Lecture 5 — Tuesday, Jan. 21

Theorem 9: $\Omega \subseteq \mathbb{C}^n$ connected open subset $(n \geq 2)$

TFAE:

(1) Ω is a domain of holomorphy.

(2) For all $K \subset\subset \Omega$ for all $f \in \mathcal{O}(\Omega)$.

$$\sup_K \frac{|f(z)|}{\operatorname{dist}(z, {}^c\Omega)} = \sup_{\hat{K}} \frac{|f(z)|}{\operatorname{dist}(z, {}^c\Omega)}$$

$$\hat{K}_\Omega = \{ z \in \Omega \mid |f(z)| \leq \sup_K |f| \quad \forall f \in \mathcal{O}(\Omega) \}$$

(3) for all $K \subset\subset \Omega$, $\hat{K}_\Omega \subset\subset \Omega$

(4) $\exists f \in \mathcal{O}(\Omega)$, $\forall p \in \partial\Omega$, $\forall r > 0$ there exists no $g \in \mathcal{O}(D^n(p, r))$ such that $g = f$ on some open

$$\emptyset \neq \omega \subseteq \Omega \cap D^n(p, r)$$

Observations: $(4) \Rightarrow (1)$

$(2) \Rightarrow (3)$

Take $f \equiv 1$ $\quad K \subset\subset \Omega$

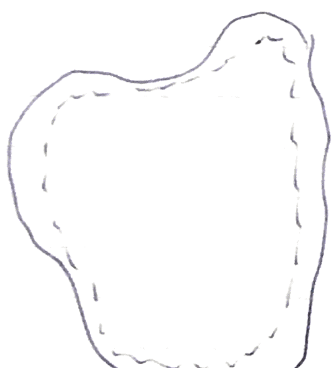
$$0 < \inf_{K} \operatorname{dist}(z, {}^c\Omega) = \inf_{\hat{K}_\Omega} \operatorname{dist}(z, {}^c\Omega)$$

$+\ \hat{K}_\Omega$ is closed in Ω

$\Rightarrow \hat{K}_\Omega$ is compact.

[Notation: $A \subset\subset B$ means A is relatively compact in B]

(3) $\Rightarrow$ (4) Build $f \in \mathcal{O}(\Omega)$ such that "$|f(z)| \to \infty$ as $z \to \partial\Omega$"



Ω

Choose an exhaustion by compact sets $K_n \subseteq K_{n+1} \subset \Omega$

Note that $\bigcup K_n = \Omega$.

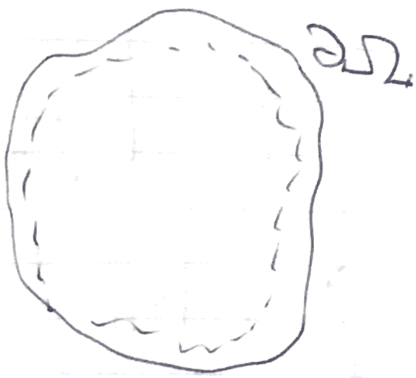
$$K_n = \overline{D(0,n)} \cap \{\operatorname{dist}(\_, {}^c\Omega) \geq \tfrac{1}{n}\}$$

By (3), $\hat{K}_{n,\Omega} \subseteq \hat{K}_{n+1,\Omega} \subseteq \Omega$

Replace K_n by $\hat{K}_{n+1}$.

$$K_n = \hat{K}_{n,\Omega} \subsetneq K_{n+1} \subset \Omega$$

Choose a sequence $\zeta_n \in K_{n+1} \setminus K_n$
such that $\{\zeta_n\} \supseteq \partial\Omega$



Take $f \in \mathcal{O}(\Omega)$ such that
$f(\zeta_n) \supseteq \partial\Omega$

$f = \sum f_n \qquad \|f_n(\zeta_n)\| \gg 1$

$$\sup_{K_n} \|f_n\| \ll 1$$

For each n, take $\tilde{f}_n \in \mathcal{O}(\Omega)$

$$|\tilde{f}_n(\zeta_n)| > \sup_{K_n} |\tilde{f}_n| \quad (\text{because } \zeta_n \notin \hat{K}_{n,\Omega})$$

Scale $\tilde{f}_n$ to get

$$|\tilde{f}_n(\zeta_n)| > 1 > \sup_{K_n} |\tilde{f}_n|$$

Take $f_n = \tilde{f}_n^{\alpha_n}$ where $\alpha_n \gg 0$.

We get: $\sup_{K_n} |f_n| \leq \frac{1}{2^n}$

$$\sum f_n(\zeta_n) \geq 1 + n + \sum_{i=0}^{n-1} |f_i(\zeta_n)|$$

Set $f = \sum f_n$. Note that:

- f converges absolutely on each K_m for every $m \geq 1 \Rightarrow f \in \mathcal{O}(\Omega)$

$$|f(\zeta_m)| = \left|\sum f_n(\zeta_m)\right|$$

$$= \Big|\sum_{n \geq m+1} + \underbrace{f_n(\zeta_m)}_{\gg 0} + \sum_{n \leq m-1}\Big|$$

$\zeta_m \in K_n$ / $n \geq m+1$. $\longrightarrow \leq 1$

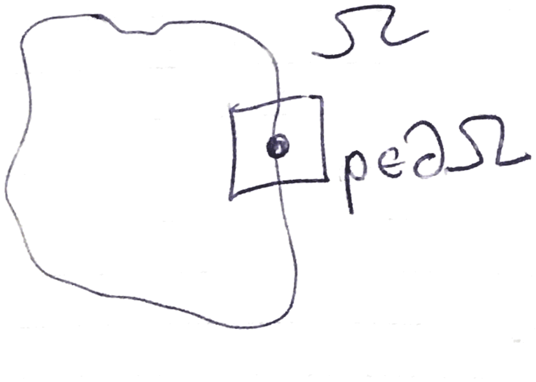
$\sum_{n \leq m-1} \longrightarrow \leq \sum_{n=0}^{m-1} |f_n(\zeta_n)|$

$$\geq |f_n(\zeta_m)| - 1 - \sum_0^{n-1} |f_n(\zeta_m)| \geq m$$

(3) $\Rightarrow$ (4)

__Aim__: Build $f \in \mathcal{O}(\Omega)$ such that $|f(\zeta_n)| \to \infty$ as $n \to \infty$.

$\Rightarrow$ f has no continuous extension through any point of $\partial\Omega$.



(1) $\Rightarrow$ (2) (Thullen)

Pick $K \subset \Omega$ compact, $f \in \mathcal{O}(\Omega)$.

Need to show that

$$\sup_{K} \underbrace{\frac{|f(z)|}{\operatorname{dist}(z, {}^c\Omega)}}_{\leq 1-\varepsilon} \geq \sup_{\hat{K}_\Omega} \frac{|f(z)|}{\operatorname{dist}(z, {}^c\Omega)} \quad (*)$$

by scaling.

<u>Lemma</u>: $\forall g \in \mathcal{O}(\Omega)$, $\forall w \in \hat{K}_\Omega$

the power series:

$$\sum_{\alpha \in \mathbb{N}^n} \frac{D^\alpha g(w)}{\alpha!}(z-w)^\alpha \quad \text{converges}$$

in $D^n(w, |f(w)|)$.

<u>Consequence</u>:

Since Ω is a domain of holomorphy,

$D^n(w, |f(w)|) \subseteq \Omega$

$\Rightarrow \operatorname{dist}(w, {}^c\Omega) \geq |f(w)|$

$$\forall w \in \hat{K}_\Omega, \quad \frac{|f(w)|}{\operatorname{dist}(w, {}^c\Omega)} \leq 1$$

Let $\varepsilon \to 0$ to get $(*)$

<u>Proof of Thullen's lemma</u>

Need to show that

$$\sup_{\alpha \in \mathbb{N}^n} \frac{|D^\alpha g(w)|}{\alpha!} |f(w)|^{|\alpha|} < C \quad \text{(bounded)}$$

$|\alpha| = \alpha_1 + \alpha_2 + \dots + \alpha_n$ where $\alpha = (\alpha_1, \dots, \alpha_n)$.

<u>Remark</u>: Only need to check that

$$\sup_K \frac{|D^\alpha g(w)|}{\alpha!} |f(w)|^{|\alpha|} < C$$

Indeed, for each α, $\frac{D^\alpha g(w)}{\alpha!} f(w)^{|\alpha|}$ is holomorphic with $\|\cdot\|_K \leq C$

$$\Rightarrow \|\cdot\|_{\hat{K}_\Omega} \leq C$$

Cauchy estimates:

$$|f(w)|^{|\alpha|} \frac{|D^\alpha g(w)|}{\alpha!} \leq \sup_{D(w, |f(w)|)} |g| \times C$$

Recall that in $\mathbb{C}$ (1-dim'l case)

$$\left|\frac{d^n g}{dz^n}\right|(w) \leq \frac{\sup_{\mathbb{D}(0,r)} |g|}{r^n} \times C$$

$$L = \bigcup_{w \in K} D^n(w, |f(w)|) \subset\subset \Omega$$

$$L \subseteq \{ \operatorname{dist}(z, \partial\Omega) \geq \varepsilon \operatorname{dist}(K, \partial\Omega) \}$$

$$\sup |f(w)|^\alpha \frac{|D g^\alpha(w)|}{\alpha!} \leq \sup_L |g| \times \text{Constant} < +\infty.$$

<u>Applications</u>:

- $\Omega, \Omega' \subset \mathbb{C}^n$
  $f: \Omega \to \Omega'$ is biholomorphism

Ω is domain of holomorphy $\underset{\text{use(3)}}{\Longleftrightarrow}$ Ω' is domain of holomorphy.

- $\Omega_1 \subseteq \mathbb{C}^{n_1}$, $\Omega_2 \subseteq \mathbb{C}^{n_2}$

Ω_1 & Ω_2 are domains of holomorphy
then $\Omega_1 \times \Omega$ is also a domain of holomorphy.

<u>Proof</u>: (Hint)

Take $K \subset\subset \Omega_1 \times \Omega_2$ | $K_1 = pr_{\Omega_1}(K) \subset\subset \Omega_1$

$K_2 = pr_{\Omega_2}(K) \subset\subset \Omega_2$

$\hat{K}_{1,\Omega_1}$ & $\hat{K}_{2,\Omega_2}$ are compact.

Check $\hat{K}_{\Omega} \subseteq \hat{K}_{1,\Omega_1} \times \hat{K}_{2,\Omega_2} \subset\subset \Omega_1 \times \Omega_2$

• $(\Omega_i)_{i \in \mathbb{N}}$ is a sequence of domains of holomorphy in $\mathbb{C}^n$.

$\Omega = \mathrm{In}\left(\bigcap_{i \in \mathbb{N}} \Omega_i\right)$ is a domain of holomorphy

<u>Proof</u> (hint): $K \subset\subset \Omega$ $\qquad \hat{K}_{\Omega} \subset\subset \Omega$

$\forall i$, $\hat{K}_{\Omega_i} \subset\subset \Omega_i$

$\mathrm{dist}(K, {}^c\Omega) \leq \mathrm{dist}(K, {}^c\Omega_i)$

$\overset{(2)}{=} \mathrm{dist}(\hat{K}_{\Omega_i}, {}^c\Omega_i)$

$\hat{K}_{\Omega} \subset \hat{K}_{\Omega_i}$ $\qquad \leq \mathrm{dist}(\cap \hat{K}_{\Omega_i}, {}^c\Omega_i)$

$\leq \mathrm{dist}(K, {}^c\Omega_i)$

Note that ${}^c\Omega = \overline{\bigcup {}^c\Omega_i}$

$0 < \mathrm{dist}(K, {}^c\Omega) \leq \mathrm{dist}(\hat{K}_\Omega, {}^c\Omega)$ //.

- Ω convex $\Rightarrow$ Ω domain of holom.

($K \subset\subset \Omega$, the usual standard convex hull

compact! $\rightarrow$ Convex hull$(K) \supseteq \hat{K}_\Omega$

Caratheodory's theorem)

- $f_1, f_2, \dots, f_m \in \mathcal{O}(\Omega)$ $\quad \Omega$ = domain of holom.

$\Omega' = \bigcap_{i=1}^{m} \{|f_i| < 1\}$ is a domain of holomorphy.

Proof: Enough to do the case $m = 1$

(for $m \geq 2$, we can apply the previous result on the intersections of domains of holomorphy)

$K \subset\subset \Omega'$

$$\hat{K}_{\Omega'} = \bigcap_{f \in \mathcal{O}(\Omega')} \{|f| \leq \sup_K |f|\}$$

$$\subseteq \{|f_1| \leq \sup_K |f_1|\}$$

$$\hat{K}_{\Omega'} \subseteq \underbrace{\{|f_1| \leq 1-\varepsilon\}}_{} \cap \hat{K}_\Omega^{1-\varepsilon}$$

If $\hat{K}_{\Omega'}$ is not compact, then it intersects $\partial\Omega'$ (in Ω), then we get a contradiction, $|f_1| = 1$

Theorem 10 (The case of Reinhardt domains)

$\mathcal{L}(z_1, \dots, z_n) = (\log|z_1|, \log|z_2|, \dots, \log|z_n|)$

Reinhardt $\Omega \subseteq \mathbb{C}^n$, $\Omega = \mathcal{L}^{-1}(D)$

where $D \subseteq \mathbb{R}^n$.

Theorem 10: Ω = Reinhardt domain $\ni 0$.

Then: Ω is a domain of holomorphy

$\Updownarrow$

Ω is complete and log-convex

$\Omega = \mathcal{L}^{-1}(D)$ where • D is convex

• $\xi \in D \Rightarrow \xi + \mathbb{R}^n_- \subseteq D$.

Proof: ($\Rightarrow$) Take $f \in \mathcal{O}(\Omega)$ that is not extendable through any point $p \in \partial\Omega$ (condition 4)

Theorem 6 $\Rightarrow$ $f = \sum a_\alpha z^\alpha$

$\exists$ $D(\sum a_\alpha z^\alpha) \supseteq \Omega$ (Reinhardt property)

$\Rightarrow$ $D(\sum a_\alpha z^\alpha) = \Omega$

$\underset{\text{thm 5}}{\Rightarrow}$ Ω is complete + log convex

$\Leftarrow$) $K \subseteq \Omega$ Can find $F \subseteq \Omega$ finite

$$K \subseteq \bigcup_{\xi \in F} \{ z \mid |z_i| \leq |\xi_j| \}$$

Ω = complete

$z \in \hat{K}_\Omega$ $\quad z = (z_1, z_2, \ldots, z_n)$

$$|z^\alpha| \leq \sup_K |\quad| \leq \sup_F |\xi|^\alpha$$

$$\sum \alpha_i \operatorname{Log}|z_i| \leq \sup_F \sum \alpha_i \operatorname{Log}|z_i| \quad \forall \alpha \in \mathbb{N}^n$$

<u>Claim</u>: $\bigcap_{\alpha \in \mathbb{N}^n} \{ (r_1, \ldots, r_n) \in \mathbb{R}^n : \sum \alpha_i r_i \leq \sup_F \sum \alpha_i \operatorname{Log}|z_i| \}$

$$= \text{Convex Hull} \bigcup_{\xi \in F} (\xi + \mathbb{R}^n_-)$$

$$= \text{Hull} (\mathcal{L}(F) + \mathbb{R}^n_-)$$

$$\Rightarrow \hat{K}_\Omega \subseteq \mathcal{L}^{-1}(\text{Hull of } \mathcal{L}(F) + \mathbb{R}^n_-) \subset\subset \Omega$$

Ω = log convex + complete. //

Lecture 6

Thursday, Jan 23

Continuing the proof from last time

Ω = Reinhardt domain in $\mathbb{C}^n$

We want to show:

Ω log-convex & complete $\Rightarrow$ Ω is a domain of holomorphy.

(we showed the reverse implication $\Leftarrow$ last time)

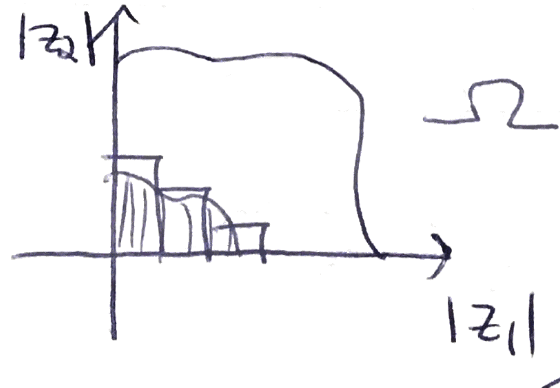
Let K be compact in Ω.
We want to show that

$\hat{K}_\Omega$ is compact in Ω.

- Take F = finite set in Ω such that

$$K \subseteq \bigcup_{\xi \in F} \{ |z_i| \le |\xi_i| \ \forall i \}$$

Ω = complete.



We will prove that the closure of $\hat{K}_\Omega$ is included in Ω (this will show that $\hat{K}_\Omega$ is compact in Ω). + bounded

$\overline{K} :=$ closure of $\hat{K}_{\Omega}$ in $\mathbb{C}^n$.

Take $z \in \overline{K}$, for all $\alpha \in \mathbb{N}^m$.

$$|z^\alpha| \leq \sup_F |\zeta|^\alpha$$

$$\sum_{i=1}^n \alpha_i \operatorname{Log}|z_i| \leq \sup_F \sum_{i=1}^n \alpha_i \operatorname{Log}|\zeta_i|$$

<u>Claim</u>: $\{ r = (r_1, r_2, \dots, r_n) \in \mathbb{R}^n$ such that

$$\sum \alpha_i r_i \leq \sup_F \sum_{i=1}^n \alpha_i \operatorname{Log}|\zeta_i| \}$$
$$\forall \alpha \in \mathbb{N}^n$$

$$= \text{Convex Hull} \left[\bigcup_{\zeta \in F} \log|\zeta| + \mathbb{R}^n_- \right]$$

$\Omega = \mathcal{L}^{-1}(D)$ where D is convex in $\mathbb{R}^n$ and complete in the sense that if $r \in D$ then $r + \mathbb{R}^n_- \subseteq D$.

$\Rightarrow \overline{K} \subseteq \mathcal{L}^{-1}(D) \subseteq \Omega$. $\square$

$\Omega \subseteq \mathbb{C}^n$ bounded domain with smooth boundary.

Suppose $\exists$ $r : \mathbb{C}^n \to \mathbb{R}$ $\mathcal{C}^\infty$-function such that $\Omega = \{r < 0\}$ and r is a submersion in a neighborhood of $r^{-1}(0)$.

We call r the defining equation.

<u>Theorem</u> (Levi, 1911)

Ω = bounded domain in $\mathbb{C}^n$ with smooth boundary.

If Ω is a domain of holomorphy, then

$$\forall p \in \partial\Omega, \quad \sum_{i,j=1,\dots,n} \frac{\partial^2 r}{\partial z_i \partial \bar{z}_j}(p)\lambda_i \bar{\lambda}_j \geq 0$$

"Levi condition"

whenever $\sum_{i=1}^{n} \frac{\partial r}{\partial z_i} \lambda_i = 0$

<u>Theorem</u> (Oka, Biermenman, Voguet)

Ω = domain with smooth boundary. Then:

Ω is a domain of holomorphy <u>if and only if</u> the Levi conditions are satisfied for any $p \in \partial\Omega$.

Our strategy will be:

Ω domain of holomorphy $\xrightarrow[\text{connect}]{}$ plurisubharmonic functions

Levi conditions $\xrightarrow[\text{connect}]{}$ Pseudo-convex functions.

Recall the following:

$\mathbb{D}^n(0,1) = \{ z = (z_1, \dots, z_n) \mid |z_i| < 1 \ \forall i \}$

unit polydisk.

$B^n(0,1) = \{ z = (z_1, \dots, z_n) \mid \sum_{i=1}^{n} |z_i|^2 < 1 \}$

unit ball.

Note that $\mathbb{D}^n(0,1) \subseteq \mathbb{C}^n$, $B^n(0,1) \subseteq \mathbb{C}^n$ are both open sets.

Note that $\mathbb{D}^1(0,1) = B^1(0,1)$, so the two sets coincide when $n=1$.

The situation is strikingly different when $n \geq 2$.

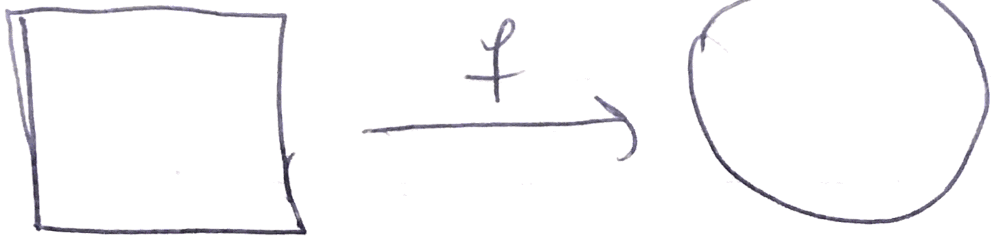
Thm (Rothstein) If $n \geq 2$, there is no proper holomorphic map $f: \mathbb{D}^n(0,1) \to B^n(0,1)$.

Cor: $\mathbb{D}^n(0,1)$ & $B^n(0,1)$ are not biholomorphic.

Proof: By contradiction, $n=2$.

$f: \mathbb{D}^n(0,1) \to B^n(0,1)$ is hol. and proper.

We want to show that f is actually constant (hence contradiction).



Intuition: boundary of polydisk is flat (ignoring corners) while boundary of the unit circle (in $\mathbb{C}^2$) is strictly smooth. (non-flat)
Let's exploit this condition.

Fix $e^{i\theta} \in S^1$. Take $\underset{D(0,1)}{\underset{\cap}{w_k}} \to e^{i\theta}$.

$g_k(\zeta) = f(\zeta, w_k)$: holom. $D(0,1) \to B^2(0,1)$

Cauchy Estimates:

$$\sup_{L \subset\subset D(0,1)} |g_k'| \leq \text{Constant} \cdot \sup_{D(0,1)} |g_k|$$

Ascola-Arzela
$\Longrightarrow$ There exist subsequence g_{k_m} that converge uniformly on compact subsets of $D(0,1)$ to $g: D(0,1) \to \overline{B^2(0,1)}$

Since f is proper, $\forall \varepsilon > 0$, $\exists N$ such that $k \geq n$, then $|g_k| \geq 1 - \varepsilon$.

$\Rightarrow g = (g_+, g_-)$. $\quad |g| = \sqrt{|g_+|^2 + |g_-|^2} = 1$

$\Rightarrow$ Claim g is constant.

<u>Proof</u>: Take an interior point where $\inf |g_+|$ is attained, by the maximum principle (applied to g_-), we get g_- is constant and so g_+ is also a constant. and so g is a constant. $\square$

Now g is a constant $\Rightarrow g' = 0$.

Take $|w_k| \to 1$. Consider $g_k(\zeta) = f(\zeta, w_k)$

Then there is a subsequence (g_{k_m}) such that $g'_{k_m} \to 0$.

- For any $\zeta \in \mathbb{D}(0,1)$,

$W \to \frac{\partial f}{\partial z_1}(\zeta, w)$ is a holomorphic function on $\mathbb{D}(0,1)$

and $\left|\frac{\partial f}{\partial z_1}(\zeta, w)\right| \xrightarrow[|w| \to 1]{} 0$.

Cauchy $\Rightarrow \frac{\partial f}{\partial z_1}(\zeta, w) = 0. \quad \forall \zeta, w$

Same argument implies $\frac{\partial f}{\partial z_2}(\zeta, w) = 0$

$\Rightarrow$ f is a constant. $\square$

Another approach to Rothstein's theorem (following Poincare - Cartan)

$\Omega \subseteq \mathbb{C}^n$ domain

$\mathrm{Bihol}(\Omega) = \{ f : \Omega \to \Omega \mid f$ is holomorphic and has a holomorphic inverse $\}$

Note: $\mathrm{Bihol}(\Omega)$ is a group under the composition law. In fact, $\mathrm{Bihol}(\Omega)$ is a topological group. The topology is given by compact-open topology: Basis of neighborhoods is given by

$$\mathcal{F}(\underset{\substack{\uparrow \\ \text{compact}}}{K}, \underset{\substack{\uparrow \\ \text{open}}}{V}) = \{ f \in \mathrm{Bihol}(\Omega) \;\; f(K) \subseteq \Omega \}$$

• $\mathrm{Bihol}(\mathbb{H}^2) = \left\{ (x,y) \mapsto \left(\frac{ax+by}{cx+d}, \frac{a'y+b'}{c'y+d'} \right) \right\}$

$\mathbb{H} = \{ \mathrm{Im} > 0 \}$ $\quad a,b,c,d,a',b',c',d' \in \mathbb{R}$

But $\mathbb{H}^2 \xrightarrow[\text{bihol.}]{} \mathbb{D}^2(0,1)$ $\quad ad-bc = a'd'-b'c' = +1$

Hence, $\rtimes \{ (x,y) \mapsto (y,x) \}$

$\mathrm{Bihol}(\mathbb{H}^2) \simeq \mathrm{Bihol}(\mathbb{D}^2(0,1))$

• Consequence,

$$\mathrm{Bihol}(\mathbb{D}^2(0,1)) \underset{\text{homeo.}}{\simeq} PSL(2,\mathbb{R}) \times PSL(2,\mathbb{R}) \times \underbrace{\{0,1\}}_{\mathbb{Z}/2\mathbb{Z}}$$

$$= (S^1)^2 \times \mathbb{D}(0,1)^2 \times \{0,1\}$$

(using the fact that $PSL(2,\mathbb{R})$ is a circle bundle: $e^{i\theta} \frac{z-a}{1-\bar{a}z}$, $|a|<1$

• $\mathrm{Bihol}(B^2(0,1)) = \{ \varphi = \underset{\substack{\text{unitary} \\ \mathcal{U}(2)}}{U} \circ \varphi_a \}$ $\quad a \in B^2(0,1)$

$\Rightarrow$ Consequence: $\mathrm{Bihol}(B^2(0,1))$ is connected

So it cannot be same as $\mathrm{Bihol}(\mathbb{D}^2(0,1))$

$$\varphi_a(z) := \frac{a - P_a(z) - s_a Q_a(z)}{1 - \langle z, a \rangle} \quad \text{for } a \neq 0$$

and $\varphi_0 = -\mathrm{id}$.

Here: $P_a(z) = a \frac{\langle z, a \rangle}{\langle a, a \rangle}$ projection

$Q_a = Id - P_a$, $s_a = (1-|a|^2)^{1/2}$.

One can compute:

$\varphi_a \in Bihol(B^2)$ ← this is the claim

$$1 - |\varphi_a|^2 = \frac{(1-|a|^2)(1-|z|^2)}{|1-\langle z, a \rangle|^2}$$

$|\varphi_a(z)| < 1 \Leftrightarrow |z| < 1$.

$\Rightarrow \varphi_a(B) \subseteq B$ and it remains to check that $\varphi_a \circ \varphi_a = id$.

Reference: Narasimhan (1980) Chicago Lectures.

Theorem ①: $\mathcal{U}$ bounded in $\mathbb{C}^n$

$f : \mathcal{U} \underset{\uparrow}{\to} \mathcal{U}$, $f(0) = 0$, $df(0) = id$
biholomorphic $\Rightarrow f = id$.

Theorem ② $\mathcal{U}$ bounded which is circular (this means that $z \in \mathcal{U} \Rightarrow e^{i\theta} \in \mathcal{U}$) ← weaker than Reinhardt.

$f : \mathcal{U} \underset{\uparrow}{\to} \mathcal{U}$, $f(0) = 0$
biholomorphic $\Rightarrow f$ is linear (complex-linear)

Dim $= 1 \to$ This is Schwarz Lemma.