32

# (9) Dynamics of polynomial maps

→ before delving to the work of Benedetto on UBC, need to ~~[illegible]~~ understand in some more depth the dynamics of rational maps

setting: $(K,|\cdot|)$ alg. closed complete $P \in K[T]$

## the archimedean case $K = \mathbb{C}$.

recall from estimates to define canonical heights

$\exists C_1$ $\left| \frac{1}{d} \operatorname{Log}^+ |P(z)| - \operatorname{Log}^+ |z| \right| \leq C_1$

$$g_P(z) = \underbrace{\lim}_{\text{uniform on } \mathbb{C}} \frac{1}{d^n} \operatorname{Log}^+ |P^n(z)| \quad : \mathbb{C} \longrightarrow \mathbb{R}_+.$$

Properties

* $g_P \circ P(z) = d\, g_P(z)$
* $g_P(z) = \operatorname{Log}^+ |z| + O(1)$

exercise: when $P$ is monic $P = z^d + \ldots$ , $g_P(z) = \operatorname{Log}^+|z| + o(1)$.

Consequences

* $\Omega_P = \{ g_P > 0 \}$ open totally invariant, contains $\mathbb{C} \setminus \mathbb{D}(0,R)$ for some $R \gg 0$

$z \in \Omega_P$ $\quad g_P(P^n(z)) = d^n g_P(z) \to \infty$

$\Rightarrow |P^n(z)| \geq \exp(-C_1 + d^n g_P(z)) \to \infty$

$\Omega_P$ = Basin of attraction of $\infty$.

* $K(P) \overset{\text{def}}{=}$ Filled-in Julia set $\overset{\text{def}}{=} \{ g_P = 0 \}$

compact totally invariant

$\supseteq \operatorname{Per}(f)$ non empty

$J(P) = \partial K(P)$ Julia set.

3/ Couple of observations.

⟶ $J(P)$ is the locus where the dynamics is chaotic

$z \in J(P)$ $\varepsilon > 0$ $\exists z'$ $|z-z'| \leq \varepsilon$ $\sup_n |P^n(z) - P^n(z')| \geq 1$

(sensitivity to the initial condition)

⟶ $K(P)$ is connected iff $J(P)$ is connected (exercise!)

$\iff$ all critical points of $P$ belong to $K(P)$

thm

See Carleson - Gamelin.

⟶ the non-archimedean case.

$K$ non-arch. complete alg. closed. (think of $K = \mathbb{C}_p$ $p$ prime)

$g_K$ defined as before

difficulty = $K$ is never locally compact.

recall $\tilde{K} = K^o / K^{oo}$ residue field is alg. closed.

$z_n \in K^o$ st $\pi(z_n) \neq \pi(z_m)$ for all $n, m$

then $|z_n - z_m| = 1$ for all $n \neq m$. !

⟶ need to work with ~~other~~ a space having more points to complete $K$ and work with a locally compact set.