ANALYSIS IN SEVERAL COMPLEX VARIABLES: EXERCICES FOR THE CHAPTER 1

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Exercice 1. Let $U \subset \mathbb{C}^n$ and $V \subset \mathbb{C}^m$ be open sets and suppose $f\colon U \to V$ and $g\colon V \to \mathbb{C}^k$ are smooth maps. Write $z = (z_1, \cdots, z_n) \in U$ and $w = (w_1, \cdots, w_n) \in V$. Show that

$$\frac{\partial g \circ f}{\partial z_j} = \sum_{l=1}^{m} \left(\frac{\partial g}{\partial w_l} \circ f \, \frac{\partial f_l}{\partial z_j} + \frac{\partial g}{\partial \bar{w}_l} \circ f \, \frac{\partial \bar{f}_l}{\partial z_j} \right)$$

$$\frac{\partial g \circ f}{\partial \bar{z}_j} = \sum_{l=1}^{m} \left(\frac{\partial g}{\partial w_l} \circ f \, \frac{\partial f_l}{\partial \bar{z}_j} + \frac{\partial g}{\partial \bar{w}_l} \circ f \, \frac{\partial \bar{f}_l}{\partial \bar{z}_j} \right)$$

Prove that the composition of two holomorphic (resp. anti-holomorphic) maps remains holomorphic.

Exercice 2. Let $f\colon U \to \mathbb{C}^n$ be any holomorphic map defined on an open set $U \subset \mathbb{C}^n$. Write $z = (z_1, \cdots, z_n) \in U$, $z_j = x_j + iy_j$ and $f_j = h_j + ig_j$, and define the real map $F(x_1, y_1, \cdots, x_n, y_n) = (h_1, g_1, \cdots, h_n, g_n)$. Show that

$$|\det(df)|^2 = \det(dF)$$

Indication: treat the case $n = 1$ first.

Exercice 3.

(a) Let $F\colon \Delta^n \to \mathbb{C}$ be any holomorphic function on the unit polydisk $\Delta^n = \{z \in \mathbb{C}^n, |z_i| \le 1\}$. Prove that for any multi-index $\alpha = (\alpha_1, \cdots, \alpha_n)$, and for any $r < 1$ one has

$$\sup_{\Delta^n(r)} |\partial^\alpha F| \le \frac{(\alpha!) \sup_{\Delta^n} |F|}{2\pi(1-r)^{|\alpha|}} \ ,$$

where $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$, and

$$|\partial^\alpha F(0)| \le \frac{(\alpha!) \sup_{\Delta^n} |F|}{2\pi} \ .$$

(b) Prove the following generalization of the previous statement. Let K be any compact subset of a connected open subset $\Omega \subset \mathbb{C}^n$. Show that for any multi-index $\alpha \in \mathbb{N}^n$, there exists a constant $C > 0$ such that

$$\sup_K |\partial^\alpha F| \le C \sup_\Omega |F| \ ,$$

for any holomorphic map $F\colon \Omega \to \mathbb{C}$.

(c) Let $F_n\colon \Omega \to \mathbb{C}$ be any sequence of holomorphic maps converging uniformly. Show that the limit function $F = \lim_n F_n$ is holomorphic.

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(d) Let $F_n\colon \Omega \to \mathbb{C}$ be any sequence of holomorphic maps with locally uniformly bounded image, i.e. $\sup_n \max_K |F_n| < +\infty$ for any compact set $K \subset \Omega$. Show the existence of a subsequence F_{n_k} which converges uniformly on compact sets of Ω (use Arzelà-Ascoli theorem).

Exercice 4. Let $F\colon \Omega \to \mathbb{C}$ be any holomorphic function on a connected open set $\Omega \subset \mathbb{C}^n$. Suppose that $|F|$ locally attains its maximum at some point $p \in \Omega$. Show that F is constant.

Deduce that there exists no non-constant holomorphic map from the unit disk Δ in $\mathbb{C}$ to the $(2n-1)$-dimensional sphere in $\mathbb{C}^n$.

Exercice 5. Let $\mathcal{O}(\Omega)$ be the set of holomorphic functions on a connected open subset $\Omega \subset \mathbb{C}^n$. Determine all units of the ring $\mathcal{O}(\Omega)$.

Exercice 6. Consider the following two holomorphic maps $F_1, F_2\colon \Delta^2(1) \to \mathbb{C}^2$.

(a) Show that $F_1(x,y) = (x^2+y^2, xy)$ is an open map. To do so compute $\det(dF_1)$; and argue that F_1 is locally open at a point (x_0,y_0) by solving $F_1(x,y) = (x_1,y_1)$ for all (x_1,y_1) close to $F_1(x_0,y_0)$.

(b) Show that $F_2(x,y) = (x, xy)$ is not an open map (compute the image of a polydisk).

Exercice 7 (Hartog's figure). Consider the following Reinhardt domain:

$$H_n = \left\{ z \in \Delta^n(1), |z_1| \leq \frac{1}{2} \Rightarrow |z_j| \leq \frac{1}{2} \text{ for all } j \right\} .$$

Show that any holomorphic function on H_n extends to the unit polydisk.

Exercice 8. Show that $\mathbb{C} \times \{0\} \subset \mathbb{C} \times \mathbb{C}$ has a fundamental system of neighborhoods that are logarithmically convex complete Reinhardt domains. Generalize the construction to treat the case $\mathbb{C}^n \times \{0\} \subset \mathbb{C}^n \times \mathbb{C}^m$.

Exercice 9 (Krantz p.79). Prove that $\Omega_a = \{(z,w) \in \mathbb{C}^2, |z| \cdot |w|^a < 1\}$ is a domain of holomorphy for any $0 < a < \infty$. Is every bounded holomorphic function on Ω_a a constant? One might consider parameterized curves of the form $t \mapsto (e^{-t}, ce^{t/a})$ with $|c| < 1$.

Exercice 10. (a) Use Cauchy estimates to prove that any holomorphic function $f\colon \mathbb{C} \to \mathbb{C}$ such that $|f(z)| \leq C(1+|z|)^k$ for some $C > 0$ and some integer $k \geq 0$ is necessarily a polynomial of degree $\leq k$.

(b) Show that any biholomorphism $f\colon \mathbb{C} \to \mathbb{C}$ is an affine map, i.e. a polynomial of degree 1. Indication: consider $1/f(1/z)$ near $z = 0$.

(c) Prove that $H(x,y) = (y, x+y^2)$ is a biholomorphism $H\colon \mathbb{C}^2 \to \mathbb{C}^2$.

(d) Show that for any $n \geq 2$ and for any $d \geq 1$ one can find a biholomorphism $H\colon \mathbb{C}^n \to \mathbb{C}^n$ whose components are polynomials, one of each has degree d.

Exercice 11. The aim of this exercice is to prove the following theorem of Rothstein. Recall that a map $f\colon X \to Y$ is proper if $f^{-1}(K)$ is compact for any compact $K \subset Y$. Let $B_2 = \{|z_1|^2 + |z_2|^2 < 1\}$ be the unit euclidean ball in $\mathbb{C}^2$, and $\Delta_2 = \{\max\{|z_1|, |z_2|\} < 1\}$ the polydisk of polyradius 1.

Any proper holomorphic map $f\colon \Delta_2 \to B_2$ is a constant.

(a) Let $f\colon \Delta_2 \to B_2$ be any continuous map. Show that f is proper iff for any $p_k \to \partial\Delta_2$ we have $f(p_k) \to \partial B_2$.

(b) Suppose $f\colon \Delta_2 \to B_2$ is holomorphic and proper. Take $e^{i\theta} \in S^1$, and any $w_k \to e^{i\theta}$.
- Define $g_k(\zeta) = f(\zeta, w_k)$. Prove that any subsequence of g_k admits a sub-subsequence converging uniformly on compact subsets to a holomorphic map g.
- Show that the image of g is included in ∂B_2.
- Prove that the sequence of derivatives g_k' converge uniformly to 0 on compact subsets.
- Prove that for a fixed ζ, the function $\frac{\partial f}{\partial z_1}(\zeta, z_2)$ tends to 0 when $|z_2| \to 1$.
- Conclude that $\frac{\partial f}{\partial z_1} = 0$.
- Prove Rothstein's theorem.

(c) Prove that there exists no biholomorphism between Δ_2 and B_2.

(d) Extend the argument to any dimension: for any $n \geq 2$, any proper holomorphic map $f\colon \Delta_n \to B_n$ is a constant.

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