Math-601D-201: Lecture 21. Pseudo-convex domains and $\bar{\partial}$-equation

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Calculus on \((p, q)\)-forms

\[ \Omega \subset \mathbb{C}^n \text{ connected open set.} \]

\[ (p, q)\text{-forms} \]

\[ u = \sum_{|I|=p, |J|=q} u_{I,J}(z) \, dz^I \wedge d\bar{z}^J \]

\[ du = \partial u + \bar{\partial} u \text{ where } \partial u \text{ is a } (p+1, q) \text{ form and } \bar{\partial} u \text{ is a } (p, q+1) \text{ form} \]

\[ \bar{\partial} u = \sum_{|I|=p, |J|=q} \bar{\partial} (a_{I,J}) \wedge dz^I \wedge d\bar{z}^J \]

\[ d^2 = 0 \text{ implies } \partial^2 = \bar{\partial}^2 = 0 \text{ and } \partial \bar{\partial} + \bar{\partial} \partial = 0 \]
Calculus on \((p, q)\)-forms

\( f : \Omega_1 \to \Omega_2 \) holomorphic, and \( \omega \) smooth \((p, q)\)-form in \( \Omega_2 \). Then

\[
f^* (\bar{\partial} \omega) = \bar{\partial} (f^* \omega)
\]

\( \longrightarrow \) the operator \( \bar{\partial} \) can be transported to any complex manifold.

**Definition**

\( \Omega \subset \mathbb{C}^n \).

\[
H^{p, q}(\Omega) = \{ \omega \in C^\infty_{p, q}(\Omega), \quad \bar{\partial} \omega = 0 \} / \bar{\partial} C^\infty_{p, q-1}(\Omega)
\]
Resolution of $\bar{\partial}$ operators on pseudo-convex domains

**Theorem**

Let $\Omega \subset \mathbb{C}^n$ be any pseudo-convex domain. For any smooth $(p, q + 1)$-forms $f$ on $\Omega$ satisfying $\bar{\partial}f = 0$, there exists a smooth $(p, q)$-form $u$ such that $\bar{\partial}u = f$. In other words,

$$H^{p,q}(\Omega) = 0 \text{ for all } q > 0.$$

→ we are going to follow Hörmander’s approach based on Hilbert spaces technics
Characterization of pseudo-convex domains

**Theorem**

$\Omega \subset \mathbb{C}^n$. The following are equivalent.

- $\Omega$ is a domain of holomorphy;
- $\Omega$ is pseudo-convex;
- $H^{p,q}(\Omega) = 0$ for all $q > 0$;
- $H^{0,q}(\Omega) = 0$ for all $0 < q < n$. 
Behnke-Thullen’s theorem

**Theorem**

\[ \Omega_j \subset \Omega_{j+1} \subset \mathbb{C}^n \text{ pseudo-convex domains.} \]

*Then* \( \bigcup_j \Omega_j \) *is pseudo-convex.*