Lecture 1 Tuesday, Jan 7.

Analysis in Several Complex Variables

Main Character: Holomorphic functions in $\mathbb{C}^n$ where $n \geq 2$.

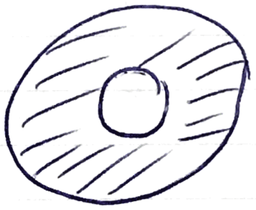
History of one variable: Cauchy, Riemann, Poincaré, Weierstrass (19th century)

German School: 1905 - 1939
Explored basic properties of hol. functions in $\mathbb{C}^n$: Hartogs, Behmke.

Theorem (Hartogs)

$z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$ $n \geq 2$.

Any holomorphic function on $\left\{ \frac{1}{r} < \sum_{i=1}^{n} |z_i|^2 < 1 \right\}$ extends to the unit ball.



⚠ Warning: not true for $n=1$. Take $f(z) = \frac{1}{z}$.

≥ 1945 Cartan (France)
Oka (Japan)
Stein, Grauert, Remmert

They introduce the notion of coherent, analytic, sheafs!
+ algebraic techniques to study analytic subsets of $\mathbb{C}^n$ (or open sets of $\mathbb{C}^n$) for $n \geq 2$.

Holomorphic subset: $\{f = 0\}$ for some hol. func. f.

We shall prove Oka's and Cartan's coherence theorems.

$\geq$ 50's - 60's

Several Complex Variables used tools from PDE's.

$\rightarrow$ resolution of the $\bar{\partial}$-equation turns out to be crucial to construct holomorphic functions in special domains.

$\bar{\partial} u = f$ ⊛

$\rightarrow$ we study subharmonic/plurisubharmonic (psh) functions.

"=" convex functions adapted to $\mathbb{C}$-structure

Examples: $\log|f|$ f=holom.

$\max\{\log|f_i|\}$ are psh functions.

Hörmander : solving $\bar{\partial}$-equation using psh weights.

$\Downarrow$

has lead to big results by Siu, Demailly in the study of complex algebraic varieties using transcendental techniques.

The course will be divided into 4 parts.

§1. Basics on holomorphic functions
analytic continuation, domains of holomorphy.

§2. Analytic Sets (algebraic in nature
with connection with AG)
Oka + Cartan Coherence Theorems.

§3. PSH functions, pseudoconvex domains.
Levi Problem: $\{\text{domains of holomorphy}\}$
$\overset{(?)}{=} \{\text{pseudo-convex domains}\}$.

§4. Solution to the Levi Problem
(Resolution of $\bar{\partial}$-equation)
Will use functional analysis + many estimates.

Uniformization Theory

Thm: $\Omega \subseteq \mathbb{C}$ convex (simply-connected sufficient)
open

Then:
$\exists\, f : \Omega \to \mathbb{D} = \{|z| < 1\}$ hol. and bijective.

This statement is false in higher dimensions!

⚠ The unit ball $\{\sum_{i=1}^{n} |z_i|^2 < 1\} \subseteq \mathbb{C}^n$ $(n \geq 2)$
is not biholomorphic to $\{\max\{z_i\} < 1\}$

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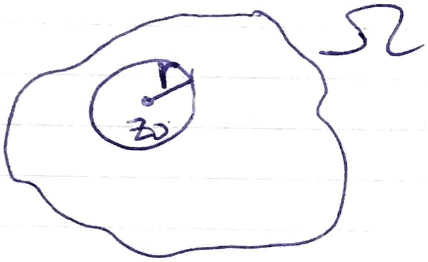
§1.1. Holomorphic Functions in one complex variable.

$\Omega \subseteq \mathbb{C}$ connected open subset.

$f: \Omega \longrightarrow \mathbb{C}$

Definition 1: f is said to be analytic iff for all $z_0 \in \Omega$, $\exists$ power series $\sum a_n z^n$ such that $\sum |a_n| r^n < +\infty$ for some $r > 0$ and $f(z+z_0) = \sum_{n \geq 0} a_n z^n$ for all $z \in \Omega \cap \mathbb{D}(0,r)$.



Definition 2: f (C^1) is conformal iff its differential $df(z_0)$ is $\mathbb{C}$-linear for any $z_0 \in \Omega$ ($\Leftrightarrow$ Cauchy-Riemann equations).

Identify $\mathbb{C}$ with $\mathbb{R}^2$. $z = x + iy$

$f: \Omega \to \mathbb{C} \rightsquigarrow f: \underbrace{\Omega}_{\subseteq \mathbb{R}^2} \longrightarrow \mathbb{R}^2$

$f = h + ig$ where $h, g: \Omega \longrightarrow \mathbb{R}$ are also C^1.

$$df(z_0) = \begin{pmatrix} \frac{dh}{dx} & \frac{dh}{dy} \\ \frac{dg}{dx} & \frac{dg}{dy} \end{pmatrix}$$

$w = u + iv$

It is clear that $df(z_0): \mathbb{R}^2 \to \mathbb{R}^2$ is $\mathbb{R}$-linear.

$$\begin{pmatrix}u\\v\end{pmatrix} \overset{df(z_0)}{\overset{\downarrow}{\longmapsto}} \begin{pmatrix}h_x & h_y\\ g_x & g_y\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}$$

$w = u + iv$

$$w \xrightarrow{df(z_0)} aw + b\bar{w}$$

where $a = \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right) \in \mathbb{C}$ (verify!)

$b = \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \in \mathbb{C}$

$df(z_0)$ is $\mathbb{C}$-linear iff $b = 0$.

We define:

$$\frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} - i\frac{\partial f}{\partial y}\right)$$

$$\frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right)$$

So, $df(z_0)$ is $\mathbb{C}$-linear $\Longleftrightarrow \frac{\partial f}{\partial \bar{z}}(z_0) = 0.$

$$df(z_0) = \frac{\partial f}{\partial z}dz + \frac{\partial f}{\partial \bar{z}}d\bar{z}$$

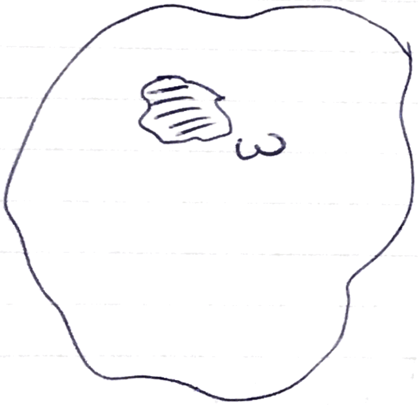
<u>Obs:</u> $\frac{\partial f}{\partial z} = 0 \underset{\substack{\text{C-R relations}\\ \text{(Cauchy-Riemann)}}}{\Longleftrightarrow} \begin{cases} h_x = g_y \\ h_y = -g_x \end{cases}$

Thm: $f : \Omega \longrightarrow \mathbb{C}$ C^1 function
f is analytic iff f is conformal.

Def: f is holomorphic if it is analytic or conformal
$(+C^1)$

Corollary: • $\mathcal{O}(\Omega) = \{ f : \Omega \to \mathbb{C} \text{ hol.} \}$
is a $\mathbb{C}$-algebra.

- $\mathcal{O}(\Omega)$ is also stable under composition.
- f holomorphic $\Rightarrow$ f is C^∞.
- analytic continuation



Ω connected, open.
$f : \Omega \xrightarrow{\text{hol}} \mathbb{C}$
such that $f|_\omega = 0$ where
$\omega \subseteq \Omega$ is open. Then $f = 0$

Proof of Theorem 1: $\Omega = \mathbb{D}(0, 1+\varepsilon)$ where $\varepsilon > 0$
without loss of generality

(analytic $\Rightarrow$ conformal) If f is analytic, then
$f(z) = \sum a_n z^n$ converges on Ω (WLOG).
We would like to compute $\frac{\partial f}{\partial \bar{z}}$.
We get:

$$\frac{\partial f}{\partial \bar{z}} = \sum_{n=0}^{\infty} a_n \frac{\partial z^n}{\partial \bar{z}}$$

$$\frac{\partial z^2}{\partial \bar{z}} = 2z \cdot \frac{\partial z}{\partial \bar{z}}$$, etc. by Leibniz.

So we just need to compute $\frac{\partial z}{\partial \bar{z}}$. We have

$$\frac{\partial z}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial z}{\partial x} + i \frac{\partial z}{\partial y}\right)$$

$$= \frac{1}{2}(1 + i \cdot i) = 0. \quad \checkmark$$

Thus, $\frac{\partial f}{\partial \bar{z}} = 0$ and we are done!

(conformal $\Rightarrow$ analytic)

Assume f is conformal. We want to show the Cauchy formula:

$$f(z) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} d\zeta$$

$$(\zeta - z)^{-1} = \zeta^{-1}\left(1 - \frac{z}{\zeta}\right) = \zeta^{-1} \sum_{n \geq 0} \left(\frac{z}{\zeta}\right)^n$$

$$f(z) = \sum_{n \geq 0} z^n \left[\frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta^{n+1}} d\zeta\right]$$

So this would show that f is analytic.

We just need to show that f satisfies the Cauchy formula.

Green-Riemann Formula

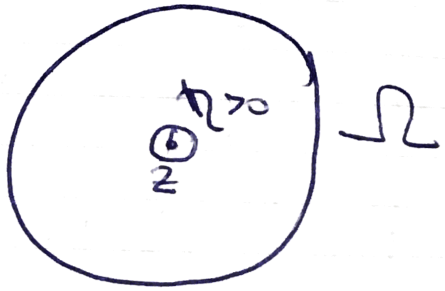
$f:\Omega\to\mathbb{C}$ is $\mathcal{C}^1$.

$$\underset{(*)}{f(z) =} \frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)}{\zeta-z}\,d\zeta + \int_{|w|<1}\frac{1}{\pi(w-z)}\left(\frac{\partial f}{\partial \bar z}\right)d\mathrm{Leb}(w)$$

This is a generalization of Cauchy formula!

⊛ follows from Stokes Theorem, applied to:

$$\omega = \frac{1}{2\pi i}\,\frac{f(w)}{w-z}\,dw$$



$$\int_{\partial\odot}\omega = \int_{\odot}d\omega$$

$\downarrow$

$$+\frac{1}{2\pi i}\int_{|\zeta|=1}\frac{f(\zeta)\,d\zeta}{\zeta-z} - \frac{1}{2\pi i}\int_{|\zeta-z|=\eta}\frac{f(\zeta)}{\zeta-z}\,d\zeta$$

$\downarrow\ \eta\to0$

$f(z)$

use $d\omega=\partial\omega+\bar\partial\omega$ $\quad ||$

$$\int_{\odot}d\omega \overset{\downarrow}{=} \int_{\odot}\frac{1}{2\pi i}\,\bar\partial\left(\frac{f(w)}{w-z}\right)\wedge dw$$

$$= \frac{1}{\pi} \int_{\odot} \left(\frac{\partial f}{\partial \bar{z}}\right) \frac{1}{w-z} \, d\,\mathrm{leb}(w)$$

<u>observation</u>: ① $\frac{1}{2i} \, dw \wedge d\bar{w} = d\,\mathrm{Leb}(w)$

$\frac{1}{w-z} \in L^1_{loc}$ (locally integrable)

$$\Rightarrow \frac{1}{\pi} \int_{\odot} \frac{\partial f}{\partial \bar{z}} \, \frac{d\,\mathrm{Leb}(w)}{w-z}$$

$$\xrightarrow{\eta \to 0} \frac{1}{\pi} \int_{|w|<1} \frac{\partial f}{\partial \bar{z}} \left(\frac{1}{w-z}\right) d\,\mathrm{Leb}(w)$$

§1.2. Holomorphic functions in $\mathbb{C}^n$ $(n>1)$

<u>Notation</u>: $z = (z_1, z_2, \dots, z_n) \in \mathbb{C}^n$

$|z| = \max\{|z_1|, |z_2|, \dots, |z_n|\}$

$\|z\|$ usually refers to $\sqrt{\sum |z_i|^2}$ (Euclidean)

$\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n$ $(\mathbb{N} = \{0, 1, \dots\})$ natural numbers

$z^\alpha := z_1^{\alpha_1} z_2^{\alpha_2} \dots z_n^{\alpha_n} = \prod_{i=1}^{n} z_i^{\alpha_i}$

$\rho = (\rho_1, \rho_2, \dots, \rho_n) \in \mathbb{R}_+^n$

$\mathbb{D}(z,\rho) = \{ w \in \mathbb{C}^n, \ |w_i - z_i| < \rho_i \}$

$\overline{\mathbb{D}}(z,\rho) =$ " " $\leq$ $\}$

$\mathbb{D}(0,r)$ is special case when $z = (0,0,\dots,0)$

$\rho = (r,r,\dots,r)$

$\Omega \subseteq \mathbb{C}^n$ open subset (connected)

<u>Def 1</u>: $f : \Omega \to \mathbb{C}$

f is analytic iff $z_0 \in \Omega$, $f(z+z_0) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$

where $\sum_{\alpha \in \mathbb{N}^m} |a_\alpha| r^{|\alpha|} < +\infty$ for some $r > 0$.

Here $|\alpha| = \sum_{i=1}^{n} \alpha_i$ is the sum of the weights of a given multi-index

$f : \underset{\mathbb{C}^n}{\underset{\cap}{\Omega}} \to \mathbb{C}$ $\quad \mathcal{C}^1$ class

$z_j = x_j + i y_j$ Define

$$\left\{ \begin{aligned} \frac{\partial f}{\partial z_j} &:= \frac{1}{2}\left[\frac{\partial f}{\partial x_j} - i \frac{\partial f}{\partial y_j} \right] \\ \frac{\partial f}{\partial \bar{z}_j} &:= \frac{1}{2}\left[\frac{\partial f}{\partial x_j} + i \frac{\partial f}{\partial y_j} \right] \end{aligned} \right.$$

$$df = \overset{\text{(1,0)-form}}{\underbrace{\sum_{j=1}^{n} \frac{\partial f}{\partial z_j}\, dz_j}_{\partial f}} + \overset{\text{(0,1)-form}}{\underbrace{\sum_{j=1}^{n} \frac{\partial f}{\partial \bar{z}_j}\, d\bar{z}_j}_{\bar{\partial} f}}$$

$$\boxed{df = \partial f + \bar{\partial} f}$$

Theorem 2: $f : \Omega \to \mathbb{C}$ of class $\mathcal{C}^1$.
f is analytic iff $\dfrac{\partial f}{\partial \bar{z}_j} = 0 \quad \forall j$

$(\Leftrightarrow \bar{\partial} f = 0)$

Def: If f is $\mathcal{C}^1$ and analytic / then f is called holomorphic.

Consequences: $\mathcal{O}(\Omega) = \{ f : \Omega \xrightarrow{\text{hol.}} \mathbb{C} \}$ is a $\mathbb{C}$-algebra.

f hol. $\Rightarrow$ f is $\mathcal{C}^\infty$. $\quad (\Omega \ni 0)$

$$f(z) = \sum_{\alpha \in \mathbb{N}^m} a_\alpha z^\alpha \qquad \alpha! = (\alpha_1!)(\alpha_2!)\cdots(\alpha_n!)$$

$$D^\alpha f(z) = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}} (z)$$

$$D^\beta f = \sum a_\alpha D^\beta(z^\alpha)$$

$$D^\beta f(0) = \alpha_\beta \cdot \beta!$$

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} \frac{D^\alpha f(0)}{\alpha!} z^\alpha$$

Lecture 2

Thursday, Jan 9

Suggested exercises: 1, 2, 3. (Chapter 1)

Reference: Hörmander, "Several Complex Variables".

Theorem 2: $f: \underset{\substack{\cap| \\ \text{Connected open} \\ \text{subset of } \mathbb{C}^n.}}{\Omega} \longrightarrow \mathbb{C}$ $\quad \mathcal{C}^1$ function $\quad n \geq 1$

TFAE:

(1) f is analytic: for all $z_0 \in \Omega$,

$$f(z+z_0) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha$$

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ $\qquad z^\alpha := z_1^{\alpha_1} \dots z_m^{\alpha_n}$

$z = (z_1, z_2, \dots, z_n)$

$|z| = \max |z_i| < r$, & $\sum |a_\alpha| r^{|\alpha|} < \infty$, $\quad |\alpha| = \sum \alpha$

(2) $\bar\partial f = 0$, i.e. for all $j = 1, 2, \ldots, n$

$$0 = \frac{\partial f}{\partial \bar z_j} = \frac{1}{2}\left(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j}\right)$$

$z_j = x_j + iy_j$

($\rightsquigarrow$ Cauchy-Riemann Equations)

Proof: (1) $\Rightarrow$ (2) The theorem is purely local, so WLOG, assume

$$\Omega = D^n(0, 1+\varepsilon) = \{\max|z_i| < 1+\varepsilon\} \text{ for } \varepsilon > 0$$

$$f(z) = \sum_{\alpha \in \mathbb{N}^n} a_\alpha z^\alpha, \quad \sum_{\alpha \in \mathbb{N}^n} |a_\alpha| (1+\varepsilon)^{|\alpha|} < \infty$$

$$\frac{\partial f}{\partial \bar z_j} = \sum_{\alpha \in \mathbb{N}^n} a_\alpha \underbrace{\left(\frac{\partial z^\alpha}{\partial \bar z_j}\right)}_{=0} = 0, \text{ as } \frac{\partial z^\alpha}{\partial \bar z_j} = 0$$ by Leibniz + induction.

(2) $\Rightarrow$ (1) We will show that

$\bar\partial f = 0 \rightsquigarrow f$ satisfies "Cauchy formula":

$$\boxed{f(z) = \frac{1}{(2\pi i)^n} \int_{|\zeta_1| = \ldots = |\zeta_n| = 1} \frac{f(\zeta_1, \ldots, \zeta_n)}{(\zeta_1 - z_1)\cdots(\zeta_n - z_n)} d\zeta_1 d\zeta_2 \ldots d\zeta_n}$$

$\Rightarrow$ f is analytic just like in the case of 1-variable. Indeed, we have

$$(\zeta_j - z_j)^{-1} = \zeta_j^{-1}\left(1 - \frac{z_j}{\zeta_j}\right)^{-1} = \zeta_j^{-1} \sum_{\alpha_j \geq 0} \left(\frac{z_j}{\zeta_j}\right)^{\alpha_j}$$

Consequently,

$$f(\zeta) = \frac{1}{(2\pi i)^n} \int\limits_{|\zeta_1| = \cdots = |\zeta_n| = 1} \frac{f(\zeta)}{\zeta_1 \cdots \zeta_n} \left(\sum_{\alpha \in \mathbb{N}^n} \frac{z^\alpha}{\zeta^\alpha} \right) d\zeta_1 \cdots d\zeta_n$$

$$= \sum_{\alpha \in \mathbb{N}^n} z^\alpha \left[\frac{1}{(2\pi i)^n} \int\limits_{|\zeta_1| = \cdots = |\zeta_n| = 1} \frac{f(\zeta)}{(\zeta_i - \zeta_n)\zeta^\alpha} d\zeta_1 \cdots d\zeta_n \right] \qquad \square$$

Theorem 3: $f : \overline{\mathbb{D}}^n(0,1) \to \mathbb{C}$ C^0 function

$\qquad \qquad \qquad \overset{\parallel}{\{\max |z_j| \leq 1\}}$

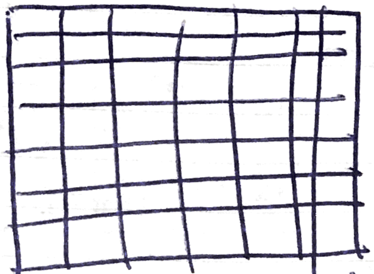
If, for all j, and for all $a \in \mathbb{C}^{n-1}$,

$(\dagger)$ $z_j \mapsto f(a_1, a_2, \ldots, a_{j-1}, z_j, a_j, \ldots, a_n)$

is holomorphic, then f satisfies ⊛.

Observation:

$\bar{\partial} f = 0 \Rightarrow \boxed{(\dagger) \Rightarrow \circledast}$



($n=2$, restriction to any line is holom.)

Proof: $n=1$. Cauchy formula on $D(0,1-\varepsilon)$

$$f(\zeta) = \frac{1}{2\pi i}\int_{|\zeta|=1-\varepsilon} \frac{f(\zeta)}{\zeta - z}\, d\zeta \qquad \varepsilon \to 0$$

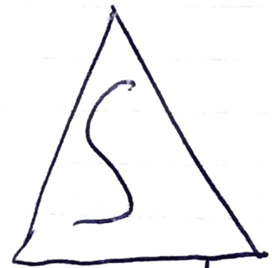
$$f\ \mathcal{C}^0 \Rightarrow \circledast$$

n arbitrary

$$f(\underbrace{z_1, z_2, \dots, z_n}_{\text{fix these}}) = \frac{1}{2\pi i}\int_{|\zeta_1|=1} \frac{\boxed{f(\zeta_1, z_2, \dots, z_n)}\, d\zeta_1}{\zeta_1 - z_1}$$

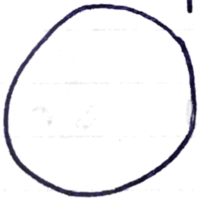
$$\frac{1}{2\pi i}\int_{|\zeta_2|=1} \frac{f(\zeta_1, \zeta_2, z_3, \dots, z_n)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\, d\zeta_2$$

and induction ... □



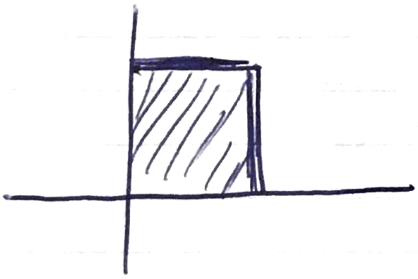
Warning!

$n=1$



f holomorphic

$n \geq 2$



$\partial \mathbb{D}^2 = \}\max\{|z_1|, |z_2|\} = 1\{$

Shilov boundary

$\partial_S \mathbb{D}^2 = \{|z_1| = |z_2| = 1\{$.

Corollary: • f holomorphic $\Rightarrow$ f is $\mathcal{C}^\infty$

- $\mathcal{O}(\Omega) = \{$hol. fun. on $\Omega\}$ is a $\mathbb{C}$-algebra.
- principle of analytic continuation

(Ω connected open subset, $f \in \mathcal{O}(\Omega)$
$f|_\omega \equiv 0$ for some open $\omega \subseteq \Omega \Rightarrow f|_\Omega = 0$.

Proof: For any $\alpha \in \mathbb{N}^n$, let

$$E_\alpha = \{D^\alpha f = 0\}, \quad \alpha = (\alpha_1, \dots, \alpha_n)$$

$$D^\alpha f = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}$$

$$\alpha! = \alpha_1! \cdots \alpha_n!$$

I

$$E = \bigcap_{\alpha \in \mathbb{N}^n} E_\alpha \quad \text{closed}$$

$E \supseteq \omega$. $z_0 \in E$. Now since f is analytic,

$$f(z + z_0) = \sum_{\alpha \in \mathbb{N}^n} \underbrace{\frac{D^\alpha f(z_0)}{\alpha!}}_{=0} z^\alpha = 0$$

$\Rightarrow$ E open.

Since E is closed & open in Ω, we conclude that $E = \Omega$ (as Ω is connected)
$\Rightarrow f|_\Omega = 0$. ▨

§1.3 The analytic implicit function theorem

$\Omega \subseteq \mathbb{C}^m$ domain

Def: $f : \Omega \to \mathbb{C}^m$ holomorphic if $f = (f_1, f_2, \dots, f_m)$ where f_j is holomorphic for all j.

Observation: $f : \Omega \to \mathbb{C}^m$ $\mathcal{C}'$ function

f is holomorphic iff $df(z)$ is $\mathbb{C}$-linear iff $\bar{\partial} f_i = 0$, i.e. $\frac{\partial f_i}{\partial \bar{z}_j} = 0$ $\forall i, j$.

$$\begin{tikzcd}
\Omega_1 \arrow[r, "f"] \arrow[d, phantom, sloped, "\subseteq"] & \Omega_2 \arrow[r, "g"] \arrow[d, phantom, sloped, "\subseteq"] & \Omega_3 \arrow[d, phantom, sloped, "\subseteq"] \\
\mathbb{C}^{n_1} & \mathbb{C}^{n_2} & \mathbb{C}^{n_3}
\end{tikzcd}$$

If f & g are holomorphic, then $g \circ f$ is hol.
[Stable under composition].

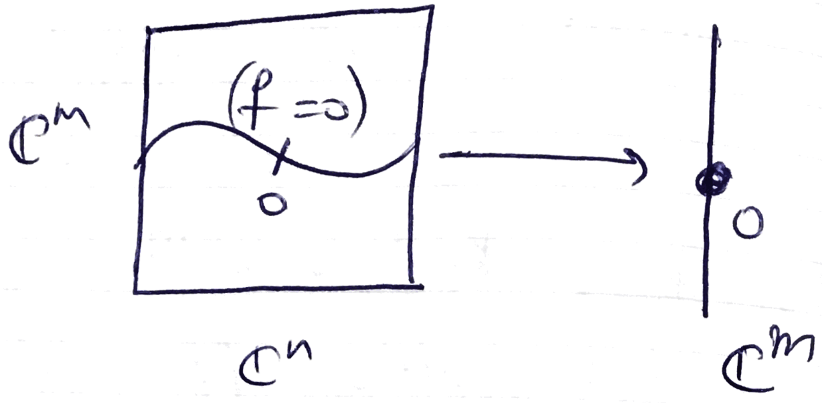
The easiest way to check this is to note that $d(g \circ f) = \underset{\substack{\uparrow \\ \mathbb{C}\text{-linear}}}{(dg) \circ f} \cdot \underset{\substack{\uparrow \\ \mathbb{C}\text{-linear}}}{df}$

$\Rightarrow d(g \circ f)$ is $\mathbb{C}$-linear $\Rightarrow$ $g \circ f$ is holomorphic.

Next, we state the implicit function theorem for holomorphic map.

Theorem 4 (Implicit Function Theorem).

$$f:\underset{\substack{\cap\!| \\ \mathbb{C}^n \times \mathbb{C}^m}}{\Omega} \to \mathbb{C}^m \text{ hol.}$$



$0 \in \Omega$, and

$\det\left(\frac{\partial f_j}{\partial w_k}\right)_{1\le j,k\le m} \neq 0$

$w \in \mathbb{C}^m$, $z \in \mathbb{C}^n$

Then $\exists \rho > 0$, $\exists h: D^n(0,\rho) \to \mathbb{C}^m$,

$\{f=0\} \cap \mathbb{D}^{m+n}(0,\rho) = \{(h(z),z) : z \in \mathbb{D}^n(0,\rho$

Corollary: (Implicit Function Theorem)

$$f:\underset{\substack{\cap\!| \\ \mathbb{C}^m}}{\Omega} \to \mathbb{C}^m \text{ holomorphic.} \quad f(0)=0$$

$df(0) \in GL(m,\mathbb{C})$

Then $\exists \rho > 0$, $\exists g: \Delta^m(0,\rho) \to \Omega$ hol.

$g(0) = 0$ $\quad g \circ f = id$ and $f \circ g = id$.

$\Omega \ni w \mapsto \det df(w) = \det\left(\frac{\partial f_j}{\partial w_k}\right)$

(We can look at the locus

__Def__: $f: \underset{\mathbb{C}^m}{\underset{\cap|_m}{\Omega}} \longrightarrow \underset{\mathbb{C}^M}{\underset{\cap|}{\Omega}}$ is __biholomorphic__

if f is holomorphic and $\exists g: \Omega' \to \Omega$ holomorphic such that

$f \circ g = id_{\Omega'}$ and $g \circ f = id_{\Omega}$

We also have an analogous concept of __local biholomorphism__. Note that the corollary of the implicit function theorem:

$df(0) \in GL(m, \mathbb{C}) \Leftrightarrow f$ is local bihol. at 0.

__Remark__: f is biholomorphic $\Leftrightarrow$ f is hol. + bijective.

(Exercise when $m = 1$, but harder in general)

__Proof of Theorem 4__: 2 approaches to Thm 4:

① analytic def: of holom. Solve $f(h(z), z) = 0$
h = power series

Krantz, Park "Implicit function theorem" Chap. 6

② Apply $\mathcal{C}^1$ IFT & check that h is holomorphic (using $\bar{\partial}$-equation).

View f as a $\mathcal{C}^1$-real map

$F: \mathbb{R}^{2m} \times \mathbb{R}^{2n} \longrightarrow \mathbb{R}^{2m}$

$w_j = \mu_j + i v_j$

$z_j = x_j + i y_j$

$$f(w_1, \dots, w_m, z_1, \dots, z_n)$$

$$\|$$

$$F(\mu_1, v_1, \mu_2, v_2, \dots, x_1, y_1, \dots, x_n, y_n)$$

$$= (\operatorname{Re} f_1, \operatorname{Im} f_1, \operatorname{Re} f_2, \operatorname{Im} f_2, \dots, \operatorname{Im} f_m)$$

Need to check that

$$A = \begin{pmatrix} \frac{\partial \operatorname{Re} f_1}{\partial \mu_1}, & \frac{\partial \operatorname{Re} f_2}{\partial v_1}, & \frac{\partial \operatorname{Re} f_1}{\partial \mu_2}, & \dots & \frac{\partial \operatorname{Re} f_1}{\partial \mu_m}, & \frac{\partial \operatorname{Re} f_1}{\partial v_m} \\ \frac{\partial \operatorname{Im} f_1}{\partial \mu_1}, & \frac{\partial \operatorname{Im} f_1}{\partial v_1}, & \dots & & & \\ \vdots & & & & & \\ \frac{\partial \operatorname{Re} f_m}{\partial \mu_1}, & \frac{\partial \operatorname{Re} f_m}{\partial v_1}, & \dots & & & \\ \frac{\partial \operatorname{Im} f_m}{\partial \mu_1}, & \frac{\partial \operatorname{Im} f_m}{\partial v_1}, & \dots & & & \end{pmatrix}$$

$A = (2m) \times (2m)$ real matrix

$M = \left(\frac{\partial f_j}{\partial w_k} \right)$ is invertible at $\vec{0}$

(this is $m \times m$ $\mathbb{C}$-matrix)

Now we will relation between the two matrices A & M.

$$D = \begin{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} & & & \\ & \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} & & \\ & & \ddots & \\ & & & \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \end{pmatrix}$$ block diagonal matrix.

Then it can be checked that

$$D^{-1}AD = 2\begin{pmatrix} \begin{pmatrix} \frac{\partial f_1}{\partial w_1} & 0 \\ 0 & \overline{\frac{\partial f_1}{\partial w_1}} \end{pmatrix} & \cdots & \\ & \ddots \quad \begin{pmatrix} \frac{\partial f_j}{\partial w_k} & 0 \\ 0 & \overline{\frac{\partial f_j}{\partial w_k}} \end{pmatrix} & \\ & \vdots & \ddots \end{pmatrix}$$

$\Rightarrow \quad 4^m \det(A) = |\det(M)|^2.$

Now we apply $\mathcal{C}^1$ Implizit function theorem.

$\Rightarrow \{f=0\} \cap \mathbb{D}^{m+n}(0,\rho) = \{(h(z),z) : z \in \mathbb{D}^n(0,\rho)\}$

h is in $\mathcal{C}^1$. Let's check that $\bar\partial h = 0$.

$$\Downarrow$$

$$\frac{\partial h_j}{\partial \bar z_\ell} = 0.$$

$$0 = f(h(z), z)$$

$$= (f_1(h_1,\dots,h_m,z),\ f_2(h_1,h_2,\dots,h_m,z),\dots)$$

$$0 = \frac{\partial}{\partial \bar z_\ell}\left(f_1(h(z),z)\right) = \sum_{j=1}^{m} \frac{\partial f_1}{\partial w_j} \cdot \frac{\partial h_j}{\partial \bar z_\ell}$$

$$\sum_{j=1}^{m} \frac{\partial f_i(0)}{\partial w_j} \frac{\partial h_j(0)}{\partial \bar z_\ell} = 0 \qquad i = 1,2,\dots,m.$$

Since $\left(\frac{\partial f_i}{\partial w_j}(0)\right) \in GL(n, \mathbb{C})$

$\Rightarrow \left(\frac{\partial f_i}{\partial w_j}(z)\right) \in GL(n, \mathbb{C})$ for any $|z| \ll 1$

$\Rightarrow \left(\frac{\partial h_1}{\partial \bar{z}_e}(z), \dots, \frac{\partial h_m}{\partial \bar{z}_e}(z)\right) = 0.$

Lecture 3

Tuesday, Jan 14

§1.4. Power Series and Reinhardt domains

Def: $\Omega \subseteq \mathbb{C}^n$ $(n \geq 1)$ is a Reinhardt domain if it is invariant by the action of $(S^1)^n \subseteq \mathbb{C}^n$, i.e.

if $z = (z_1, \dots, z_n) \in \Omega$, and

$t = (t_1, \dots, t_n) \in (S^1)^n$ with $|t_i| = 1$

$\Rightarrow tz = (t_1 z_1, \dots, t_n z_n) \in \Omega$

Another way to write this is:

$z \in \Omega \Rightarrow (e^{i\theta_1} z_1, e^{i\theta_2} z_2, \dots, e^{i\theta_n} z_n) \in \Omega$

for all $\theta_1, \theta_2, \dots, \theta_n \in \mathbb{R}$.

Ex: $n = 1$

Ω connected open Reinhardt $\Leftrightarrow$ it is an annulus or a disk (centered at 0)