, \text{ } a_r = 0 \text{ on } \Delta^d \]

(\( \pi(A) = \Delta^d \)).
But then we get to deal by either induction on $r$ or a proof by contradiction (pick $r$ minimal, ...).

$\lambda(z'')$ = linear form in $z_{d+1}, \ldots, z_n$

so it is of the form $c_{d+1}z_{d+1} + \ldots + c_n z_n$.

So generic

\[ p_{\lambda(z', t)} = \prod_{(z', t'') \in \mathcal{A}} (t - \lambda(z'')) \]

[coefficients $\in O(\Delta d/5) + $ bounded

$\Rightarrow$ so extends to $\Delta d$]

$= T^q + \ldots$ where coefficients are in $O(\Delta d) [T^q]$

Claim: $p_{\lambda} | A = 0$.

Argument 1: redo previous argument

$\otimes \oplus$ with $z_n = \lambda(z'')$

Argument 2: $\gamma(A, 0) = 0$

$\prod_{\lambda \mid A} \prod_{\lambda' \mid \mathcal{A}^{\otimes}} = 0$

$\delta \cdot p_{\lambda} | A = 0$
\[ \delta P_{\lambda} \alpha \Rightarrow P_{\lambda} \alpha \text{ prime} \]

Proceed by contradiction \( A \setminus \pi'(s) \neq A \).

Local near the origin, get \( z_j = (z_j', z_j'') \to 0 \) s.t. \( z_j \in A \)

but \( z_j \notin A \setminus \pi^{-1}(s) = A_0 \)

(here \( A_0 := A \setminus \pi^{-1}(s) \)).

\[ z_j'' \notin F_j := \text{pr}'' (\overline{A_0 \cap \pi^{-1}(z_j')} ) \]

finite of cardinality \( \leq 9 \).

Roots of \( P_{\lambda}(z_j', T) \in \lambda (F_j) \)

continuity of roots.

should be true for any \( \lambda \). Now, choose \( \lambda \) s.t.

\[ z_j'' = (z_j'', d_j, \ldots, z_j'' n) \]

\[ P_{\lambda}(z_j', z_j''n) = 0, \quad z_j''n \in \lambda (F_j) \]
Cartan's Coherence Theorem → next Tuesday

A is a complex manifold.

Then \( \text{Sing}(A) = A \setminus \text{Reg}(A) \) is an analytic subset of \( A \) such that for \( x \in A \): \( \dim(\text{Sing}(A), x) \leq \dim(A) - 1 \).

Claim: Suppose \( A \) is irreducible germ of an analytic subset in \( (\mathbb{C}^n, 0) \).

\( \mathfrak{I}(A, 0) = \alpha \), \( \alpha \) is prime.

\( \dim(A, 0) = d \)

\( \dim(A \cap \pi^{-1}(s), 0) \leq d - 1 \).

\( \dim(A, 0) = \limsup_{x \to 0} \dim(A, x) \)

By @, \( A \cap \pi^{-1}(s) \in \text{Reg}(A) \)

It is a local biholomorphism near any point in \( A \setminus \pi^{-1}(s) \), but \( \dim(A \setminus \pi^{-1}(s)) = 0 \)
\[ \Rightarrow \dim_c(A \setminus \tilde{\pi}(s)) = d \Rightarrow \dim(A, o) \geq d. \]

A \( \pi^{-1}(s) \) analytic \( y \sim 0 \) 
\( y \in \text{Reg}(A \pi^{-1}(s)) \).

\( k = \dim(A \pi^{-1}(s), 0) = \dim(A, \pi^{-1}(s), y) \)

\( \pi : (A \pi^{-1}(s), y) \rightarrow A_d \)

Perturbing \( y \), if necessary, we can assume that \( \text{rank}(d\tilde{\pi}) \) is locally constant.

\[ \dim(A \pi^{-1}(s), y) \]

\[ = \dim(\tilde{\pi}(A \pi^{-1}(s)), y) + \dim(\text{fib of } \pi) \]

\[ \leq d-1 + 0 \]

\[ = d-1. \]
Proof:

\[ \text{Obs: } \text{Sing}(A) \subseteq A \cap \overline{\pi^{-1}(S)} \]

proper nowhere dense analytic subset of \( A \).

\[ \text{Obs: } d = n - 1, \text{ exercise.} \]

We need Cartan's Coherence Theorem.

Cartan's Coherence Theorem:

\[ \exists f_1, \ldots, f_N \in \mathcal{O}(\Delta^n) \text{ such that} \]

\[ \forall x \in \Delta^n, \exists (A, x) = \langle f_1, x \rangle, \ldots, \langle f_N, x \rangle \]

\[ \subseteq \mathcal{O}(\Delta^n, x) \]

\[ x \in \text{Reg}(A) \iff A \text{ is locally defined by } n - d \text{ equations} \]

\[ (g_1, g_2, \ldots, g_{n-d}) \]

such that \( \partial g_i(x) \) are linearly independent.
Here, \( g_i \in \mathcal{I}(A, x) \).

\[ \iff \exists \mathbf{i} = (i_1, \ldots, i_{n-d}) \subseteq \{1, \ldots, N\} \text{ such that } df_{i_1}(x), \ldots, df_{i_{n-d}}(x) \text{ are linearly independent.} \]

\[ \text{Sing}(A) \ni x \iff \det(df_{\mathbf{i}}(x)) = 0 \text{ for all } |\mathbf{i}| = n-d. \text{ analytic!} \]