⑧ Dynamical heights

$d \geq 2 \qquad M_d(z) = z^d$

obs. $h(M_d(x)) = d\,h(x)$ for all $x \in \bar{\mathbb{Q}}$.

proof. $\mathrm{Log}^+|M_d(z)| = \mathrm{Log}\max\{1, |z|^d\} = d\,\mathrm{Log}^+|z|$

$\Rightarrow \quad K = \mathbb{Q}(x) \qquad x, M_d(x) \in K$

$$h(M_d(x)) = \frac{1}{[K:\mathbb{Q}]} \sum_{v \in M_K} n_v \mathrm{Log}^+|M_d(x)| = d\,h(x) \qquad ///.$$

~~Theorem~~ (Kronecker)

Any algebraic integers whose conjugates have all euclidean norm ≤ 1 is a root of units

proof.

x alg. integer $\overset{def}{=}$ solution of an monic (irreducible) polynomial

$P \in \mathbb{Z}[T]$: $T^d + a_1 T^{d-1} + \dots + a_d \qquad a_i \in \mathbb{Z}$.

$\rightsquigarrow$ p prime $\quad y$ solution of $P(T)$ in $\mathbb{C}_p$.

$|a_i|_p \leq 1$ for all i

$\Rightarrow |y|_p \leq 1$.

$$\rightsquigarrow \quad h(x) = \frac{1}{\deg(x)} \sum_{v \in M_{\mathbb{Q}}} \sum_{\substack{y \sim x \\ \in \mathbb{C}_v}}' \mathrm{Log}^+|y|_v = 0.$$

$\rightarrow$ for each n

$$h(x^{2^n}) = 0$$

Northcott $\Rightarrow \{x^{2^n}\}_{n \geq 0}$ is finite $\qquad ///$

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rem. We have proved that

$$h_{\text{Gpu}}(\Gamma_d) = \{h = 0\}$$

$\to$ generalize this to any rational map

Theorem K number field $f \in K(T)$ $\deg f \geq 2$

$\exists!\ h_f : \mathbb{P}^1(\mathbb{Q}^{alg}) \to \mathbb{R}_+$ s.t.

① $h_f \circ f = d\, h_f$

② $\sup |h_f - h| < \infty$

proof

uniqueness h_1 & h_2 satisfying ① & ②

$|h_1 - h_2| \leq C$ $\quad \left| \underset{h_1}{\underset{\shortparallel}{\dfrac{h_1 \circ f^n}{d^n}}} - \underset{h_2}{\underset{\shortparallel}{\dfrac{h_2 \circ f^n}{d^n}}} \right| \leq \dfrac{C}{d^n}$

existence

lemma $|h \circ f - f| \leq C$ on $\mathbb{P}^1(\mathbb{Q}^{alg})$

$\Rightarrow \left| \dfrac{h \circ f^{n+1}}{d^{n+1}} - \dfrac{h \circ f^n}{d^n} \right| \leq \dfrac{C}{d^n}$ hence $\lim \dfrac{h \circ f^n}{d^n}$ exists and satisfies ① & ②

~~proof of the lemma~~ $f = \dfrac{P}{Q}$

$P = \sum a_i T^i$ $\quad Q = \sum b_j T^j$ $\quad d = \max\{\deg(P), \deg(Q)\}$ $\quad P \wedge Q = 1$.

$v \in M_K$ write $A_v = \max\{1, |a_i|, |b_j|\} \times \begin{cases} 1 & v \text{ non arch} \\ d+1 & v \text{ arch.} \end{cases}$

$z \in \mathbb{C}_v$ $\quad |P(z)|_v \begin{cases} \underset{\text{non-arch}}{\leq} \max |a_i| \max\{1, |z|\}^d \\ \underset{\text{arch}}{\leq} (d+1) \max |a_i| \max\{1, |z|\}^d \end{cases} \leq A_v \max\{1, |z|\}^d.$

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$$\frac{1}{d} \operatorname{Log} \max |P|, |Q| \leq \operatorname{Log}^+ |z| + \frac{1}{d} \operatorname{Log} A_v$$

Lower bound : $U, V \in \mathbb{Q}[T]$ $\quad PU + QV = 1$. $\delta = \max \deg U, \deg V$

$$A_v = \max \{1, \text{norm of all coefficients of } U \& V\} \left\{ \begin{array}{ll} 1 & v \text{ non arch} \\ \delta+1 & v \text{ arch.} \end{array} \right.$$

The non-archimedean case WLOG $\deg P = d \geq \deg Q$

$$R_v = \max \left\{ 1, \left| \frac{a_i}{a_d} \right| , \right\}$$

$|z| > R_v$ $\quad \max\{|P(z)|, |Q(z)|\} \geq |P(z)| = |a_d| |z|^d$

since $|a_d| |z|^d > |a_i| |z|^{d-1} \geq |a_i| |z|^i$ $\quad i \leq d-1$

$|z| \leq R_v$ $\quad 1 \leq \max\{|P|, |Q|\} \; B_v R_v^{\delta}$

$$\frac{1}{d} \operatorname{Log} \max |P|, |Q| \geq \operatorname{Log}^+ |z| + \operatorname{Log} C_v$$

$$C_v = \min \left\{ \frac{1}{B_v R_v^{\delta}}, |a_d|^{1/d} \right\}.$$

exercise : Done in the archimedean case.

We have proved for each $v \in M_K$ $\exists C_v \geq 1$ s.t.

$$\left| \frac{1}{d} \operatorname{Log} \max (|P|, |Q|) - \operatorname{Log}^+ |z| \right| \leq \operatorname{Log} C_v$$

Moreover $C_v = 1$ for all but finitely many places. $\mathcal{B} = \{v \in M_K, C_v \neq 1\} \supseteq M_{K,\infty}$

$$h(f(x)) = \frac{1}{[K:\mathbb{Q}]} \sum_{M_K} \operatorname{Log} \max \left(\frac{|P|}{|Q|}(x), 1 \right) \underset{\text{formula}}{\overset{\text{Product}}{=}} \frac{1}{[K:\mathbb{Q}]} \sum \operatorname{Log} \max |P|, |Q|$$

$$\left| \frac{1}{d} h(f(x)) - h(x) \right| \leq \frac{1}{[K:\mathbb{Q}]} \sum_{\mathcal{B}} \operatorname{Log} C_v.$$

///

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Applications

$f \in K(T)$ $\deg(f) = 2$ K number field

$\mathrm{Preper}(f) = \{ h_f = 0 \}$ is a set of bounded height (for h!)

In particular, the set of preperiodic points of f of degree (over K) $\leq N$ is bounded.

$\Rightarrow$ Thm B.

Final remark : there are stronger form of Thm B
conjectural

- $f \in K(T)$ $d \geq 2$

$\exists \kappa \geq 1$ $h_f(x) \geq \frac{\kappa}{\deg(x)}$ if $x \notin \mathrm{Preper}(f)$

unknown even for $h_f = h$ (Lehmer's conjecture!)

- conjecture uniform lower bounds (Conjecture 4.98 Silverman)

~~$\forall f \in K(T)$ (K fixed) $\forall x \in \mathbb{P}(K) \setminus \mathrm{Preper}(f)$ $h_f(x) \geq c$~~