Dynamical system on valuation space

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The setting

$P : \mathbb{C}^2 \to \mathbb{C}^2$ polynomial

- dominant $\Leftrightarrow P(\mathbb{C}^2) = \mathbb{C}^2 \setminus Z$, Z alg. curve $\Leftrightarrow \mathrm{Jac}(P) \neq 0$
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Dynamical problem

- Describe $\{P^n(z)\}_{n\geq 0}$ for all z
- Describe $\mathcal{B} = \{z,\ P^n(z) \to \infty\}$
- Speed: $\{|P^n(z)|\}_{n\in\mathbb{N}}$

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Plan

1. Introduction
2. Behaviour at ∞ (no iteration)
3. Dynamical heuristic
4. Statements
5. A first approach
6. Action on valuation space
7. Next

Basics

- $P = P_0 + P_1 + \cdots + P_d$,
  $P_i : \mathbb{C}^2 \to \mathbb{C}^2$ homogeneous of degree i.
- $\|z\| := 1 + \max\{|x|, |y|\}$
- $\|P(z)\| \leq C_2 \|z\|^d$ everywhere
- $C_1 \|z\|^d \leq \|P(z)\|$ on the cone $\Omega := \{z, \ |P_d(z)| \geq \varepsilon |z|^d\}$

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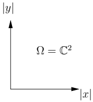

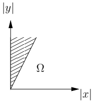

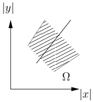

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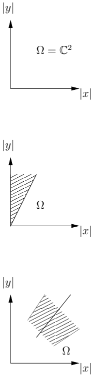

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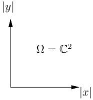

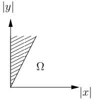

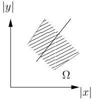

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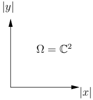

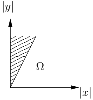

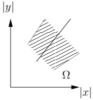

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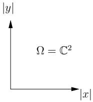

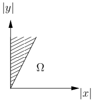

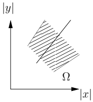

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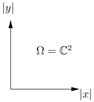

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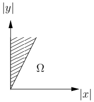

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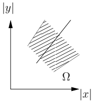

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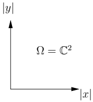

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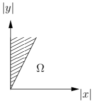

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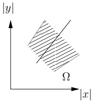

A few iterations

- n fixed, $d_n := \deg(P^n)$

$$C_{1,n}\|z\|^{d_n} \underset{\text{in } \Omega_n}{\leq} \|P^n(z)\| \underset{\text{in } \mathbb{C}^2}{\leq} C_{2,n}\|z\|^{d_n} .$$

- $d_{n+m} \leq d_n \times d_m$.

Definition (Dynamical degree)

$d_\infty = \lim_n d_n^{1/n} \geq 1$

Remark

$d_\infty(\psi^{-1} \circ P \circ \psi) = d_\infty(P)$, $\psi : \mathbb{C}^2 \to \mathbb{C}^2$ birational.

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Our hope

Conjecture

- $d_n \simeq d_\infty^n$?
- $\Omega = \bigcap \Omega_n \neq \emptyset$?
- *Uniform constant* $C_{1,n}, C_{2,n}$?

If YES, then

$$(C_1\|z\|)^{d_\infty^n} \underset{in\ \Omega}{\leq} \|P^n(z)\| \underset{in\ \mathbb{C}^2}{\leq} (C_2\|z\|)^{d_\infty^n} .$$

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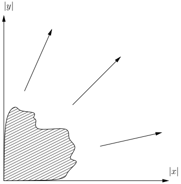

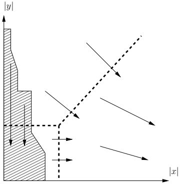

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  $\mathbb{C}^2 \setminus \Omega$ compact
- $P = (x^2 + y + c, x)$, $d_n = 2^n$,

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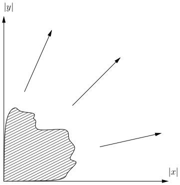

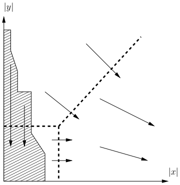

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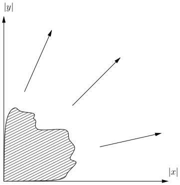

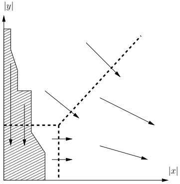

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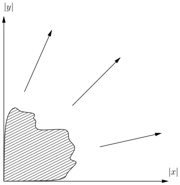

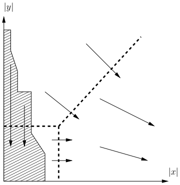

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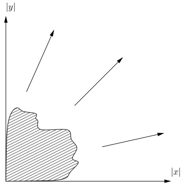

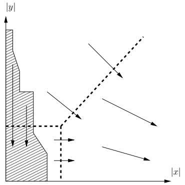

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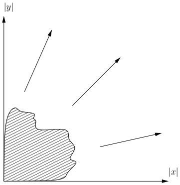

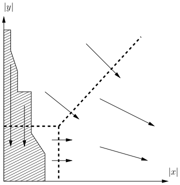

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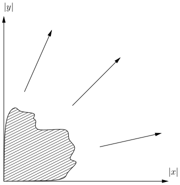

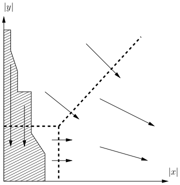

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  $(C_1\|z\|)^{d^n} \leq \|P^n(z)\| \leq (C_2\|z\|)^{d^n}$

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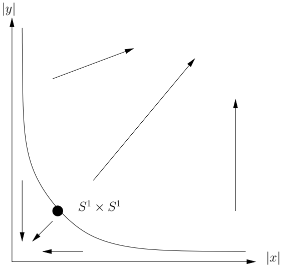
  $\mathbb{C}^2 \setminus \Omega$ unbounded

Monomial maps



When $d_\infty^2 > |\det(M)|$

- $M = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$
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- $M \cdot (v_1, v_2) = d_\infty(v_1, v_2)$
  $\Omega = \{|x|^{v_1}|y|^{v_2} > 1\}$
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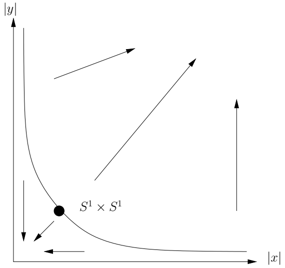

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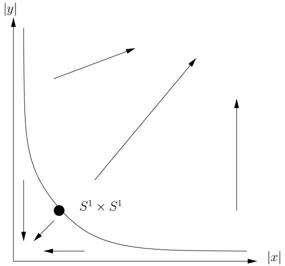

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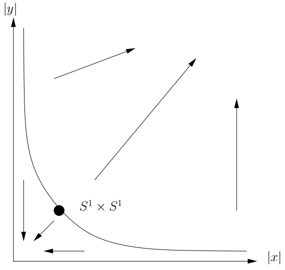

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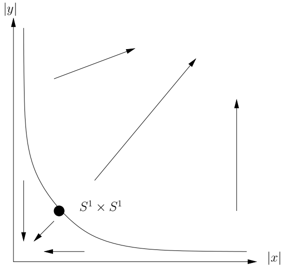

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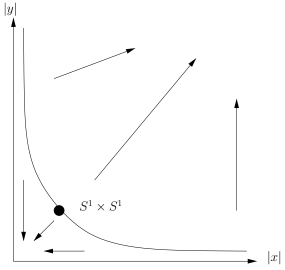

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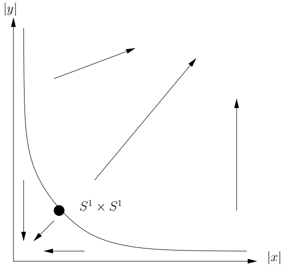

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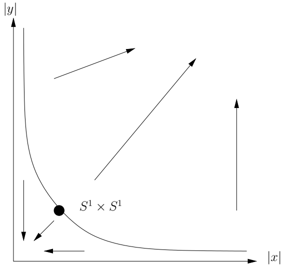

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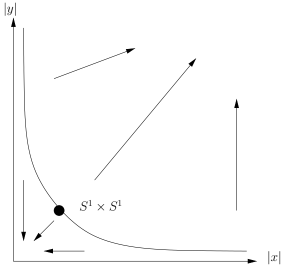

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Skew products

Definition

P is a skew product iff it preserves a *rational* fibration of $\mathbb{P}^2$

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- $d_\infty = \max\{k, l\}$

$$k \neq l \quad \Rightarrow \quad d_n \simeq d_\infty^n$$

$$k = l \quad \Rightarrow \quad \begin{cases} \deg(A_l) = 0 & d_n \simeq d_\infty^n \\ \deg(A_l) \geq 1 & d_n \simeq n d_\infty^n \end{cases}$$

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Statements

Theorem (Favre-Jonsson, 2004)

$P : \mathbb{C}^2 \to \mathbb{C}^2$ polynomial, dominant.
Then $d_\infty^2 = a d_\infty + b$, $a, b \in \mathbb{Z}$.

- *Either $d_\infty^n \leq d_n \leq C \cdot d_\infty^n$*
- *Or P is a skew product*

Theorem (Speed estimates)

$$g_n = \frac{1}{d_\infty^n} \log \|P^n(z)\| \longrightarrow g \not\equiv 0 \ .$$

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Remarks

- Outside Ω, pts may tend to infinity, at different speed (Dinh-Dujardin-Sibony)
- g continuous $\Rightarrow$ Ω is open
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Look at the action of F on the set of valuations on $\mathbb{C}[x, y]$ centered at infinity

- Explain why this is the right thing to do!
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Sequence of degrees

Why $d_{n+m} < d_n \times d_m$?

- Compactification: $\mathbb{P}^2 = \mathbb{C}^2 \cup L_\infty$
  $P : \mathbb{C}^2 \dashrightarrow \mathbb{C}^2$
- $P^* L_\infty = d_1 L_\infty$.... $(P^2)^* L_\infty \neq P^* P^* L_\infty$?
- $P[x : y : t] = [y^2 t : x^3 : t^3]$, $d_1 = 3$, $d_2 = 6$
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On surfaces

- $X \xrightarrow{f} Y \xrightarrow{g} Z \supset C$
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- Otherwise $f(C) \in I_g$ for some curve C.

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Suppose there exists $\pi : X \to \mathbb{P}^2$ s.t. $\forall C \subset \pi^{-1}(L_\infty)$, $P^n(C) \not\subset I_P$.

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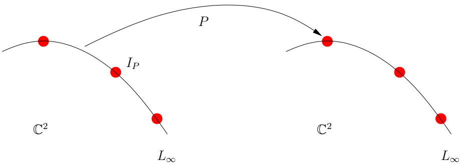
Example



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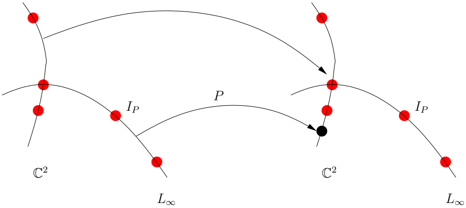

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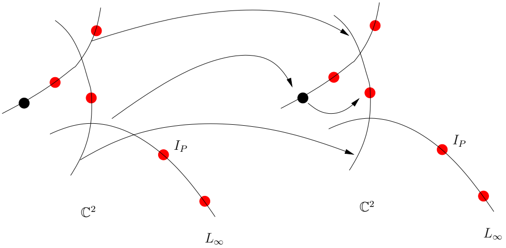

Conclusive remarks

Understand the dynamics of P on the set of irreducible curves C centered at infinity $\pi : X \to \mathbb{P}^2$, $C \subset \pi^{-1}(L_\infty)$

Basics

- Identify $C \rightsquigarrow \mathrm{div}_C : \mathbb{C}[x, y] \to \mathbb{R}$
  $L_\infty \rightsquigarrow -\deg$
- $\mathcal{V}_{\mathrm{div}} = \{$Divisorial valuations on $\mathbb{C}[x, y]$ centered at infinity i.e. $\nu(\phi) < 0$ for some $\phi\}$
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Monomial maps

$P(x, y) = (x^a y^b, x^c y^d)$

- $\nu_{s,t}$ monomial with $\nu_{s,t}(x) = s$, $\nu_{s,t}(y) = t$
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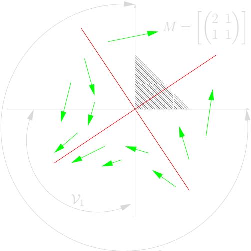
- $\begin{pmatrix} s \\ t \end{pmatrix} \xrightarrow{P_*} \begin{pmatrix} as + bt \\ cs + dt \end{pmatrix}$



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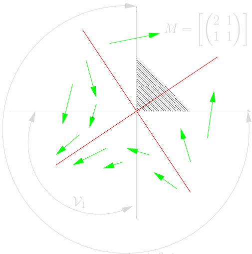

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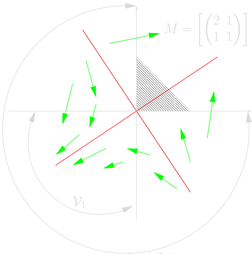

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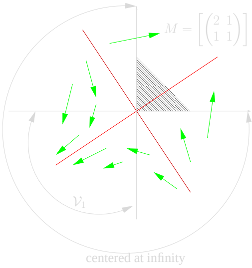

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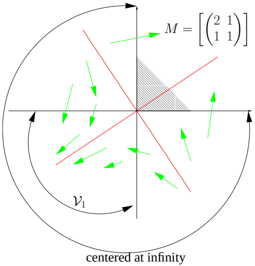

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$$\mathcal{V}_1 = \{\nu : \mathbb{C}[x,y] \to \mathbb{R}, \}$$

Fact

Define $P_*\nu(\phi) = \nu(\phi \circ P)$. Then $P_*(\mathcal{V}_1) \subset \mathcal{V}_1$

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Theorem (Eigenvaluation)

$P_*\nu = \lambda\nu$ *for some* $\nu \in \mathcal{V}_1$

Theorem

Suppose $\nu \in \mathcal{V}_1$

- *Either* $C_1(-\deg) \leq \nu \leq C_2(-\deg)$
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$\mathcal{V}_1$ is a tree

- First theorem: fixed pt theorem on tree
- Second theorem: structure result for valuation in $\mathcal{V}_1$
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2. global results for valuations in $\mathcal{V}_1$
3. fixed point thm
4. desingularization of polynomial maps.

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