A non-archimedean Montel's theorem

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Work in progress with J. Kiwi and E. Trucco

Montel's Theorem

$\Omega \subset \mathbb{C}$ an open set.

Theorem

For any sequence of holomorphic maps $f_n : \Omega \to \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, there exists a subsequence f_{n_j} that converges uniformly on compact subsets of Ω to a holomorphic function f.

- either $f(\Omega) \subset \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$;
- or f is a constant map.

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Proof of Montel's theorem: the bounded case.

Sur les suites de fonctions infinies: (Annales de l'ENS 1907)
`http://www.numdam.org/`

Assume $f_n : \Omega \to B(0,1)$.

- Cauchy's estimates imply the equicontinuity of the f_n's;
- Arzelà-Ascoli's theorem: the family $\{f_n\}$ is relatively compact;
- Ω is separable: one can make a diagonal extraction argument

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Proof of Montel's theorem: the general case.

Sur les familles normales de fonctions analytiques: (Annales de l'ENS 1916): `http://www.numdam.org/`

Assume $f_n : \Omega \to \mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$.

- f_n contracts the hyperbolic metric which implies the equicontinuity of the family $\{f_n\}$.

Applications of Montel's theorem

Fatou, Julia, Montel. Michèle Audin.

$\geq$ 1918 Fatou and Julia give the first applications in one variable complex dynamics.

Definition

A family $\mathcal{F}$ of holomorphic functions $\Omega \to \mathbb{P}^1(\mathbb{C})$ is normal if any sequence $\{f_n\} \subset \mathcal{F}$ admits a converging subsequence.

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Dynamical applications of Montel's theorem

f rational map on $\mathbb{P}^1(\mathbb{C})$ or an entire map of $\mathbb{C}$.

- $\mathrm{Fatou}(f) = \{z \in \mathbb{P}^1(\mathbb{C}),\ s.t.\ \{f^n\}$ is normal near $z\}$
- $\mathrm{Julia}(f) = \mathbb{P}^1(\mathbb{C}) \setminus \mathrm{Fatou}(f)$

Theorem

Repelling periodic orbits are dense in the Julia set.

- λ-lemma (Mané-Sad-Sullivan)

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Non-archimedean fields

$(k, |\cdot|)$ complete non-archimedean valued field:

- $|z| = 0$ iff $z = 0$;
- $|zw| = |z|\,|w|$;
- $|z + w| \le \max\{|z|, |w|\}$.

- Ring of integers: $\mathcal{O}_k = \{z,\ |z| \le 1\}$;
- Unique (maximal) ideal: $\mathfrak{m}_k = \{z,\ |z| < 1\}$;
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Examples

- $k = \mathbb{C}((T))$ with $|R(T)| = \exp(-\mathrm{ord}_0(R))$.

$$\mathcal{O}_k = \mathbb{C}[[T]],\ \mathfrak{m}_k = (T),\ \tilde{k} = \mathbb{C}$$

$p > 0$ a prime number.

- $k = \mathbb{F}_p((T))$ with $|R(T)| = \exp(-\mathrm{ord}_0(R))$. Here $\tilde{k} = \mathbb{F}_p$.
- $k = \mathbb{Q}_p$ with the p-adic norm.

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Why doing non-archimedean dynamics?

- It's fun! It mixes ideas closely related to complex analysis, and more number theoretic or algebraic ideas.
- Degeneracies of holomorphic objects lead to non-archimedean objects (e.g. Morgan-Shalen, Kiwi, DeMarco-McMullen)

$$z \mapsto z^2 + c,\ c \in \mathbb{C}$$

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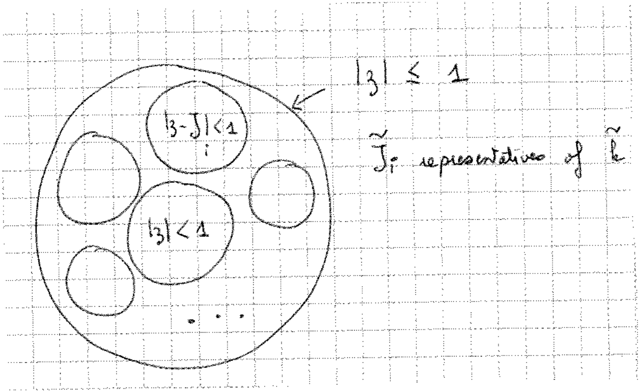
The unit ball

Definition

A closed ball (in k) is a set $\bar{B}(z, r) = \{w \in k,\ |z - w| \leq r\}$.



H'sia and Hu-Yang's theorem

Definition

An analytic map f on a ball is given by a converging power series $f(z) = \sum_{j\geq 0} a_j z^j$.

Counter-example

Take $|\zeta_n| = 1$ *such that* $|\zeta_n - \zeta_m| = 1$ *if* $n \neq m$ *(possible if* $\tilde{k}$ *is infinite).*

Theorem (H'sia and Hu-Yang)

Any family of analytic maps on a ball avoiding 0 *is equicontinuous for the projective metric.*

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The Berkovich affine line: definition

k is algebraically closed.

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The Berkovich line $\mathbb{A}^1_k$ "is" the set of all closed balls in k (together with some extra points).

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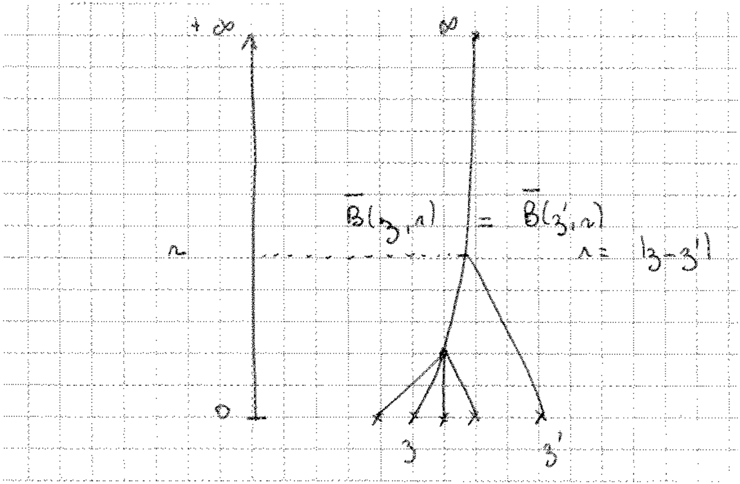

The Berkovich affine line: topology

- $\mathbb{A}^1_k$ has a natural tree structure;
- $P \in k[T]$, $|P(x)| := \sup_x |P| : \mathbb{A}^1_k \to \mathbb{R}$;
- The weakest topology on $\mathbb{A}^1_k$ such that $x \mapsto |P(x)|$ is continuous is locally compact (Tychonov),
- but if $\tilde{k}$ is uncountable, it is non-metrizable.

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Analytic functions on balls

A ball in $\mathbb{A}^1_k$ is $\beth(z,r) = \{x \in \mathbb{A}^1_k, x \subset B(z,r)\}$.

$$f(z) = \sum_j a_j z^j$$

Claim

The image of a ball by an analytic function remains a ball.

- f induces a (continuous) map from $\beth(z,r)$ to $\mathbb{A}^1_k$.

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The counter example in the Berkovich framework

The Gauss point x_g corresponds to $\bar{B}(0,1)$.

Example

Take $|\zeta_n| = 1$ such that $|\zeta_n - \zeta_m| = 1$ if $n \neq m$. Then $\zeta_n \to x_g$.

But x_g is not an analytic function.

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$\Omega \subset \mathbb{A}^1_k$ any open set.

Theorem

Any sequence of analytic functions $f_n : \Omega \to \mathbb{A}^1_k \setminus \{0, 1\}$ admits a subsequence that is pointwise converging.

Theorem

Assume $\mathrm{char}(\tilde{k}) = 0$. Any sequence of analytic functions $f_n : \Omega \to \mathbb{A}^1_k \setminus \{0, 1\}$ admits a subsequence converging pointwise to a continuous function.

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Example

If $\mathrm{char}(k) = p > 0$, take $f_n(z) = z^{p^{n!}}$, and $\Omega = \mathbb{A}^1_k \setminus \{0, 1\}$. Then f_n converges pointwise to a non-continuous function.

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The Fatou set of a polynomial/entire function $f : \mathbb{A}^1_k \to \mathbb{A}^1_k$ is the set of points at which $\{f^n\}$ forms a normal family.

Theorem

If $\mathrm{char}(\tilde{k}) = 0$ then the closure of the set of periodic cycles contains the Julia set.

Over $\mathbb{C}_p$: Bezivin.

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Reduction to the bounded case

The image of a ball remains a ball hence

$$f_n(\Omega) \subset \beth(\zeta_n, 1)$$

for some ζ_n (with $\beth(\zeta) = \{z, |z| > 1\}$ if $|\zeta| > 1$).

1. Either there exists a subsequence $|\zeta_n - \zeta_m| = 1$ if $n \neq m$;
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Sequential compactness

Case (2): Power series:

$$f_n(z) = \sum_{j \geq 0} a_j^{(n)} z^j, \text{ such that } |a_j^{(n)}| \leq 1 \ .$$

Polynomials of uniform bounded degree:

$$f_n(z) = \sum_0^d a_j^{(n)} z^j, \text{ such that } |a_j^{(n)}| \leq 1 \ .$$

- $a^{(n)} = (a_0^{(n)}, \ldots, a_d^{(n)}) \in \bar{B}(0,1)^{d+1}$
- Embed $\bar{B}(0,1)^{d+1}$ in the Berkovich polydisk $\bar{\mathbb{D}}^{d+1}(0,1)$.
- Extract $a^{(n)} = (a_0^{(n)}, \ldots, a_d^{(n)}) \to \alpha \in \bar{\mathbb{D}}^{d+1}(0,1)$;

Theorem (F.- Poineau)

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- $f_n(z) = \sum_0^d a_j^{(n)} z^j$, with $(a_0^{(n)}, \ldots, a_d^{(n)}) \to \alpha \in \bar{\beth}^{d+1}(0)$;
- For any $P \in k[T_0, \ldots, T_d]$, $|P(a_0^{(n)}, \ldots, a_d^{(n)})|$ converges;
- $z, w \in k$, then $|f_n(z) - w| = |(\sum_0^d T_j z^j - w)(a_0^{(n)}, \ldots, a_d^{(n)})|$ converges;
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