ANALYSIS IN SEVERAL COMPLEX VARIABLES: ANALYTIC SETS.
EXERCICES FOR THE CHAPTER 2

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Exercise 1.
• Show that the unit sphere $S^2$ can be endowed with a structure of complex manifold.
• Prove that no sphere of odd dimension can be endowed with a structure of complex manifold.

Comment: it is a theorem of Borel and Serre that if $S^n$ can be endowed with a structure of complex manifold, then $n \in \{2, 6\}$, and it is still open whether $S^6$ can be endowed with a structure of complex manifold, see https://mathoverflow.net/questions/101888

Exercise 2 (Hopf manifold). Pick any $\lambda \in \mathbb{C}^\ast$ such that $0 < |\lambda| < 1$, and consider the linear map $\Lambda(z_1, \cdots, z_n) = (\lambda z_1, \cdots, \lambda z_n)$.
• Prove that $\Lambda$ acts properly discontinuously on $\mathbb{C}^n \setminus \{0\}$ (ie for any compact set $K$, the set $\{n \in \mathbb{Z}, \Lambda^n(K) \cap K \neq \emptyset\}$ is finite).
• Prove that the quotient space $S := (\mathbb{C}^n \setminus \{0\})/\langle \Lambda \rangle$ can be endowed with a unique structure of complex manifold such that the canonical projection map $\pi: (\mathbb{C}^n \setminus \{0\}) \to S$ is holomorphic. Here we identify two points $p, p' \in \mathbb{C}^n \setminus \{0\}$ iff $p' = \Lambda^n(p)$ for some $n \in \mathbb{Z}$.
• Show that $S$ is compact.
• Prove that $S$ is homeomorphic to $S^1 \times S^{2n-1}$.

Comment: it is possible to endow any product $S^{2m-1} \times S^{2n-1}$ with a structure of complex manifold (Calabi-Eckmann manifolds).

Exercise 3. A lattice $\Lambda$ in $\mathbb{C}^n$ is an additive subgroup which is discrete.
• Prove that the quotient space $T = \mathbb{C}^n/\Lambda$ can be endowed with a unique structure of complex manifold such that the canonical projection map $\pi: \mathbb{C}^n \to T$ is holomorphic.
• Prove that for any integer $k \in \mathbb{Z}$, the linear map $\phi_k(z) = kz$ from $\mathbb{C}^n$ to $\mathbb{C}^n$ induces a holomorphic self-map $\Phi_k: T \to T$.

Exercise 4. Let $\Omega$ be any open subset of $\mathbb{C}^n$ and let $f: \Omega \to \mathbb{C}^k$ be any holomorphic function. Prove that for any $l \leq \min\{n, k\}$ the set of point $z \in \Omega$ such that the rank of $dF(z)$ is $\leq l$ is an analytic subset.
• Construct an example with $k = n = 2$ where $\{\text{rank } dF(z) = 0\}$ is a point, and $\{\text{rank } dF(z) \leq 1\}$ is a line.
• Construct an example with $k = n = 2$ where $\{\text{rank } dF(z) = 0\}$ is a line, and $\{\text{rank } dF(z) = 1\}$ is empty.

Exercise 5. Let $f: \Omega \to \mathbb{C}$ be any holomorphic function defined on an open subset of $\mathbb{C}^n$.

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Exercise 6.  
• Prove that \( \mathcal{O}_{\mathbb{C},0} \) is a principal ideal domain (i.e. every ideal is a power of the maximal ideal).  
• Prove that \( \mathcal{O}_{\mathbb{C}^n,0} \) is not a principal ideal domain for any \( n \geq 2 \).

Exercise 7. Let \( f \) be a holomorphic map defined in a neighborhood of \( 0 \in \mathbb{C}^n \) with values in \( \mathbb{C} \) such that \( f(0) = 0 \) and \( df(0) \neq 0 \).

• Prove that \( f \) is a distinguished polynomial of degree 1 in one of the variables \( z_1, \ldots, z_n \).
• Using Weierstrass preparation theorem, prove that \( \{f = 0\} \) is a submanifold in a neighborhood of 0.

Exercise 8. Suppose that \( f \in \mathcal{O}_{\mathbb{C}^n,0} \) is a Weierstrass polynomial in the variables \( z = (z', w) \). Prove that if \( f \) is irreducible in \( \mathcal{O}_{\mathbb{C}^n-1,0}[w] \) then it is irreducible in \( \mathcal{O}_{\mathbb{C}^n,0} \).

Exercise 9. Let \( Z \) be any analytic subset of a complex manifold \( X \).

A continuous function \( f: Z \to \mathbb{C} \) is said to be holomorphic if for any \( x \in Z \) there exists a open neighborhood \( x \in V \subset X \) and a holomorphic function \( F: V \to \mathbb{C} \) such that \( F|_Z = f \).

For any open subset \( \Omega \) of \( V \), we let \( \mathcal{O}_V(\Omega) \) be the set of all holomorphic functions on \( \Omega \).

• Prove that \( \mathcal{O}_V \) is a sheaf of local \( \mathbb{C} \)-algebras (i.e. each stalk \( \mathcal{O}_{V,x} \) is a local ring).
• Prove that for each \( x \in V \), the stalk \( \mathcal{O}_{V,x} \) is isomorphic to \( \mathcal{O}_{X,x}/\mathfrak{m}_{V,x} \).
• Deduce that \( \mathcal{O}_{V,x} \) is Noetherian, and that it is a domain iff the germ \( (V,x) \) is irreducible.
• Suppose \( Z = \{x^2 = y^3\} \subset \mathbb{C}^2 \). Using the normalization map \( t \in \mathbb{C} \to (t^3, t^2) \in Z \) show that \( \mathcal{O}_{Z,0} \) is not isomorphic to \( \mathcal{O}_{\mathbb{C},0} \) as a \( \mathbb{C} \)-algebra (consider the quotient \( \mathcal{O}_{Z,0} \) by its maximal ideal).

Exercise 10. Let \( \mathcal{F} \) be any sheaf of \( \mathcal{O}_V \)-modules on a complex manifold \( V \). Prove that the support of \( \mathcal{F} \) (defined as the set of points for which \( \mathcal{F}_x \neq (0) \)) is an analytic subset whenever \( \mathcal{F} \) is of finite type.

Exercise 11. Let \( \mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3 \) be sheaves of \( \mathcal{O}_V \)-modules on a complex manifold \( V \). Suppose that we have a morphism \( \rho: \mathcal{F}_1 \to \mathcal{F}_2 \).

• Prove that \( \ker(\rho) \) is a coherent sheaf if \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are coherent.
• Prove that \( \mathcal{I}(\rho) \) is a coherent sheaf if \( \mathcal{F}_1 \) and \( \mathcal{F}_2 \) are coherent.
• Suppose that we have an exact sequence \( 0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0 \). Prove that if any two of the sheaves are coherent then the third one is also coherent.
Exercice 12. Consider the sheaves $\mathcal{F} = \mathcal{O}^{\oplus 3}_V$ and $\mathcal{G} = \mathcal{O}_V$ on $V = \mathbb{C}^3$, and define the morphism

$$\varphi : \mathcal{F} \to \mathcal{G}, \quad (f_1, f_2, f_3) \mapsto z_1f_1 + z_2f_2 + z_3f_3$$

Show that $\varphi$ is a surjective sheaf morphism, and compute $\ker(\varphi)$ (the difficulty is in computing the stalk over $0$).

Exercice 13 (Hilbert syzygies theorem). Let $V$ be any complex manifold of dimension $n$, and let $\mathcal{F}$ be any coherent analytic sheaf of $\mathcal{O}_V$-modules.

- Using Oka's theorem, prove that there exists an exact sequence of sheaves of $\mathcal{O}_V$-algebras
  $$\cdots \to \mathcal{O}^{\nu_n}_V \xrightarrow{\sigma_n} \mathcal{O}^{\nu_{n-1}}_V \xrightarrow{\sigma_{n-1}} \cdots \to \mathcal{O}^{\nu_1}_V \xrightarrow{\sigma_1} \mathcal{F} \to 0 \, .$$

- Fix a point $x \in V$, and choose local coordinates $(z_1, \cdots, z_n)$ centered at that point. For each $j = 1, \cdots, n$, let $m_j \cdot \mathcal{O}_{V,x}$ be the ideal generated by $z_1, \cdots, z_j$, and denote by $S_k \subset \mathcal{O}^{\nu_k}_{V,x}$ the kernel of $\sigma_k$ (at $x$).

  We shall prove that

  $$(1) \quad S_k \cap m_j \mathcal{O}^{\nu_k}_{V,x} = m_j \cdot S_k \text{ for all } 1 \leq j \leq k \, .$$

  - Prove the inclusion $\supset$.
  - Prove the statement for $k = 1$.
  - Proceed by induction on $j$, and pick $F = z_1G_1 + \cdots + z_{j+1}G_{j+1} \in S_k \cap m_{j+1} \mathcal{O}^{\nu_k}_{V,x}$ for some $k \geq j + 1$. Prove that $\sigma_k(G_{j+1}) = z_1G_1' + \cdots + z_jG_j'$ for some other $G_1', \cdots, G_j' \in \mathcal{O}^{\nu_k}_{V,x}$.
  - Apply $\sigma_{k-1}$ to the previous equality and show there exist $H_1, \cdots, H_j \in \mathcal{O}^{\nu_k}_{V,x}$ such that $G_{j+1} - (z_1H_1 + \cdots + z_jH_j) \in S_k$.
  - Conclude the induction step.

- Let $F_1, \cdots, F_r$ be a minimal set of generators of $S_{n-1}$, and consider the natural morphism $\rho : \mathcal{O}^{\nu}_{V,x} \to S_{n-1}$ they induce.

  Let $\mathcal{K}$ be the kernel of $\rho$. Show that $\mathcal{K} \subset m_{V,x} \cdot \mathcal{O}^{\nu}_{V,x}$, and use (1) to conclude that $\mathcal{K} = m_{V,x} \cdot \mathcal{O}^{\nu}_{V,x}$.

- Apply Nakayama’s result to prove that $\mathcal{K} = 0$.

- What did we prove about coherent sheaves?

PIMS

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