# Potential theory on trees and applications

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[1] . Introduction to the series of talks

- trees and application → usually on group theory
  of Bestvina.

  here two applications
  - → $\mathbb{C}$ dynamics w. Jonsson
  - → height theory w. Rivera-Letelier

  focus on opposition isometries on trees / non-invertible maps

. General facts on trees
- 4 definitions
- tgt vectors → branch pts
- weak topology
- compactness

. Example
- of increasing complexity
- of the valuative tree at infinity.

[2] . Asymptotic degrees

[3] . p-adic dynamics + potential theory on trees

[4] . application to height theory → dynamics of maps $R \in \mathbb{Q}(T)$
adelic approach.

# Talk 1

## Thanks

I'd like first to thank the organizers for having brought all of us to this very beautiful place of Chile. I would like to thank them too for giving me the opportunity of talking about a project I've been working on quite intensively for 5 years on with different collaborators (including IRL and OYS), ~~and which~~ I think that the theory I'll talk about can be expected to have a number of other applications than the ones I'll describe here. My hope is that other people will apply this theory to different pbms.

## General presentation

Let me first give a general overview of my series of talks.

- central object = real tree.
- main goal = develop ~~one kind of~~ a general analysis of R-trees (may call) dendrology, and give applications of this analysis to 2 very ~~quite~~ ~~different~~ problems of very different ~~subjects~~ nature.

2/

× rough def of trees = union of real intervals patched together in such a way that the resulting space contains no cycle.

× very common in math → considerable attention in <u>geometric gp theory</u>

reference to Bestvina
Morgan
Shalen   dendrologies and its applications

aim = look actions of groups by isometry on trees essentially to get information on the structure of the group.

× As opposed to the "classical" approach, I'll present two instances where one is naturally lead to study the dynamics of a (non-invertible) self-map on a tree.

Quick / Rough presentation of these pbms.

1. <u>$\mathbb{C}$ dynamics</u>  $F : \mathbb{C}^2 \to \mathbb{C}^2$ polynomial
Question is to describe $\{\deg(F^n)\}_{n \geq 0}$.
To do so, let $F$ act on $\mathcal{V}_\infty(\mathbb{C}^2)$ set of valuations centered at $\infty$ in $\mathbb{P}^2$.
(J. w M Jonsson)

2. <u>p-adic dynamics</u>
start with $R \in K(T)$   $K$ = extension of $\mathbb{Q}_p$
usually $\mathbb{C}_p$
and study $R : \mathbb{P}^1(K) \to \mathbb{P}^1(K)$.
Focus on ergodic properties of $R$ → construct inv. measure

in the $\mathbb{C}$ case, one way to do that is through potential theory.

here, I will follow the same path

① $R : \mathbb{P}^1(K) \circlearrowleft \xrightarrow[\text{1st step}]{} R : \mathbb{P}^1_{Ber}(K) \circlearrowleft$

$\mathbb{R}$-tree containing $\mathbb{P}^1(K)$ in its boundary

② develop general potential theory on trees, apply it to $\mathbb{P}^1_{Ber}(K)$ to construct and study Brolin measure.

→ I will also give applications of all this to the pbm of equidistribution of pts of small heights in an arithmetical context

Let me insist on the fact that both trees we shall encounter

$\mathcal{V}_\infty(\mathbb{C}^2)$ and $\mathbb{P}^1_{Ber}(K)$

have a very similar description as sets of valuations. They thus share many ~~properties~~ of the same caracteristic.

**Plan**

1. Generalities on trees (fix terminology)

   example of $\mathbb{R}$-tree $= \mathcal{V}_\infty(\mathbb{C}^2)$

2. Asymptotic degrees for $F : \mathbb{C}^2 \to \mathbb{C}^2$

3. Potential theory on trees and application to p-adic dynamics

4. Equidistribution of pts of small heights.

4/

# 1. Generalities on trees.

## • 4 different definitions of trees.

| | − metric | + metric |
|---|---|---|
| + marked point | rooted ℝ-tree | parameterized ℝ-tree |
| − marked pt | ℝ-tree | metric ℝ-tree |

**rooted ℝ-tree** $(\mathcal{T}, \leq)$ poset
- unique minimal element $\tau_0$ called the root
- $\{\sigma \in \mathcal{T}, \sigma \leq \tau\} \simeq ([0,1], \leq)$ $\forall \tau$
+ additional condition allows to avoid long lines in $\mathcal{T}$.

$\tau_1 \wedge \tau_2 = \min\{\tau_1, \tau_2\}$

$[\tau_1, \tau_2] = \{\tau_1 \wedge \tau_2 \leq \sigma \leq \tau_1\} \cup \{\tau_1 \wedge \tau_2 \leq \sigma \leq \tau_2\}$


**ℝ-tree** = "rooted ℝ-tree" without root!

formally $P(\mathcal{T})$ = set of partial ordering on $\mathcal{T}$ s.t. $(\mathcal{T}, \leq)$ is a tree

$\leq_1 \sim \leq_2 \in P(\mathcal{T})$ if they define the same segments

structure of ℝ-tree on $\mathcal{T}$ is an equivalence class in $P(\mathcal{T})$

**metric ℝ-tree** = more concrete def. than "ℝ-tree + metric"!

$(\mathcal{T}, d)$ metric space connected uniquely pathwise connected
↓
unique topological arc
joining any 2 pts $\tau_1, \tau_2 \in \mathcal{T}$

$([\tau_1, \tau_2], d)$ isometric to a real segment ($I$, standard metric)
$I \subset \mathbb{R}$

$t : I \xrightarrow{\tau} \mathcal{T}$ $\quad d(\tau(t), \tau(t')) = |t - t'|$

metric ℝ-tree $\xrightarrow[\text{+ root } \tau_0]{}$ rooted ℝ-tree

$\tau \leq_{\tau_0} \tau' \iff [\tau_0, \tau'] \supseteq [\tau_0, \tau]$

change the root → all $\leq_{\tau_0}$ are equivalent → ℝ-tree

<u>parametrized tree</u>

metric $\mathbb{R}$-tree + a root $\tau_0 \in \mathcal{T}$

$\Updownarrow$

$(\mathcal{T}, \varepsilon)$ rooted $\mathbb{R}$-tree

+ $\alpha : \mathcal{T} \to \mathbb{R}$ s.t. $\alpha : [\tau_0, \tau] \to [\alpha(\tau_0), \alpha(\tau)]$

is a bijection $\forall \tau$

$\alpha$ = parametrization of $\mathcal{T}$.

- <u>Dendrology</u> $\mathcal{T}$ $\mathbb{R}$-tree

we fix $\tau \in \mathcal{T}$


$\sigma \sim_\tau \sigma' \iff ]\tau, \sigma] \cap ]\tau, \sigma'] \neq \emptyset$

Equivalence class = tgt vector (direction) $\vec{v}$

Set of tangent vectors = tgt space at $\tau$ denoted by $T\tau$

endpt $\# T\tau = 1$

regular pt $\# T\tau = 2$

branched pt $\# T\tau \geq 3$

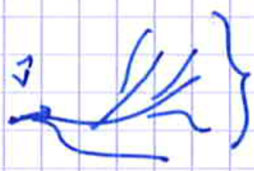
$[0,1]$,

endpts $= \{0,1\}$

regular pts $]0,1[$

- <u>Topology</u> $\mathcal{T}$ $\mathbb{R}$-tree $\to$ weak topology

- $\tau \in \mathcal{T}$ $\vec{v} \in T\tau$ $U(\vec{v}) = \{\sigma \in \mathcal{T},$ determines $\vec{v}$ at $\tau\}$




$U(\vec{v})$.

- weak topology for which $U(\vec{v})$ is a basis of open sets

open set = arbitrary union of finite intersection of $U(\vec{v})$'s.

6/ suppose now that $\mathcal{T}$ is a <u>metric</u> $\mathbb{R}$-tree

two topologies on $\mathcal{T}$ < weak topo / metric topo

I'd like to explain that these two topologies can be very different. In practice, both topologies are useful but in different contexts.

To understand the difference between both topol., we look at the when $\mathcal{T}$ is compact.

- weak sense = only obstruction is when $\mathcal{T}$ does not contain its ends.

$$\mathcal{T} \xrightarrow[\text{completion}]{} \overline{\mathcal{T}} = \mathcal{T} + \text{its ends}$$

ends = sequence of pts $\tau_i$, $\tau_j \in [\tau_0, \tau_i]$ $j \le i$ and $\tau_i$ not converging in $\mathcal{T}$ modulo suitable equivalence relation

$\overline{\mathcal{T}}$ is still a $\mathbb{R}$-tree.

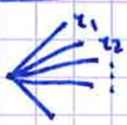
<u>ex</u> $\mathcal{T} = (\mathbb{R}, |.|) \longrightarrow \overline{\mathcal{T}} = \{-\infty\} \cup \mathbb{R} \cup \{+\infty\}$

**Thm** $\mathcal{T}$ metric $\mathbb{R}$-tree

then $\overline{\mathcal{T}}$ is a $\mathbb{R}$-tree which is weakly compact

**Prop** $(\mathcal{T}, d)$ metric $\mathbb{R}$-tree

Suppose $T_\tau$ is uncountable, then $(\mathcal{T}, d)$ is <u>not</u> locally compact

<u>Pf</u>: 

$d(\tau, \tau_i) \geq \varepsilon_0 > 0$ $\tau_i$ determine distinct tgt vectors

## 2. A valuative tree.

I shall spend the rest of my time to describe and study an important example of tree that I shall denote by $\mathcal{V}_\infty(\mathbb{C}^2)$ and which shall play a key role tomorrow in the study of degrees of iterates of a $F : \mathbb{C}^2 \to \mathbb{C}^2$.

- valuation on the ring $\mathbb{C}[X,Y]$

$$v : \mathbb{C}[X,Y] \to \mathbb{R} \cup \{+\infty\}$$

$$v(PQ) = v(P) + v(Q)$$

$$v(P+Q) \geq \min v(P), v(Q)$$

$$v(\lambda) = 0 \quad \lambda \in \mathbb{C}^* \qquad v(0) = +\infty$$

- interested in valuations "centered" at infinity. (shall explain this geom. later on)

$$v(P) < 0 \text{ for some } P.$$

- normalization:

if $v$ centered at $\infty$ either $v(X) < 0$ or $v(Y) < 0$

$v$ and $tv$ $t > 0$ have essent. same geom. content.

$$\min v(X), v(Y) = -1$$

$\mathcal{V}_\infty(\mathbb{C}^2) = \{ v$ val. on $\mathbb{C}[X,Y]$ centered at $\infty$ $\min v(X), v(Y) = -1 \}$.

<u>aim</u> = show that this set is a tree

Before that $\to$ describe a few set of examples.

8/

**1** $-\deg \in \mathcal{V}_\infty(\mathbb{C}^2)$

geom. interpretation $\qquad \mathbb{C}^2 \subseteq \mathbb{P}^2(\mathbb{C})$

$\mathbb{C}^2 \,\big)\, L_\infty \qquad P \in \mathbb{C}[X,Y] \to \tilde{P} : \mathbb{P}^2(\mathbb{C}) \to \mathbb{C}\cup\{\infty\}$ meromorphic

$$-\deg(P) = \mathrm{ord}_{L_\infty}(\tilde{P}).$$

**2** $\pi : X \to \mathbb{P}^2(\mathbb{C}) \qquad \pi$ : composition of pt blow-up.

$E \subseteq X$ irreducible component

then $\mathrm{ord}_E$ is a valuation on $\mathbb{C}[X,Y] \to$ <u>divisorial</u>.

$\mathrm{ord}_E$ centered at infinity $\Leftrightarrow$ $\pi(E) \subseteq L_\infty$.

$$\mathrm{ord}_E \rightsquigarrow \nu_E \in \mathcal{V}_\infty(\mathbb{C}^2)$$

**3** Not all val. are divisorial

$s = (s_1, s_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$\nu_s(P) = \min\{ i s_1 + j s_2,\ a_{ij} \neq 0 \quad P = \sum a_{ij} X^i Y^j \}$.

monomial valuations.

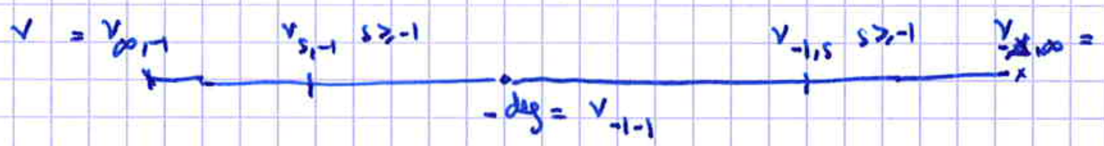
when $[s_1 : s_2] \in \mathbb{P}^1(\mathbb{Q})$ then $\nu_s$ is divisorial

$$\nu_{(1,0)} = \mathrm{ord}_{\{x=0\}}$$

when $[s_1 : s_2] \notin \mathbb{P}^1(\mathbb{Q})$ we say that $\nu_s$ is irr. monomial.

<u>rem.</u> $\nu_s \in \mathcal{V}_\infty(\mathbb{C}^2)$ iff $\min\{s_1, s_2\} = -1$.

?



4 Quasimonomial valuations = cocktail 50% div 50% mon.

$$\pi : X \to \mathbb{P}^2 \qquad p \in \pi^{-1}(L_\infty)$$

$(z,w)$ coord. at $p$.

$(s_1, s_2) \in \mathbb{R}_+^2$ weights.

$$P \longrightarrow v_{s_1 s_2}^{z,w}(\widetilde{P} \circ \pi) = \pi_* v_{s_1 s_2}^{z,w}(\widetilde{P})$$

$\widetilde{P}$ mero. fct on $X$.

$p \in \pi^{-1}(L_\infty)$ + $s_1, s_2 \geq 0$ $\Rightarrow$ then $v$ is centered at infinity.

$\to$ other example of valuations but they won't play an essential role in what follows.

## Tree structure

$$v \leq v' \iff v(P) \leq v'(P) \quad \forall P.$$

minimal element $= -\deg$.

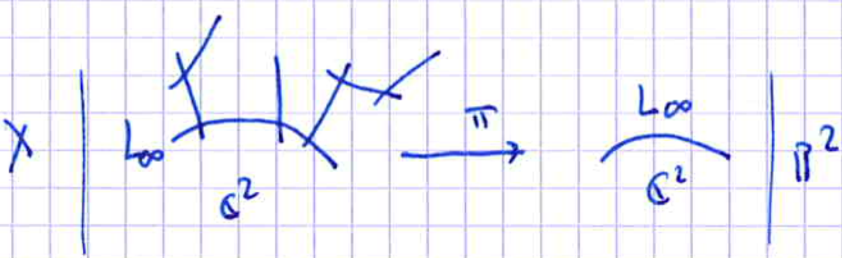
**Thm** $(\mathcal{V}_\infty(\mathbb{C}^2), \leq)$ is a rooted tree whose root is $-\deg$.

## Idea of proof

Look first at the set of qm valuations.

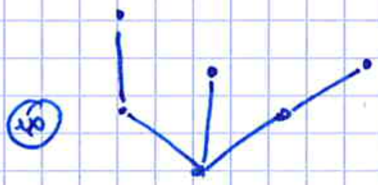
Fix $\pi : X \to \mathbb{P}^2$ centered at $\infty$

$\pi$ (Exc$(\pi)$) $\subseteq L_\infty$.



9/ Configuration of curves ⇝ attach its dual graph $\Gamma_\pi$

| vertices = irred. comp. of $\pi^{-1}(L_\infty)$.
| edge when $E, E'$ intersect.

(ex) 

$\Gamma_\pi$ = simplicial tree (by induction)

dual graph embedds naturally in $\mathcal{V}_\infty(\mathbb{C}^2)$

. vertex $\ni E \longrightarrow v_E$ divisorial val. in $\mathcal{V}_\infty(\mathbb{C}^2)$

. edges $e$ 


Choose coord. $\{zw=0\} = E \cup E'$

look at $\pi_* v^{z,w}_{s_1,s_2}$ $\quad s_1, s_2 \geq 0$.

val. centered at $\infty$
+
normalization $a_1 s_1 + a_2 s_2 = 1$

→ get a segment inside $\mathcal{V}_\infty(\mathbb{C}^2)$

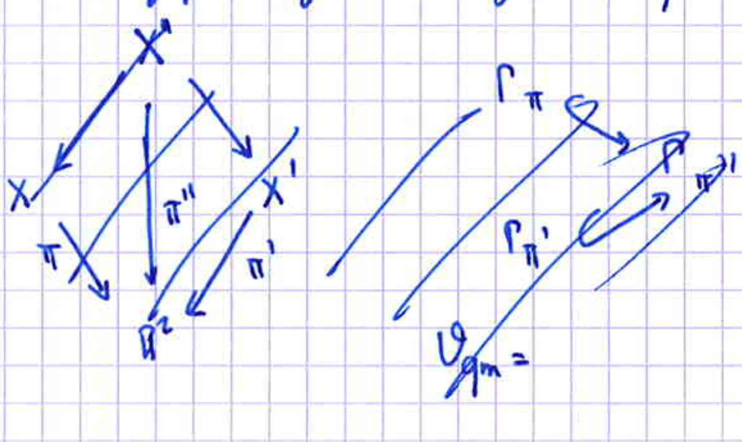
joining $\pi_* v^{z,w}_{0,1/a_2} = v_{E'}$

$\pi_* v^{z,w}_{1/a_1,0} = v_E$

$i_\pi : \Gamma_\pi \longrightarrow \mathcal{V}_\infty(\mathbb{C}^2)$

bijection onto the set of monomial valuations at each intersection pts $E \cap E'$ $\quad E, E' \subseteq \pi^{-1}(L_\infty)$.

* put all these graphs together to get all gm valuations.



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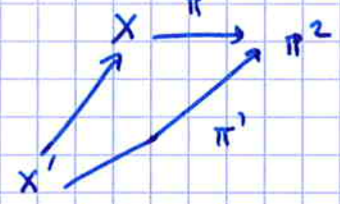
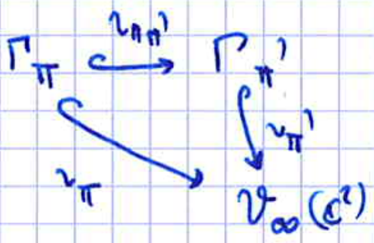
$\pi'$ dominates $\pi$ iff




classical fact that $\forall \pi, \pi'$ $\exists \pi''$ dominating both of them

so $\{\pi : X \to \mathbb{P}^2\}$ has an inductive structure

$\varinjlim \Gamma_\pi$ (= union of all $\Gamma_\pi$ patched together with the $\nu_{\pi\pi'}$ maps)

↓ one hand: it is clearly an $\mathbb{R}$-tree

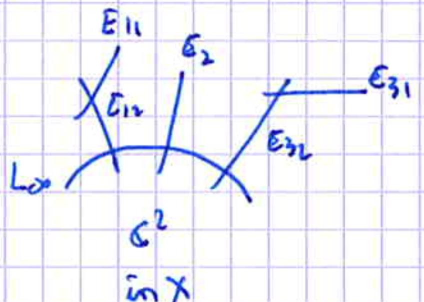
↓ other hand: by def $\mathcal{V}_{qm} \subseteq \mathcal{V}_\infty(\mathbb{C}^2)$

x to conclude <
- prove that completion of $\mathcal{V}_{qm} \cong \mathcal{V}_\infty(\mathbb{C}^2)$
- relate the tree structure to $\leq$

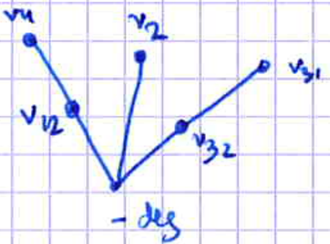
very delicate but the construction above gives a good geometric picture and explains why $\mathcal{V}_\infty(\mathbb{C}^2)$ has a chance to be a tree! □

<u>dendrology</u> = understand the tangent space at a divisorial valuation at $-\deg$.

Fix $\pi : X \to \mathbb{P}^2$
$L_\infty \subset X$




<u>in X</u>




<u>in $\mathcal{V}_\infty(\mathbb{C}^2)$</u>

blow up a pt $p \to E_p$
- if $p \notin L_\infty$ say in $E_2$ — no new tgt vectors
- if $p \in L_\infty \cap E$ — same
- if $p \in L_\infty \setminus \cup E$ — go a new tgt vector

11/

**Prop** | $p \in L_\infty \longmapsto \vec{v}_p$ = tgt determined by $v_{E_p}$ at - deg.

this map is a <u>bijection</u> onto $T$(-deg)

→ True at any div. val

→ $T_v$ uncountable = $\mathbb{P}^1(\mathbb{C})$ !

~~~~ for any tree metric, $\mathcal{V}_\infty(\mathbb{C}^2)$ is uncountable !~~

| | |
|---|---|
| branched pt | div. val |
| regular pt | irrat. qm val |
| end points | others |

<u>Topologies</u>.

**Thm** | weak topology on $\mathcal{V}_\infty(\mathbb{C}^2)$ coincides with the pointwise convergence topology $v_k \to v \Leftrightarrow v_k(P) \to v(P)\ \forall P$

→ compact.

**Prop** | For any tree metric on $\mathcal{V}_\infty(\mathbb{C}^2)$, the space $(\mathcal{V}_\infty(\mathbb{C}^2), d)$ is <u>NOT</u> locally compact.
~~~~

Talk 2 j.w. M. Jonsson

- concern with dynamics of maps $F: \mathbb{C}^2 \to \mathbb{C}^2$ polynomial and dominant.

ask many questions here → ergodic properties of these maps. especially concerning their entropy

∃ various conjectures: to explain ∃ need to introduce 2 invariants

× topo. degree $e \geq 1$ inv. of conj. by polyn. aut.

× $d(F)$ = degree of $F$ $F = (P_d, Q_d) +$ l.o.t. hom. of degree d.

NOT invariant $d(F^{k+l}) \leq d(F^k)\, d(F^l)$

$d_\infty(F) = \lim d(F^k)^{1/k}$ asympt. degree invariant

<u>Conj 1</u> $h_{top}(F) = \log \max\{d_\infty(F), e\}$ ≤ Gromov

<u>Conj 2</u> When $d_\infty(F) \neq e$, ∃! max entropy $\mu$

<u>Guedj</u> | proved $d_\infty(F) < e$
| sketch an attack $d_\infty(F) > e$.

interesting = first needs to construct an ≥0 closed (1,1) inv. current T s.t. $F^* T = d_\infty(F)\, T$.

**Thm 1** ∃ ≥0 closed (1,1) current T which does not charge curve and $F^* T = d_\infty(F)\, T$.

cheating ⟶ would like to sketch.

**Thm 2** $d_\infty(F)$ is a quadratic integer ① either $F = (P(X), Q(X,Y))$ after chge of coord.
② or $d_\infty(F)^n \leq d(F^n) \leq C_1 \cdot d_\infty(F)^n$ ∀n ⟺

methods used to prove 2 ⇒ 1

~~let me first recall~~

Let me first add a few comments on the valuation space we looked at yesterday.

- $\mathcal{V}_\infty(\mathbb{C}^2) = \{ v \text{ valuation } \min v(x), v(y) = -1 \}$.

  order relation $\leq$ — Rooted tree at a dog.

- $\mathcal{V}_{qm} \subseteq \mathcal{V}_\infty(\mathbb{C}^2)$

  | divisorial → branched pt
  | irrat. qm → regular pt
  | others → endpts.

- I'll turn now to the application of the tree $\mathcal{V}_\infty$ $\mathbb{C}$ dyn.

2/ <u>idea of proof</u> : lift action $F$ on $\mathcal{V}_\infty(\mathbb{C}^2)$ = set of valuations at $\infty$ in $\mathbb{C}^2$!

- rough justification: $\mathbb{C}^2 \circlearrowleft F$, $L_\infty$ — assume that $L_\infty$ is not contracted

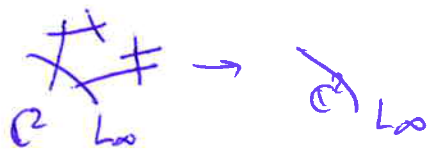
$$F = (P_d, Q_d) + \dots$$

$[X:Y] \to [P_d : Q_d]$ is a non constant rational map

$$F^n = (P_d, Q_d)^n + \dots \neq 0.$$

$$\deg(F^n) = d^n \quad \forall n.$$

- might hope that $\pi : X \to \mathbb{P}^2$




if you understand the dynamics of $f$ on the irreducible components of $\pi^{-1}(L_\infty)$ → will be able to understand $\{\deg(F^n)\}_{n \geq 0}$

## 1 Action on $\mathcal{V}_\infty(\mathbb{C}^2)$

$$v \in \mathcal{V}_\infty(\mathbb{C}^2) \longrightarrow F_* v(P) = v(P \circ F).$$

(ex) $F = (X, XY)$ $\quad v = v_{(s_1, s_2)}$ $\quad F_* v_{(s_1,s_2)} = v_{(s_1, s_1+s_2)}$

$F_* v_{(1/2,-1)} = v_{(1/2,-1/2)}$ not normalized

$F_* v_{(2,-1)} = v_{(2,1)}$ not centered at infinity.

→ First find a subtree on which $F_*$ is well-defined

3/ · $\mathcal{V}^- = \{v,\ v(P) < 0\} \ni -\deg$

all monomial valuations $\max\{s_1, s_2\} < 0$.

$v \in \mathcal{V}^-$ then $F_\bullet v \in \mathcal{V}^-$.

* $\min\{F_\bullet v(x), F_\bullet v(y)\} = \min\{v(P), v(Q)\} < 0$.

$$d(F,v) = -\min\{v(P), v(Q)\}$$

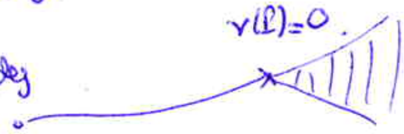
$v \in \mathcal{V}^-$ $\quad F_\bullet v = d(F,v)\, F\cdot v$. $\quad d(F,-\deg) = \deg(F)$.

$$F_\bullet : \mathcal{V}^- \cap \mathcal{V}_\infty(\mathbb{C}^2) \circlearrowleft.$$

* Claim: $\mathcal{V}^- \cap \mathcal{V}_\infty(\mathbb{C}^2)$ is a subtree of $\mathcal{V}_\infty(\mathbb{C}^2)$

proof $\mathcal{T} = \bigcap_P \{v \in \mathcal{V}_\infty(\mathbb{C}^2),\ v(P) < 0\}$

↑ subtree




$(\mathcal{V}_\infty(\mathbb{C}^2), \leq)$ is a rooted tree □.

$F_\bullet : \mathcal{V}^- \cap \mathcal{V}_\infty(\mathbb{C}^2) \circlearrowleft$ continuous tree map

## 2 Sketch of proof

1. Find a valuation $v$ fixed by $F_\bullet$ : general fixed pt thm on trees.
2. Prove that $v$ is comparable to $-\deg$, or attached to a rational fibration.

$$-\deg \le v \le C(-\deg).$$

$$\deg(F^n) = d(-\deg, F^n) \geq d(v, F^n) \geq C \deg(F^n).$$
$$\| \qquad d(v,F)^n.$$

$d_\infty(F) = d(v,F)$ + estimate.

~~- basic rational fibration $\{x = cst\} \to v_{(0,-1)} = -\deg_y$~~

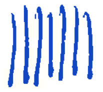
~~⇝ TSVP general rational fibration (thm AMS) = $\{x = cst\}$ after polyn. change of coord.~~

. We shall prove that in the second case $v$ is associated to a pencil of smooth affine lines ($\simeq \mathbb{C}$) covering $\mathbb{C}^2$.

× basic example of such valuation

- $v_{(0,-1)} \in \mathcal{V}_\infty(\mathbb{C}^2)$ $\quad v_{(0,-1)}(P) = -\deg_y(P)$.
- $v_{(0,-1)}(P) = \#\, P^{-1}(0) \cdot \{x = cte\}$ in $\mathbb{C}^2$

geo. alg.



val. $v_{(0,-1)}$

$x = cte$

$F_* v_{(0,-1)} = d\, v_{(0,-1)}$ $\quad F_* v_{(0,-1)}(x) = 0$ $\quad F = (P(x), Q(y,x))$.

**Def** × in general $v$ is associated to a rational pencil iff $v = \phi_* v_{(0,-1)}$ after a suitable chge of coord.

$\phi \in \mathrm{Aut}[\mathbb{C}^2]$.

$\mathcal{T}$ = complete $\mathbb{R}$-tree.
$h : \mathcal{T} \to \mathcal{T}$ continuous on segments.
then $\exists\, \tau \in \mathcal{T}$ $h(\tau) = \tau$.

probably hidden in the literature otherwise JRL, FJ.


follow your image like a rocket following its target. □.

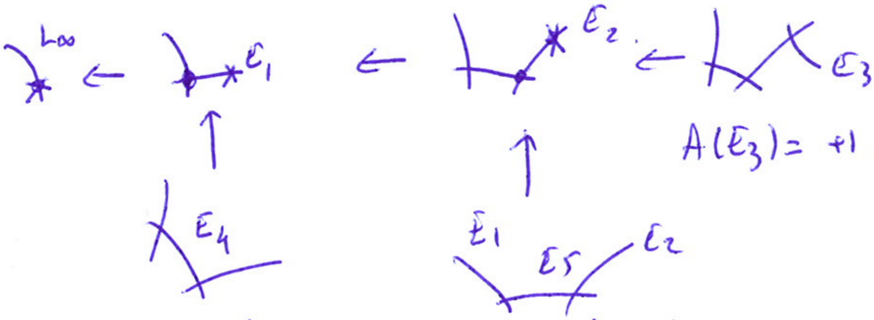
## tree $\mathcal{V}_1$

definition

- the thinness function $E \subseteq \pi^{-1}(L_\infty)$.

$$A(E) = 1 + \mathrm{ord}_E(\pi^* dX \wedge dY).$$

$$A(L_\infty) = 1 - 3 = -2 \qquad A(E_1) = -1 \qquad A(E_2) = 0$$




$A(E_3) = +1$.

$A(E_4) = -3$ $\qquad$ $A(E_5) = -1$.

~~[illegible]~~ $\exists!\, A : \mathcal{V}_{div} \to \mathbb{Q}$.

$$\left| \begin{array}{l} A(\pi_* \mathrm{ord}_E) = A(E) \\ A(t v) = t\, A(v) \end{array} \right.$$

$A(v_{c,c_2}) = s_1 + s_2$. $\rightsquigarrow$ guess that $A$ can be extended

Prop. A extends to a continuous function along segments on $\mathcal{V}_\infty(\mathbb{C}^2)$

~~It follows from the formula~~

~~$A(F_* v) = A(v) + v(JF)$~~

on qm valuations → coq of the preceding computation.

$A : \mathcal{V}_\infty(\mathbb{C}^2) \longrightarrow [-2, +\infty]$ is an increasing parameterization

$\mathcal{V}' = \overline{\{v, \ v(P) < 0 \ \forall P, \ A(v) \leq 0\}}$.

- complete tree → w. compact
- $F_\circ : \mathcal{V}' \to \mathcal{V}'$ $\quad A(F_* v) = A(v) + v(JF)$.

$\in \mathcal{V}'$
- either $-\deg \leq v \leq C(-\deg)$
- or $v$ is attached to a rational fibration.

(*)

summary
- $F_* = d \times F_\circ$
- $F_\circ : \mathcal{V}' \to \mathcal{V}'$ find a fixed pt $v_\infty$.
- $d_\infty = d(F, v_\infty)$ + apply dual. thm for $v_\infty \in \mathcal{V}'$

aim = show that $d_\infty$ is a quadratic integer.

This last part is very delicate and rely on the proof of contraction properties near $v_\infty$.

Proof of thm 2

?

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## (*) comments on the pf of thm 2

- deep result: says that under suitable conditions
  $\nu$ of purely local nature $\to$ can be controlled globally.

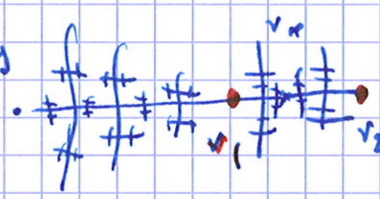
- proof relies on a very delicate technique called key polynomials which is a way of encoding valuation.
  - MacLane
  - Abhyankar-Moh work in affine space $\to$ Jacobian conj.
    approximate roots
  - used by Reguera-Piltant-Campillo
    to prove Nef cone of some rat. surface is polyhedral

really a very powerful tool.

## §4. Contraction properties of $\nu_*$.

$\nu_*$ end point
$\nu_*$ divisorial } $d_\infty \in \mathbb{N}$
$\nu_*$ qm irrat. $\longrightarrow$ least interesting case.

**Thm** Assume $\nu_* \in \mathcal{V}'$ is irrat. qm and fixed by $F_\bullet$.
$\exists$ two divisorial valuations
$\nu_1 < \nu_* < \nu_2 \in \mathcal{V}'$ s.t.

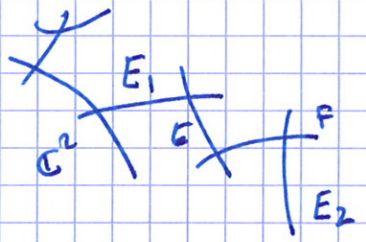
- dsg



& either $F_\bullet^2 \big|_{[\nu_1,\nu_2]} = \mathrm{Id}$

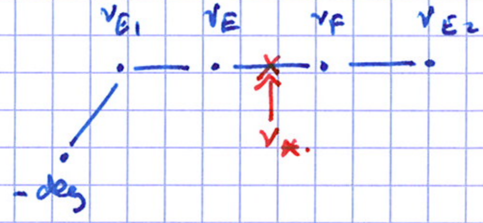
& or $U = \{\nu_1 < \mu\} \setminus \{\nu_2 \leq \mu\}$ is $F_\bullet$-invariant
and $\forall \mu \in U \quad F_\bullet^n \mu \longrightarrow \nu_*$.

- **Thm ⇒ $d_\infty$ quadratic integer.**

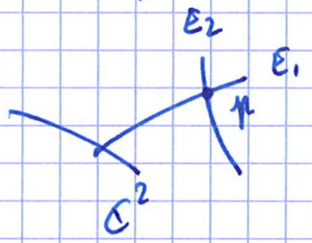
- blow-up to let $E_1, E_2$ appear. $\pi : X \to \mathbb{P}^2$

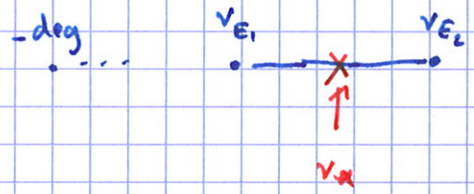



in $\mathcal{V}_\infty(\mathbb{C}^2)$



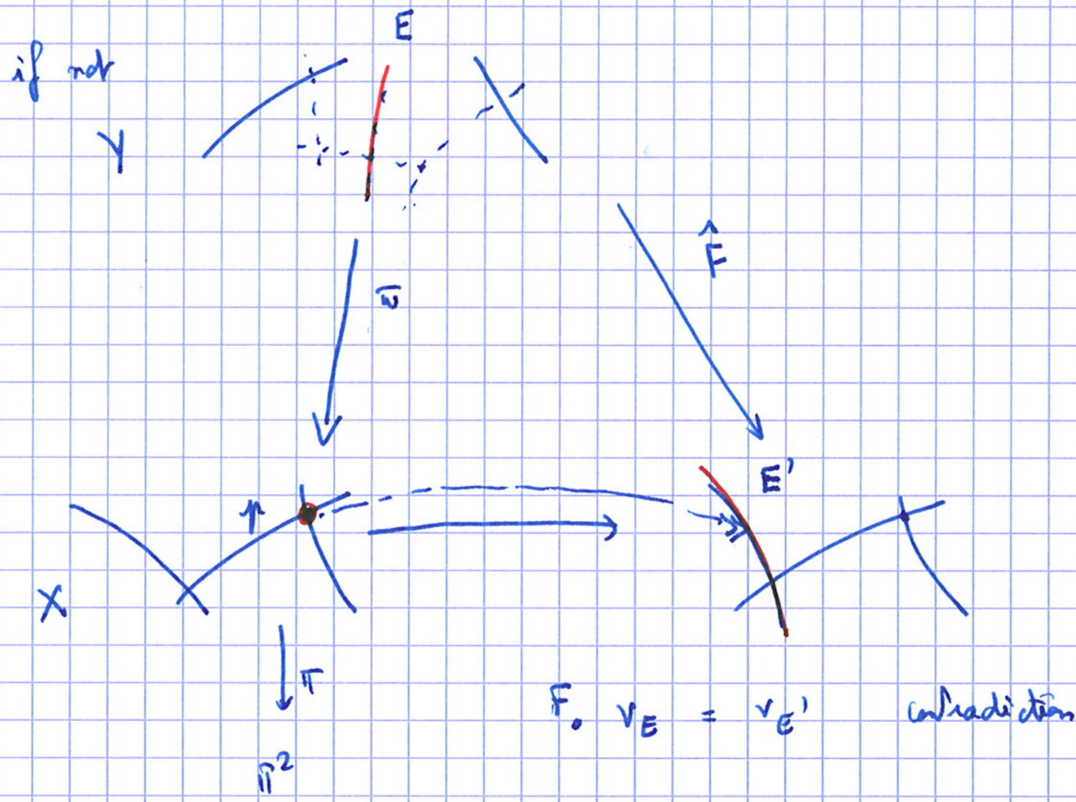

- first tricky pt: one can take $E_1, E_2$ in the thm above s.t.




- Claim: F is holomorphic at p

if not




$F_\bullet \nu_E = \nu_{E'}$ contradiction

- Claim: the local map $(F, p)$ is super special!

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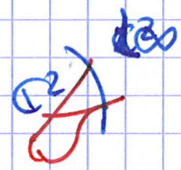
- as before, in the thm one can take $v_{E_1}$ and $v_{E_2}$ arbitrarily close to $v_\infty$ → ie blowup more above the pt $p$.

in $\mathbb{P}^2$  finite number of critical components

- each analytic branch at $\infty$ defines a valuation
- take $v_{E_1}, v_{E_2}$ s.t. no branche belongs to $U$

a geom. $\overline{\mathrm{Crit}(F)} \not\ni p$.

$\partial(\hat{F}, p) = \mathrm{Crit}(\hat{F}) = E_1 \cup E_2 = \mathrm{Crit}(\hat{F}^n)$.

$E_2$ $E_1$

rigid $\Rightarrow$ $(\hat{F}, p) \simeq (z^a w^b, z^c w^d)$.

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- $d(F^n, v_*) = -\min\{v(X \circ F^n), v(Y \circ F^n)\}$.

$v = \pi_* v^{z,w}_{s_1, s_2}$ $\quad X = z^{-h} w^{-l}$ $\quad h, l > 0$. $\quad Y = z^{-h'} w^{-l'}$

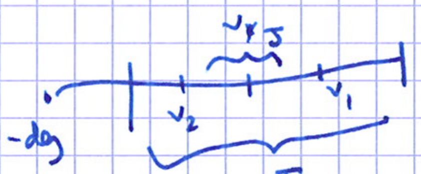
$= * \max\left\{(s_1, s_2)\begin{bmatrix} a & b \\ c & d \end{bmatrix}^n \binom{h}{l}, (s_1, s_2)\begin{bmatrix} a & b \\ c & d \end{bmatrix}^n \binom{h'}{l'}\right\}$.

$\simeq \delta^n$ $\quad \delta$ = spec radius of $M$

<u>idea of proof of the thm</u>

$v_\infty$ is a regular pt

$F_* : J \to I$



in a suitable parametrization $\alpha$.

$\alpha(F_* v) = \dfrac{a\alpha + b}{c\alpha + d}$

$a, b, c, d \in \mathbb{N}^*$

→ any such map has an attracting fixed pt. □

Y

TALK 3

I first need to apologize for people who expected / wanted to have more details on the sketch of proof I explained yesterday. But if you want more information about this

- × arxiv Eigenvaluations F-Jonsson
- × ask me directly

In this talk and the next, I'll discuss a totally different pbm than the growth of degrees and look at

ergodic properties of p-adic rational maps.

and (if time allows) talk about application to height theory.

## 1. The complex case

$R \in \mathbb{C}(T) \quad \deg(R) = D \geq 2$

$R : \mathbb{P}^1(\mathbb{C}) \circlearrowleft$

**Thm** $\exists !$ measure $\rho$ of maximal entropy equal to $\log D$.

- mixing
- describes distribution of preimages: $D^{-n} \sum_{R^n(w)=z} [w] \longrightarrow \rho$

(at most 2 exceptions)

- ———————— periodic pts: $D^{-n} \sum_{R^n(w)=w} [w] \longrightarrow \rho$.

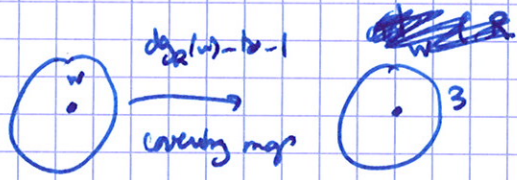
construction of the measure = polyn. Brolin '60

= proved Lyubich + FLM '80

— description of the construction of $\rho$.

* Pull-back operator on ≥ 0 measure.

$$R_* \varphi(z) = \sum_{R(w)=z} \varphi(w) \times \deg_R(w).$$




$C^0$ operator on $C^0$ ~~measure~~ fcts

$$|R_* \varphi|_\infty \le D\,|\varphi|_\infty .$$

duality $\langle R^* \rho, \varphi \rangle = \langle \rho, R_* \varphi \rangle$.

Mass $R^* \rho$ = Mass $\rho \times D$.

Supp $R^* \rho = R^{-1}$ Supp $\rho$.

Start $\omega$ = Leb. measure on $\mathbb{P}^1(\mathbb{C})$.

$D^{-1} R^* \omega - \omega = \Delta g$   $g : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$, $C^0$

Put here some complex analysis.

iterate $D^{-k} R^{*k} \omega - \omega = \Delta g_k$

↓ $\rho$

$$g_k = \sum_{j=0}^{k-1} \frac{g \circ R^j}{D^j} \qquad \left|\frac{g \circ R^j}{D^j}\right|_\infty \le \frac{|g|_\infty}{D^j}$$

unif CVG ↓ $g_\infty$

$D^{-k} R^{k n} \omega \to \rho$ $\quad R^* \rho = D\rho$ does not change pts

+ satisfies the thm above.

<u>Goal</u> = explain to generalize all this to the p-adic setting.

## 2. Basics on p-adic dynamics

$R \in \mathbb{C}_p(T)$. $\quad R : \mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\mathbb{C}_p)$. $D \geq 2$

- $\mathbb{Q}$ $\quad | \ |_\infty$

$$\left|p^k \frac{a}{b}\right|_p = p^{-k} \qquad \gcd\{a, p\} = \gcd\{b, p\} = +1.$$

$$\begin{array}{ccccccc} \mathbb{Q} & \xrightarrow{|\cdot|_\infty} & \mathbb{R} & \xrightarrow{\text{alg. clos.}} & \mathbb{C} & & \\ | & & & & & & \\ \mathbb{Q} & \xrightarrow{|\cdot|_p} & \mathbb{Q}_p & \xrightarrow{\text{alg. clos.}} & \overline{\mathbb{Q}_p} & \xrightarrow{\text{completion}} & (\mathbb{C}_p, |\cdot|_p) \end{array}$$

Not use the alg. structure of $\mathbb{C}_p$ that is we don't care about

$\mathrm{Gal}(\overline{\mathbb{Q}_p} / \mathbb{Q}_p)$ far from simple

$\to$ use the topological structure.

- $(\mathbb{C}_p, |\cdot|_p)$ ultrametric $\longrightarrow$ $\overline{B}(\zeta, r) = \{ |z - \zeta| \leq r \}$. $B = \{<\}$

any pt is a center.

Structure of balls.

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$\overline{B}(0,1) \ni z$.

$\|$

$\mathcal{O}_p$ is a ring of integers of $\mathbb{C}_p$.

$\cup$

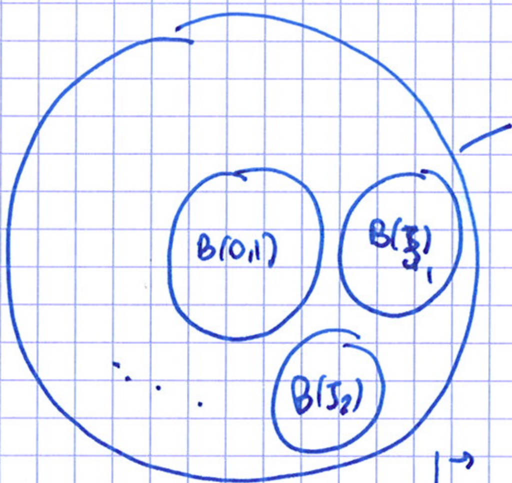
$m_{\mathcal{O}_p} = \{ |z| < 1 \}$. maximal ideal

$\mathcal{O}_p \longrightarrow \mathcal{O}_p / m_{\mathcal{O}_p}$ field.

in the case of
- $\mathbb{Q}_p$ $\quad \mathcal{O}_p / m_p \simeq \mathbb{F}_p$.
- $\mathbb{C}_p$ $\quad \mathcal{O}_p / m_{\mathcal{O}_p} \simeq \overline{\mathbb{F}_p}$ countable.

$$\overline{B}(0,1) \xrightarrow{\ \pi\ } \mathcal{O}_p / m_{\mathcal{O}_p} \simeq \overline{\mathbb{F}_p}$$

$$\pi(z) = \pi(z') \Leftrightarrow |z - z'| < 1$$




$J_1, J_2 \in \mathcal{O}_p / m_{\mathcal{O}_p}$.

$B(J_1) = \{ \xi,\ |\xi - z| < 1 \}$.

→ $\mathbb{C}_p$ not locally compact.

→ same picture for any other ball

→ ~~def of balls in $\mathbb{P}^1(\mathbb{C}_p)$~~.

- Look at $R : \mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\mathbb{C}_p)$

Now we have a better picture of what $\mathbb{C}_p$ and $\mathbb{P}^1(\mathbb{C}_p)$ we can look at the dynamics of $R$ on these space.

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## 3. Projective Berkovich line over $\mathbb{C}_p$.

good measure theory → better with a compact space.

$\mathbb{P}^1(\mathbb{C}_p) \hookrightarrow \mathbb{P}^1_{Ber}(\mathbb{C}_p)$ idea due to JRL.

$\mathbb{R}$-tree.

$\mathbb{P}^1(\mathbb{C}_p)$ = ends of $\mathbb{P}^1_{Ber}(\mathbb{C}_p)$

→ different presentation than Juan

- Set of semi-norms

$$S : \mathbb{C}_p[T] \to \mathbb{R}_+$$

$$\begin{cases} S(PQ) = S(P)\,S(Q) \\ S(P+Q) \leq \max\{S(P), S(Q)\} \\ S|_{\mathbb{C}_p} = |\cdot|_p \end{cases}$$

+ fort $S = S_\infty$ sends all non-cst polyn. to $+\infty$

endow this set of ptwise convergence → compact.

- $z \in \mathbb{C}_p$ $\quad S_z(P) = |P(z)|_p$.

not a norm.

$\mathbb{P}^1(\mathbb{C}_p) \hookrightarrow \mathbb{P}^1_{Ber}(\mathbb{C}_p)$.

- $B = \overline{B}(z, r)$ $\quad S_B(P) = \sup_B |P(z)|_p$

$$\mathbb{H}_p^{\mathbb{R}} = \{ S_B,\ \text{diam} > 0 \}.$$

$$= \mathbb{H}_p^{\mathbb{Q}} \sqcup \mathbb{H}_p^{\mathbb{R}/\mathbb{Q}}.$$

diam $\in p^{\mathbb{Q}}$ or $\in p^{\mathbb{R}/\mathbb{Q}}$

... other pts.

→ Start with the Lebesgue measure on $\mathbb{P}^1(\mathbb{C})$

+ use then elementary facts from potential theory

• $g : \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{R}(\mathbb{C})$ $\mathcal{C}^\infty$

$dg$ $\quad \partial g = (1,0)$-part of $dg$ $\quad \mathbb{C}$ 1-form

$= \frac{\partial g}{\partial z} dz \qquad \frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$

$\bar\partial g = (0,1)$ part of $dg$ $\quad \mathbb{C}$-antilinear 1-form

$= \frac{\partial g}{\partial \bar z} d\bar z \qquad \frac{\partial}{\partial \bar z} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$

$\Delta g := \frac{i}{\pi} \partial\bar\partial g$ fct $\longmapsto$ (1,1) form on $\mathbb{P}^1(\mathbb{C})$

||| by duality.

smooth measure on $\mathbb{P}^1(\mathbb{C})$.

• $g : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ $L^1_{loc}$

$\Delta g := \frac{i}{\pi} \partial\bar\partial g$ well-defined as a distribution (real)

× **Fact** $\rho_1$ $\rho_2$ are two probability measures

then $\exists\, g : \mathbb{P}^1(\mathbb{C}) \to \mathbb{R}$ $L^1_{loc}$

$\rho_1 - \rho_2 = \Delta g$.

$D^{-1} R^* \omega - \omega = \Delta g$

$\omega = \pi_* dd^c \operatorname{Log} |X|^2 + |Y|^2$

$\mathbb{C}^2$ $(P, Q)$ ↺

$g = \frac{\operatorname{Log}(|P|^2 + |Q|^2)}{D} - \log(|X|^2 + |Y|^2)$

$\downarrow \pi$

$\mathbb{P}^1$ ↺ $R$.

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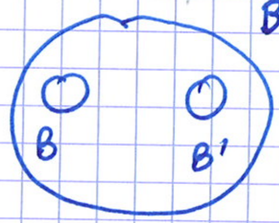
× On $\mathbb{H}_p^{\mathbb{R}}$ define a metric.

$$d(S_B, S_{B'}) = \log_p \left| \frac{\mathrm{diam}(B)}{\mathrm{diam}(B')} \right|_p . \qquad B' \subseteq B.$$




$$d(S_B, S_{B'}) = d(S_B, S_{BB'}) + d(S_{BB'}, S_{B'}).$$

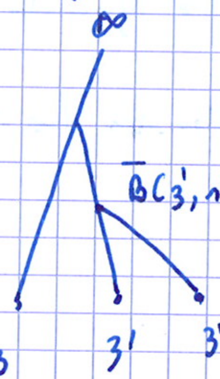
**def** $\mathbb{H}_p$ = completion as a metric space of $(\mathbb{H}_p^{\mathbb{R}}, d)$

**thm**

- $(\mathbb{H}_p, d)$ is a metric $\mathbb{R}$-tree
- $\mathbb{H}_p \hookrightarrow \mathbb{P}^1_{Ber}(\mathbb{C}_p)$ := completion of $\mathbb{H}_p$ as a tree, weak topo ≡ pointwise cvg.

$$\mathbb{P}^1_{Ber}(\mathbb{C}_p) = \underbrace{\mathbb{H}_p^{\mathbb{Q}} \sqcup \mathbb{H}_p^{\mathbb{R}\setminus\mathbb{Q}}}_{\text{closed balls}} \sqcup \underbrace{\mathbb{P}^1(\mathbb{C}_p)}_{\text{standard pts}} \sqcup \underbrace{(\mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{H}_p^{\mathbb{R}})}_{\text{singular pts}}$$

**proof** segments are exactly $\{ B(z,r),\ 0 < r < +\infty \}$.




$\bar{B}(z', r) = \bar{B}(z'', r) \qquad r = |z' - z''|_p.$

□.

- $S \in \mathbb{H}_p^{\mathbb{Q}}$ → branches in bijection with $\mathbb{P}^1(\overline{\mathbb{F}}_p)$. ← EXPLAIN.
- $S \in \mathbb{H}_p^{\mathbb{R}\setminus\mathbb{Q}}$ = regular pts as $|z - z'| \in p^{\mathbb{Q}}$.
- $S \in \mathbb{P}^1_{Ber}(\mathbb{C}_p) \setminus \mathbb{H}_p^{\mathbb{R}}$ = end pts.

// $\mathcal{V}_\infty(\mathbb{C}^2)$ | divisorial / irr. qm / other pts

TALK 4

$R \in \mathbb{C}_p(T) \quad \deg R \geq 2$

$R : \mathbb{P}^1(\mathbb{C}_p) \circlearrowleft$

interested in ergodic properties.

1) compute the topological entropy
2) construct good inv. measure

$\mathbb{P}^1(\mathbb{C}_p)$ not locally compact

$\Downarrow$

Better work on a bigger space

$R \quad \mathbb{P}^1(\mathbb{C}_p) \hookrightarrow \mathbb{P}^1_{Ber}(\mathbb{C}_p) \circlearrowleft R$

Let me emphasize that this idea of looking at the action of R on $\mathbb{P}^1_{Ber}(\mathbb{C}_p)$ in order to ~~study~~ deduce informations on the dynamics over $\mathbb{P}^1(\mathbb{C}_p)$ is due to J Rivera-Letelier in his PhD

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## 4. Action of $R$ on $\mathbb{P}^1_{Ber}(\mathbb{C}_p)$.

$\times$ on $\mathbb{P}^1(\mathbb{C}_p)$ clear

$\times$ on $\mathbb{H}_p \ni S$ norm extends to $\mathbb{C}_p(T)$.

$$R.(S)(P) = S(P \circ R)$$

$R : \mathbb{P}^1_{Ber}(\mathbb{C}_p) \to \mathbb{P}^1_{Ber}(\mathbb{C}_p)$ weakly continuous. Preserves the type!

To build measure → pull-back operator

$$\varphi \in \mathcal{B}^0\left(\mathbb{P}^1_{Ber}(\mathbb{C}_p)\right) \longrightarrow R_* \varphi(S) = \sum_{R(S')=S'} \deg_R(S')\, \varphi(S'').$$

pbm of local degree later.

| if $R(B) = B'$ then $\deg_R(S_B) = \deg(R: B \to B')$

$$|R_* \varphi|_\infty \leq D |\varphi|_\infty$$

$\downarrow$

$$\langle R^* \rho, \varphi \rangle = \langle \rho, R_* \varphi \rangle .$$

**Thm** JRL + F. $R \in \mathbb{C}_p(T)$.

$\exists$ proba measure $\rho \in \mathbb{P}^1_{Ber}(\mathbb{C}_p)$ does not charge pts in $\mathbb{P}^1(\mathbb{C}_p)$

① $\left.\begin{array}{l}\forall S \in \mathbb{H}_p \\ \forall \zeta \in \mathbb{P}^1(\mathbb{C}_p) \setminus \mathcal{E}\end{array}\right\} D^{-n} R^{n*}[S] \longrightarrow \rho$

② $\rho$ is mixing (ergodic). $R^* \rho = D \rho$.

| comments = also equid. ; not measure of max entr. ; entr. not $= \log D$. (see examples.)

| proof = as in the $\mathbb{C}$ case

→ define $\Delta$ on $\mathbb{P}^1_{Ber}(\mathbb{C}_p)$.

• $\deg_R(S)$ = same as in the $\mathbb{C}$ case

local topological degree of $R_* : U \to R(U)$

(small weak open sets).

$$\sum_{R(S')=S} \deg_R(S') = \deg_R(S)$$

• <u>comment</u> on entropy

× $z \mapsto z^2$ $\quad P^{-1} B(0,1) = B(0,1)$

so $S_{\mathfrak{0}}$ is $P$ - totally invariant

$\rho = [S_{\mathfrak{0}}]$.

→ in general $\rho$ charges pts (in $\mathbb{H}_p$)

→ $h_\rho(R) = 0 = h_{top}(R)$

× $R = \dfrac{\sum a_k T^k}{\sum b_k T^k}$ $\quad \max\{|a_k|, |b_k|\} = +1$

↓

$\bar{R} \in \bar{\mathbb{F}}_p(T)$ $\quad$ in general $\bar{R}$ can be a cst.

$\deg(\bar{R}) \geq 1 \iff R^{-1} S_{B(0,1)} = S_{B(0,1)}$

$[\deg_{S_{\mathfrak{0}}} = \deg(\bar{R})]$.

**def** Silverman: $R$ has good reduction if $\deg(\bar{R}) = \deg(R)$

⇕

$R^{-1} S_{\mathfrak{0}} = S_{\mathfrak{0}} \Rightarrow \rho = [S_0]$

**Thm**

× $R$ has good red. (after Möbius chge of coord.)

× $R$ charges a pt

× $h_\rho(R) = 0$

× $h_{top}(R) = 0$

Other examples → $h_{top}(R) \leq \log D$ but $\rho$ might not be of max. entropy

$\exists R \quad 0 < h_\rho = h_{top} < \log D$

## 5. Potential theory on trees = use to construct $\mu$.

situation is as follows.

$(\mathcal{T}, d)$ metric $\mathbb{R}$-tree $\qquad\qquad (\mathbb{H}_p, d)$

$\downarrow \qquad\qquad\qquad\qquad \downarrow$

$\overline{\mathcal{T}}$ completion $\qquad\qquad \mathbb{P}^1_{Berk}(\mathbb{C}_p)$.

<u>Aim</u> = define $\mathcal{S}$ class of fcts on $\mathcal{T}$

$\Delta : \mathcal{S} \to$ signed measures on $\overline{\mathcal{T}}$.

do not present an axiomatic view of $\Delta, \mathcal{S}$

I shall try to give examples of fcts on $\mathbb{H}_p$, with $\Delta$

in order to motivate the def of these spaces in the general case.

[0] $\Delta$ cst fct $\equiv 0$.

[1] $s_0$ •———• $s_1$ $\qquad g|_{[s_0, s_1]}(s) = d(s, s_0)$

locally ct outside

$$\Delta g = [s_0] - [s_1].$$

$\to$ let $s_1 \to \infty$ and take $s_0 = \overline{B}(0,1)$

$$g(z) = \mathrm{Log} \max\{|z|, 0\}.$$

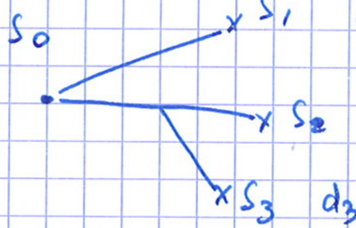
$$\Delta g = [s_0] - [\infty]$$

<u>recall</u> in the $\mathbb{C}$ case $\Delta g = \lambda_{S^1} - [\infty]$.

$\to$ $s_1 \to z_0$ $\quad s_0$ $\quad s_1 \to \infty$ $\qquad \Delta \mathrm{Log}|z - z_0| = [z_0] - [\infty]$

(Poincaré-Lelong formula)

$\boxed{3}$ $\quad D^{-1} R^{a} [S_0] - [S_0] = \Delta g$




$d_1 = d_{S_0}(S_1)$ $\quad -D \times g = d_1 \times d(\ , S_0)$ on $[S_0, S_1]$

$= (d_2 + d_3) \times d(\ . S_0)$

$$D^{-k} R^{*k} [S_0] - [S_0] = \Delta \sum_{0}^{k-1} \frac{g \circ R^{j}}{D^{j}} \searrow g_\infty$$

**Thm** — $g_k \in \mathcal{S}$ s.t. $[S_0] + \Delta g_k \geq 0$.

$\downarrow$

$g_\infty$ ptwise $\quad$ then $\quad$ 1) $g_\infty \in \mathcal{S}$

2) $[S_0] + \Delta g_k \to [S_0] + \Delta g_\infty$.

How to construct $(\Delta, \mathcal{S})$ satisfying all nice properties we want

3 approaches.

<u>A. Thuillier</u> $\quad$ on finite subtrees and "smooth" functions

$\downarrow$

(On any curves) $\quad$ by duality $\Delta$ on any function

like a distribution. $\mathcal{S}$ = set for which $\Delta g \geq 0$.

<u>B - Rumely</u> $\quad$ on finite subtrees define $\mathcal{S}$ and $\Delta$.

(appl. to graph theory) $\quad$ goes to $\mathcal{C}$ by viewing it as a union of its

finite subtree.

<u>F- Jonsson</u> $\quad$ work directly on $\mathcal{C}$.

~~Finite subtree~~

~~metrized.~~

~~$\Delta \rightsquigarrow$ mix of two operators~~

~~| · on segments looks like $\frac{d^2}{dx^2}$~~

~~| · at branched pts = discrete laplace operator~~

Segment = choose an orientation

(Look for $\int$ ... $\int \xrightarrow{\frac{d}{dx}} \mathcal{V} \xrightarrow{\frac{d}{dm}}$ measure

$\uparrow \quad |\cdot|$

$\rho \to f_\rho\{\sigma\} = \rho\{\tau \geq \sigma\}$.

Take the real line $\mathbb{R}_+$, standard metric

fct of bdd variation + left $C^0$ + 0 $\quad \sum |f(\sigma_i) - f(\sigma_{i+1})| \leq \infty$

$\Delta g$ measure on $\mathbb{R}_+ \cup \{\infty\}$ $\quad \int$ = integrals of fcts in $\mathcal{V}$.

$$g = cte + \int f\, dt. \qquad f \in \mathcal{V}.$$

$$\delta g = f \qquad \Delta g = d \circ \delta g$$

Main remark: change orientation does not change $\int$ !

General case ~~...~~ Work locally → define $\int, \Delta$ on a distinguished open set.

$$\int \; \underset{d}{\overset{S}{\rightleftarrows}} \; \mathcal{V} \; \underset{\delta}{\overset{I}{\leftrightarrows}} \; \mathcal{M}_0$$

bdd variation in a tree sense.

left $C^0$.

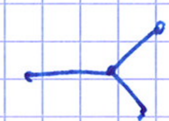
choose one scheme pt (as a root (or an orientation))

$$\Delta = - \delta \circ d$$

ex.

$f\, \mathbb{1}_{[s_0, s_1]} = 1$ and 0 outside.

ex. on a finite tree



| on each segment $-\frac{d^2}{dx^2}$

| at each branched pt $\sum$ outward derivative