1/

TALK 2 Valuative methods

- Motivation 15 min
- Valuative space [topology].
- Action of F_*
- Skewness and thinness

(I). $F : \mathbb{C}^2 \to \mathbb{C}^2$ dominant $F^n = \underbrace{F \circ \dots \circ F}_{n \text{ times}}$

$\deg(F^{n+m}) \leq \deg(F^n) \times \deg(F^m)$ $\lambda = \lim \deg(F^n)^{1/n}$

Thm Either $\deg(F^n) \simeq \lambda^n$
Or F is a skew product.

rmk: same method gives λ is a quadratic integer.

main idea = look at the action of F on a suitable valuation space.

Explain

× principle
construct $\mathcal{V}$ compact space + $F_\bullet : \mathcal{V} \to \mathcal{V}$ continuous
$d(F, \cdot)$: continuous fct.

$$\deg(F^n) = \prod_{k=0}^{n-1} d(F, F_\bullet^k v_0) \quad \text{cocycle}.$$

suppose $F_\bullet^k v_0 \to v_*$ then $\lambda = d(F, v_*)$.
if $d(F, \cdot)$ locally cst then $\deg(F^{n+N}) = \lambda \deg(F^{n+N})$

motivation $F : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 = [\tilde{P} : \tilde{Q} : x_2^d]$

suppose $\deg(F \circ f) < \deg(F^2)$

$$\Updownarrow$$

$$x_2 \mid \tilde{P}(\tilde{P}, \tilde{Q}, x_2^d) \ \& \ \tilde{Q}(\tilde{P}, \tilde{Q}, x_2^d)$$

$$\Updownarrow$$

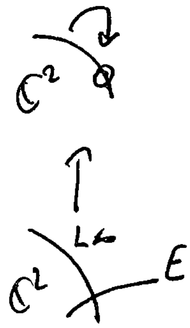
$$\tilde{P}\left(\tilde{P}(x_0,x_1,0), \tilde{Q}(x_0,x_1,0), 0\right), \ \tilde{Q}(\cdots) \equiv 0$$

$$\Updownarrow$$

$$F(L_\infty) = [\tilde{P}(x_0,x_1,0) : \tilde{Q}(x_0,x_1,0) : 0] \in I(F)$$

Obs $\deg(F^2) < \deg(F)^2 \iff F(L_\infty) \in I(F)$.

rmk $\deg(F^2) = \deg(F)^2 \Rightarrow \deg(F^n) = \deg(F)^n \ \forall n$.



What to do = blow-up indeterminacy points

new model **hope**

1. curves are not contracted to indeterminacy
2. this helps understand $\{\deg(F^n)\}$.

* In most cases 1. can be achieved but this is usually hard to prove

* what will appear = the understanding of where curves resp. are mapped by F gives sufficient info to prove the thm.

2/ (II) (30 min)

(def) $\hat{\mathcal{V}} = \{ v : \mathbb{C}[X,Y] \to \mathbb{R} \cup \{+\infty\}$

$v(PQ) = v(P) + v(Q)$

$v(P+Q) \geq \min\{v(P), v(Q)\}$

$v|_{\mathbb{C}^*} \equiv 0 \qquad v(0) = +\infty$

$\exists P \quad v(P) < 0 \qquad \}$

rmk 1) $v(X), v(Y) \geq 0 \Rightarrow \forall P \quad v(P) \geq 0$.

$v \in \hat{\mathcal{V}} \iff v$ valuation and $\min v(X), v(Y) < 0$.

2) $v \in \hat{\mathcal{V}} \Rightarrow tv \in \hat{\mathcal{V}} \quad \forall t > 0$.

$\mathcal{V} = \{ v \in \hat{\mathcal{V}}, \min\{v(X), v(Y)\} = -1 \}$

[depends on the choice of affine coordinates]

Basic example

$-\deg \qquad \mathbb{C}^2 \,)\, L_\infty$

$P \in \mathbb{C}[X,Y]$ view as a meromorphic map on $\mathbb{P}^2$

$-\deg(P) = \mathrm{ord}_{L_\infty}(P)$.

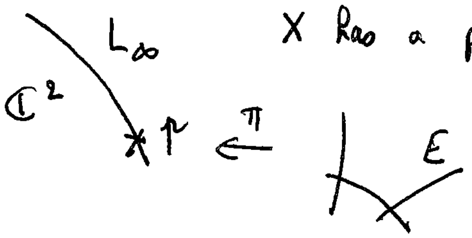
divisorial valuations.

$\mathbb{P}^2 \xleftarrow{\pi} X \supseteq E \qquad E$ irreducible

$v_E(P) := \mathrm{ord}_E(P \circ \pi)$

* if $\pi(E) \cap \mathbb{C}^2 \neq \emptyset$ then $P \circ \pi$ hol. at generic point of $\pi(E)$ hence $v_E(P) \geq 0$

* if $\pi(E) \subseteq L_\infty$
$\quad\;\; \| \quad p$



X has a pole at $p \rightsquigarrow \frac{1}{X}$ has a zero at p

$\Rightarrow \mathrm{ord}_E(\frac{1}{X} \circ \pi) > 0$.

Prop Let v be a divisorial valuation. Then $\pi(E) = L_\infty \iff \exists\, \ell \quad v_E(\ell) < 0$. [ie $v \in \hat{\mathcal{V}}$]

$\to$ say in this case that v_E is centred at infinity.

quasi monomial valuations

monomial valuations.

- $s,t \in \mathbb{R}$ $\quad v_{s,t}\left(\sum a_{ij} x^i y^j\right) = \min \{ is + tj,\ a_{ij} \neq 0 \}$

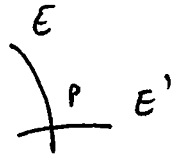
- $v_{s,t}$ centred at infinity $\iff \min\{s,t\} < 0$

  $v_{s,t} \in \mathcal{V} \iff \min\{s,t\} = -1$.

- $v_{s,t}$ is divisorial $\iff st = 0$ or $\frac{s}{t} \in \mathbb{Q}^*$.

quasi-monomial

$$\mathbb{P}^2 \xleftarrow{\ \pi\ } X \supseteq E, E'$$



pick coordinates (w,w') at p $\quad \{w=0\} = E \quad \{w'=0\} = E'$

$s,t \geq 0$.

$$v(\ell) := v_{(s,t)}^{(w,w')}(\ell \circ \pi)$$

$v \in \hat{\mathcal{V}} \iff \pi(p) \in L_\infty$.

other type of valuations.

might have $v(\ell) = +\infty$

(ex) $\quad \ell \longmapsto -\deg_y \ell(0,y)$

3/

Topology on $\bar{\mathcal{V}}/\mathcal{V}$ (20 min)

Product topology = top of ptwise cvgence

in terms of sequences $v_n \to v$ iff $v_n(\ell) \to v(\ell)$ $\forall \ell$.

Prop $\mathcal{V}$ is compact

proof fcts $\mathbb{C}[x,y] \to \mathbb{R} \cup \{\pm\infty\}$ $v(\ell\ell') = v(\ell) + v(\ell')$ $v|_{\mathbb{C}^*} = 0$ $v(0) = +\infty$

$\min\{v(x), v(y) = -1\}$ is compact.

$\ell \in \mathbb{C}[x,y]$ $v(\ell) \geq -\deg(\ell) > -\infty$

$\Rightarrow$ $\mathcal{V}$ is compact □.

Thm $\mathcal{V}$ is an $\mathbb{R}$-tree

two definitions [no circuit]

1. order relation on $\mathcal{V}$ $v_1 \leq v_2 \Leftrightarrow v_1(\ell) \leq v_2(\ell)$ $\forall \ell$

$v_1 \geq -\deg$ minimal unique elt.

|| $\forall v \in \mathcal{V}$ $\exists$ increasing bij. $\{\mu, -\deg \leq \mu \leq v\} \to ([0,1], \leq)$

2. path = $\mathcal{C}^0$ map $\gamma : [0,1] \to \mathcal{V}$

inj path γ inj.

- $\forall v, v'$ $\exists$ inj path γ $\gamma(0) = v$ $\gamma(1) = v'$
- $\forall$ inj. path γ, γ' as above $\gamma[0,1] = \gamma'[0,1]$

→ we shall prove that $\mathcal{V}_{qm} \subseteq \mathcal{V}$ is a $\mathbb{R}$-tree
[second course].

① $\mathbb{P}^2 \xleftarrow{\pi} X$ bo above $\mathbb{C}^2$
$\pi^{-1}\{L_\infty\} = \{E_i\}_{i=1}^{N}$ irred. components

Γ_X = dual graph

× vertices ⇔ E_i's   edges E_iE_j ⇔ intersection $E_i \cap E_j \neq \emptyset$

× embed $\Gamma_X \subseteq \mathbb{R}_+^N$

- $e_i = (0, \frac{1}{b_i}, 0)$ (i-th place) $b_i = \min\{\mathrm{ord}_{E_i}(X\circ\pi), \mathrm{ord}_{E_j}(Y\circ\pi)\}$

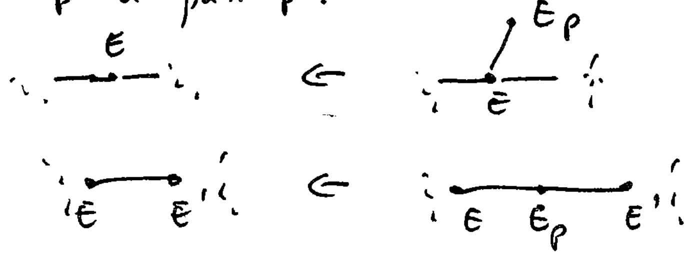
- σ_{ij} edge joining E_i to E_j
  = segment joining e_i to e_j.
  = $s e_i + (1-s) e_j$ $s \in [0,1]$.

② $\Gamma_{\mathbb{P}^2} = \circ$

$X \xleftarrow{\mu} \hat{X}$ blw-up a point p.

if $p \in E$

if $p = E \cap E'$



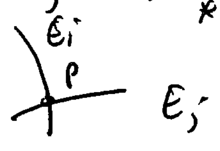
Cor Γ_X is a tree as a graph.

③ there is a natural map $\Gamma_X \hookrightarrow \mathcal{V}_{qm}$.

× $\Gamma_X \subseteq \mathbb{R}_+^N$

- map e_i to $\frac{1}{b_i}\pi_* \mathrm{ord}_{E_i}$

- map $s e_i + (1-s) e_j$ to $\pi_* \nu^{w,w'}_{s,(1-s)}$



(4). $\mathcal{V}_{qm} = \bigcup_{\text{all } X} \Gamma_X$

- $v, v' \in \mathcal{V}_{qm}$ $\quad X$ s.t. $v, v' \in \Gamma_X$ $\rightsquigarrow$ unique inj path γ
- if γ' is other inj. path. suppose $\gamma'(t) \notin [v, v']$ for some t as $\gamma'(t)$ is qm find X' when v, v' $\gamma'(t) \in \Gamma_{X'}$. contradiction.

~~Comments . how to go from $\mathcal{V}_{qm}$ to $\mathcal{V}$.?~~

Comments

- how to get the tree structure in terms of the order relation in $\mathcal{V}_{qm}$

  - $\deg < v_0 \leq v_1$ $\Rightarrow$ v_0 & v_1 have same center in $\mathbb{P}^2$
  (qm) [illegible]) say $[1:0:0]$. choose coord $[1:z:w]$

  $\Rightarrow \forall\ \tilde{P} \in \mathbb{C}[z,w]$

  $\tilde{P}\left(\frac{y}{x}, \frac{1}{x}\right) x^{\deg \tilde{P}}$ polynomial in (x,y)

  hence $v_0(\tilde{P}(z,w)) \leq v_1(\tilde{P}(z,w))$

  $\Rightarrow$ using curvettes get $\frac{1}{b_{v_0}} Z_{v_0} \ \cancel{\geq}\ \frac{1}{b_{v_1}} Z_{v_1}$

  (see below) $\Rightarrow$ $v_0 \in [-\deg, v_1]$ on Γ_X for suitable X.

- how to go $\mathcal{V}_{qm} \rightsquigarrow \mathcal{V}$

  Berkovich $\forall v \in \mathcal{V}$ $\exists$ increasing sequence $v_n \in \mathcal{V}_{qm} \to v$.

5/

(III) Action of F_*. (30 min)

$v \in \hat{\mathcal{V}}$ then $F_* v(\ell) = v(\ell \circ F)$ is a valuation.

(ex) $f = (x, xy)$ $F_* v_{(2,-1)} = v_{(2,1)}$!

(def) $\hat{\mathcal{V}}_0 \ni v \iff \forall \ell \quad v(\ell) \leq 0$.

$\mathcal{V}_0 = \hat{\mathcal{V}}_0 \cap \mathcal{V}$.

Thm $\mathcal{V}_0$ is an $\mathbb{R}$-tree

proof $\forall \ell \quad v \longmapsto v(\ell)$ is increasing $\square$.

F_* sends $\mathcal{V}_0$ to $\mathcal{V}_0$.

proof.

$\forall \in \mathcal{V}_0$ $F_* v \notin \mathcal{V}_0 \Rightarrow F_* v \equiv 0$ on $\mathbb{C}[x,y]$.

$$\mathbb{C}(x,y) \xrightarrow{F^*} \mathbb{C}(x,y)$$
$$\varphi \longmapsto \varphi \circ F$$

image $\mathbb{C}(\ell, \ell) \subseteq \mathbb{C}(x,y)$ finite extension.

as f is dominant

$$\varphi \in \mathbb{C}(x,y) \qquad \varphi^N = \sum_{i=0}^{N-1} \psi_i(\ell, \ell)\, \varphi^i$$

$$v(\psi_i \varphi^i) = v(\psi_j \varphi^j) \Rightarrow v(\varphi) = 0 \qquad \square.$$

Local degree.

$v \in \mathcal{V}_0$ write $F_* v = d(F,v) \times F_\bullet v$ ($F_\bullet v \in \mathcal{V}_0$)

- $d(F,v) = -\min v(P), v(Q)$ $F = (P, Q)$

  $\mathcal{C}^0$ map on $\mathcal{V}_0$

- $F_\bullet : \mathcal{V}_0 \to \mathcal{V}_0$ is $\mathcal{C}^0$

Prop

$$d(F, -\deg) = \deg(F)$$

$$d(F \circ G, v) = d(G, v) \times d(F, G_\bullet v)$$

$$\deg(F^n) = \prod_{k=0}^{n-1} d(F, F_\bullet^k \, -\deg).$$

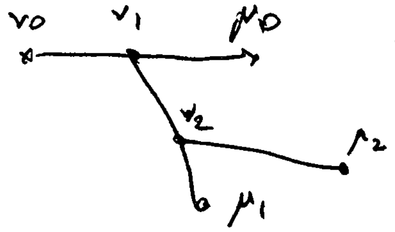
6/

Thm $F_\bullet : \mathcal{V}_0 \to \mathcal{V}_0$ admits a fixed point

(10 min)

proof.

- $\nu_0 = -\deg \qquad \mu_0 = F_\bullet - \deg$.



$\nu \mapsto r_{[\nu_0,\mu_0]} F_\bullet \nu \qquad [\nu_0,\mu_0] \circlearrowleft$

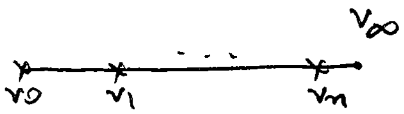
$\nu_1 =$ fixed pt

etc...

$\nu_n \to \nu_\infty$

$F_\bullet \nu_n \geq \nu_n \Rightarrow F_\bullet \nu_\infty = \nu_\infty$.

$F_\bullet \nu_n \wedge \nu_\infty = \nu_n$



$F_\bullet \nu_\infty \wedge \nu_\infty = \nu_\infty \Rightarrow F_\bullet \nu_\infty = \nu_\infty$ D.

Cogs for the main thm

$\Downarrow$

assume $\nu_* \in \mathcal{V}_0$ and $-\deg \leq \nu_* \leq c\,(-\deg) \qquad c > 0$.

then $\qquad \deg(F^n) \geq d_*^n \geq c \deg(F^n)$

$d_* = d(F, \nu_*)$

$\Rightarrow$ Have to understand which valuation are comparable to $-\deg$.

(Rmk) $\nu \simeq -\deg \Rightarrow \nu(l) < 0 \quad \forall l$

[converse not true]

$\nu_{(-1,0)} \in \mathcal{V}_0$ but not ~~xxxx~~ comparable to $-\deg$.

7/ (W) Skewness. (20 min)

$v \in \mathcal{V}_0 \rightsquigarrow \alpha(v) \in \mathbb{R}_+$ so that $v \leq \alpha(v)\,(-\deg)$.

1. ~~[struck out]~~

$\pi : X \to \mathbb{P}^2 \qquad \pi^{-1}(L_\infty) = \bigcup_i^N E_i$

intersection form on E_i's.

Prop $\langle , \rangle$ on $\bigoplus_{i\geq 1}^N \mathbb{Z}[E_i] = NS(X)$.

$\exists$ a basis $\hat{E}_i$ of the module s.t.
$\hat{E}_0^2 = +1 \quad \hat{E}_i^2 = -1 \quad \hat{E}_i \cdot \hat{E}_j = 0$

signature $(+1, -1, \cdots -1)$.

proof

induction $X \overset{\mu}{\leftarrow} \hat{X}$ blow-up of a pt.

basis $\mu^* \hat{E}_0 \cdots \mu^* \hat{E}_N$ of X + E new exept. divisor □

2. v divisorial in $\hat{\mathcal{V}}$ & $v = c\,\pi_* \mathrm{ord}_E$

in $X \supseteq E$ $\exists!\, Z_E \in \bigoplus \mathbb{Z}[E_i]$ s.t.
$Z_E \cdot Z = \mathrm{ord}_E(Z) \quad \forall Z \in \bigoplus \mathbb{Z}[E_i]$

x $\ell \in \mathbb{C}[x,y]$

$\mathrm{div}(\ell) = Z_\ell + C_\ell$
$\quad\quad Z_\ell \in \bigoplus \mathbb{Z}[E_i], \quad C_\ell \subset \mathbb{C}^2$

$0 = Z_E \cdot \mathrm{div}(\ell) = Z_\ell \cdot Z_E + C_\ell \cdot Z_E$

Cor $\boxed{\mathrm{ord}_E(\ell) = -\,C_\ell \cdot Z_E}$

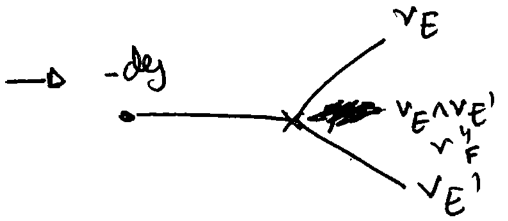
(def) $\alpha(v_E) = \frac{1}{b_E^2} Z_E^2$. $\left[\alpha(\mathrm{ord}_E) = Z_E^2 \right]$.

Thm → α extends in a unique way to $\vec{\mathcal{V}}$ s.t.

~~[illegible] on~~

α is continuous on segments, affine on edges in $\Gamma_X \vee X$, strictly decreasing.

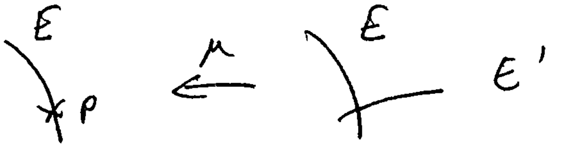
→ -deg



$$\frac{1}{b_E b_{E'}} Z_E \cdot Z_{E'} = \frac{1}{b_F} Z_F^2$$

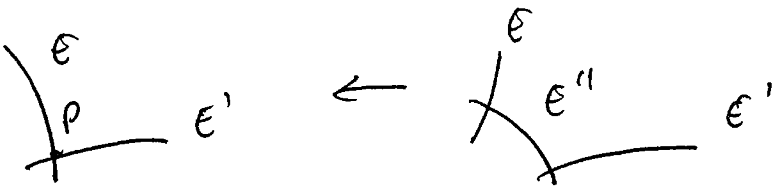
proof = subtle induction on the number of blow-ups for Γ_X.

•



$$\left| \begin{array}{l} Z_{E'} = \mu^* Z_E - E' \\ b_{E'} = b_E \end{array} \right.$$

•



$$\left| \begin{array}{l} Z_{E''} = \mu^* (Z_E + Z_{E'}) - E'' \\ b_{E''} = b_E + b_{E'} \end{array} \right.$$

□.

~~(grd) α is decreasing in ϑ~~

8/

G1 $v \in \mathcal{V}_0$ iff $\alpha(v) \geq 0$.

proof v divisorial

$\Rightarrow$ $Z_E \cdot Z = Z_E \cdot Z_\infty + Z_E \cdot Z_0 \geq 0$.

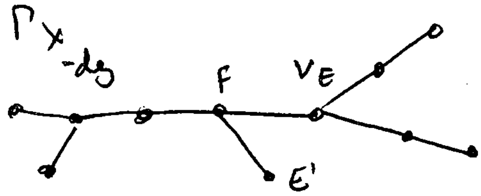
Z effective $Z = Z_\infty + Z_0$, $\pi^* L_\infty$ $\mathbb{C}^2$

$Z_E \cdot Z_\infty = \mathrm{ord}_E(Z_\infty) \geq 0$

$Z_E \cdot Z_0 = -v(Z_0) \geq 0$

$\leadsto v \in \mathcal{V}_0$.

$\Leftarrow$ in $\Gamma_{X,-dg}$



$\to$ $\mathrm{ord}_{E'}(Z_E) = Z_E \cdot Z_{E'} = \frac{1}{b_F} Z_F^2$

$Z_F^2 \geq 0$ $\leadsto$ $\alpha \downarrow$ hence Z_E effective.

$\mathrm{ord}_E(\ell) = - C_D \cdot Z_E \geq 0$ □.

G2 $\alpha > 0$ $\Rightarrow$ $v \leq \alpha_X(-dg)$. $c > 0$

proof $Z_E = \sum \alpha_{EE'} E'$ $\alpha_{EE'} > 0$. $\pi^* L_\infty = \sum b_{E'} E'$

[illegible]

$Z_E \geq c\, \pi^* L_\infty$ □.

9/ <u>thinness</u> (20 min)

→ to study further when $d=0$

$dx\wedge dy$ in $\mathbb{C}^2$ $\qquad \pi : X \to \mathbb{P}^2$, $X \supset E$

$$\mathrm{div}(\pi^*\omega) = \sum (a_E - 1)\, E \qquad a_E \in \mathbb{N}.$$

(ex)
- $a_{L_\infty} = -2$.
- $a_{E_p} = a_E + 1$ (free blow-up)
- $a_{E_p} = a_E + a_{E'}$ (satellite blow-up).

def $\nu \in \mathcal{V}$ divisorial $\quad \nu = t\,\pi_* \mathrm{ord}_E$

$A(\nu) = t \times a_E$.

thm A extends continuously to $\mathcal{V}$
1. strictly increasing on segment
2. $A(F_*\nu) = A(\nu) + \nu(JF)$

$JF = \det DF$

<u>proof</u>
- use (ex) above
- $X' \xrightarrow{F} X$ hol

$$F^*\omega = \omega \times JF \quad \text{in } \mathbb{C}^2.$$ □.

<u>Cor</u> $\mathcal{V}_0 \cap \{A \leq 0\}$ is an $\mathbb{R}$-tree.
$F_\bullet$ - inv.

Proof the main thm [$\deg(F^n) \simeq \lambda^n$ a F skew product]

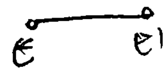
$F_\circ : \mathcal{V}_0 \cap \{A \leq 0\} \circlearrowleft \bar{\mathcal{B}}^\circ \rightsquigarrow v_*$ fixed pt.

assume $v_* \in \mathcal{V}_{qm}$

× $\alpha(v) > 0 \Rightarrow \deg(F^n) \asymp d(F, v_*)^n$.

× $\alpha(v) = 0$

+ v is divisorial. α is affine on segment and take values ∈ ℚ for divisorial val. ⇒ $v \in [v_E, v_{E'}]$



$v = v^{w,w'}_{s,1-s}$ $\alpha(v) = c_0 s + c_1$

$c_1 = \alpha(v_{E'}) \in \mathbb{Q}$

$c_0 + c_1 = \alpha(v_E) \in \mathbb{Q}$

$\alpha \in \mathbb{Q} \Leftrightarrow \alpha$ div □.

+ $Z_E^2 = 0$ R-R

$h^0(Z_E) \geq 1 - \frac{1}{2} Z_E \cdot K_X$.

$K_X = \sum \mathrm{ord}_{E'}(\pi^* \omega)\, E'$

$Z_E \cdot K_X = a_E - 1 < 0$.

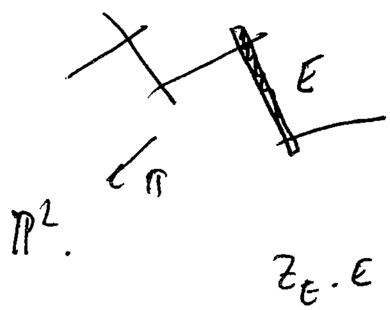
⇒ $h^0(Z_E) \geq 2$

σ = quotient of 2 sections $X \to \mathbb{P}^1$.

– hol. as $Z_E^2 = 0$

– may take $\sigma^{-1}(\infty) = Z_E$

~~$\sigma^{-1}(c)$ intersects Z_E at E for c generic.~~



– except for finitely many $\Theta \in \mathbb{P}^1$ $C_\Theta = \sigma^{-1}(\Theta)$ does not contain components of $\pi^{-1}(L_\infty)$

$Z_E \cdot E = +1$ $Z_E \cdot E' = 0$ ⇒

~~Proof of the main thm.~~

4/ C_Θ cuts $\pi^{-1}(L_\infty)$ at a single pt $\in E$ (finitely many exceptions)

$g(C_\Theta) = 1 + \frac{1}{2}(Z_C^2 + t_E \cdot K) = 0 \Rightarrow C_\Theta \simeq \mathbb{P}^1$ smooth

Thm Suzuki Abh-Roh

Let $C \subseteq \mathbb{P}^2$ be a curve

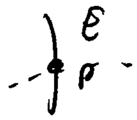
1. C has a single branch at ∞
2. $C \simeq \mathbb{P}^1$

$\exists\, \phi \in \mathrm{Aut}(\mathbb{C}^2)$ $\quad \phi(C) = \{x = 0\}$

$\Rightarrow$ $v = \deg x$ and F is a skew product $\square$.

Note on the proof of the thm concerning existence of sheaves

satellite case - [induction on ~~dim~~ NS(X) wall on Γ_X]



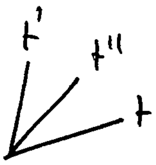
$E = \{x = 0\}$ $\qquad b = b_E$

v_t monomial $\qquad v_t(x) = 1/b_E \qquad v_t(y) = t$

$b(v_t) = b \times q \qquad t = \frac{p}{q} \qquad p \wedge q = +1$

$t \leq t'$ ~~[crossed out]~~ $\qquad \boxed{\frac{1}{b_t} z_t \cdot \frac{1}{b_{t'}} z_{t'} = \alpha - \frac{t}{b^2}}$

proof



$t' \geq t \qquad t' = \frac{p'}{q'} \qquad t = \frac{p}{q} \qquad p'q - pq' = +1$

$t'' = \frac{p+p'}{q+q'}$

$t' - t'' = \frac{1}{q'(q+q')} \qquad t'' - t = \frac{1}{q(q+q')}$

$\bullet\ \alpha(v_{t''}) = z_{t''}^2 \times (b + b')^{-2}$

$= (q+q')^{-2} b^{-2} \left(z_t^2 + z_{t'}^2 + 2 z_t \cdot z_{t'} - 1 \right).$

$= \frac{1}{b^2 (q+q')^2} \left[q^2 b^2 \left(\alpha - \frac{t}{b^2}\right) + q'^2 b^2 \left(\alpha - \frac{t'}{b^2}\right) + \frac{2}{qq' b^2} \left(\alpha - \frac{t}{b^2}\right) - 1 \right].$

$= \alpha$ ~~[crossed out]~~

$- \frac{1}{b^2 (q+q')^2} \left[+ q^2 \left(t'' - \frac{1}{q(q+q')} \right) + q'^2 \left(t'' + \frac{1}{q'(q+q')} \right) \right.$

$\left. + 2qq' \left(t'' - \frac{1}{q(q+q')} \right) + 1 \right]$

$\alpha - \frac{1}{b^2} t'' - \frac{1}{b^2 (q+q')^2} \left[- \frac{q}{q+q'} + \frac{q'}{q+q'} - \frac{2q'}{q+q'} + \frac{q+q'}{q+q'} \right]$

$\shortparallel$

$0.$

$\bullet\ \frac{1}{b_{t''}} z_{t''} \cdot \frac{1}{b_{t'}} z_{t'} = \frac{1}{b^2} \frac{1}{q'(q+q')} \left(z_{t'}^2 + z_t \cdot z_{t'} \right) = \frac{1}{b^2 (q'+q) q'} \left[\left(\alpha - \frac{t'}{b^2} \right) b^2 q'^2 \right.$

$\left. + b^2 qq' \left(\alpha - \frac{t}{b^2} \right) \right]$

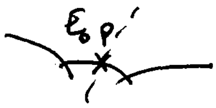
$$= \frac{1}{b^2 q'(q+q')}\left[\alpha\left(b^2 q'^2 + qq'\right) - \frac{q'^2}{b^2}\left(t'' - \frac{1}{q'(q+q')}\right) - \frac{qq'}{b^2}\left(t'' + \frac{1}{q(q+q')}\right)\right]$$

$$= \alpha - \frac{1}{q'(q+q')b^2}\left[(q'^2+qq')\, t''\right] = \alpha - \frac{t''}{b^2} \qquad \square$$

<u>Consequences</u>

- $\alpha(v)$ affine and decreasing on edges in Γ_X for all X.
- $\frac{1}{b_E} z_E \cdot \frac{1}{b_{E'}} z_{E'} = \frac{1}{b_{E\wedge E'}} z^2_{E\wedge E'}$ (*)

. Induction on the number of free blow-ups.



for any toric blowup at p. to check (*)

E, E' above p → previous computation.

E above p, E' not → reduce to E_0 and E'

E E' not above p → ok. $\square$

$\Gamma_X \hookrightarrow \mathcal{V}$ preserves the order.